Coherence time of a Bose-Einstein condensate in an isolated harmonically trapped gas

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Abstract

We study the condensate phase dynamics in a low-temperature equilibrium gas of weakly interacting bosons, harmonically trapped and isolated from the environment. We find that at long times, much longer than the collision time between Bogoliubov quasiparticles, the variance of the phase accumulated by the condensate grows with a ballistic term quadratic in time and a diffusive term affine in time. We give the corresponding analytical expressions in the limit of a large system, in the collisionless regime and in the ergodic approximation for the quasiparticle motion. When properly rescaled, they are described by universal functions of the temperature divided by the Thomas-Fermi chemical potential. The same conclusion holds for the mode damping rates. Such universality class differs from the previously studied one of the homogeneous gas.

Keywords: Bose gases; Bose-Einstein condensate; temporal coherence; trapped gases; ultracold atoms

1. Introduction and overview

We consider here an unsolved problem of the theory of quantum gases : the coherence time of a spinless boson gas in the weakly interacting regime, in a harmonic trap. The gas is prepared at thermal equilibrium at a temperature T much lower than the critical temperature T_c, that is in the strongly condensed regime, and it is perfectly isolated in its subsequent evolution. The coherence time of the bosonic field is then intrinsic and dominated by that of the condensate. In view of recent technical developments [1, 2, 3], this question could soon receive an experimental response in cold gases of atoms confined in non-dissipative magnetic potentials [4, 5, 6] and, unlike other solid state systems [7, 8, 9, 10], well decoupled from their environment and showing only low particle losses. Our theoretical study is also important for future applications in atom optics and matter wave interferometry.

Following the pioneering work of references [11, 12, 13], our previous studies [14, 15, 16, 17] performed in a spatially homogeneous boson gas, rely on the Bogoliubov method, which reduces the system to a weakly interacting quasiparticle gas. They have identified two mechanisms limiting the coherence time, both involving the dynamics of the phase operator $\hat{\theta}(t)$ of the condensate :

— *phase blurring* : when the conserved quantities (the energy E of the gas and its number of particles N) fluctuate from one experimental realization to another, the average rate of evolution of the phase $[\hat{\theta}(t) - \hat{\theta}(0)]/t$ in one realization, as a function of these conserved quantities, fluctuates too. After averaging over realizations, this induces a ballistic spread of the phase shift $\hat{\theta}(t) - \hat{\theta}(0)$, that is a quadratic divergence of its variance, with a ballistic coefficient A [14] :

$$\mathrm{Var}[\hat{\theta}(t) - \hat{\theta}(0)] \sim At^2 \tag{1}$$

this holds at long times with respect to $\gamma_{\mathrm{coll}}^{-1}$, where γ_{coll} is the typical collision rate between thermal Bogoliubov quasiparticles ;

— *phase diffusion* : even if the system is prepared in the microcanonical ensemble, where E and N are fixed, the interactions between quasiparticles cause their occupation numbers, and therefore the instantaneous speed $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta}$ of the phase, which depends on them, to fluctuate. This induces a diffusive spread of $\hat{\theta}(t) - \hat{\theta}(0)$ at the times $t \gg \gamma_{\mathrm{coll}}^{-1}$, with a coefficient D [15, 16] :

$$\mathrm{Var}_{\mathrm{mc}}[\hat{\theta}(t) - \hat{\theta}(0)] \sim 2Dt \tag{2}$$

In the general case, both mechanisms are present and the variance of the phase shift admits (1) as the dominant term, and (2) as a subdominant term. The condensate phase spreading directly affects its first-order temporal coherence function

$$g_1(t) = \langle \hat{a}_0^\dagger(t)\hat{a}_0(0)\rangle \tag{3}$$

where $\hat{a}_0$ is the annihilation operator of a boson in the condensate mode, thanks to the the approximate relation

$$g_1(t) \simeq \mathrm{e}^{-\mathrm{i}\langle\hat{\theta}(t)-\hat{\theta}(0)\rangle}\mathrm{e}^{-\mathrm{Var}[\hat{\theta}(t)-\hat{\theta}(0)]/2} \tag{4}$$

admitted in reference [17] under the hypothesis of a Gaussian distribution of $\hat{\theta}(t)-\hat{\theta}(0)$, then justified in reference [18] at sufficiently low temperature under fairly general assumptions. [1] We propose here to generalize these first studies to the experimentally more usual case of a harmonically trapped system (see however reference [19]). As the dependence of the damping rates of the Bogoliubov modes on the energy or the temperature are already very different from those of the homogeneous case, as it was shown in reference [20], it will certainly be the same for the spreading of the condensate phase. The trapped case is non-trivial, since the Bogoliubov modes are not known analytically, and there is no local density approximation applicable to the phase evolution (as verified by reference [21]). Fortunately we have the possibility to consider :

— the classical limit for the Bogoliubov quasiparticles motion in the trapped gas. Indeed, at the thermodynamic limit ($N \to +\infty$ with constant Gross-Pitaevskii's chemical potential μ_{GP} and constant temperature), the trapping angular frequencies ω_α, $\alpha \in \{x, y, z\}$, tend to zero as $1/N^{1/3}$:

$$\hbar\omega_\alpha \ll \mu_{\mathrm{GP}}, k_B T \tag{5}$$

so that we can cleverly reinterpret the thermodynamic limit as a classical limit $\hbar \to 0$;

— the limit of very weak interactions between Bogoliubov quasiparticles :

$$\gamma_{\mathrm{coll}} \ll \omega_\alpha \tag{6}$$

This implies that all the modes, even those of weaker angular frequency $\approx \omega_\alpha$, are in the collisionless regime (by opposition to hydrodynamics), and makes it possible to make a secular approximation on the kinetic equations describing the collisions between the quasiparticles ;

— ergodicity in a completely anisotropic trap : as shown by references [22, 23], the classical motion of quasiparticles in a non isotropic harmonic trap with cylindrical symmetry is highly chaotic at energies $\epsilon \approx \mu_{\mathrm{GP}}$ but almost integrable when $\epsilon \to 0$ or $\epsilon \to +\infty$. In a completely anisotropic trap, at temperatures neither too small nor too large with respect to μ_{GP}/k_B, we can hope to complete the secular approximation by the hypothesis of ergodicity, which we will endeavor to show.

Our article is articulated as follows. In section 2, after a few reminders about Bogoliubov's theory in a trap, we specify the state of the system and introduce the quantities to formally describe the phase spread, namely the derivative of the condensate phase operator and its time correlation function. In section 3, we give an expression of the ballistic coefficient A in the thermodynamic limit in any harmonic trap (including isotropic), first in the most general state of system considered here, then in the simpler case of a statistical mixture of canonical ensembles of the same temperature T. In the long section 4, we tackle the heart of the problem, calculating the correlation function $C_{\mathrm{mc}}(\tau)$ of $\mathrm{d}\hat{\theta}/\mathrm{d}t$ in the microcanonical ensemble, which gives access in general terms to the sub-ballistic

1. Let us recall the assumptions used in reference [18] to establish equation (4). (*i*) The relative fluctuations of the modulus of $\hat{a}_0$ are small, the system being strongly Bose condensed. (*ii*) The system is close enough to the thermodynamic limit, with normal fluctuations and asymptotically Gaussian laws for the energy and the number of particles. This is used in particular to put the ballistic contribution of the phase shift to $g_1(t)$ in the form (4). (*iii*) The diffusion coefficient of the phase (of order $1/N$) must be much smaller than the typical collision rate γ_{coll} between Bogolioubov quasi-particles (of order N^0) but much larger than the spacing of the energy levels (of order N^{-2}) of quasiparticle pairs created or annihilated during Beliaev collision processes. This is used, for a system prepared in the microcanonical ensemble, to show that $g_1(t)$ is of the form (4) on time intervals $t = O(N^0)$ and $t = O(N^1)$, with the same diffusion coefficient. (*iv*) The correlation function of $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta}$ is real, as predicted by kinetic equations. (*v*) We ignore the commutator of $\hat{\theta}(t)$ with $\hat{\theta}(0)$, which introduces a $O(t/N)$ phase error into the factor $\exp[-\mathrm{i}\langle\hat{\theta}(t) - \hat{\theta}(0)\rangle]$. This is an error of order unity at times $t \approx N$ but $g_1(t)$ then began to decrease under the effect of phase diffusion in the microcanonical ensemble (and is otherwise already very strongly damped under the effect of ballistic phase blurring after a time $t \approx N^{1/2}$).

spreading terms of the phase, since they are independent of the state of the system in the thermodynamic limit at fixed average energy and fixed average number of particles. We first introduce the semiclassical limit in subsection 4.1, the Bogoliubov quasiparticles motion being treated classically but the bosonic field of quasiparticles remaining quantum ; the semiclassical form of $\mathrm{d}\hat{\theta}/\mathrm{d}t$ is deduced from a correspondence principle. We then write, in subsection 4.2, kinetic equations on the quasiparticle occupation numbers in the classical phase space $(\mathbf{r}, \mathbf{p})$ and we show how, once linearized they formally lead to $C_{\mathrm{mc}}(\tau)$. The problem remains formidable, since the occupation numbers depend on the six variables $(\mathbf{r}, \mathbf{p})$ and time. In the secular limit $\gamma_{\mathrm{coll}} \ll \omega_\alpha$ and in the ergodic approximation for the quasiparticles motion (which excludes isotropic or cylindrically-symmetric traps), we reduce in subsection 4.3 to occupation numbers that are functions only of the energy of the classical motion ϵ and the time, which leads to explicit results on $C_{\mathrm{mc}}(\tau)$, on phase diffusion and, an interesting by-product, on the damping rate of Bogoliubov's modes in the trap, in subsection 4.4 where we also evaluate the condensate phase shift due to particle losses. Finally, we make a critical discussion of the ergodic approximation in subsection 4.5, estimating in particular the error that it introduces on the quantities controlling the phase diffusion of the condensate. We conclude in section 5.

2. Summary of the formalism and results

The derivative of the phase - As we recalled in the introduction, the coherence time of a condensate is controlled by the dynamics of its phase operator $\hat{\theta}(t)$ at times long with respect to the typical collision time $\gamma_{\mathrm{coll}}^{-1}$ of quasiparticles. The starting point of our study is therefore the expression of the temporal derivative of $\hat{\theta}(t)$, smoothed temporally (that is, coarse-grained over a short time with respect to $\gamma_{\mathrm{coll}}^{-1}$ but long with respect to the typical inverse frequency $\epsilon_k^{\mathrm{th}}/\hbar$ of thermal quasiparticles). As it has been established in all generality in reference [18], to order one in the non-condensed fraction :

$$-\hbar\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \mu_0(\hat{N}) + \sum_{k\in\mathcal{F}_+}\frac{\mathrm{d}\epsilon_k}{\mathrm{d}N}\,\hat{n}_k \equiv \hat{\mu} \tag{7}$$

Here $\mu_0(N)$ is the chemical potential of the gas in the ground state and $\hat{N}$ is the total number of particles operator. The sum over the generic quantum number k (it's not a wavenumber) deals with the Bogoliubov modes of eigenenergy ϵ_k, and $\hat{n}_k$ is the operator number of quasiparticles in the k mode. The expression (7) is a quantum version of the second Josephson relation : its right-hand side is a chemical potential operator $\hat{\mu}$ of the gas, since it is the adiabatic derivative (with the occupation numbers $\hat{n}_k$ fixed) with respect to N of the Bogoliubov Hamiltonian

$$\hat{H}_{\mathrm{Bog}} = E_0(\hat{N}) + \sum_{k\in\mathcal{F}_+}\epsilon_k\hat{n}_k \tag{8}$$

The Bogoliubov modes are of the are family $\mathcal{F}_+$, according to the terminology of reference [24] in the sense that their modal functions $(u_k(\mathbf{r})v_k(\mathbf{r}))$ are solutions of the eigenvalue equation

$$\epsilon_k\begin{pmatrix}|u_k\rangle\\|v_k\rangle\end{pmatrix} = \begin{pmatrix} H_{\mathrm{GP}} + Qg\rho_0(\hat{\mathbf{r}})Q & Qg\rho_0(\hat{\mathbf{r}})Q \\ -Qg\rho_0(\hat{\mathbf{r}})Q & -[H_{\mathrm{GP}} + Qg\rho_0(\hat{\mathbf{r}})Q]\end{pmatrix}\begin{pmatrix}|u_k\rangle\\|v_k\rangle\end{pmatrix} \equiv \mathcal{L}(\hat{\mathbf{r}},\hat{\mathbf{p}})\begin{pmatrix}|u_k\rangle\\|v_k\rangle\end{pmatrix} \tag{9}$$

with the normalization condition $\int \mathrm{d}^3r\,(|u_k(\mathbf{r})|^2 - |v_k(\mathbf{r})|^2) = 1 > 0$. We took the wave function $\phi_0(\mathbf{r})$ of the condensate real, normalized to one ($\int \mathrm{d}^3r\,\phi_0^2(\mathbf{r}) = 1$), and written to the order zero in the non-condensed fraction, that is to the Gross-Pitaevskii approximation :

$$H_{\mathrm{GP}}|\phi_0\rangle = 0 \quad \text{with} \quad H_{\mathrm{GP}} = \frac{\hat{\mathbf{p}}^2}{2m} + U(\hat{\mathbf{r}}) + g\rho_0(\hat{\mathbf{r}}) - \mu_{\mathrm{GP}} \tag{10}$$

so that at this order, the condensed density is $\rho_0(\mathbf{r}) = N\phi_0^2(\mathbf{r})$. Here, $g = 4\pi\hbar^2 a/m$ is the coupling constant, proportional to the s-wave scattering length a between bosons of mass m, and $U(\mathbf{r}) = \sum_\alpha m\omega_\alpha^2 r_\alpha^2/2$ is the trapping potential. The projector Q projects orthogonally to $|\phi_0\rangle$ and ensures that $|\phi_0\rangle\perp|u_k\rangle$ and $|\phi_0\rangle\perp|v_k\rangle$ as it should be [24]. Since the condensate is in its ground mode (ϕ_0 minimizes the Gross-Pitaevskii energy functional), the ϵ_k are positive.

The state of the system - Evaporative cooling in cold atom gases does not lead *a priori* to any of the usual ensembles of statistical physics. To cover all reasonable cases, we therefore suppose that the gas is prepared at time 0 in a

generalized ensemble, statistical mixture of eigenstates $|\psi_\lambda\rangle$ of the complete Hamiltonian $\hat{H}$, with N_λ particles and energy E_λ, hence of density operator

$$\hat{\sigma} = \sum_\lambda \Pi_\lambda |\psi_\lambda\rangle\langle\psi_\lambda| \tag{11}$$

with, as the only restriction, the existence of narrow laws on E_λ and N_λ, of variances and covariance not growing faster than the averages $\bar{E}$ and $\bar{N}$ in the thermodynamic limit.

Average phase shift - Let's average expression (7) in the steady state $|\psi_\lambda\rangle$. At the right-hand side appears the expectation in $|\psi_\lambda\rangle$ of the chemical potential operator. Because of the interactions between Bogoliubov quasiparticles, the N body system is expected to be ergodic in the quantum sense of the term, that is, to obey to the, so called in the Anglo-American literature, "Eigenstate Thermalisation Hypothesis" (see references [25, 26, 27]),

$$\langle\psi_\lambda|\hat{\mu}|\psi_\lambda\rangle = \mu_{\rm mc}(E_\lambda, N_\lambda) \tag{12}$$

where $\mu_{\rm mc}(E, N)$ is the chemical potential in the microcanonical ensemble of energy E with N particles. For a large system, it suffices to expand to first order in the fluctuations, to obtain :

$$\mu_{\rm mc}(E_\lambda, N_\lambda) = \mu_{\rm mc}(\bar{E}, \bar{N}) + (E_\lambda - \bar{E})\partial_E\mu_{\rm mc}(\bar{E}, \bar{N}) + (N_\lambda - \bar{N})\partial_N\mu_{\rm mc}(\bar{E}, \bar{N}) + O(1/\bar{N}) \tag{13}$$

It remains to average on the states $|\psi_\lambda\rangle$ with the weights Π_λ as in equation (11) to get the first brick of the time coherence function (4), that is the average phase shift :

$$\langle\hat{\theta}(t) - \hat{\theta}(0)\rangle = -\mu_{\rm mc}(\bar{E}, \bar{N})t/\hbar \tag{14}$$

with an error $O(1/\bar{N})$ on the coefficient of t.

Average quadratic phase shift - Proceeding in the same way for the second moment of the phase shift of the condensate, we find as it is written implicitly in [16, 18] that

$$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] = At^2 + 2\int_0^t \mathrm{d}\tau\,(t-\tau)\,\mathrm{Re}\,C_{\rm mc}(\tau) \tag{15}$$

with the ballistic coefficient

$$A = \mathrm{Var}[(N_\lambda - \bar{N})\partial_N\mu_{\rm mc}(\bar{E}, \bar{N}) + (E_\lambda - \bar{E})\partial_E\mu_{\rm mc}(\bar{E}, \bar{N})]/\hbar^2 \tag{16}$$

and the correlation function of the phase derivative in the microcanonical ensemble of energy $\bar{E}$ and $\bar{N}$ particles :

$$C_{\rm mc}(\tau) = \left\langle \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}(\tau)\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}(0)\right\rangle_{\rm mc} - \left\langle\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}\right\rangle^2_{\rm mc} \tag{17}$$

This completes our formal knowledge of $g_1(t)$.

In view of future experimental observations, however, it remains to calculate *explicitly* A and $C_{\rm mc}(\tau)$ for a harmonically trapped system. It will be necessary in particular to verify that $C_{\rm mc}(\tau)$ in the trapped case decreases fast enough so that one finds a diffusive law (2) as in the spatially homogeneous case.

3. Calculation of the ballistic coefficient in the phase shift variance

In the generalized statistical ensemble - To calculate the average phase shift (14) and the ballistic coefficient (16) in the general case, we must know the microcanonical chemical potential $\mu_{\rm mc}(\bar{E}, \bar{N})$ and its derivatives in the harmonic trap. At the thermodynamic limit, $\mu_{\rm mc}$ coincides with the chemical potential $\mu_{\rm can}$ in the canonical ensemble of temperature T and number of particles $\bar{N}$, more convenient to calculate, provided that the temperature T is adjusted so that there is equality of mean energies $E_{\rm can}(T, \bar{N})$ and $\bar{E}$. In other words,

$$\mu_{\rm mc}(E_{\rm can}(T, \bar{N}), \bar{N}) \sim \mu_{\rm can}(T, \bar{N}) \tag{18}$$

one just takes the derivative of this relation with respect to T or $\bar{N}$ to get the useful derivatives of $\mu_{\rm mc}$, then one replaces $E_{\rm can}$ by $\bar{E}$, to obtain :

$$\partial_E \mu_{\rm mc}(\bar{E},\bar{N}) \sim \frac{\partial_T \mu_{\rm can}(T,\bar{N})}{\partial_T E_{\rm can}(T,\bar{N})} \tag{19}$$

$$\partial_N \mu_{\rm mc}(\bar{E},\bar{N}) \sim \partial_N \mu_{\rm can}(T,\bar{N}) - \frac{\partial_N E_{\rm can}(T,\bar{N})}{\partial_T E_{\rm can}(T,\bar{N})}\partial_T \mu_{\rm can}(T,\bar{N}) \tag{20}$$

At the first order in the non-condensed fraction, the canonical chemical potential is deduced from the free energy F of the ideal gas of Bogoliubov quasiparticles of Hamiltonian (8) by the usual thermodynamic relation $\mu_{\rm can} = \partial_N F$. The free energy is a simple functional of the density of states $\rho(\epsilon)$ of quasiparticles,

$$F(T,\bar{N}) = E_0(\bar{N}) + k_B T \int_0^{+\infty} {\rm d}\epsilon\, \rho(\epsilon) \ln\left(1 - e^{-\beta\epsilon}\right) \tag{21}$$

with $\beta = 1/k_B T$. At the thermodynamic limit, the ground state energy E_0 of the gas in the harmonic trap is deduced from that of the homogeneous system [28] by a local density approximation, and the density of states $\rho(\epsilon)$ is obtained by taking the classical limit $\hbar \to 0$, thanks to inequality (5) [6] :

$$\rho(\epsilon) = \int \frac{{\rm d}^3 r {\rm d}^3 p}{(2\pi\hbar)^3} \delta(\epsilon - \epsilon(\mathbf{r},\mathbf{p})) \tag{22}$$

The classical Hamiltonian $\epsilon(\mathbf{r},\mathbf{p})$ is the positive eigenvalue of the Bogoliubov's 2×2 matrix of equation (9) with the position $\mathbf{r}$ and the momentum $\mathbf{p}$ treated classically [2] and the condensed density $\rho_0(\mathbf{r})$ written at the classical limit that is in the Thomas-Fermi approximation :

$$g\rho_0^{\rm TF}(\mathbf{r}) = \begin{cases} \mu_{\rm TF} - U(\mathbf{r}) \equiv \mu_{\rm loc}(\mathbf{r}) & \text{if} \quad U(\mathbf{r}) < \mu_{\rm TF} \\ 0 & \text{elsewhere} \end{cases} \tag{23}$$

Here, the Thomas-Fermi chemical potential, classical limit of $\mu_{\rm GP}$ of Gross-Pitaevskii, is

$$\mu_{\rm TF} = \frac{1}{2}\hbar\bar{\omega}[15\bar{N}a(m\bar{\omega}/\hbar)^{1/2}]^{2/5} \tag{24}$$

and $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$ is the geometric average of the trapping angular frequencies. We deduce that

$$\epsilon(\mathbf{r},\mathbf{p}) = \begin{cases} \left\{\frac{p^2}{2m}\left[\frac{p^2}{2m} + 2\mu_{\rm loc}(\mathbf{r})\right]\right\}^{1/2} & \text{if} \quad U(\mathbf{r}) < \mu_{\rm TF} \\ \frac{p^2}{2m} + U(\mathbf{r}) - \mu_{\rm TF} & \text{elsewhere} \end{cases} \tag{25}$$

The six-fold integral (22) has been calculated in reference [29]. [3] Here we give the result in a somewhat more compact form :

$$\rho(\epsilon) = \frac{\mu_{\rm TF}^2}{(\hbar\bar{\omega})^3} f(\check{\epsilon} \equiv \epsilon/\mu_{\rm TF}) \tag{26}$$

$$f(\check{\epsilon}) = \frac{1}{\pi}\left[-2\sqrt{2}\check{\epsilon}^2 \,{\rm acos}\,\frac{\check{\epsilon}-1}{(1+\check{\epsilon}^2)^{1/2}} + 2\sqrt{2}\check{\epsilon}\ln\frac{1+\sqrt{2\check{\epsilon}}+\check{\epsilon}}{(1+\check{\epsilon}^2)^{1/2}} + \sqrt{\check{\epsilon}}(5\check{\epsilon}-1) + (1+\check{\epsilon})^2\,{\rm acos}\,\frac{1}{(1+\check{\epsilon})^{1/2}}\right] \tag{27}$$

We finally obtain the canonical chemical potential

$$\mu_{\rm can}(T,\bar{N}) = \mu_0(\bar{N}) + \frac{6k_BT}{5\bar{N}}\left(\frac{\mu_{\rm TF}}{\hbar\bar{\omega}}\right)^3 \int_0^{+\infty} {\rm d}\check{\epsilon} f(\check{\epsilon}) \ln\left(1 - {\rm e}^{-\check{\beta}\check{\epsilon}}\right) + \frac{2\mu_{\rm TF}}{5\bar{N}}\left(\frac{\mu_{\rm TF}}{\hbar\bar{\omega}}\right)^3 \int_0^{+\infty} {\rm d}\check{\epsilon} \frac{f(\check{\epsilon})\check{\epsilon}}{{\rm e}^{\check{\beta}\check{\epsilon}} - 1} \tag{28}$$

2. The projector Q, projecting on a space of codimension one, can be omitted at the thermodynamic limit.

3. The case of an anisotropic harmonic trap comes down to the isotropic case treated in [29] by performing the change of variable (with unit Jacobian) $r_\alpha = \lambda_\alpha r'_\alpha$, with $\omega_\alpha\lambda_\alpha = \bar{\omega}$, such that $U(\mathbf{r}) = \frac{1}{2}m\bar{\omega}^2 r'^2$.

with the contribution of the ground state [6]

$$\mu_0(\bar{N}) = \mu_{\rm TF}\left[1 + \pi^{1/2}\left(\mu_{\rm TF}a^3/g\right)^{1/2}\right] \tag{29}$$

When one takes the derivative of (28) with respect to T and $\bar{N}$ to evaluate expressions (19) and (20), one will remember that $\check{\beta} = \mu_{\rm TF}/k_BT$ depends on $\bar{N}$ through $\mu_{\rm TF}$. For brevity, we do not give the result here.

In a slightly less general ensemble - A simpler expression[4] of the ballistic coefficient A can be obtained when the state of the system is a statistical mixture of canonical ensembles of the same temperature T but of variable number of particles. By expressing the various coefficients in (16, 19, 20) as derivatives of the free energy $F(T,\bar{N})$ with respect to $\bar{N}$ and T, and remembering the expression $\mathrm{Var}_{\rm can}E = k_BT^2\partial_T E_{\rm can}$ of the variance of the energy in the canonical ensemble, we find to the dominant order $1/\bar{N}$ that

$$A(T) = (\mathrm{Var}\,N)\left(\frac{\partial_N\mu_{\rm can}(T,\bar{N})}{\hbar}\right)^2 + \frac{k_BT^2\left[\partial_T\mu_{\rm can}(T,\bar{N})\right]^2}{\hbar^2\partial_T E_{\rm can}(T,\bar{N})} \tag{30}$$

At zero temperature, only the first term contributes, and we find the prediction of references [30, 31] pushed to order one in the non-condensed fraction $f_{\rm nc}$. A $T \neq 0$ but in the absence of fluctuations of N, only the second term contributes; it is none other than the ballistic coefficient $A_{\rm can}(T)$ in the canonical ensemble. In the validity regime of the Bogoliubov approximation, $f_{\rm nc} \ll 1$, the chemical potential $\mu_{\rm can}(T,\bar{N})$ of the gas remains close to that of a Thomas-Fermi pure condensate, so that

$$\partial_N\mu_{\rm can}(T,\bar{N}) = \partial_N\mu_{\rm TF} + O\left(\frac{f_{\rm nc}}{\bar{N}}\right) \tag{31}$$

$\partial_T\mu_{\rm can}(T,\bar{N})$ is immediately first-order in $f_{\rm nc}$, and the same goes for the second term in equation (30). It is therefore only for strongly subpoissonian fluctuations of N ($\mathrm{Var}\,N \ll \mathrm{Var}_{\rm Pois}N \equiv \bar{N}$) that the second term of (30), that is the effect of thermal fluctuations, is not dominated by the first one. Assuming this condition satisfied in the experiment, we represent in figure 1 the canonical coefficient $A_{\rm can}(T)$ scaled by the $A_{\rm Pois}$ value of A in a pure condensate with Poissonian fluctuations of N,

$$A_{\rm Pois} = \bar{N}\left(\frac{\partial_N\mu_{\rm TF}}{\hbar}\right)^2 \tag{32}$$

all divided by the small parameter of the Bogoliubov theory at zero temperature,[5] proportional to $f_{\rm nc}(T=0)$:

$$[\rho_0(\mathbf{0})a^3]^{1/2} = \frac{2\sqrt{2}}{15\pi^{1/2}\bar{N}}\left(\frac{\mu_{\rm TF}}{\hbar\bar{\omega}}\right)^3 \tag{33}$$

The ratio thus formed is a universal function of $k_BT/\mu_{\rm TF}$. From the low and high energy expansions of the quasiparticle density of states,

$$f(\check{\epsilon}) \underset{\check{\epsilon}\to 0}{=} \frac{32}{3\pi}\check{\epsilon}^{3/2} - 2\sqrt{2}\,\check{\epsilon}^2 + O(\check{\epsilon}^{5/2}) \tag{34}$$

$$f(\check{\epsilon}) \underset{\check{\epsilon}\to +\infty}{=} \frac{1}{2}\check{\epsilon}^2 + \check{\epsilon} + \frac{1}{2} + O(\check{\epsilon}^{-1/2}) \tag{35}$$

4. The general expression (16) of A is a little tricky to grasp. Since the energy of the ground state depends on N, fluctuations of N mechanically cause energy fluctuations. For example, if N fluctuates at $T=0$ (in each subspace of fixed N, the system is in the ground state), we can, to find $A(T=0)$ of equation (30) from equation (16), use the fact that $E_\lambda - \bar{E} = (N_\lambda - \bar{N})\mu_0(\bar{N}) + O(\bar{N}^0)$ and that $\partial_E\mu_{\rm mc}(\bar{E},\bar{N}) \underset{T\to 0}{\sim} -2/(25\bar{N})$, whose report in (20) gives $\partial_N\mu_{\rm mc}(\bar{E},\bar{N}) \underset{T\to 0}{\sim} \partial_N\mu_0(\bar{N}) + 2\mu_0(\bar{N})/(25\bar{N})$.

5. One sometimes prefers to take as a small parameter $1/[\rho_0(\mathbf{0})\xi^3]$, where the healing length ξ of the condensate at the center of the trap is such that $\hbar^2/(m\xi^2) = \mu_{\rm TF}$. One can easily go from one small parameter to the other using the relation $[\rho_0(\mathbf{0})a^3]^{1/2}\rho_0(\mathbf{0})\xi^3 = 1/(8\pi^{3/2})$.

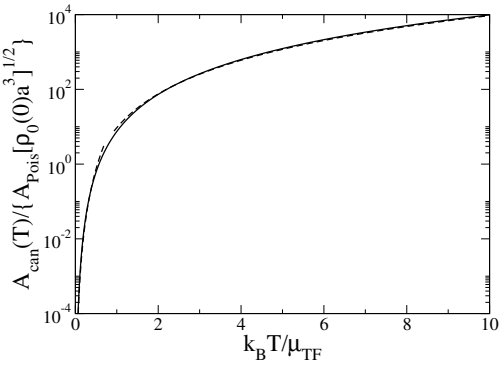

FIGURE 1: Coefficient of ballistic spreading (1) of the condensate phase in the long time limit with respect to the collision time γ_{coll}^{-1} of quasiparticles, for a gas of $\bar{N}$ bosons prepared in the canonical ensemble in an harmonic trap (isotropic or not), depending on the temperature. The result is at the thermodynamic limit where the trapping angular frequencies ω_α are negligible compared to Thomas-Fermi's chemical potential μ_{TF} (24). Full line : second term of equation (30), deduced from the canonical chemical potential (28) to the Bogoliubov approximation (weak interactions, $T \ll T_c$). Dashes : equivalents at low and high temperature (dominant terms of equations (36, 37)). The division of $A_{can}(T)$ by the small parameter (33) of Bogoliubov's theory and by the value (32) of the ballistic coefficient for Poissonian fluctuations of N leads to a universal function of k_BT/μ_{TF}.

we obtain low and high temperature expansions ($\check{T} = k_BT/\mu_{TF} = 1/\check{\beta}$)

$$\frac{A_{can}(T)}{A_{Pois}[\rho_0(\mathbf{0})a^3]^{1/2}} \underset{\check{T}\to 0}{=} \frac{21\zeta(7/2)}{\sqrt{2}}\check{T}^{9/2}\left[1+\frac{4\sqrt{2}\pi^{9/2}}{525\zeta(7/2)}\check{T}^{1/2}+O(\check{T})\right] \tag{36}$$

$$\underset{\check{T}\to+\infty}{=} \frac{15\pi^{1/2}}{2\sqrt{2}}\frac{3\zeta(3)^2}{4\zeta(4)}\check{T}^3\left[1+\check{\beta}\left(\frac{4\zeta(2)}{3\zeta(3)}-\frac{\zeta(3)}{2\zeta(4)}\right)+O(\check{\beta}^{3/2})\right] \tag{37}$$

whose dominant terms [6] are shown as dashed lines on figure 1. Let us note a particularly simple and beautiful reworking of the high temperature equivalent, accidentally already operational at $k_BT/\mu_{TF} \geq 2$:

$$\frac{A_{can}(T)}{A_{Pois}} \underset{k_BT\gg\mu_{TF}}{\sim} \frac{3\zeta(3)}{4\zeta(4)}\left(\frac{T}{T_c^{(0)}}\right)^3 \tag{38}$$

where $T_c^{(0)}$ is the critical temperature of an ideal gas of bosons in a harmonic trap at the thermodynamic limit, $k_BT_c^{(0)} = \hbar\bar{\omega}[\bar{N}/\zeta(3)]^{1/3}$. In this limit, $A_{can}(T)$ is therefore lower than A_{Pois} by a factor proportional to the non-condensed fraction $(T/T_c^{(0)})^3 \ll 1$.

4. Variance of the condensate phase shift in the microcanonical ensemble

Here we calculate the correlation function of $\mathrm{d}\hat{\theta}/\mathrm{d}t$, namely $C_{mc}(\tau)$, for a system prepared in the microcanonical ensemble, using at the thermodynamic limit $\frac{\hbar\omega_\alpha}{\mu_{TF}} \to 0$ a semiclassical description of the quasiparticles and taking into account the effect of their interaction by quantum Boltzmann kinetic equations on their classical phase space distribution $n(\mathbf{r},\mathbf{p})$.

4.1. Semi-classical form of Bogoliubov's Hamiltonian and $\mathrm{d}\hat{\theta}/\mathrm{d}t$

In the semiclassical description, the motion of Bogoliubov quasiparticles is treated *classically*, that is that they have at each moment a well defined position $\mathbf{r}$ and momentum $\mathbf{p}$ [6], whose evolution in phase space derives from the Hamiltonian $\epsilon(\mathbf{r},\mathbf{p})$ given in equation (25) [22] :

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \partial_{\mathbf{p}}\,\epsilon(\mathbf{r},\mathbf{p}) \tag{39}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\partial_{\mathbf{r}}\,\epsilon(\mathbf{r},\mathbf{p}) \tag{40}$$

6. In the value window of figure 1, in practice $1/10 \leq \check{T} \leq 10$, the inclusion of subdominant terms does not usefully approximate the exact result.

but we treat *in a quantum way* the bosonic field of quasiparticles by introducing their occupation numbers operators $\hat{n}(\mathbf{r},\mathbf{p})$ in the phase space, which allows us to take into account the discrete nature of the numbers of quasiparticles and the quantum statistical effects (with the Bose law rather than the equipartition law of the classical field at equilibrium).

In this semiclassical limit, the Bogoliubov Hamiltonian (8) (without interaction between quasiparticles) is written immediately

$$H_{\rm Bog}^{\rm sc} = E_0(\hat{N}) + \int \frac{{\rm d}^3r\,{\rm d}^3p}{(2\pi\hbar)^3}\,\epsilon(\mathbf{r},\mathbf{p})\,\hat{n}(\mathbf{r},\mathbf{p}) \tag{41}$$

One might think, given formula (7), that ${\rm d}\hat{\theta}/{\rm d}t$ admits a similar writing, with $\epsilon(\mathbf{r},\mathbf{p})$ replaced by $\frac{\rm d}{{\rm d}N}\epsilon(\mathbf{r},\mathbf{p})$. This is not so, the reason being that the derivative $\frac{\rm d}{{\rm d}N}\epsilon(\mathbf{r},\mathbf{p})$ is not constant on the classical trajectory. The ${\rm d}\hat{\theta}/{\rm d}t$ operator is part of a general class of so-called Fock quantum observables (diagonal in the Fock basis of quasiparticles thus - here linear - functionals of Bogoliubov's occupation numbers) :

$$\hat{A} = \sum_{k\in\mathcal{F}_+} a_k\hat{n}_k \qquad \text{with} \quad a_k = (\langle u_k|,\langle v_k|)\,\mathcal{A}(\hat{\mathbf{r}},\hat{\mathbf{p}})\begin{pmatrix}|u_k\rangle\\|v_k\rangle\end{pmatrix} \tag{42}$$

where $\mathcal{A}(\hat{\mathbf{r}},\hat{\mathbf{p}})$ is a 2×2 hermitian matrix operator and a_k its average in the Bogoliubov mode of eigenenergy ϵ_k. The observable ${\rm d}\hat{\theta}/{\rm d}t$ corresponds to the choice $\mathcal{A}_{\dot\theta} = \sigma_z\frac{\rm d}{{\rm d}N}\mathcal{L}$ where σ_z is the third Pauli matrix and $\mathcal{L}(\hat{\mathbf{r}},\hat{\mathbf{p}})$ is the operator appearing in equation (9). By using Hellmann-Feynman's theorem [7], we have indeed

$$(\langle u_k|,-\langle v_k|)\left(\frac{\rm d}{{\rm d}N}\,\mathcal{L}\right)\begin{pmatrix}|u_k\rangle\\|v_k\rangle\end{pmatrix} = \frac{{\rm d}\epsilon_k}{{\rm d}N} \tag{43}$$

For these Fock operators we use the semiclassical correspondence principle

$$\hat{A}^{\rm sc} = \int \frac{{\rm d}^3r\,{\rm d}^3p}{(2\pi\hbar)^3}\,\overline{a(\mathbf{r},\mathbf{p})}\,\hat{n}(\mathbf{r},\mathbf{p}) \tag{44}$$

where $a(\mathbf{r},\mathbf{p}) = (U(\mathbf{r},\mathbf{p}),V(\mathbf{r},\mathbf{p}))\,\mathcal{A}(\mathbf{r},\mathbf{p})\binom{U(\mathbf{r},\mathbf{p})}{V(\mathbf{r},\mathbf{p})}$, $\mathcal{A}(\mathbf{r},\mathbf{p})$ being the classical equivalent of $\mathcal{A}(\hat{\mathbf{r}},\hat{\mathbf{p}})$, and $\overline{a(\mathbf{r},\mathbf{p})}$ represents the time average of $a(\mathbf{r},\mathbf{p})$ on the only classical trajectory passing through $(\mathbf{r},\mathbf{p})$ at time $t=0$:

$$\overline{a(\mathbf{r},\mathbf{p})} \equiv \lim_{t\to+\infty}\frac{1}{t}\int_0^t {\rm d}\tau\,a(\mathbf{r}(\tau),\mathbf{p}(\tau)) \tag{45}$$

The vector $(U(\mathbf{r},\mathbf{p}),V(\mathbf{r},\mathbf{p}))$, normalized according to the condition $U^2-V^2=1$, is eigenvector of the classical equivalent $\mathcal{L}(\mathbf{r},\mathbf{p})$ of $\mathcal{L}(\hat{\mathbf{r}},\hat{\mathbf{p}})$ with eigenvalue $\epsilon(\mathbf{r},\mathbf{p})$:

$$\begin{pmatrix}U(\mathbf{r},\mathbf{p})\\V(\mathbf{r},\mathbf{p})\end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{1}{2}\left[\left(\frac{p^2/2m}{\epsilon(\mathbf{r},\mathbf{p})}\right)^{1/2}+\left(\frac{p^2/2m}{\epsilon(\mathbf{r},\mathbf{p})}\right)^{-1/2}\right] \\ \frac{1}{2}\left[\left(\frac{p^2/2m}{\epsilon(\mathbf{r},\mathbf{p})}\right)^{1/2}-\left(\frac{p^2/2m}{\epsilon(\mathbf{r},\mathbf{p})}\right)^{-1/2}\right] \end{pmatrix} & \text{if} \quad U(\mathbf{r})<\mu_{\rm TF} \\ \begin{pmatrix}1\\0\end{pmatrix} & \text{elsewhere} \end{cases} \tag{46}$$

At the basis of this correspondence principle lies the idea that the equivalent of a stationary quantum mode $(|u_k\rangle,|v_k\rangle)$ in the classical world is a classical trajectory of the same energy, itself also stationary as a whole by temporal evolution. To the quantum expectation a_k of the observable $\mathcal{A}(\hat{\mathbf{r}},\hat{\mathbf{p}})$ in the mode $(|u_k\rangle,|v_k\rangle)$ thus we must associate *an average*

7. The theorem is here generalized to the case of a non-Hermitian operator $\mathcal{L}$, $(\langle u_k|,-\langle v_k|)$ being the dual vector of the eigenvector $(|u_k\rangle,|v_k\rangle)$ of $\mathcal{L}$.

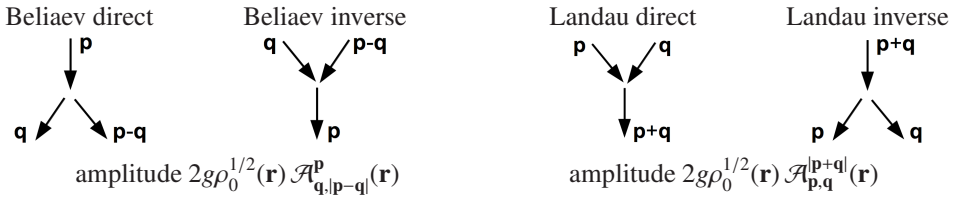

FIGURE 2: Beliaev and Landau process involving three quasiparticles and the corresponding coupling amplitudes.

over a trajectory of the expectation $a(\mathbf{r},\mathbf{p})$ of the classical equivalent $\mathcal{A}(\mathbf{r},\mathbf{p})$ in the local mode $(U(\mathbf{r},\mathbf{p}), V(\mathbf{r},\mathbf{p}))$. We therefore retain for the semiclassical version of the derivative of the condensate phase operator :

$$-\hbar\frac{\mathrm{d}\hat{\theta}^{\mathrm{sc}}}{\mathrm{d}t} = \mu_0(\hat{N}) + \int \frac{\mathrm{d}^3r\,\mathrm{d}^3p}{(2\pi\hbar)^3}\,\overline{\frac{\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})}{\mathrm{d}N}}\,\hat{n}(\mathbf{r},\mathbf{p}) \tag{47}$$

Here, let us repeat it, the expectation $a(\mathbf{r},\mathbf{p}) = \frac{\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})}{\mathrm{d}N}$ is not a constant of motion, unlike $\epsilon(\mathbf{r},\mathbf{p})$, so we can not omit as in (41) the temporal average.

4.2. About the usefulness of kinetic equations in calculating the correlation function of $\mathrm{d}\hat{\theta}/\mathrm{d}t$

We must determine, in the semiclassical limit, the correlation function of $\mathrm{d}\hat{\theta}/\mathrm{d}t$, for a system prepared in the microcanonical ensemble. Given equations (17) and (47) we must calculate

$$C^{\mathrm{sc}}_{\mathrm{mc}}(\tau) = \int \frac{\mathrm{d}^3r\,\mathrm{d}^3p}{(2\pi\hbar)^3}\int \frac{\mathrm{d}^3r'\,\mathrm{d}^3p'}{(2\pi\hbar)^3}\,\overline{\frac{\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})}{\hbar\mathrm{d}N}}\,\overline{\frac{\mathrm{d}\epsilon(\mathbf{r}',\mathbf{p}')}{\hbar\mathrm{d}N}}\,\langle\delta\hat{n}(\mathbf{r},\mathbf{p},\tau)\,\delta\hat{n}(\mathbf{r}',\mathbf{p}',0)\rangle \tag{48}$$

where $\langle\ldots\rangle$ represents the mean in the state of the system and where we introduced the fluctuations of the occupation number operators in phase space at time τ,

$$\delta\hat{n}(\mathbf{r},\mathbf{p},\tau) = \hat{n}(\mathbf{r},\mathbf{p},\tau) - \bar{n}(\mathbf{r},\mathbf{p}) \tag{49}$$

The microcanonical ensemble can be seen in the semiclassical phase space as a constant-energy statistical mixture of Fock states $|\mathcal{F}\rangle = |n(\mathbf{r}'',\mathbf{p}'')_{(\mathbf{r}'',\mathbf{p}'')\in\mathbb{R}^6}\rangle$, eigenstates of $H^{\mathrm{sc}}_{\mathrm{Bog}}$, where all $n(\mathbf{r}'',\mathbf{p}'')$ are integers. It is assumed at first that the system is prepared in one of such Fock states $|\mathcal{F}\rangle$ at the initial time $t = 0$, eigenstate of $\delta\hat{n}(\mathbf{r}',\mathbf{p}',0)$ with the eigenvalue $n(\mathbf{r}',\mathbf{p}') - \bar{n}(\mathbf{r}',\mathbf{p}')$; it remains then to calculate in equation (48) the quantity

$$\langle\mathcal{F}|\delta\hat{n}(\mathbf{r},\mathbf{p},\tau)|\mathcal{F}\rangle = n(\mathbf{r},\mathbf{p},\tau) - \bar{n}(\mathbf{r},\mathbf{p}) \equiv \delta n(\mathbf{r},\mathbf{p},\tau) \tag{50}$$

at $\tau > 0$, that is the evolution of the mean occupation numbers $n(\mathbf{r},\mathbf{p},\tau)$ in phase space, their initial values being known, taking into account (*i*) the Hamiltonian quasiparticle transport and (*ii*) the effect of quasiparticle collisions by the Beliaev or Landau three-quasiparticles processes [8] represented in figure 2. This is exactly what the usual Boltzmann-type quantum kinetic equations can do, with the difference that the semiclassical distribution function $n(\mathbf{r},\mathbf{p},\tau)$ does not correspond here to a local thermal equilibrium state of the system, but to the mean occupation number at time τ knowing that the initial state of the system is a quasiparticle Fock state. The evolution equation of the average occupation numbers $n(\mathbf{r},\mathbf{p},\tau)$ is of the form

$$\frac{\mathrm{D}}{\mathrm{D}\tau}n(\mathbf{r},\mathbf{p},\tau) + I_{\mathrm{coll}}(\mathbf{r},\mathbf{p},\tau) = 0 \tag{51}$$

The first term is the convective derivative resulting from the classical Hamilton equations :

$$\frac{\mathrm{D}}{\mathrm{D}\tau} = \partial_\tau + \partial_{\mathbf{p}}\epsilon(\mathbf{r},\mathbf{p})\cdot\partial_{\mathbf{r}} - \partial_{\mathbf{r}}\epsilon(\mathbf{r},\mathbf{p})\cdot\partial_{\mathbf{p}} \tag{52}$$

8. Four-quasiparticle processes, of higher order in the non-condensed fraction, are assumed here negligible.

It preserves the density in the phase space along a classical trajectory (Liouville's theorem). The second term describes the effect of collisions between quasiparticles, local in position space, and which can only occur, at the order of Beliaev-Landau, at points where the Thomas-Fermi density of the condensate $\rho_0(\mathbf{r})$ is nonzero (see the diagrams in figure 2) : [9]

$$\begin{aligned} I_{\rm coll}(\mathbf{r},\mathbf{p},\tau) &= \frac{1}{2}\int \frac{\mathrm{d}^3q}{(2\pi\hbar)^3}\,\frac{2\pi}{\hbar}\left[2g\rho_0^{1/2}(\mathbf{r})\,\mathcal{A}^{\mathbf{p}}_{\mathbf{q},|\mathbf{p}-\mathbf{q}|}(\mathbf{r})\right]^2\delta\left(\epsilon(\mathbf{r},\mathbf{q})+\epsilon(\mathbf{r},\mathbf{p}-\mathbf{q})-\epsilon(\mathbf{r},\mathbf{p})\right) \\ &\quad\times\{-n(\mathbf{r},\mathbf{p},\tau)[1+n(\mathbf{r},\mathbf{q},\tau)][1+n(\mathbf{r},\mathbf{p}-\mathbf{q},\tau)]+n(\mathbf{r},\mathbf{q},\tau)n(\mathbf{r},\mathbf{p}-\mathbf{q},\tau)[1+n(\mathbf{r},\mathbf{p},\tau)]\} \\ &+ \int \frac{\mathrm{d}^3q}{(2\pi\hbar)^3}\,\frac{2\pi}{\hbar}\left[2g\rho_0^{1/2}(\mathbf{r})\,\mathcal{A}^{|\mathbf{p}+\mathbf{q}|}_{\mathbf{p},\mathbf{q}}(\mathbf{r})\right]^2\delta\left(\epsilon(\mathbf{r},\mathbf{p})+\epsilon(\mathbf{r},\mathbf{q})-\epsilon(\mathbf{r},\mathbf{p}+\mathbf{q})\right) \\ &\quad\times\{-n(\mathbf{r},\mathbf{p},\tau)n(\mathbf{r},\mathbf{q},\tau)[1+n(\mathbf{r},\mathbf{p}+\mathbf{q},\tau)]+n(\mathbf{r},\mathbf{p}+\mathbf{q},\tau)[1+n(\mathbf{r},\mathbf{p},\tau)][1+n(\mathbf{r},\mathbf{q},\tau)]\} \end{aligned} \tag{53}$$

In this process are involved, at point $\mathbf{r}$, a quasiparticle of momentum $\mathbf{p}$ (whose evolution of the average occupation number $n(\mathbf{r},\mathbf{p}\,\tau)$ has to be determined), a second outgoing or incoming quasiparticle of momentum $\mathbf{q}$ on which it is necessary to integrate, and a third quasiparticle whose momentum is fixed by momentum conservation. In equation (53) the first integral takes into account the Beliaev processes; it shows a factor of $1/2$ to avoid double counting of the final or initial two quasiparticle states $(\mathbf{q},\mathbf{p}-\mathbf{q})$ and $(\mathbf{p}-\mathbf{q},\mathbf{q})$; the second integral takes into account the Landau processes. Note in both cases : (*i*) the factor $\frac{2\pi}{\hbar}$, from Fermi's golden rule, (*ii*) the $-$ sign taking direct processes into account (they depopulate the $\mathbf{p}$ mode at point $\mathbf{r}$) and the $+$ sign for inverse processes, with the bosonic amplification factors $1+n$, (*iii*) the presence of an energy conservation Dirac function at $\mathbf{r}$. The reduced coupling amplitudes for three-quasiparticles processes at point $\mathbf{r}$ are given by [14, 32]

$$\mathcal{A}^{\mathbf{p}_1}_{\mathbf{p}_2,\mathbf{p}_3}(\mathbf{r}) = \frac{s^2(\mathbf{r},\mathbf{p}_2)+s^2(\mathbf{r},\mathbf{p}_3)-s^2(\mathbf{r},\mathbf{p}_1)}{4s(\mathbf{r},\mathbf{p}_1)s(\mathbf{r},\mathbf{p}_2)s(\mathbf{r},\mathbf{p}_3)}+\frac{3}{4}s(\mathbf{r},\mathbf{p}_1)s(\mathbf{r},\mathbf{p}_2)s(\mathbf{r},\mathbf{p}_3) \tag{54}$$

with $s(\mathbf{r},\mathbf{p}) = U(\mathbf{r},\mathbf{p})+V(\mathbf{r},\mathbf{p})$. As expected, the kinetic equations admit the average thermal equilibrium occupation numbers as a stationary solution [10]

$$\bar{n}(\mathbf{r},\mathbf{p}) = \frac{1}{\mathrm{e}^{\beta\epsilon(\mathbf{r},\mathbf{p})}-1} \tag{55}$$

The well-known property of the Bose law $1+\bar{n} = \mathrm{e}^{\beta\epsilon}\bar{n}$ allows to verify it easily : supplemented with energy conservation, it leads to the perfect compensation in all points of direct and inverse processes, that is to the cancellation of the quantities between curly brackets in equation (53), following the principle of microreversibility; we also have $\frac{\mathrm{D}}{\mathrm{D}\tau}\bar{n} = 0$ since $\bar{n}(\mathbf{r},\mathbf{p})$ is a function of $\epsilon(\mathbf{r},\mathbf{p})$, a quantity that is conserved by the Hamiltonian transport.

As our system fluctuates weakly around the equilibrium, we linearise the kinetic equations around $n=\bar{n}$ as in reference [16] to get

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\delta n(\mathbf{r},\mathbf{p},\tau) = -\Gamma(\mathbf{r},\mathbf{p},\tau)\delta n(\mathbf{r},\mathbf{p},\tau)+\int\frac{\mathrm{d}^3q}{(2\pi\hbar)^3}K(\mathbf{r},\mathbf{p},\mathbf{q})\delta n(\mathbf{r},\mathbf{q},\tau) \tag{56}$$

the diagonal term comes from the fluctuation $\delta n(\mathbf{r},\mathbf{p},\tau)$ in the right-hand side of equation (53), and the non-local momentum term comes from fluctuations $\delta n(\mathbf{r},\mathbf{q},\tau)$ and $\delta n(\mathbf{r},\mathbf{p}\pm\mathbf{q},\tau)$ whose contributions are collected by changing the variables $\mathbf{q}' = \mathbf{p}\pm\mathbf{q}$ in $\int \mathrm{d}^3q$. The expression of $K(\mathbf{r},\mathbf{p},\mathbf{q})$ is not useful for the following, so let us only give the expression of the local damping rate of the Bogoliubov quasiparticles of momentum $\mathbf{p}$ at point $\mathbf{r}$:

$$\begin{aligned} \Gamma(\mathbf{r},\mathbf{p}) &= \frac{4\pi\rho_0(\mathbf{r})g^2}{\hbar}\int\frac{\mathrm{d}^3q}{(2\pi\hbar)^3}\,\left[\mathcal{A}^{\mathbf{p}}_{\mathbf{q},|\mathbf{p}-\mathbf{q}|}(\mathbf{r})\right]^2\delta\left(\epsilon(\mathbf{r},\mathbf{q})+\epsilon(\mathbf{r},\mathbf{p}-\mathbf{q})-\epsilon(\mathbf{r},\mathbf{p})\right)\left[1+\bar{n}(\mathbf{r},\mathbf{q})+\bar{n}(\mathbf{r},\mathbf{p}-\mathbf{q})\right] \\ &+ \frac{8\pi\rho_0(\mathbf{r})g^2}{\hbar}\int\frac{\mathrm{d}^3q}{(2\pi\hbar)^3}\,\left[\mathcal{A}^{|\mathbf{p}+\mathbf{q}|}_{\mathbf{p},\mathbf{q}}(\mathbf{r})\right]^2\delta\left(\epsilon(\mathbf{r},\mathbf{p})+\epsilon(\mathbf{r},\mathbf{q})-\epsilon(\mathbf{r},\mathbf{p}+\mathbf{q})\right)\left[\bar{n}(\mathbf{r},\mathbf{q})-\bar{n}(\mathbf{r},\mathbf{p}+\mathbf{q})\right] \end{aligned} \tag{57}$$

9. These diagrams imply a hidden process of absorption or stimulated emission in the condensate mode.

10. Strictly speaking, this stationary solution corresponds to the average occupation numbers in the canonical ensemble, rather than in the microcanonical one. The difference, computable as in Appendix C of reference [16], but out of reach of our kinetic equations, tends to zero at the thermodynamic limit and is negligible here. It should also be noted that the non-conservation of the total number of quasiparticles by the Beliaev-Landau processes requires the Bose's $\bar{n}$ law to have unit fugacity.

This expression coincides with the damping rate of a momentum $\mathbf{p}$ mode in a spatially homogeneous condensed gas of density $g\rho_0(\mathbf{r})$ [32]. Just like $\delta n(\mathbf{r},\mathbf{p},\tau)$, $\langle\mathcal{F}|\delta n(\mathbf{r},\mathbf{p},\tau)\delta n(\mathbf{r}',\mathbf{p}',0)|\mathcal{F}\rangle$ considered as a function of $(\mathbf{r},\mathbf{p},\tau)$, obeys equation (56); the same is true for its average $\langle\delta n(\mathbf{r},\mathbf{p},\tau)\delta n(\mathbf{r}',\mathbf{p}',0)\rangle$ over all initial Fock states $|\mathcal{F}\rangle$, since the coefficients Γ and K do not depend on $|\mathcal{F}\rangle$. Let's contract the latter by the quantity

$$B(\mathbf{r}',\mathbf{p}') \equiv \frac{1}{\hbar}\overline{\frac{\mathrm{d}\epsilon(\mathbf{r}',\mathbf{p}')}{\mathrm{d}N}} \tag{58}$$

as in equation (48) to form the auxiliary unknown

$$X(\mathbf{r},\mathbf{p},\tau) = \int \frac{\mathrm{d}^3r'\mathrm{d}^3p'}{(2\pi\hbar)^3}\, B(\mathbf{r}',\mathbf{p}')\, \langle\delta n(\mathbf{r},\mathbf{p},\tau)\delta n(\mathbf{r}',\mathbf{p}',0)\rangle \tag{59}$$

Then $X(\mathbf{r},\mathbf{p},\tau)$ evolves according to the linear kinetic equations (56) with the initial condition

$$X(\mathbf{r},\mathbf{p},0) = \int \frac{\mathrm{d}^3r'\mathrm{d}^3p'}{(2\pi\hbar)^3}\, Q(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}')\, B(\mathbf{r}',\mathbf{p}') \tag{60}$$

where the matrix of covariances at equal times of the number of quasiparticles has been introduced :

$$Q(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') = \langle\delta n(\mathbf{r},\mathbf{p},0)\delta n(\mathbf{r}',\mathbf{p}',0)\rangle \tag{61}$$

whose expression in the microcanonical ensemble will be connected to that in the canonical ensemble in due time, in sub-section 4.3. The sought microcanonical correlation function of $\mathrm{d}\hat{\theta}^{\mathrm{sc}}/\mathrm{d}t$ is then

$$C^{\mathrm{sc}}_{\mathrm{mc}}(\tau) = \int \frac{\mathrm{d}^3r\mathrm{d}^3p}{(2\pi\hbar)^3}\, B(\mathbf{r},\mathbf{p})X(\mathbf{r},\mathbf{p},\tau) \tag{62}$$

4.3. Solution in the secular-ergodic approximation

Our study restricts to the collisionless regime $\Gamma_{\mathrm{th}} \ll \omega_\alpha$ where Γ_{th} is the typical thermal value of the quasiparticles damping rate $\Gamma(\mathbf{r},\mathbf{p})$ and ω_α are the trapping angular frequencies. The quasiparticles then have time to perform a large number of Hamiltonian oscillations in the trap before undergoing a collision. We can therefore perform the secular approximation that consists in replacing the coefficients of the linearized kinetic equation (56) by their temporal average over a trajectory. Thus,

$$\Gamma(\mathbf{r},\mathbf{p}) \underset{\text{approx.}}{\overset{\text{secular}}{\to}} \overline{\Gamma(\mathbf{r},\mathbf{p})} = \lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathrm{d}\tau\,\Gamma(\mathbf{r}(\tau),\mathbf{p}(\tau)) \tag{63}$$

the auxiliary unknown $X(\mathbf{r},\mathbf{p},\tau)$ of equation (59), just like the fluctuations of the occupation numbers $\delta n(\mathbf{r},\mathbf{p},t)$, depend only on the trajectory $\tau \mapsto (\mathbf{r}(\tau),\mathbf{p}(\tau))$ passing through $(\mathbf{r},\mathbf{p})$ and on time. Still the problem remains formidable.

Fortunately, as we have said, in a completely anisotropic trap, the Hamiltonian dynamics of quasiparticles should be highly chaotic, except within the limits of very low energy $\epsilon \ll \mu_{\mathrm{TF}}$ or very high energy $\epsilon \gg \mu_{\mathrm{TF}}$ [22, 23]. We thus use the ergodic hypothesis, by identifying the temporal average on an trajectory of energy ϵ to the "uniform" mean in the phase space on the energy shell ϵ :

$$\overline{\Gamma(\mathbf{r},\mathbf{p})} \underset{\text{hypothesis}}{\overset{\text{ergodic}}{=}} \Gamma(\epsilon) = \langle\Gamma(\mathbf{r},\mathbf{p})\rangle_\epsilon \equiv \frac{1}{\rho(\epsilon)}\int \frac{\mathrm{d}^3r\mathrm{d}^3p}{(2\pi\hbar)^3}\Gamma(\mathbf{r},\mathbf{p})\delta(\epsilon-\epsilon(\mathbf{r},\mathbf{p})) \tag{64}$$

where the density of states $\rho(\epsilon)$ is given by equation (22). We will come back to this hypothesis in section 4.5. In this case, the function $X(\mathbf{r},\mathbf{p},\tau)$ depends only on the energy $\epsilon = \epsilon(\mathbf{r},\mathbf{p})$ and time :

$$X(\mathbf{r},\mathbf{p},\tau) \underset{\text{hypothesis}}{\overset{\text{ergodic}}{=}} X(\epsilon,\tau) \tag{65}$$

We obtain the evolution equation of $X(\epsilon,\tau)$ by averaging that of $X(\mathbf{r},\mathbf{p},\tau)$ on the energy shell ϵ : [11]

$$\partial_\tau X(\epsilon,\tau) = -\Gamma(\epsilon)X(\epsilon,\tau) - \frac{1}{2\rho(\epsilon)}\int_0^\epsilon \mathrm{d}\epsilon' L(\epsilon-\epsilon',\epsilon')\{X(\epsilon',\tau)[\bar n(\epsilon)-\bar n(\epsilon-\epsilon')] + X(\epsilon-\epsilon',\tau)[\bar n(\epsilon)-\bar n(\epsilon')]\}$$
$$- \frac{1}{\rho(\epsilon)}\int_0^{+\infty} \mathrm{d}\epsilon'\, L(\epsilon,\epsilon')\{X(\epsilon',\tau)[\bar n(\epsilon)-\bar n(\epsilon+\epsilon')] - X(\epsilon+\epsilon',\tau)[1+\bar n(\epsilon)+\bar n(\epsilon')]\} \quad (66)$$

with

$$\Gamma(\epsilon) = \frac{1}{2\rho(\epsilon)}\int_0^\epsilon \mathrm{d}\epsilon'\, L(\epsilon-\epsilon',\epsilon')[1+\bar n(\epsilon')+\bar n(\epsilon-\epsilon')] + \frac{1}{\rho(\epsilon)}\int_0^{+\infty}\mathrm{d}\epsilon'\, L(\epsilon,\epsilon')[\bar n(\epsilon')-\bar n(\epsilon+\epsilon')] \quad (67)$$

In these expressions, the first integral, limited to energies ϵ' lower than the energy of the quasiparticle ϵ considered, corresponds to Beliaev processes, and the second integral to Landau processes. The integral kernel [12]

$$L(\epsilon,\epsilon') = \int \frac{\mathrm{d}^3r\,\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi\hbar)^6}\frac{8\pi g^2\rho_0(\mathbf{r})}{\hbar}\left[A^{\epsilon+\epsilon'}_{\epsilon,\epsilon'}(\mathbf{r})\right]^2\delta(\epsilon-\epsilon(\mathbf{r},\mathbf{p}))\delta(\epsilon'-\epsilon(\mathbf{r},\mathbf{q}))\delta(\epsilon+\epsilon'-\epsilon(\mathbf{r},\mathbf{p}+\mathbf{q})) \quad (68)$$

$$= \frac{32\sqrt{2}}{\pi^{1/2}}\frac{[\rho_0(\mathbf{0})a^3]^{1/2}}{\hbar\mu_{\mathrm{TF}}}\left(\frac{\mu_{\mathrm{TF}}}{\hbar\bar\omega}\right)^3\int_0^{\mu_{\mathrm{TF}}}\frac{\mu_0\mathrm{d}\mu_0(\mu_{\mathrm{TF}}-\mu_0)^{1/2}\epsilon\epsilon'(\epsilon+\epsilon')\left[A^{\epsilon+\epsilon'}_{\epsilon,\epsilon'}(\mu_0)\right]^2}{\mu_{\mathrm{TF}}^{5/2}(\epsilon^2+\mu_0^2)^{1/2}(\epsilon'^2+\mu_0^2)^{1/2}[(\epsilon+\epsilon')^2+\mu_0^2]^{1/2}} \quad (69)$$

uses the reduced coupling amplitude (54) at point $\mathbf{r}$, reparametrized in terms of energies $\epsilon_i = \epsilon(\mathbf{r},\mathbf{p}_i)$ $(1\le i\le 3)$ or even in terms of the local Gross-Pitaevskii chemical potential $\mu_0 = g\rho_0(\mathbf{r})$. It has the symmetry property $L(\epsilon,\epsilon') = L(\epsilon',\epsilon)$.

We write the result before giving some indications on its obtention (one will also consult reference [16]). In the secularo-ergodic approximation, the microcanonical correlation function of $\mathrm{d}\hat\theta^{\mathrm{sc}}/\mathrm{d}t$ is

$$C^{\mathrm{ergo}}_{\mathrm{mc}}(\tau) = \int_0^{+\infty}\mathrm{d}\epsilon\,\rho(\epsilon)B(\epsilon)X(\epsilon,\tau) \quad (70)$$

Here $B(\epsilon)$ is the ergodic average of the quantity $B(\mathbf{r},\mathbf{p})$ introduced in equation (58) :

$$B(\epsilon) = \frac{1}{\rho(\epsilon)}\int\frac{\mathrm{d}^3r\,\mathrm{d}^3p}{(2\pi\hbar)^3}\frac{\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})}{\hbar\mathrm{d}N}\delta(\epsilon-\epsilon(\mathbf{r},\mathbf{p})) \quad (71)$$

$$= \frac{\mathrm{d}\mu_{\mathrm{TF}}/\mathrm{d}N}{\hbar\pi f(\check\epsilon)}\left[2\check\epsilon^{1/2}(\check\epsilon+1) - \sqrt{2}(\check\epsilon^2+1)\,\mathrm{argsh}\,\frac{(2\check\epsilon)^{1/2}}{(1+\check\epsilon^2)^{1/2}} - \check\epsilon^{1/2}(\check\epsilon-1) - (1+\check\epsilon)^2\,\mathrm{acos}\,\frac{1}{(1+\check\epsilon)^{1/2}}\right] \quad (72)$$

$$B(\epsilon) \underset{\epsilon\to 0}{=} \frac{\mathrm{d}\mu_{\mathrm{TF}}}{\hbar\mathrm{d}N}\left[-\frac{\check\epsilon}{5} - \frac{3\pi}{40\sqrt{2}}\check\epsilon^{3/2} + O(\check\epsilon^2)\right], \quad B(\epsilon) \underset{\epsilon\to+\infty}{=} \frac{\mathrm{d}\mu_{\mathrm{TF}}}{\hbar\mathrm{d}N}\left[-1 + \frac{32}{3\pi}\check\epsilon^{-3/2} + O(\check\epsilon^{-5/2})\right] \quad (73)$$

with $\check\epsilon = \epsilon/\mu_{\mathrm{TF}}$ and $f(\check\epsilon)$ the reduced density of states (27). The auxiliary unknown $X(\epsilon,\tau)$ is a solution of the linear equation (66) with the initial condition

$$X(\epsilon,0) = \bar n(\epsilon)[1+\bar n(\epsilon)][B(\epsilon)-\Lambda\epsilon] \quad (74)$$

where $\hbar\Lambda$ is the derivative of the microcanonical chemical potential with respect to to the total energy E of the gas [13], as in equation (19) :

$$\Lambda = \frac{\int_0^{+\infty}\mathrm{d}\epsilon\,\rho(\epsilon)\epsilon B(\epsilon)\bar n(\epsilon)[1+\bar n(\epsilon)]}{\int_0^{+\infty}\mathrm{d}\epsilon\,\rho(\epsilon)\epsilon^2\bar n(\epsilon)[1+\bar n(\epsilon)]} \quad (75)$$

The equation (70) is the ergodic rewriting of equation (62). The initial condition (74) is the difference of two contributions :

11. The simplest is to average the complete kinetic equations (51), then linearize the result around the stationary solution (55).

12. To get (69), we reduced equation (68) to a single integral on the r modulus (after having formally reduced to the case of an isotropic trap as in note 3) by integrating in spherical coordinates on p, q and u, the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$. In $\int_{-1}^{1}\mathrm{d}u$, the argument of the third Dirac vanishes at a point u_0 and only one, given the inequalities $\epsilon^{\mathrm{Bog}}_{|p-q|} \le \epsilon^{\mathrm{Bog}}_p + \epsilon^{\mathrm{Bog}}_q \le \epsilon^{\mathrm{Bog}}_{p+q}$ satisfied by the Bogoliubov dispersion relation $\epsilon^{\mathrm{Bog}}_p = [\frac{p^2}{2m}(\frac{p^2}{2m}+2\mu_0)]^{1/2}$, $\forall\mu_0\ge 0$.

13. The deep reason for the appearance of this derivative is given in reference [16]. It explains why the kinetic equations allow to find in the canonical ensemble the ballistic term At^2 of equation (15) with the correct expression of the coefficient $A = (\partial_E\mu_{\mathrm{mc}}/\hbar)^2\mathrm{Var}\,E$.

— the first is the one that one would obtain in the canonical ensemble. The ergodic mean of the covariance matrix (61) would then be simply $Q_{\rm can}(\epsilon,\epsilon') = \bar{n}(\epsilon)[1+\bar{n}(\epsilon)]\delta(\epsilon-\epsilon')/\rho(\epsilon)$;

— the second comes from a projection of δn canonical fluctuations on the subspace of the δn fluctuations of zero energy, $\int_0^{+\infty} \mathrm{d}\epsilon\, \rho(\epsilon)\epsilon\delta n(\epsilon) = 0$, the only one eligible in the microcanonical ensemble. Only subtle point, this projection must be carried out parallel to the stationary solution $e_0(\epsilon) = \epsilon\bar{n}(\epsilon)[1+\bar{n}(\epsilon)]$ of the linearized kinetic equations (66). [14] We then check that, for the value of Λ given, $X(\epsilon, 0)$ is in the subspace of zero energy fluctuations.

4.4. Results and discussion

We present some results in graphic form, after a clever scaling making them independent of the trapping angular frequencies (provided they are quite distinct two by two to allow the ergodic hypothesis) and of the strength of the interactions [15] ; it is enough to know the temperature in units of Thomas-Fermi's chemical potential $\mu_{\rm TF}$. These results illustrate the universality class of the completely anisotropic harmonic traps, different from that of the spatially homogeneous systems of reference [16].

An interesting by-product of our study is shown in figure 3 : it is the damping rate $\Gamma(\epsilon)$ in the secularo-ergodic approximation of the Bogoliubov modes of energy ϵ. In a cold atom experiment it is possible to excite such modes and to follow their decay in time. The rate we predict is then measurable and it can be compared to the experiments, at least in its validity regime of classical motion $\epsilon \gg \hbar\omega_\alpha$ (deviations from the ergodic hypothesis are discussed in section 4.5). The limiting behaviors

$$\hbar\Gamma(\epsilon) \underset{\epsilon\to 0}{\sim} \frac{3\mathcal{I}}{4}\left(\frac{\epsilon}{\mu_{\rm TF}}\right)^{1/2} k_B T[\rho_0(\mathbf{0})a^3]^{1/2} \quad \text{with} \quad \mathcal{I} = 4.921\,208\ldots \tag{76}$$

$$\hbar\Gamma(\epsilon) \underset{\epsilon\to+\infty}{\sim} \frac{128\sqrt{2}}{15\sqrt{\pi}}\frac{\mu_{\rm TF}^2}{\epsilon}[\rho_0(\mathbf{0})a^3]^{1/2} \tag{77}$$

shown in dashed lines in figure 3, [16] are derived in Appendix A. They are very different from the spatially homogeneous case, where the damping rate vanishes linearly in ϵ at low energy and diverges as $\epsilon^{1/2}$ at high energy. In particular, the behavior (76) in $\epsilon^{1/2}$ results from the existence of the Thomas-Fermi edge of the condensate.

Let's go back to the condensate phase spreading in the microcanonical ensemble. In figure 4, we represent in black solid line the variance of the phase shift $\hat{\theta}(t)-\hat{\theta}(0)$ of the condensate as a function of time t in the ergodic approximation (70) at the temperatures $T = \mu_{\rm TF}/k_B$ and $T = 10\mu_{\rm TF}/k_B$. The variance has a parabolic departure in time, which corresponds to the precollisional regime $t \ll t_{\rm coll}$, where $t_{\rm coll}$ is the typical collision time between quasiparticles : we can then assume that $C_{\rm mc}(\tau) \simeq C_{\rm mc}(0)$, so that the integral contribution to equation (15) is $\simeq C_{\rm mc}(0)t^2$. At long times, $t \gg t_{\rm coll}$, the correlation function of $\mathrm{d}\hat{\theta}/\mathrm{d}t$ seems to quickly reach zero (red solid line) ; a more detailed numerical study (see the inset in figure 4 b) reveals however the presence of a power law tail $t^{-\alpha}$,

$$C_{\rm mc}(t) \underset{t\to+\infty}{\sim} \frac{C}{t^5} \tag{78}$$

the exponent $\alpha = 5$ is greater than the one $\alpha_h = 3$ of the decay law of $C_{\rm mc}(t)$ in the spatially homogeneous case [16]. Its value can be found by a rough heuristic approximation, called *rate approximation* or *projected Gaussian* [15], already used for α_h with success in this same reference [16] : we keep in the linearized kinetic equations (66) only

14. For this projection to be compatible with linearized kinetic evolution, it is necessary that the projection direction and the hyperplane on which we project be invariant by temporal evolution, the second point being ensured by conservation of energy. The form of $e_0(\epsilon)$ derives from the fact that (55) remains a stationary solution for an infinitesimal variation of $\beta, \beta \to \beta + \delta\beta$, around its physical value.

15. In a first step, we show that the results can depend on the trapping frequencies ω_α only through their geometric mean $\bar{\omega}$. This is a direct consequence of the ergodic hypothesis and the fact that the observables involved here, including the Hamiltonian, depend only on the position $\mathbf{r}$ of the quasiparticles *via* the trapping potential $U(\mathbf{r}) = \frac{1}{2}m\sum_\alpha \omega_\alpha^2 r_\alpha^2$. In the integral $\int \mathrm{d}^3 r$ participating in the ergodic mean, one can then perform the isotropising change of variables of note 3.

16. For $k_B T = \mu_{\rm TF}$, $\Gamma(\epsilon)/\epsilon^{1/2}$ has a deceptive maximum in the neighborhood of $\epsilon/\mu_{\rm TF} = 0.02$ of about 5% above its limit in $\epsilon = 0$.

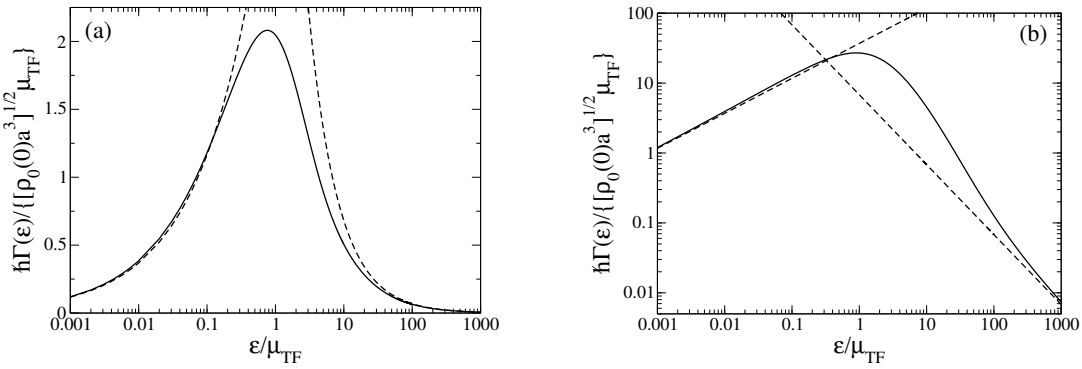

FIGURE 3: Beliaev-Landau damping rate $\Gamma(\epsilon)$ of Bogoliubov modes of a condensate in a completely anisotropic harmonic trap as a function of the mode energy ϵ, at the thermodynamic limit, in the secularo-ergodic approximation (63, 64, 67), at temperature (a) $k_BT = \mu_{\rm TF}$ and (b) $k_BT = 10\mu_{\rm TF}$, where $\mu_{\rm TF}$ is the Thomas-Fermi's chemical potential of the condensate. Thanks to the chosen units, the curve is universal ; in particular, it does not depend on the trapping angular frequencies ω_α. The Bogoliubov modes considered must be in the classical motion regime $\epsilon \gg \hbar\omega_\alpha$ and the system must be in the regime of an almost pure condensate, $[\rho_0(\mathbf{0})a^3]^{1/2} \ll 1$ and $T \ll T_c$, where $\rho_0(\mathbf{0}) = \mu_{\rm TF}/g$ is the density of the condensate in the center of the trap and T_c the critical temperature. In dashed lines, the equivalents (76) and (77) of $\Gamma(\epsilon)$ at low and high energy.

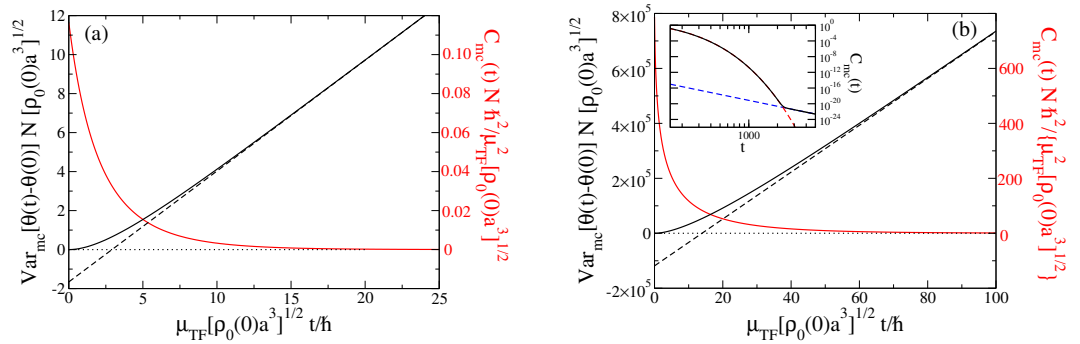

FIGURE 4: In the conditions of figure 3, for a system prepared at $t = 0$ in the microcanonical ensemble at temperature (a) $k_BT = \mu_{\rm TF}$ or (b) $k_BT = 10\mu_{\rm TF}$, and isolated from its environment in its subsequent evolution, variance of the condensate phase shift $\hat\theta(t) - \hat\theta(0)$ as a function of time t (solid line black) and its asymptotic diffusive behavior (80) (dashed). The correlation function $C_{\rm mc}(t)$ of $\mathrm{d}\hat\theta/\mathrm{d}t$ is shown on the same figure in the secular-ergodic approximation (70) as a function of time (red solid line, ticks on the right) and, for (b), in an inset in log-log scale at long times (black solid line) to show that after a quasi-exponential decay in the square root of time (fit with $t^6\exp(-C\sqrt{t})$ in red dashed line) it follows a power law $\propto t^{-5}$ (blue dashed). As in figure 3, the multiplication of the quantities on the axes by well-chosen factors makes these results universal.

the term of pure decay $-\Gamma(\epsilon)X(\epsilon,\tau)$ in the right-hand side, which makes them immediately integrable and leads to the estimate [17]

$$C_{\rm mc}(t) \approx \int_0^{+\infty} \mathrm{d}\epsilon\, \rho(\epsilon)[B(\epsilon) - \Lambda\epsilon]^2 \bar n(\epsilon)[1 + \bar n(\epsilon)]\mathrm{e}^{-\Gamma(\epsilon)t} \tag{79}$$

The power law behavior of the density of states $\rho(\epsilon)$ at low energy [see (34)], of the coefficients $B(\epsilon)$ in $\mathrm{d}\hat\theta/\mathrm{d}t$ [see (73)], of the occupation numbers $n(\epsilon) \sim k_BT/\epsilon$ and of the damping rate $\Gamma(\epsilon)$ [see (76)] then reproduce the exponent $\alpha = 5$ found numerically. [18] Since $C_{\rm mc}(t)$ tends to zero faster than $1/t^{2+\eta}$, for some $\eta > 0$, we obtain the following important result : the variance of the condensate phase shift $\mathrm{Var}_{\rm mc}[\hat\theta(t) - \hat\theta(0)]$ exhibits at long times the typical affine growth of a diffusive regime with delay :

$$\mathrm{Var}_{\rm mc}[\hat\theta(t) - \hat\theta(0)] \underset{t\gg t_{\rm coll}}{=} 2D(t - t_0) + o(1) \tag{80}$$

17. Care has been taken to account for the projection on the microcanonical subspace of zero energy fluctuations not only in the initial condition (74), but also in the contraction by $B(\epsilon)$ in (70), replacing $B(\epsilon)$ with $B(\epsilon) - \Lambda\epsilon$; this precaution, optional in the exact formulation, is necessary here since the rate approximation violates conservation of energy.

18. In contrast, the predicted value for the coefficient C in (78) for $k_BT = 10\mu_{\rm TF}$, that is $\simeq 10^{-5}$, differs significantly from the numerical value $\simeq 7\times10^{-5}$.

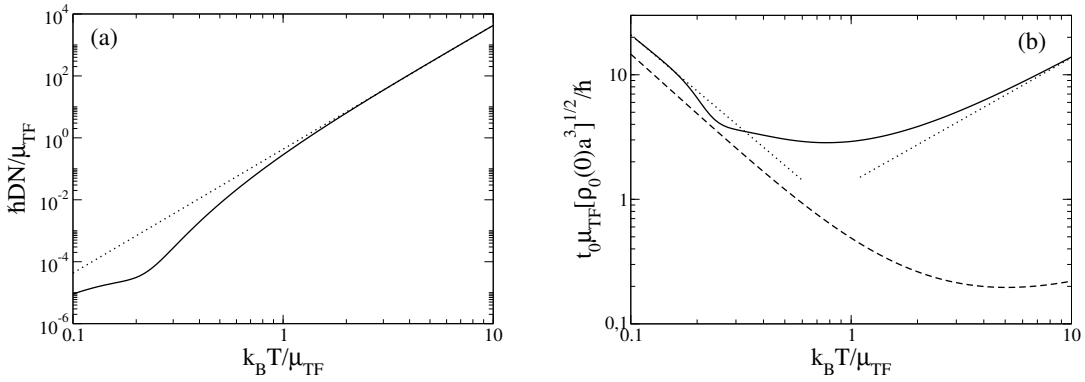

FIGURE 5: In the conditions of the previous figures 3 and 4, (a) diffusion coefficient D of the phase of the harmonically trapped condensate and (b) delay time t_0 to the phase diffusion as a function of temperature (in black solid line) deduced from equations (81, 82) and (70). These quantities are independent of the statistical ensemble. With the chosen scalings, the curves are universal. Dotted lines : in (a) fit by a T^4 law at high temperature and in (b) fit by a law linear in T at high temperature and by a $T^{-3/2}$ law at low temperature. The power laws represented at high temperature are only indicative because they certainly omit logarithmic factors in k_BT/μ_{TF}. In (b) the dashed line gives the estimate $1/\Gamma(\epsilon = k_BT)$ of the typical collision time t_{coll} between quasiparticles.

represented by a dashed line on figure 4. The delay t_0 is due to the non-zero width of the correlation function $C_{mc}(\tau)$:

$$D = \int_0^{+\infty} d\tau\, C_{mc}(\tau) \tag{81}$$

$$t_0 = \frac{\int_0^{+\infty} d\tau\, \tau C_{mc}(\tau)}{\int_0^{+\infty} d\tau\, C_{mc}(\tau)} \tag{82}$$

We represent the diffusion coefficient D of the condensate phase as a function of temperature in figure 5 a. It exhibits a high temperature growth ($k_BT > \mu_{TF}$) much faster than in the spatially homogeneous case : it was only linear (up to logarithmic factors), it seems to scale here as T^4 (dotted in the figure). The diffusion delay time t_0 is plotted as a function of temperature in figure 5 b. We compare it to the estimate $t_{coll} \simeq 1/\Gamma(\epsilon = k_BT)$ of the collision time between quasiparticles, in dashed line : this one gives a good account of the sudden rise of t_0 at low temperatures, but reproduces with much delay and greatly underestimating that at high temperatures. The rise of t_0 is well represented by a $T^{-3/2}$ low temperature law, and appears to be linear in T at high temperature (see dashed line).

Let us find by a simple reasoning the observed power laws. If a scaling law exists, it should survive the rate approximation on the linearized kinetic equations ; we can therefore take the approximate expression (79) of $C_{mc}(t)$ as a starting point and put it in the expressions (81) and (82) of D and t_0.

At high temperature, the integrals on ϵ giving D and t_0 in the rate approximation are dominated by energies of order k_BT ; we set $\epsilon = k_BT\bar{\epsilon}$ and send T to $+\infty$ at fixed $\bar{\epsilon}$ under the integral. The behaviors of $\rho(\epsilon)$ and $B(\epsilon)$ at high energy are known. Only that of $\Gamma(k_BT\bar{\epsilon})$ is missing ; to get it, we notice on (69) that $L(k_BT\bar{\epsilon}, k_BT\bar{\epsilon}')$ tends to a constant when $T \to +\infty$. The approximation $L(\epsilon, \epsilon') \simeq L(\epsilon - \epsilon', \epsilon') \simeq$ const, however, triggers a logarithmic infrared divergence in the integrals on ϵ' in (67), which stops at $\epsilon' \lesssim \mu_{TF}$, so that [19]

$$\frac{\hbar\Gamma(k_BT\bar{\epsilon})}{\mu_{TF}[\rho_0(\mathbf{0})a^3]^{1/2}} \underset{k_BT/\mu_{TF}\to+\infty}{\sim} \frac{512\sqrt{2}}{15\pi^{1/2}} \frac{1}{\bar{\epsilon}^2} \frac{\mu_{TF}}{k_BT} \ln\frac{k_BT}{\mu_{TF}} \tag{83}$$

All this leads to the scaling laws $D \approx T^4$ and $t_0 \approx T$ at high temperature, up to logarithmic factors.

At low temperatures, we proceed in the same way. The behavior of $\Gamma(k_BT\bar{\epsilon})$ is as $T^{3/2}$ when $T \to 0$ at fixed $\bar{\epsilon}$, as one could expect it from the equivalent (76) and as it is confirmed by a calculation. The only trap to avoid is that

19. A more accurate calculation leads to replace in (83) the symbol $\sim$ by $=$ and the factor $\ln\frac{k_BT}{\mu_{TF}}$ by $[\ln\frac{k_BT}{\mu_{TF}} + \frac{\bar{\epsilon}}{4} + \ln(1 - e^{-\bar{\epsilon}}) + \frac{31}{15} - 3\ln 2 + O(\mu_{TF}/k_BT)]$.

$B(k_BT\bar{\epsilon}) - \Lambda k_BT\bar{\epsilon}$ scales as $T^{3/2}$ when $T \to 0$, not as T as one might think, because the dominant terms of $B(k_BT\bar{\epsilon})$ and $\Lambda k_BT\bar{\epsilon}$, both linear in $k_BT\bar{\epsilon}$, exactly compensate each other, see equation (75). This leads to the exact power laws (without logarithmic corrections) $D \propto T^4$ and $t_0 \propto T^{-3/2}$ at low temperature ; only the second one is accessible on the temperature interval of figure 5, but we checked the first one numerically on a larger temperature range.

To encourage an experimental study with cold atoms, let's finish with a small study of the fundamental limits to the observability of phase diffusion of a trapped condensate. There are, of course, several practical difficulties to overcome, such as (*i*) significant reduction of the fluctuations in the energy and number of particles in the gas to mitigate the ballistic blurring of the phase, which is a dangerous competitor of the diffusion, (*ii*) the introduction of a sensitive and unbiased detection scheme for the condensate phase shift or coherence function $g_1(t)$, of the Ramsey type as proposed in references [17, 18], (*iii*) reduction of the technical noise of the experimental device, (*iv*) trapping of the atoms in a cell with a sufficiently high vacuum to make cold atom losses by collision with the residual hot gas negligible (one-body losses) : lifetimes of the order of the hour are possible under cryogenic environment [33, 34]. These practical aspects vary according to the experimental groups and are beyond the scope of this article. In contrast, particle losses due to three-body collisions, with the formation of a dimer and a fast atom, are intrinsic to alkaline atoms and constitute a fundamental limit. Each atom loss changes, at a random time, the rate of variation of the phase $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta}$, since this is a function of N, which adds a stochastic component to its evolution [16, 35]. To calculate the variance of the phase shift of the condensate induced by the three-body losses, we place ourselves at zeroth order in the uncondensed fraction, that is in the case of a pure condensate at zero temperature prepared at time 0 with an initially well defined number of particles N, as in reference [16] from which we can recycle (adapting them to the trapped case and to three-body losses) expressions (G7) and (64) :

$$\mathrm{Var}_{\mathrm{losses}}[\hat{\theta}(t) - \hat{\theta}(0)] = \left(\frac{\mathrm{d}\mu_{\mathrm{TF}}}{\hbar \mathrm{d}N}\right)^2 \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \left[\langle \hat{N}(\tau)\hat{N}(\tau')\rangle - \langle \hat{N}(\tau)\rangle\langle \hat{N}(\tau')\rangle\right] \underset{\Gamma_3 t \to 0}{\sim} \left(\frac{\mathrm{d}\mu_{\mathrm{TF}}}{\hbar \mathrm{d}N}\right)^2 N\Gamma_3 t^3 \tag{84}$$

We introduced the Γ_3 particle number decay rate, related as follows to the K_3 rate constant of the three-body losses and to the Thomas-Fermi $\rho_0(\mathbf{r})$ density profile of the condensate :

$$\frac{\mathrm{d}}{\mathrm{d}t}N \equiv -\Gamma_3 N = -K_3 \int \mathrm{d}^3 r \, [\rho_0(\mathbf{r})]^3 \tag{85}$$

We obtain a more meaningful writing, directly comparable to our results without losses, by writing (84) in adimensioned form :

$$\overline{\mathrm{Var}}_{\mathrm{losses}}[\hat{\theta}(t) - \hat{\theta}(0)] \underset{\Gamma_3 t \to 0}{\sim} \frac{8}{525\pi} \bar{K}_3 \bar{t}^{\,3} \tag{86}$$

where $\overline{\mathrm{Var}}$ and $\bar{t}$ are the variance of the phase shift and the elapsed time in the units of figure 4, and $\bar{K}_3 = mK_3/(\hbar a^4)$. The reduced constant $\bar{K}_3$ is an intrinsic property of the atomic species used in the experiment (even if it can be varied using a magnetic Feshbach resonance [36]). To estimate the order of magnitude of $\bar{K}_3$ in a gas of cold atoms, consider the example of rubidium 87 in the ground hyperfine sublevel $|F = 1, m_F = -1\rangle$ at vanishing magnetic field : measurements give $K_3 = 6 \times 10^{-42}$ m ^{6}/s and $a = 5.31$nm [37] so $\bar{K}_3 \simeq 10$. In figure 4a ($k_BT = \mu_{\mathrm{TF}}$), at the reduced entrance time $\bar{t} = 5$ in the asymptotic regime of phase diffusion we see that the loss-induced parasitic variance for this value of $\bar{K}_3$ is about three times the useful variance ; their very different time dependencies should, however, make it possible to separate them. The situation is much more favorable at higher temperature, $k_BT \gg \mu_{\mathrm{TF}}$, the effect of the losses on the phase shift variance being for example still negligible at the reduced time $\bar{t} = 100$ in figure 4b ($k_BT = 10\mu_{\mathrm{TF}}$).

4.5. Discussion of ergodic hypothesis

As shown by references [22, 23] in the case of a harmonic trap with cylindrical symmetry, the classical motion of Bogoliubov quasiparticles is highly chaotic at energies $\epsilon \simeq \mu_{\mathrm{TF}}$ but even at this energy, the Poincaré sections reveal patches of stability in the phase space, which are not crossed by the trajectories of the chaotic sea : there is no ergodicity in the strict sense.

What about the case of a completely anisotropic trap ? We want to test the ergodic hypothesis for two physical quantities. The first one appears in our linearized kinetic equations, it is the $\Gamma(\mathbf{r}, \mathbf{p})$ damping rate. The second appears

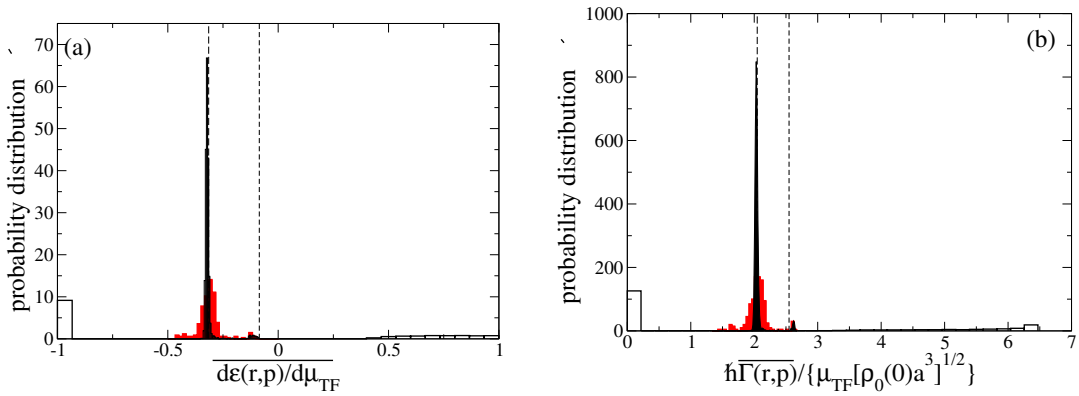

FIGURE 6: Histogram of time averages of the physical quantities (a) $\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})/\mathrm{d}\mu_{\mathrm{TF}}$ and (b) $\Gamma(\mathbf{r},\mathbf{p})$ for 10^4 independent initial values $(\mathbf{r}(0),\mathbf{p}(0))$ drawn uniformly on the energy shell $\epsilon = \mu_{\mathrm{TF}}$ and a Hamiltonian evolution (39, 40) of Bogoliubov quasiparticles for a variable duration $t : t = 0$ (hollow bars, black line), $t = 5\times10^3/\bar{\omega}$ (solid red bars), $t = 2.5\times10^5/\bar{\omega}$ (black solid bars). The harmonic potential is completely anisotropic, with incommensurate trapping frequencies (ratios $\omega_x : \omega_y : \omega_z = 1 : \sqrt{3} : \sqrt{5}-1$). The temperature, which enters in the $\Gamma(\mathbf{r},\mathbf{p})$ damping rate, is $T = \mu_{\mathrm{TF}}/k_B$. Black vertical dashed lines : on the left, the ergodic value (average of the physical quantity on the energy shell ϵ) ; on the right, the temporal average of the quantity over a period of the linear trajectory of energy ϵ along the direction Oy (most confining axis of the trap), obtained analytically for (a) (see note 21) and numerically for (b).

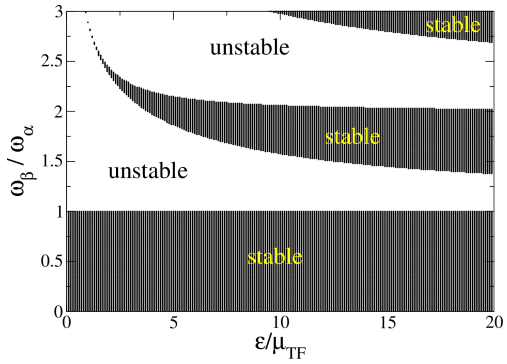

FIGURE 7: For the classical Hamiltonian dynamics of Bogoliubov quasiparticles in a harmonic potential, stability of the linear motion along an eigenaxis $O\alpha$ of the trap for an infinitesimal initial perturbation (a displacement) along another eigenaxis $O\beta$, as a function of the energy ϵ of the trajectory and the ratio $\omega_\beta/\omega_\alpha$ of the trapping frequencies (the shaded areas are stable).

in the initial conditions of the $C_{\mathrm{mc}}(\tau)$ correlation function of $\mathrm{d}\hat{\theta}/\mathrm{d}t$, it is $\mathrm{d}\epsilon(\mathbf{r},\mathbf{p})/\mathrm{d}\mu_{\mathrm{TF}}$. For a uniform sampling of the energy surface ϵ, that is with the probability distribution $\delta(\epsilon-\epsilon(\mathbf{r},\mathbf{p}))/\rho(\epsilon)$ in the phase space, we show in figure 6 the histograms of these quantities after time averaging on each trajectory over times $t = 0$, $t = 5000/\bar{\omega}$ and $t = 250\,000/\bar{\omega}$, at the energy $\epsilon = k_BT = \mu_{\mathrm{TF}}$ for incommensurable trapping frequencies.[20] The temporal averaging leads to a spectacular narrowing of the probability distribution, which peaks around the ergodic mean (dashed line on the left), which goes in the direction of the ergodic hypothesis. This narrowing dynamics continues over very long times, but never eliminates a small lateral peak far from the ergodic average.

An inspection of the trajectories contributing to the lateral peak shows that they are small perturbations of the stable linear trajectories along the most confining trap axis. The temporal average value of the two quantities considered on these linear trajectories is represented by the right dashed vertical lines in figure 6, it is actually close to the peak in

20. Equations of motion (39, 40), put together as $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X} = \mathbf{f}(\mathbf{X})$, are numerically integrated with a semi-implicit scheme of the second order, $\mathbf{X}(t+\mathrm{d}t) = \mathbf{X}(t) + \mathrm{d}t[1-\frac{\mathrm{d}t}{2}M]^{-1}\mathbf{f}(\mathbf{X}(t))$ where M is the first differential of $\mathbf{f}(\mathbf{X})$ in $\mathbf{X}(t)$ [38]. If the trajectory crosses the surface of the condensate between t and $t+\mathrm{d}t$, we must determine the crossing time t_s with an error $O(\mathrm{d}t)^3$, then apply the diagram semi-implicit successively on $[t,t_s]$ and $[t_s,t+\mathrm{d}t]$, to overcome the discontinuity of $\mathbf{f}(\mathbf{X})$ and its derivatives.

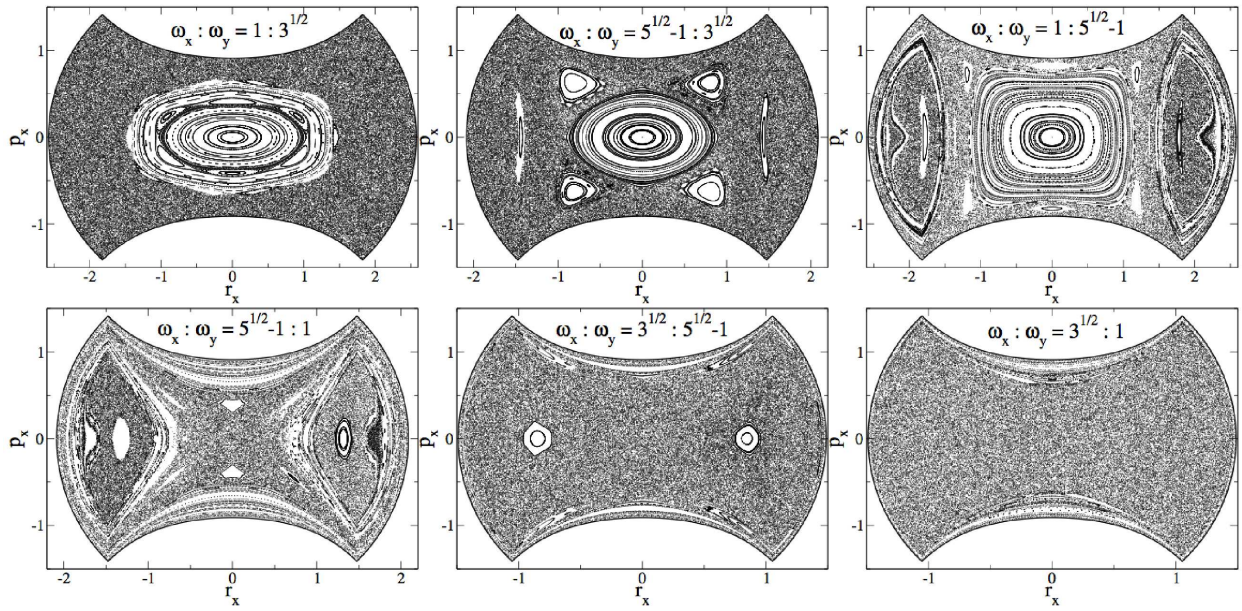

FIGURE 8: For the classical Hamiltonian dynamics of Bogoliubov quasiparticles in a harmonic potential, Poincaré sections in the plane ($r_y = 0, p_y(\epsilon) > 0$) of planar trajectories in xOy (200 independent trajectories, evolution time $5000/\bar{\omega}$), with a ratio $\omega_x : \omega_y$ taking all possible values in the trap of figure 6 : $1 : \sqrt{3}$, $\sqrt{3} : 1$, $1 : \sqrt{5}-1$, $\sqrt{5}-1 : 1$, $\sqrt{3} : \sqrt{5}-1$ and $\sqrt{5}-1 : \sqrt{3}$. The sections are ordered by increasing ratio ω_x/ω_y from left to right and from top to bottom (given that $1/\sqrt{3} < (\sqrt{5}-1)/\sqrt{3} < 1/(\sqrt{5}-1) < 1$). This shows that the Poincaré section is more chaotic as the ratio ω_x/ω_y is larger. r_x is in units of $(\mu_{\rm TF}/m\bar{\omega}^2)^{1/2}$ and p_x in units of $(m\mu_{\rm TF})^{1/2}$.

question. The stability diagram of a linear trajectory along an eigenaxis α of the trap, with respect to a perturbation along another eigenaxis β is shown in figure 7, in the plane (energy, ratio $\omega_\beta/\omega_\alpha$). It shows that the linear trajectory along the most confining axis is stable at all energies.[21] The Poincaré sections of the planar trajectories in the $\alpha O\beta$ planes in figure 8 specify the width of the stability island and reveal the existence of secondary islands, etc. There is therefore no full ergodicity of our classical dynamics, even at the energies $\epsilon \approx \mu_{\rm TF}$, even in the completely anisotropic case.

To quantitatively measure the error made by the ergodic hypothesis in the computation of $C_{\rm mc}(0)$ and $C_{\rm mc}(\tau > 0)$, we consider the differences between

$$\left\langle \overline{\frac{{\rm d}\epsilon(\mathbf{r},\mathbf{p})}{{\rm d}\mu_{\rm TF}}}^{\,2} \right\rangle_\epsilon \quad \text{and} \quad \left\langle \frac{{\rm d}\epsilon(\mathbf{r},\mathbf{p})}{{\rm d}\mu_{\rm TF}} \right\rangle_\epsilon^2 \tag{87}$$

$$\left\langle \frac{1}{\overline{\Gamma(\mathbf{r},\mathbf{p})}} \right\rangle_\epsilon \quad \text{and} \quad \frac{1}{\langle \Gamma(\mathbf{r},\mathbf{p}) \rangle_\epsilon} = \frac{1}{\Gamma(\epsilon)} \tag{88}$$

where we recall that the horizontal bar $\overline{O(\mathbf{r},\mathbf{p})}$ above a physical quantity represents the time average on the trajectory

21. The linear trajectory of a quasiparticle of energy ϵ along the proper axis $O\alpha$ of the trap is written $m^{1/2}\omega_\alpha r_\alpha(t) = |\mu_{\rm TF} + {\rm i}\epsilon| \sin(\sqrt{2}\omega_\alpha t)/\sqrt{G(t)}$ and $p_\alpha(t)/(2m)^{1/2} = \epsilon/\sqrt{G(t)}$ with $G(t) = \mu_{\rm TF} + |\mu_{\rm TF} + {\rm i}\epsilon| \cos(\sqrt{2}\omega_\alpha t)$. This corresponds to the choice $r_\alpha(0) = 0$, $p_\alpha(0) \geq 0$ and is for $-t_s \leq t \leq t_s$, where the time to reach the surface of the condensate is given by $\sqrt{2}\omega_\alpha t_s = \operatorname{acos} \frac{\epsilon - \mu_{\rm TF}}{|\mu_{\rm TF} + {\rm i}\epsilon|}$. Outside the condensate, the quasiparticle oscillates harmonically like a free particle for a time $2\omega_\alpha^{-1} \operatorname{atan}[(\epsilon/\mu_{\rm TF})^{1/2}]$ before regaining the condensate, to cross it in a time $2t_s$, and so on. The knowledge of the trajectory makes immediate the linear analysis of numerical stability. It also allows to calculate analytically the time average of ${\rm d}\epsilon(\mathbf{r},\mathbf{p})/{\rm d}\mu_{\rm TF}$ on the linear trajectory; if we put $\check{\epsilon} = \epsilon/\mu_{\rm TF}$, the result is written

$$\frac{\overline{{\rm d}\epsilon(\mathbf{r},\mathbf{p})}}{{\rm d}\mu_{\rm TF}} = \frac{\ln \frac{1+\check{\epsilon}+\sqrt{2\check{\epsilon}}}{(1+\check{\epsilon}^2)^{1/2}} - \sqrt{2}\operatorname{atan}\sqrt{\check{\epsilon}}}{\operatorname{acos}\frac{\check{\epsilon}-1}{(1+\check{\epsilon}^2)^{1/2}} + \sqrt{2}\operatorname{atan}\sqrt{\check{\epsilon}}}$$

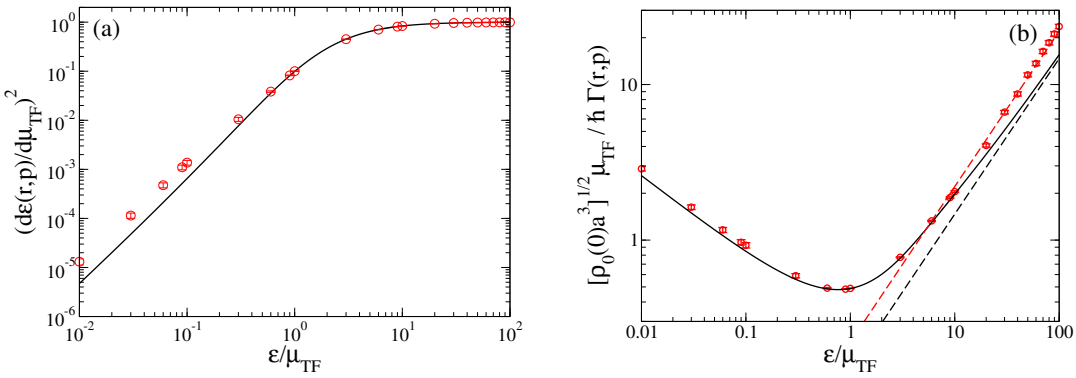

FIGURE 9: Visualization of the error introduced by the ergodic hypothesis $\bar{O} = \langle O \rangle_\epsilon$ on two physical quantities $O(\mathbf{r}, \mathbf{p})$ involved in the diffusion of the condensate phase, as a function of the energy ϵ : (a) for $O(\mathbf{r}, \mathbf{p}) = \mathrm{d}\epsilon(\mathbf{r}, \mathbf{p})/\mathrm{d}\mu_{\mathrm{TF}}$, we compare as in equation (87) $\langle \bar{O}^2 \rangle_\epsilon$ (red circles) to its ergodic approximation $\langle O \rangle_\epsilon^2$ from (72) (solid line black) ; (b) for $O(\mathbf{r}, \mathbf{p}) = \Gamma(\mathbf{r}, \mathbf{p})$, we compare as in equation (88) $\langle 1/\bar{O} \rangle_\epsilon$ (red circles) to its ergodic approximation $1/\langle O \rangle_\epsilon$ (solid line black). The time average $\bar{O}$ is computed over an evolution time $t = 5 \times 10^4/\bar{\omega}$ of the quasiparticles in the trap of figure 6 ; the average on the energy shell ϵ is taken on 200 independent trajectories, with initial conditions $(\mathbf{r}, \mathbf{p})$ drawn according to the uniform law $\delta(\epsilon - \epsilon(\mathbf{r}, \mathbf{p}))/\rho(\epsilon)$, which leads to a statistical uncertainty represented by the error bars in the figure. Dashed lines in (b) : asymptotic equivalents (89) of $\langle 1/\bar{\Gamma} \rangle_\epsilon$ (in red) and (77) of $1/\langle \Gamma \rangle_\epsilon$ (in black).

going through $(\mathbf{r}, \mathbf{p})$ in phase space, as in equation (63), and the brackets $\langle O(\mathbf{r}, \mathbf{p}) \rangle_\epsilon$ represent the uniform mean over the energy shell ϵ as in equation (64). In equations (87, 88), the left column contains the quantities appearing in $C_{\mathrm{mc}}(0)$ or in the secular kinetic equations before the ergodic approximation, and the right column what they become after ergodic approximation. Importantly, we consider in equation (88) $1/\bar{\Gamma}$ rather than $\bar{\Gamma}$ because it is the inverse M^{-1} and M^{-2} that appear in expressions (81, 82) of the diffusion coefficient D and the delay time t_0, M being the operator representing the right-hand side of linearized kinetic equations (66). [22] The quantities to be compared (87, 88) are represented as functions of the energy ϵ in figure 9 at temperature $T = \mu_{\mathrm{TF}}/k_B$. There is a remarkable agreement on a wide range of energies around $\epsilon = \mu_{\mathrm{TF}}$. Deviations from the ergodic approximation at very low energy and very high energy were expected : within these limits, the classical dynamics becomes integrable [22]. At high energy, we obtain for the quantity $\Gamma(\mathbf{r}, \mathbf{p})$ the following analytic prediction : [23]

$$\left\langle \frac{1}{\hbar \overline{\Gamma(\mathbf{r}, \mathbf{p})}} \right\rangle_\epsilon \underset{\epsilon \to +\infty}{\sim} \frac{\pi^{5/2}}{56\sqrt{2}} \frac{\epsilon}{\mu_{\mathrm{TF}}^2} \frac{1}{[\rho_0(\mathbf{0}) a^3]^{1/2}} \tag{89}$$

It differs from the ergodic prediction (77) by a numerical coefficient, and reproduces well the results of the numerical simulations (see the red dashed line in figure 9 b). This prevents us from calculating the diffusion coefficient D and the diffusion delay t_0 in the secularo-ergodic approximation at a too high temperature. Concerning $\mathrm{d}\epsilon(\mathbf{r}, \mathbf{p})/\mathrm{d}\mu_{\mathrm{TF}}$, which tends to -1 at high energy, the deviation can only be significant at low energy ; in fact it does so only at very low energy, and it would be a problem for our ergodic calculation of D and t_0 only at temperatures $k_B T \ll \mu_{\mathrm{TF}}$ rarely reached in cold atom experiments.

22. Should it be recalled, $\langle \overline{\Gamma(\mathbf{r}, \mathbf{p})} \rangle_\epsilon = \Gamma(\epsilon)$, the uniform mean being invariant by time evolution. So, the inequality between arithmetic mean and harmonic mean imposes $\langle 1/\overline{\Gamma(\mathbf{r}, \mathbf{p})} \rangle_\epsilon \geq 1/\Gamma(\epsilon)$.

23. At the dominant order in ϵ, we get $\overline{\Gamma(\mathbf{r}, \mathbf{p})}$ using the equivalent (97) (in which $\mu_0 = g\rho_0(\mathbf{r})$) on a harmonic trajectory undisturbed by the condensate, $r_\alpha(t) = A_\alpha \cos(\omega_\alpha t + \phi_\alpha)$, $\forall \alpha \in \{x, y, z\}$. Let us consider cleverly the quantity $g\rho_0(\mathbf{r})$ to be averaged as a function $f(\boldsymbol{\theta})$ of the angles $\theta_\alpha = \omega_\alpha t + \phi_\alpha$. It is a periodic function of period 2π in each direction, expandable in Fourier series, $f(\boldsymbol{\theta}) = \sum_{\mathbf{n} \in \mathbb{Z}^3} c_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$. In the incommensurable case, $\mathbf{n} \cdot \boldsymbol{\omega} \neq 0$ and the time average of $\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}}$ is zero $\forall \mathbf{n} \in \mathbb{Z}^{3*}$, so that $\overline{f(\boldsymbol{\theta})} = c_{\mathbf{0}}$. In the usual integral expression of $c_{\mathbf{0}}$, we make the change of variable $x_\alpha = X_\alpha \cos\theta_\alpha$, where $X_\alpha = (\epsilon_\alpha/\mu_{\mathrm{TF}})^{1/2}$ and ϵ_α is the energy of the motion along $O\alpha$. It remains to take the limit of all X_α tending to $+\infty$ under the integral sign to get

$$\frac{\hbar \overline{\Gamma(\mathbf{r}, \mathbf{p})}}{\mu_{\mathrm{TF}} [\rho_0(\mathbf{0}) a^3]^{1/2}} \underset{\epsilon \to +\infty}{\sim} \frac{32\sqrt{2}}{15\pi^{3/2}} \left(\frac{\epsilon}{\mu_{\mathrm{TF}}} \right)^{1/2} \prod_\alpha \left(\frac{\mu_{\mathrm{TF}}}{\epsilon_\alpha} \right)^{1/2}$$

By averaging the inverse of this equivalent over the probability distribution $2\epsilon^{-2}\delta(\epsilon - \sum_\alpha \epsilon_\alpha)$ of the energies per direction for a harmonic oscillator of total energy ϵ, we find (89).

5. Conclusion

Motivated by recent experimental advances in the manipulation of trapped cold atom gases [1, 2, 3], we theoretically studied the coherence time and the phase dynamics of a Bose-Einstein condensate in an isolated and harmonically trapped boson gas, a fundamental problem important for interferometric applications. The variance of the phase shift experienced by the condensate after a time t increases indefinitely with t, which limits the intrinsic coherence time of the gas. For $t \gg t_{\rm coll}$, where $t_{\rm coll}$ is the typical collision time between Bogoliubov quasiparticles, it becomes a quadratic function of time,

$$\mathrm{Var}[\hat\theta(t) - \hat\theta(0)] = At^2 + 2D(t - t_0) + o(1) \tag{90}$$

where $\hat\theta$ is the phase operator of the condensate. This asymptotic law has the same form as in the spatially homogeneous case previously studied [16], which was not guaranteed, but the coefficients of course differ. To calculate them, we consider the thermodynamic limit in the trap, in which the number of particles tends to infinity, $N \to +\infty$, at fixed temperature T and fixed Gross-Pitaevskii chemical potential $\mu_{\rm GP}$. This requires that the reduced trapping frequencies tend to zero, $\hbar\omega_\alpha/\mu_{\rm GP} \to 0$, which we reinterpret as a classical limit $\hbar \to 0$.

The dominant term At^2 is due to the fluctuations in the initial state of the quantities conserved by temporal evolution, N and E, where E is the total energy of the gas. We give an explicit expression (16) - (19) - (20) of the coefficient A in a generalized ensemble, any statistical mixture of microcanonical ensembles with at most normal fluctuations of N and E. In this case, $A = O(1/N)$. We obtain a simpler form (30) in the case of a statistical mixture of canonical ensembles of the same temperature but of variable number of particles. At usual temperatures, larger than $\mu_{\rm GP}/k_B$, and for Poissonian particle number fluctuations, the contribution to A of thermal fluctuations of E is rendered negligible by a factor of order the non-condensed fraction $\propto (T/T_c)^3$. The variance of N must be reduced to see the effect of thermal fluctuations on the ballistic spread of the condensate phase.

The subdominant term $2D(t - t_0)$ does not depend on the ensemble in which the system is prepared, at least to the first non-zero order $1/N$ at the thermodynamic limit, and it is the only one that remains in the microcanonical ensemble. The calculation of its two ingredients, the diffusion coefficient D of the phase and the diffusion delay t_0, requires the knowledge at all times of the correlation function of $\mathrm{d}\hat\theta/\mathrm{d}t$ in the microcanonical ensemble, and thus the resolution of linearized kinetic equations on the Bogoliubov quasiparticle occupation numbers. It is indeed the temporal fluctuations of these occupation numbers for a given realization of the system which stochastise the evolution of the phase of the condensate. To this end, we adopt a semiclassical description, in which the motion of quasiparticles in the trapped gas is treated classically in the phase space $(\mathbf{r}, \mathbf{p})$, but the quasiparticle bosonic field is still quantum, through the occupation number operators $\hat n(\mathbf{r}, \mathbf{p})$. In quantum observables of the form $\hat A = \sum_k a_k \hat n_k$, such as $\mathrm{d}\hat\theta/\mathrm{d}t$, the average a_k and the sum on the Bogoliubov quantum modes k are then replaced, according to a correspondence principle, by a temporal mean and an integral on the classical trajectories (see equations (42)-(44)). The linearized kinetic equations on the fluctuations $\delta\hat n(\mathbf{r}, \mathbf{p})$ include a transport part, according to the classical Hamiltonian motion of quasiparticles, and a collision integral, local in position, which describes the Beliaev-Landau interaction processes among three quasiparticles. They take the same form as the linearized quantum Boltzmann equations on the semiclassical distribution function $n(\mathbf{r}, \mathbf{p}, t)$ of quasiparticles in phase space. We simplify them in the secular limit $\omega_\alpha t_{\rm coll} \gg 1$ and under the assumption of a classical ergodic motion of quasiparticles. This hypothesis, according to which the fluctuations $\delta\hat n(\mathbf{r}, \mathbf{p})$ averaged on a trajectory depend only on the energy of the trajectory, only holds if the trap is completely anisotropic ; in this case we give a careful numerical justification.

The desired quantities D and t_0, correctly adimensioned, are universal functions of $k_BT/\mu_{\rm TF}$ where $\mu_{\rm TF}$ is the Thomas-Fermi limit of $\mu_{\rm GP}$, and are in particular independent of the ratios $\omega_\alpha/\omega_\beta$ of the trapping frequencies. They are represented on figure 5. An interesting and more directly measurable by-product of our study are the $\Gamma(\epsilon)$ damping rates of the Bogoliubov modes of energy ϵ in the trap. Once the adimensioned temperature $k_BT/\mu_{\rm TF}$ is fixed, the rate is also described by a universal function of $\epsilon/\mu_{\rm TF}$ independent of the trapping frequencies, see figure 3. These results are part of a new class of universality, that of the completely anisotropic harmonic traps, very different from that, theoretically better explored, of spatially homogeneous systems, and will hopefully receive an experimental confirmation soon.

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Appendix A. Behavior of $\Gamma(\epsilon)$ at low and high energy

To get the limiting behaviors (76) and (77) of the $\Gamma(\epsilon)$ damping rates of the Bogoliubov modes in a trap in the secularo-ergodic approximation, we rewrite the integral in phase space (64) as an average on the local Gross-Pitaevskii chemical potential $\mu_0 = g\rho_0(\mathbf{r})$ of the damping rate $\Gamma_h(\epsilon, \mu_0, k_BT)$ of a mode of energy ϵ in a homogeneous system of density μ_0/g and temperature T :

$$\Gamma(\epsilon) = \int_0^{\mu_{\rm TF}} \mathrm{d}\mu_0 P_\epsilon(\mu_0)\Gamma_h(\epsilon, \mu_0, k_BT) \tag{91}$$

with

$$P_\epsilon(\mu_0) \equiv \frac{1}{\rho(\epsilon)} \int \frac{\mathrm{d}^3r\,\mathrm{d}^3p}{(2\pi\hbar)^3}\delta(\epsilon - \epsilon(\mathbf{r},\mathbf{p}))\delta(\mu_0 - g\rho_0(\mathbf{r})) = \frac{4}{\pi\rho(\epsilon)}\frac{1}{(\hbar\bar{\omega})^3}\frac{\epsilon^2(\mu_{\rm TF}-\mu_0)^{1/2}}{(\mu_0^2+\epsilon^2)^{1/2}[(\mu_0^2+\epsilon^2)^{1/2}+\mu_0]^{1/2}} \tag{92}$$

In the $\epsilon \to 0$ limit, we first heuristically replace the integrand in equation (91) with a low energy equivalent, using :

$$P_\epsilon(\mu_0) \underset{\epsilon\to 0}{\sim} \frac{3}{8\sqrt{2}}\frac{\epsilon^{1/2}(\mu_{\rm TF}-\mu_0)^{1/2}}{\mu_{\rm TF}^{1/2}\mu_0^{3/2}} \tag{93}$$

$$\frac{\hbar\Gamma_h(\epsilon,\mu_0,k_BT)}{2} \underset{\epsilon\to 0}{\sim} \epsilon\left(\frac{\mu_0 a^3}{g}\right)^{1/2} F(k_BT/\mu_0) \tag{94}$$

In equation (93), we used equation (34); the result (94) is in reference [32], where the function F is computed and studied. As $F(\theta) \underset{\theta\to+\infty}{\sim} \frac{3\pi^{3/2}}{4}\theta$, this causes in equation (91) the divergent integral $\epsilon^{3/2}\int_0^{\mu_{\rm TF}} \mathrm{d}\mu_0/\mu_0^2$ to appear. It is clear, however, that one should cut this integral to $\mu_0 > \epsilon$ so that the equivalent (94) remains usable, hence the scaling law $\Gamma(\epsilon) \approx \epsilon^{1/2}$, dominated by the edge of the trapped condensate and very different from the linear law of the homogeneous case. To find the prefactor in the law, we simply make the change of scale $\mu_0 = \epsilon\nu_0$ in the integral and use the "high temperature " approximation of reference [39] on Γ_h, uniformly valid near the edge of the trapped condensate,

$$\frac{\hbar\Gamma_h(\epsilon,\mu_0,k_BT)}{2} \underset{k_BT\gg\epsilon,\mu_0}{\sim} k_BT\left(\frac{\mu_0 a^3}{g}\right)^{1/2}\phi(\epsilon/\mu_0) \tag{95}$$

before going to the $\epsilon \to 0$ limit under the integral sign, which leads to the sought equation (76) with [24]

$$\mathcal{I} = \int_0^{+\infty} \mathrm{d}\nu_0 \frac{\nu_0^{1/2}\phi(1/\nu_0)}{(1+\nu_0^2)^{1/2}[\nu_0+(1+\nu_0^2)^{1/2}]^{1/2}} = 4.921\,208\ldots \tag{96}$$

In the limit $\epsilon \to +\infty$, we use the fact that, in the homogeneous case, the damping rate of quasiparticles is reduced to the collision rate $\rho_0\sigma v$ of a particle of velocity $v = (2\epsilon/m)^{1/2}$ with condensate particles, with zero velocity and density ρ_0, with the cross section $\sigma = 8\pi a^2$ for indistinguishable bosons (this is a Beliaev process) :

$$\frac{\hbar\Gamma_h(\epsilon,\mu_0,k_BT)}{2} \underset{\epsilon\to+\infty}{\sim} \mu_0 a\frac{(2m\epsilon)^{1/2}}{\hbar} \tag{97}$$

Using the same high energy expansion (35) for $\rho(\epsilon)$, we find that $P_\epsilon(\mu_0) \sim (8/\pi)(\mu_{\rm TF}-\mu_0)^{1/2}\epsilon^{-3/2}$. The insertion of these equivalents in equation (91) gives (77).

24. In practice, the function ϕ is deduced from equation (57) by doing the classical field approximation $1 + \bar{n}(\mathbf{r},\mathbf{q}) \simeq \bar{n}(\mathbf{r},\mathbf{q}) \simeq k_BT/\epsilon(\mathbf{r},\mathbf{q})$. In the numerical calculation of $\mathcal{I}$, done by taking $\check{\epsilon} = 1/\nu_0$ as the integration variable, we reduce the effect of digital truncation with the help of the development asymptotic $\phi(\check{\epsilon}) \underset{\check{\epsilon}\to+\infty}{=} 4\left(\frac{2\pi}{\check{\epsilon}}\right)^{1/2}\left[2\ln\frac{\check{\epsilon}}{2} + \frac{1-\ln(\check{\epsilon}/2)}{\check{\epsilon}} + \frac{23+6\ln(\check{\epsilon}/2)}{8\check{\epsilon}^2} + O(\frac{\ln\check{\epsilon}}{\check{\epsilon}^3})\right]$, which corrects and improves equation (35) of reference [39].

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