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Weighted Hölder continuity of Riemann-Liouville fractional integrals - Application to regularity of solutions to fractional Cauchy problems with Carathéodory dynamics

Loïc Bourdin

Abstract

This paper is dedicated to several original (weighted) Hölder continuity results for Riemann-Liouville fractional integrals of weighted integrable functions. As an application, we prove a new weighted continuity result for solutions to nonlinear Riemann-Liouville fractional Cauchy problems with Carathéodory dynamics.

Keywords: Fractional calculus; Riemann-Liouville integral; Riemann-Liouville derivative; Caputo derivative; Cauchy problem; Carathéodory dynamic; existence/uniqueness of a solution; weighted integrability; weighted Hölder continuity; Bielecki norm.

AMS Classification: 26A33; 34A08; 34A12; 34A34.

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1 Introduction

Let $n \in \mathbb{N}^*$ denote a positive integer and let $a < b$ be two real numbers. In this introduction, we will use standard notations for the functional framework. For example, L^r will denote the usual Lebesgue space of r -integrable functions, C will denote the space of continuous functions and H^λ the space of λ -Hölder continuous functions, etc. These notations are all detailed in Section 2.1.

The initial motivation of the present paper is to investigate the weighted continuity of the solutions x to nonlinear Riemann-Liouville fractional Cauchy problems of the form

$$\begin{cases} D^\alpha[x](t) = f(x(t), t), & \text{a.e. } t \in [a, b], \\ I^{1-\alpha}[x](a) = x_a, \end{cases} \quad (\text{CP})$$

where $\alpha \in (0, 1]$ and $x_a \in \mathbb{R}^n$ are fixed, where D^α and $I^{1-\alpha}$ are the standard (left) Riemann-Liouville fractional operators (see Section 3 for basic recalls on fractional calculus), and where $f : \mathbb{R}^n \times [a, b] \rightarrow \mathbb{R}^n$, $(x, t) \mapsto f(x, t)$ is a Carathéodory dynamic, in the sense that f is continuous in its first variable x and (only) measurable in its second variable t . In the present work it is not our aim to deal with local (but nonglobal) solutions. As a consequence, for simplicity, we will assume that f is globally Lipschitz in its first variable, in the sense that there exists a nonnegative constant $L \geq 0$ such that $\|f(x_2, t) - f(x_1, t)\|_{\mathbb{R}^n} \leq L\|x_2 - x_1\|_{\mathbb{R}^n}$ for all $x_1, x_2 \in \mathbb{R}^n$ and for almost every $t \in [a, b]$.

A wide literature already deals with Riemann-Liouville fractional Cauchy problems and it is not our aim to give here a complete overview of this topic. For this purpose we refer the reader to the detailed historical reviews in [12, Chapter 3] and [9, 13].

A well-known contraction mapping strategy. Recall that the main method (initiated in [14]) of investigation of (CP) consists in its reduction to the integral representation

$$x(t) = \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}x_a + I^\alpha[f(x, \cdot)](t), \quad (\text{IR})$$

where Γ denotes the usual Gamma function. Therefore, in order to establish the existence and uniqueness of a solution to (CP), a common strategy in the literature (see, e.g., [1, 4, 5, 7, 8, 10, 11]) is based on the contraction mapping theorem. Precisely, the first step is to find two functional spaces $A, B \subset L^1$ such that:

- (i) x is a solution to (CP) if and only if x satisfies (IR), for all $x \in A$;
- (ii) $t \mapsto \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}x_a \in A$;
- (iii) $f(A) \subset B$ (in the sense that $t \mapsto f(x(t), t) \in B$ for all $x \in A$);
- (iv) $I^\alpha[B] \subset A$ (in the sense that $I^\alpha[y] \in A$ for all $y \in B$).

In that case, the functional

$$\begin{aligned} \mathcal{F} : A &\longrightarrow A \\ x &\longmapsto \mathcal{F}(x) : [a, b] \longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathcal{F}(x)(t) := \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}x_a + I^\alpha[f(x, \cdot)](t), \end{aligned}$$

is well-defined, in the sense that $\mathcal{F}(x) \in A$ for all $x \in A$. Then, the second step is to find a norm $\|\cdot\|_A$ on A such that:

- (v) $(A, \|\cdot\|_A)$ is a Banach space;
- (vi) $\mathcal{F} : (A, \|\cdot\|_A) \rightarrow (A, \|\cdot\|_A)$ is a contractive map.

At this stage, one can conclude that \mathcal{F} admits a unique fixed point and thus there exists a unique solution x to (CP) in the functional space A . Thus, in view of getting regularity properties on this solution, one should look for a functional space A satisfying (i)-(vi) as small as possible. On the other hand, another method consists in finding a subset $\tilde{A} \subset A$ such that $I^\alpha[f(A)] \subset \tilde{A}$. In that case, one can conclude from (IR) that x belongs to the set

$$\frac{1}{\Gamma(\alpha)}(\cdot - a)^{\alpha-1}x_a + \tilde{A}$$

included in A .

Weighted Hölder continuity of Riemann-Liouville fractional integrals. It is a very well-known fact that singularities at $t = a$ emerge from the use of (left) Riemann-Liouville fractional operators. The first term in the integral representation (IR) is an illustration of this feature. As a consequence, it was natural for authors to introduce and use weighted functional spaces in order to deal with this phenomenon. For example, we say that a function $x \in L^r_\beta$ is weighted integrable if the function $t \mapsto \Gamma(\beta)(t-a)^{1-\beta}x(t) \in L^r$. The weighted functional spaces C_β and H^λ_β are defined similarly. We refer to Section 2.2 for recalls on weighted functional spaces.

The inclusions of the form $I^\alpha[B] \subset A$ play a crucial role in the above contraction mapping strategy. In the literature (see, e.g., [6, 10]), numerous inclusions of this form have been proved, in particular in the cases where A and B are (weighted or not) Lebesgue, continuous, Hölder spaces, etc. We refer for instance to [12, Chapter 2] and [16, Chapter 1 Paragraph 3] for overviews of these results. In particular, a well-known statement is concerned with the case where B is a Lebesgue space and A is a Hölder space. Precisely, it can be proved that the inclusion

$$I^\alpha[L^r] \subset H^{\alpha-(1/r)} \tag{1}$$

holds true for all $0 \leq (1/r) < \alpha \leq 1$ (see, e.g., [16, Theorem 3.6] or [3, Property 4]).

The first major contribution of the present work is to provide several generalizations of (1) to the case of weighted functional spaces. Precisely, we will derive in Section 4 several inclusions of the form

$$I^\alpha[L^r_\beta] \subset H^\eta_\gamma, \tag{2}$$

with different assumptions on the values of α , β , r , η and γ . We refer to Theorems 2 and 3 for details. As explained in the next paragraph, these new results are of great interest in order to investigate the weighted continuity of solutions to (CP).

We conclude this paragraph by mentioning that some generalizations of (1) to the case of weighted functional spaces have already been explored in the literature. We refer for instance to [16, Theorem 3.8] and [15, 17]. Nevertheless, to the best of our knowledge, the inclusions derived in Theorems 2 and 3 are all new and cannot be recovered from previous statements in the literature.

Carathéodory dynamic versus continuous dynamic. In the present work, recall that f is a Carathéodory dynamic assumed to be globally Lipschitz in its first variable. Assuming moreover that $t \mapsto f(0_{\mathbb{R}^n}, t) \in L^1$, the above contraction mapping strategy can be applied with $A = B = L^1$ and by considering a Bielecki norm [2] on L^1 which is equivalent to the usual norm $\|\cdot\|_{L^1}$ (see Section 2.3 for recalls about Bielecki norms). In that situation, the existence and uniqueness of a solution to (CP) in the functional space L^1 is established. This approach has been explored in [8] (and has been adapted to the multi-fractional order case in [4]). However, to the best of our knowledge, no information is known about the weighted continuity of the solution in that framework.

On the other hand, assuming that $f(C_\alpha) \subset C_\alpha$, the contraction mapping strategy can also be applied with $A = B = C_\alpha$ and an appropriate Bielecki norm on C_α . In that situation, the existence and uniqueness of a solution to (CP) in the functional space C_α is established. In that case, the weighted continuity of the solution is proved. We refer for instance to [10] (see also [12, Theorem 3.11]) where this result has been stated from another method called Kolmogorov-Fomin. However, in that framework, the weighted continuity of the solution to (CP) is indeed guaranteed but under the (quite restrictive) assumption $f(C_\alpha) \subset C_\alpha$, which implies in particular that the dynamic f has to be continuous with respect to its second variable t . Note that the assumption $f(C_\alpha) \subset C_\alpha$ is particularly restrictive in the field of fractional control theory. For example, in the one-dimensional setting $n = 1$, we know that the linear control system

$$\begin{cases} D^\alpha[x](t) = u(t)x(t), & \text{a.e. } t \in [a, b], \\ I^{1-\alpha}[x](a) = x_a, \end{cases}$$

where the control $u \in L^\infty$, admits a unique solution x in the functional space L^1 . Unfortunately, since the control u is not continuous *a priori*, the assumption $f(C_\alpha) \subset C_\alpha$ is not satisfied and we cannot conclude that x belongs to the weighted functional space C_α .

The second major contribution of the present paper is to fill this gap in the literature. Precisely, we will prove in Section 5, under an assumption of the form $t \mapsto f(0_{\mathbb{R}^n}, t) \in L^r_\beta$ and without the assumption $f(C_\alpha) \subset C_\alpha$, that (CP) admits a unique solution that belongs to the weighted functional space C_α . We refer to Theorem 6 for details. This theorem will be derived from the above contraction mapping strategy and by applying the new inclusions of the form $I^\alpha[L^r_\beta] \subset H^\eta_\gamma$ stated in Section 4.

Organization of the paper. Section 2 is dedicated to the functional framework of the present paper. Section 3 is devoted to basic recalls on (left) Riemann-Liouville and Caputo fractional operators. Section 4 contains the major contributions of the present work. Precisely, several new inclusions of the form $I^\alpha[L^r_\beta] \subset H^\eta_\gamma$ are stated in Theorems 2 and 3. Finally, Section 5 can be seen as an application of these results, in which we investigate the weighted continuity of the solution to (CP). We refer to Theorem 6 for details.

2 Functional framework

This section is devoted to the functional framework of the present paper. All notions recalled below are very standard and usual in the literature. Throughout this work, $n \in \mathbb{N}^*$ denotes a

positive integer and $a < b$ are two real numbers. In the whole paper, for all $1 \leq r \leq \infty$, we denote by $1 \leq r' \leq \infty$ the classical Lebesgue conjugate of r defined by $r' := \frac{r}{r-1}$ and satisfying the equality $\frac{1}{r} + \frac{1}{r'} = 1$.

2.1 Classical functional spaces

In the whole paper we denote by:

- $L^r := L^r([a, b], \mathbb{R}^n)$ the classical Lebesgue space of r -integrable functions on $[a, b]$ with values in \mathbb{R}^n , endowed with its usual norm $\|\cdot\|_{L^r}$, for all $1 \leq r < \infty$;
- $L^\infty := L^\infty([a, b], \mathbb{R}^n)$ the classical Lebesgue space of essentially bounded functions on $[a, b]$ with values in \mathbb{R}^n , endowed with its usual norm $\|\cdot\|_{L^\infty}$;
- $C := C([a, b], \mathbb{R}^n)$ the classical space of continuous functions on $[a, b]$ with values in \mathbb{R}^n , endowed with the classical uniform norm $\|\cdot\|_C$;
- $AC := AC([a, b], \mathbb{R}^n)$ the classical subspace of C of all absolutely continuous functions;
- $H^\lambda := H^\lambda([a, b], \mathbb{R}^n)$ the classical subspace of C of all λ -Hölder continuous functions, for all $0 < \lambda \leq 1$.

If E denotes one of the three last above functional spaces, we denote by E^0 the functional subspace defined by

$$E^0 := \{x \in E \mid x(a) = 0_{\mathbb{R}^n}\}.$$

In particular we endow $H^{\lambda,0}$ with its usual norm

$$\|x\|_{H^{\lambda,0}} := \sup_{a \leq t_1 < t_2 \leq b} \frac{\|x(t_2) - x(t_1)\|_{\mathbb{R}^n}}{(t_2 - t_1)^\lambda},$$

for all $x \in H^{\lambda,0}$ and all $0 < \lambda \leq 1$. Finally we recall that all above normed spaces are Banach spaces.

2.2 Weighted functional spaces

In this paper, if E denotes one of the above classical functional spaces, the associated weighted functional space E_α is defined by

$$E_\alpha := \{x \in L^1 \mid \rho_\alpha x \in E\},$$

where the weight function $\rho_\alpha \in C$ is defined by

$$\rho_\alpha(t) := \Gamma(\alpha)(t - a)^{1-\alpha},$$

for all $t \in [a, b]$ and all $0 < \alpha \leq 1$, where Γ denotes the classical Gamma function. Moreover, if $(E, \|\cdot\|_E)$ is one of the above Banach spaces, we endow E_α with the norm $\|\cdot\|_{E_\alpha}$ defined by

$$\|x\|_{E_\alpha} := \|\rho_\alpha x\|_E,$$

for all $x \in E_\alpha$ and all $0 < \alpha \leq 1$. In that case note that $(E_\alpha, \|\cdot\|_{E_\alpha})$ is also a Banach space.

Remark 1. If E is one of the above classical functional spaces, note that $E_1 = E$.

Remark 2. The inclusions $C \subset C_\alpha \subset L^\infty_\alpha \subset L^1_\alpha$ and $L^\infty \subset L^\infty_\alpha \subset L^1$ hold true for all $0 < \alpha \leq 1$. Other simple inclusions can be easily derived.

Remark 3. From the classical Hölder inequality (recalled in Appendix A, see Lemma 3), the inclusion $L^r_\alpha \subset L^1$ holds true for all $0 \leq (1/r) < \alpha \leq 1$.

Example 1. Let us consider the one-dimensional setting $n = 1$ and let $0 < \alpha, \beta \leq 1$. One can easily see that $\frac{1}{\rho_\alpha} \in H^{\alpha-\beta,0}_\beta$ if $0 < \beta < \alpha \leq 1$, and that $\frac{1}{\rho_\alpha} \in H^1_\beta$ if $\beta = \alpha$. On the other hand, it holds that $\frac{1}{\rho_\alpha} \in L^r_\beta$ for all $0 < \alpha < \beta \leq 1$ and all $1 \leq r < \frac{1}{\beta-\alpha}$.

2.3 Equivalent Bielecki norms

Let $(E, \|\cdot\|_E)$ be one of the above Banach spaces. The Bielecki norm $\|\cdot\|_{E,k}$ (see [2]) defined on E by

$$\|x\|_{E,k} := \|e_k x\|_E,$$

for all $x \in E$, where the weight function $e_k \in C$ is defined by

$$e_k(t) := e^{-k(t-a)},$$

for all $t \in [a, b]$ and all $k \in \mathbb{N}$, is an equivalent norm on E to $\|\cdot\|_E$. In particular $(E, \|\cdot\|_{E,k})$ is also a Banach space for all $k \in \mathbb{N}$.

3 Basics on fractional calculus

Throughout the paper, the abbreviation R-L stands for Riemann-Liouville. This section is dedicated to basic recalls about R-L and Caputo fractional operators. All definitions and results recalled below are very standard in the literature and are mostly extracted from the monographs [12, 16].

3.1 R-L fractional integrals

The R-L fractional integral $I^\alpha[x]$ of order $\alpha > 0$ of $x \in L^1$ is defined on $[a, b]$ by

$$I^\alpha[x](t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau,$$

provided that the right-hand side term exists. For $\alpha = 0$ and $x \in L^1$, we define $I^0[x] := x$. For the two next results we refer to [12, Lemmas 2.1 and 2.3].

Proposition 1. *The continuous embedding*

$$I^\alpha[L^1] \subset L^1$$

holds true for all $\alpha \geq 0$, and a constant of continuity is given by $\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}$.

Proposition 2. *The equalities*

$$I^{\alpha_1} \left[I^{\alpha_2} [x] \right] = I^{\alpha_1+\alpha_2} [x] = I^{\alpha_2+\alpha_1} [x] = I^{\alpha_2} \left[I^{\alpha_1} [x] \right]$$

hold true for all $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and all $x \in L^1$.

Let $\alpha \geq 0$ and $x \in L^1$. From Proposition 1, $I^\alpha[x](t)$ exists for almost every $t \in [a, b]$. Throughout the paper, if $I^\alpha[x]$ is equal almost everywhere to a continuous function on $[a, b]$, then $I^\alpha[x]$ is automatically identified to its continuous representative. In that case $I^\alpha[x](t)$ exists for every $t \in [a, b]$. One of the most well-known results in this sense is recalled in the next proposition. We refer to [16, Theorem 3.6] (see also [3, Property 4]).

Proposition 3. *The continuous embedding*

$$I^\alpha[L^r] \subset H^{\alpha-(1/r),0}$$

holds true for all $0 \leq \alpha \leq 1$ and all $1 \leq r \leq \infty$ such that $0 \leq (1/r) < \alpha \leq 1$, and a constant of continuity is given by $\frac{2}{\Gamma(\alpha)(r'(\alpha-1)+1)^{1/r'}}$.

Remark 4. Let $0 \leq \alpha \leq 1$ and $1 \leq r \leq \infty$ such that $0 \leq (1/r) < \alpha \leq 1$. In that case, note that

$$0 < \alpha \leq 1, \quad 1 < r \leq \infty \quad \text{and} \quad 1 \leq r' < \frac{1}{1-\alpha}.$$

In particular it holds that $r'(\alpha-1)+1 > 0$, and thus the constant of continuity in Proposition 3 is well-defined.

Remark 5. Section 4 is devoted to several generalizations of Proposition 3, replacing L^r by L^r_β . As a consequence, a proof of Proposition 3 can be found in Appendix A.2, by considering the case $\beta = 1$.

3.2 R-L and Caputo fractional derivatives

We say that $x \in L^1$ admits a R-L fractional derivative $D^\alpha[x]$ of order $0 \leq \alpha \leq 1$ if and only if $I^{1-\alpha}[x] \in AC$. In that case $D^\alpha[x]$ is defined by

$$D^\alpha[x](t) := \frac{d}{dt} \left[I^{1-\alpha}[x] \right](t),$$

for almost every $t \in [a, b]$. In particular $D^\alpha[x] \in L^1$. We denote by AC^α the set of all functions $x \in L^1$ possessing a R-L fractional derivative $D^\alpha[x]$ of order $0 \leq \alpha \leq 1$.

Remark 6. If $\alpha = 1$, $AC^1 = AC$ and $D^1[x] = \dot{x}$ for all $x \in AC$. If $\alpha = 0$, $AC^0 = L^1$ and $D^0[x] = x$ for all $x \in L^1$.

We say that $x \in C$ admits a Caputo fractional derivative ${}_cD^\alpha[x]$ of order $0 \leq \alpha \leq 1$ if and only if $x - x(a) \in AC^\alpha$. In that case ${}_cD^\alpha[x]$ is defined by

$${}_cD^\alpha[x](t) := D^\alpha[x - x(a)](t),$$

for almost every $t \in [a, b]$. In particular ${}_cD^\alpha[x] \in L^1$. We denote by ${}_cAC^\alpha$ the set of all functions $x \in C$ possessing a Caputo fractional derivative ${}_cD^\alpha[x]$ of order $0 \leq \alpha \leq 1$.

Remark 7. If $\alpha = 1$, ${}_cAC^1 = AC$ and ${}_cD^1[x] = \dot{x}$ for all $x \in AC$. If $\alpha = 0$, ${}_cAC^0 = C$ and ${}_cD^0[x] = x - x(a)$ for all $x \in C$.

Remark 8. From Propositions 1 and 2, one can easily see that the inclusion $AC \subset {}_cAC^\alpha$ holds true with ${}_cD^\alpha[x] = I^{1-\alpha}[\dot{x}]$ for all $x \in AC$ and all $0 \leq \alpha \leq 1$.

4 Weighted Hölder continuity of R-L fractional integrals

This section contains the major contributions of the present paper. Proposition 3 gives a sufficient condition on $0 \leq \alpha \leq 1$ and $1 \leq r \leq \infty$ for the Hölder continuity of $I^\alpha[x]$ for $x \in L^r$. In this section, our main goal is to provide sufficient conditions on $0 \leq \alpha, \beta \leq 1$ and $1 \leq r \leq \infty$ for the (weighted) Hölder continuity of $I^\alpha[x]$ for $x \in L_\beta^r$. Precisely our aim is to derive several generalizations of Proposition 3 replacing L^r by L_β^r .

Following the framework of Proposition 3, we will assume that $0 \leq (1/r) < \alpha \leq 1$. Moreover, in order to guarantee the inclusion $L_\beta^r \subset L^1$ (and thus that $I^\alpha[x]$ is well-defined for all $x \in L_\beta^r$), we will also assume that $0 \leq (1/r) < \beta \leq 1$ (see Remark 3). Hence, in the whole paper, we will only deal with the following framework:

$$0 \leq \alpha, \beta \leq 1, \quad 1 \leq r \leq \infty \quad \text{and} \quad 0 \leq (1/r) < \min(\alpha, \beta) \leq 1. \quad (\mathcal{A}_r^{\alpha, \beta})$$

Similarly to Remark 5, if $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied, note that

$$0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 1 < r \leq \infty \quad \text{and} \quad 1 \leq r' < \min\left(\frac{1}{1-\alpha}, \frac{1}{1-\beta}\right).$$

In particular it holds that $r'(\alpha - 1) + 1 > 0$ and $r'(\beta - 1) + 1 > 0$. In that case we introduce $C(\alpha, \beta, r) > 0$ the positive constant defined by

$$C(\alpha, \beta, r) := \frac{2}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\Gamma(r'(\alpha - 1) + 1)\Gamma(r'(\beta - 1) + 1)}{\Gamma(r'(\alpha + \beta - 2) + 2)} \right)^{1/r'}.$$

In the case $\beta = 1$, note that $C(\alpha, \beta, r) = C(\alpha, 1, r)$ recovers the constant of continuity given in Proposition 3.

If $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied, note that

$$-1 < \alpha - (1/r) + (\beta - 1) \leq 1.$$

Proposition 4 below highlights that the sign of $\alpha - (1/r) + (\beta - 1)$ plays a crucial role in the singularity at $t = a$ of $I^\alpha[x]$ for $x \in L_\beta^r$. As a consequence, this section will be divided in two cases:

$$\alpha - (1/r) + (\beta - 1) > 0 \quad \text{and} \quad \alpha - (1/r) + (\beta - 1) \leq 0.$$

In the first case, we will obtain the Hölder continuity of $I^\alpha[x]$ for $x \in L_\beta^r$ (see Theorems 1 and 2), while the weighted Hölder continuity will be derived in the second case (see Theorem 3).

To the best of our knowledge, all results presented in this section (except Theorem 1, which coincides with Proposition 3) are new in the literature. For the reader's convenience, the technical proofs are all provided and detailed in Appendix A.

Proposition 4. *If $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied, then it holds that*

$$\|I^\alpha[x](t)\|_{\mathbb{R}^n} \leq \frac{1}{2}C(\alpha, \beta, r)\|x\|_{L_\beta^r}(t-a)^{\alpha-(1/r)+(\beta-1)},$$

for almost every $t \in [a, b]$ and all $x \in L_\beta^r$. In particular, if moreover $\alpha - (1/r) + (\beta - 1) \geq 0$, the continuous embedding

$$I^\alpha[L_\beta^r] \subset L^\infty$$

holds true, and a constant of continuity is given by $\frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)}C(\alpha, \beta, r)$.

Proof. See Appendix A.1. □

4.1 Hölder continuity in the case $\alpha - (1/r) + (\beta - 1) > 0$

We first deal with the case $\beta = 1$ (and thus $L_\beta^r = L^r$) in order to recover Proposition 3. Precisely, Proposition 3 can be stated as follows.

Theorem 1. *If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta = 1$, then the continuous embedding*

$$I^\alpha[L_\beta^r] \subset H^{\alpha-(1/r)+(\beta-1),0}$$

holds true, and a constant of continuity is given by $C(\alpha, \beta, r)$.

Proof. See Appendix A.2. □

In this section our aim is to provide a generalization of Theorem 1 to the case $\beta \in (0, 1)$. For this purpose we first enunciate the following crucial lemma.

Lemma 1. *If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta \in (0, 1)$, then the continuous embedding*

$$L_\beta^r \subset L^{1/(\alpha-\lambda)}$$

holds true for all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$, and a constant of continuity is given by

$$\frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)-\lambda} C\left(1, \beta, \frac{1}{1+(1/r)-(\alpha-\lambda)}\right).$$

Proof. See Appendix A.3. □

We are now in a position to state a generalization of Theorem 1 to the case $\beta \in (0, 1)$ as follows.

Theorem 2. *If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta \in (0, 1)$, then the continuous embedding*

$$I^\alpha[L_\beta^r] \subset H^{\lambda,0},$$

holds true for all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$, and a constant of continuity is given by

$$\frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)-\lambda} C\left(\alpha, 1, \frac{1}{\alpha-\lambda}\right) C\left(1, \beta, \frac{1}{1+(1/r)-(\alpha-\lambda)}\right).$$

Proof. See Appendix A.4. □

We conclude from Theorems 1 and 2 that, if $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$, then $I^\alpha[x]$ is Hölder continuous on $[a, b]$, and vanishes at $t = a$, for all $x \in L_\beta^r$.

4.2 Weighted Hölder continuity in the case $\alpha - (1/r) + (\beta - 1) \leq 0$

If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, note that necessarily $\alpha, \beta \in (0, 1)$. In that situation, one cannot expect from $I^\alpha[x]$ for $x \in L_\beta^r$ to be Hölder continuous on $[a, b]$ in general. For example, in the one-dimensional setting $n = 1$, one can easily see that

$$\frac{1}{\rho_\beta} \in L_\beta^\infty, \quad \text{while} \quad I^\alpha\left[\frac{1}{\rho_\beta}\right] = \frac{1}{\rho_{\alpha+\beta}} \notin C,$$

for any $0 < \alpha, \beta \leq 1$ such that $\alpha + \beta < 1$ (take $\alpha = \beta = \frac{1}{4}$ for example). Nevertheless Theorem 3 below asserts that $I^\alpha[x]$ for $x \in L_\beta^r$ is at least weighted Hölder continuous on $[a, b]$.

Theorem 3. If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, then the continuous embedding

$$I^\alpha[L_\beta^r] \subset H_\gamma^{\eta,0},$$

holds true for all $0 < \gamma < \alpha - (1/r) + \beta$ and for all $0 < \eta < \alpha - (1/r)$ such that

$$0 < \eta \leq \min\left(\alpha - (1/r) + (\beta - \gamma), 1 - \alpha\right).$$

A constant of continuity is given by

$$\frac{\Gamma(\gamma)}{2}(b-a)^{\alpha-(1/r)+(\beta-\gamma)-\eta} \times \left(\frac{\Gamma(\beta+\mu)}{\Gamma(\beta)} C(\alpha, \beta + \mu, r) + \frac{\Gamma(\alpha-\eta)}{\Gamma(\alpha)} C(\alpha - \eta, \beta, r) \right),$$

where $\mu := \min(1 - \beta, 1 - \gamma) = 1 - \max(\beta, \gamma)$.

Proof. See Appendix A.5. □

The inclusion $I^\alpha[L_\beta^r] \subset H_\gamma^{\eta,0}$ and the assumptions in Theorem 3 can be declined in several cases. The next corollary is in this sense.

Corollary 1. If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, then two cases:

(i) If $0 < \alpha \leq \frac{r+1}{2r}$, then the inclusion

$$I^\alpha[L_\beta^r] \subset \left(H_\beta^{\eta,0} \cap H_\gamma^{\alpha-(1/r)+(\beta-\gamma),0} \right),$$

holds true for all $0 < \eta < \alpha - (1/r)$ and all $\beta < \gamma < \alpha - (1/r) + \beta$;

(ii) If $\frac{r+1}{2r} < \alpha < 1$, then the inclusion

$$I^\alpha[L_\beta^r] \subset \left(H_{2\alpha-(1/r)+(\beta-1)}^{1-\alpha,0} \cap H_\gamma^{\alpha-(1/r)+(\beta-\gamma),0} \right),$$

holds true for all $2\alpha - (1/r) + (\beta - 1) < \gamma < \alpha - (1/r) + \beta$.

If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, Theorem 3 (and Corollary 1) states that $I^\alpha[x]$ for $x \in L_\beta^r$ is weighted Hölder continuous on $[a, b]$. In fact, it can be proved that $I^\alpha[x]$ is Hölder continuous on all intervals $[a + \varepsilon, b]$ with $\varepsilon > 0$. Indeed, from the proof of Theorem 3 in Appendix A.5 (precisely from Proposition 14), the next proposition can be deduced.

Proposition 5. If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, then two cases:

(i) If $0 < \alpha \leq \frac{r+1}{2r}$, then $I^\alpha[x]$ is η -Hölder continuous on $[a + \varepsilon, b]$ for all $0 < \eta < \alpha - (1/r)$;

(ii) If $\frac{r+1}{2r} < \alpha < 1$, then $I^\alpha[x]$ is $(1 - \alpha)$ -Hölder continuous on $[a + \varepsilon, b]$;

for all $x \in L_\beta^r$ and all $\varepsilon > 0$.

4.3 Descriptions of some typical situations

The aim of this section is to describe the results of Theorems 1, 2 and 3 (precisely Corollary 1) and Proposition 5 in some typical situations of values for α , β and r .

Proposition 6 ($\beta = \alpha$). Let $0 \leq \alpha \leq 1$ and $1 \leq r \leq \infty$ such that $0 \leq \frac{1}{r} < \alpha \leq 1$. Three cases:

(i) If $\alpha = 1$, then the inclusion $I^1[L^r] \subset H^{1-(1/r),0}$ holds true.

(ii) If $\frac{r+1}{2r} < \alpha < 1$, then the inclusion $I^\alpha[L_\alpha^r] \subset H^{\lambda,0}$ holds true for all $0 < \lambda < 2\alpha - (\frac{r+1}{r})$.

(iii) If $0 < \alpha \leq \frac{r+1}{2r}$, then the inclusion

$$I^\alpha[L_\alpha^r] \subset \left(H_\alpha^{\eta,0} \cap H_\gamma^{2\alpha-(1/r)-\gamma,0} \right)$$

holds true for all $0 < \eta < \alpha - (1/r)$ and all $\alpha < \gamma < 2\alpha - (1/r)$.

In the last case, for all $\varepsilon > 0$, it holds that $I^\alpha[x]$ is η -Hölder continuous on $[a + \varepsilon, b]$ for all $0 < \eta < \alpha - (1/r)$ and all $x \in L_\alpha^r$.

Proposition 7 ($\beta = \alpha$ and $r = \infty$). Let $0 < \alpha \leq 1$. Three cases:

(i) If $\alpha = 1$, then the inclusion $I^1[L^\infty] \subset H^{1,0}$ holds true.

(ii) If $\frac{1}{2} < \alpha < 1$, then the inclusion $I^\alpha[L_\alpha^\infty] \subset H^{\lambda,0}$ holds true for all $0 < \lambda < 2\alpha - 1$.

(iii) If $0 < \alpha \leq \frac{1}{2}$, then the inclusion

$$I^\alpha[L_\alpha^\infty] \subset \left(H_\alpha^{\eta,0} \cap H_\gamma^{2\alpha-\gamma,0} \right)$$

holds true for all $0 < \eta < \alpha$ and all $\alpha < \gamma < 2\alpha$.

In the last case, for all $\varepsilon > 0$, it holds that $I^\alpha[x]$ is η -Hölder continuous on $[a + \varepsilon, b]$ for all $0 < \eta < \alpha$ and all $x \in L_\alpha^\infty$.

Proposition 8 ($\beta = 1 - \alpha$ and $r = \infty$). Let $0 < \alpha < 1$. Two cases:

(i) If $\frac{1}{2} < \alpha < 1$, then the inclusion

$$I^\alpha[L_{1-\alpha}^\infty] \subset \left(H_\alpha^{1-\alpha,0} \cap H_\gamma^{1-\gamma,0} \right),$$

holds true for all $\alpha < \gamma < 1$;

(ii) If $0 < \alpha \leq \frac{1}{2}$, then the inclusion

$$I^\alpha[L_{1-\alpha}^\infty] \subset \left(H_{1-\alpha}^{\eta,0} \cap H_\gamma^{1-\gamma,0} \right),$$

holds true for all $0 < \eta < \alpha$ and all $1 - \alpha < \gamma < 1$.

5 Regularity of solutions to fractional Cauchy problems

This section is dedicated to the application of the new results stated in Section 4 to the regularity of solutions to fractional Cauchy problems with Carathéodory dynamics. Precisely we will focus on the two following nonlinear fractional Cauchy problems:

$$\begin{cases} {}_c D^\alpha[x](t) = f(x(t), t), & \text{a.e. } t \in [a, b], \\ x(a) = x_a, \end{cases} \quad ({}_c\text{CP})$$

and

$$\begin{cases} D^\alpha[x](t) = f(x(t), t), & \text{a.e. } t \in [a, b], \\ I^{1-\alpha}[x](a) = x_a, \end{cases} \quad (\text{CP})$$

where $\alpha \in (0, 1]$ and $x_a \in \mathbb{R}^n$ are fixed, and where $f : \mathbb{R}^n \times [a, b] \rightarrow \mathbb{R}^n$, $(x, t) \mapsto f(x, t)$ is a Carathéodory function, in the sense that f is continuous in its first variable x and (only) measurable in its second variable t .

Remark 9. Note that (cCP) involves a Caputo fractional derivative ${}_cD^\alpha$ and the initial condition $x(a) = x_a$, while (CP) involves a R-L fractional derivative D^α and the initial condition $I^{1-\alpha}[x](a) = x_a$.

Following the framework introduced in Section 3, we will naturally be looking for solutions in ${}_cAC^\alpha$ for (cCP) , and in AC^α for (CP) . Our objective in this section is to investigate the (weighted) continuity of these solutions from the statements of Section 4.

In the present work it is not our aim to deal with local (but nonglobal) solutions. As a consequence, for simplicity, we will assume that f is globally Lipschitz, in the sense that there exists a nonnegative constant $L \geq 0$ such that

$$\|f(x_2, t) - f(x_1, t)\|_{\mathbb{R}^n} \leq L\|x_2 - x_1\|_{\mathbb{R}^n}, \quad (\mathcal{H}_{\text{glob}})$$

for all $x_1, x_2 \in \mathbb{R}^n$ and for almost every $t \in [a, b]$. For $0 < \beta \leq 1$ and $1 \leq r \leq \infty$, we will denote by $(\mathcal{H}_{r,\beta})$ the hypothesis

$$t \mapsto f(0_{\mathbb{R}^n}, t) \in L^r_\beta. \quad (\mathcal{H}_{r,\beta})$$

Remark 10. In the literature (see, e.g., [8]), the authors usually consider that $(\mathcal{H}_{1,1})$ (or $(\mathcal{H}_{\infty,1})$) is satisfied. Note that Hypothesis $(\mathcal{H}_{r,\beta})$ introduced in this paper constitutes a (modest) generalization which is suitable for the functional framework considered in this paper.

Remark 11. Note that, if $(\mathcal{H}_{\infty,1})$ is satisfied, then $(\mathcal{H}_{r,\beta})$ is satisfied for all $0 < \beta \leq 1$ and all $1 \leq r \leq \infty$.

Let $A, B \subset L^1$ be two functional spaces. In the whole section, we will use the notation

$$f(A) \subset B \iff \forall x \in A, \quad t \mapsto f(x(t), t) \in B.$$

The next proposition gives an example.

Proposition 9. *Let $0 < \beta_1, \beta_2 \leq 1$ and $1 \leq r_1, r_2 \leq \infty$. Let us assume that $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{r_2, \beta_2})$ are satisfied. If $r_2 \leq r_1$, with $\beta_2 - \beta_1 < \frac{1}{r_2} - \frac{1}{r_1}$ if $\beta_2 - \beta_1 > 0$, then the inclusion*

$$f(L^{r_1}_{\beta_1}) \subset L^{r_2}_{\beta_2}$$

holds true.

Proof. It holds that

$$\|\rho_{\beta_2}(t)f(x(t), t)\|_{\mathbb{R}^n} \leq L \frac{\Gamma(\beta_2)}{\Gamma(\beta_1)} (t-a)^{\beta_1-\beta_2} \|\rho_{\beta_1}(t)x(t)\|_{\mathbb{R}^n} + \rho_{\beta_2}(t) \|f(0_{\mathbb{R}^n}, t)\|_{\mathbb{R}^n},$$

for almost every $t \in [a, b]$ and for all $x \in L^{r_1}_{\beta_1}$. This concludes the proof in the case $\beta_1 - \beta_2 \geq 0$. In the case $\beta_1 - \beta_2 < 0$, the proof is concluded by applying the classical Hölder inequality (recalled in Appendix A, see Lemma 3). \square

5.1 Hölder continuity of solutions to (cCP)

Assuming for example that $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{\infty,1})$ are satisfied, it is well-known, and easy to prove from Proposition 3, that (cCP) admits a unique solution that is α -Hölder continuous on $[a, b]$. Hence, the only contribution of this section is to derive a similar result, but under the generalized and weaker hypothesis $(\mathcal{H}_{r,\beta})$ instead of $(\mathcal{H}_{\infty,1})$.

Note that if x is a solution to (cCP) , then $x \in {}_cAC^\alpha \subset C$ and $t \mapsto f(x(t), t) \in L^1$. From Section 3 and Theorems 1 and 2, one can easily derive the following integral representation.

Proposition 10 (Integral representation). *Let $(\mathcal{A}_r^{\alpha,\beta})$ be satisfied with $\alpha - (1/r) + (\beta - 1) > 0$. Let $x \in L^1$ such that $t \mapsto f(x(t), t) \in L_\beta^r$. Then x is a solution to (cCP) if and only if it holds that*

$$x(t) = x_a + \mathbb{I}^\alpha[f(x, \cdot)](t),$$

for every $t \in [a, b]$.

From Propositions 9 and 10 and from Theorems 1 and 2, we get the next theorem.

Theorem 4. *Let $(\mathcal{A}_r^{\alpha,\beta})$ be satisfied with $\alpha - (1/r) + (\beta - 1) > 0$. If $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{r,\beta})$ are satisfied, then (cCP) admits a unique solution. Moreover this solution belongs to $H^{\alpha - (1/r)}$ if $\beta = 1$, and to H^λ for all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$ if $\beta \in (0, 1)$.*

Proof. From Proposition 9, we know that the inclusion $f(L^\infty) \subset L_\beta^r$ holds true. As a consequence, the inclusion $f(C) \subset L_\beta^r$ also holds true. From Theorems 1 and 2, we know that the application

$$\begin{aligned} \mathcal{F}: C &\longrightarrow C \\ x &\longmapsto \mathcal{F}(x): [a, b] \longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathcal{F}(x)(t) := x_a + \mathbb{I}^\alpha[f(x, \cdot)](t), \end{aligned}$$

is well-defined, in the sense that $\mathcal{F}(x) \in C$ for all $x \in C$. Then, considering $k \in \mathbb{N}$ sufficiently large for $\frac{L}{k^\alpha} < 1$, we endow C with the Bielecki norm $\|\cdot\|_{C,k}$ (see Section 2.3 for details). It holds that

$$\|e_k(t)(\mathcal{F}(y)(t) - \mathcal{F}(x)(t))\|_{\mathbb{R}^n} \leq \frac{L}{\Gamma(\alpha)} e^{-k(t-a)} \int_a^t (t-\tau)^{\alpha-1} \|y(\tau) - x(\tau)\|_{\mathbb{R}^n} d\tau,$$

and thus

$$\|e_k(t)(\mathcal{F}(y)(t) - \mathcal{F}(x)(t))\|_{\mathbb{R}^n} \leq \frac{L}{\Gamma(\alpha)} \|y - x\|_{C,k} \int_a^t (t-\tau)^{\alpha-1} e^{-k(t-\tau)} d\tau,$$

for all $t \in [a, b]$ and all $x, y \in C$. With a simple change of variable, it can be proved that $\int_a^t (t-\tau)^{\alpha-1} e^{-k(t-\tau)} d\tau \leq \frac{\Gamma(\alpha)}{k^\alpha}$. Since $\frac{L}{k^\alpha} < 1$, we conclude that \mathcal{F} is a contractive map on the Banach space $(C, \|\cdot\|_{C,k})$ and thus \mathcal{F} admits a unique fixed point. From Proposition 10 and the inclusion $f(C) \subset L_\beta^r$, we deduce that (cCP) admits a unique solution. The regularity of this solution is obtained from the inclusion $f(C) \subset L_\beta^r$ and from Theorems 1 and 2. \square

Considering $(r, \beta) = (\infty, 1)$ in Theorem 4, we recover the following well-known result as a corollary.

Corollary 2. *If $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{\infty,1})$ are satisfied, then (cCP) admits a unique solution. Moreover this solution belongs to H^α .*

5.2 Weighted Hölder continuity of solutions to (CP)

The major contribution of the present work about regularity of solutions to fractional Cauchy problems with Carathéodory dynamics concerns the weighted continuity of solutions to (CP) in this paragraph.

Note that if x is a solution to (CP) , then $x \in AC^\alpha \subset L^1$ and $t \mapsto f(x(t), t) \in L^1$. From Section 3, one can easily derive the following well-known integral representation.

Proposition 11 (Integral representation). *Let $x \in L^1$ such that $t \mapsto f(x(t), t) \in L^1$. Then x is a solution to (CP) if and only if it holds that*

$$x(t) = \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha-1} x_a + \mathbb{I}^\alpha[f(x, \cdot)](t),$$

for almost every $t \in [a, b]$.

From Propositions 9 and 11, we recover the following theorem already well-known in the literature (see, e.g., [8]). The proof is recalled for the reader's convenience.

Theorem 5. *If $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{1,1})$ are satisfied, then (CP) admits a unique solution.*

Proof. From Proposition 9, we know that the inclusion $f(\mathbb{L}^1) \subset \mathbb{L}^1$ holds true. We deduce from Proposition 1 that the application

$$\begin{aligned} \mathcal{F} : \mathbb{L}^1 &\longrightarrow \mathbb{L}^1 \\ x &\longmapsto \mathcal{F}(x) : [a, b] \longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathcal{F}(x)(t) := \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}x_a + \text{I}^\alpha[f(x, \cdot)](t), \end{aligned}$$

is well-defined, in the sense that $\mathcal{F}(x) \in \mathbb{L}^1$ for all $x \in \mathbb{L}^1$. Then, considering $k \in \mathbb{N}$ sufficiently large for $\frac{L}{k^\alpha} < 1$, we endow \mathbb{L}^1 with the Bielecki norm $\|\cdot\|_{\mathbb{L}^1, k}$ (see Section 2.3 for details). Similarly to the proof of Theorem 4 we prove that

$$\|e_k(t)(\mathcal{F}(y)(t) - \mathcal{F}(x)(t))\|_{\mathbb{R}^n} \leq \frac{L}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} e^{-k(t-\tau)} \|e_k(\tau)(y(\tau) - x(\tau))\|_{\mathbb{R}^n} d\tau,$$

for almost every $t \in [a, b]$ and all $x, y \in \mathbb{L}^1$. Using the classical Fubini formula and the same change of variable than in the proof of Theorem 4, one can get that

$$\|\mathcal{F}(y) - \mathcal{F}(x)\|_{\mathbb{L}^1, k} \leq \frac{L}{k^\alpha} \|y - x\|_{\mathbb{L}^1, k}$$

for all $x, y \in \mathbb{L}^1$. Since $\frac{L}{k^\alpha} < 1$, we conclude that \mathcal{F} is a contractive map on the Banach space $(\mathbb{L}^1, \|\cdot\|_{\mathbb{L}^1, k})$ and thus \mathcal{F} admits a unique fixed point. From Proposition 11 and the inclusion $f(\mathbb{L}^1) \subset \mathbb{L}^1$, we deduce that (CP) admits a unique solution. \square

Theorem 5 is a well-known result [8] in the literature in order to deal with R-L fractional Cauchy problems with Carathéodory dynamics. However, no information is provided on the weighted continuity of the solution in that framework. The major aim of the next theorem is to fill this gap in the literature. This new result is derived from Propositions 9 and 11, from Theorems 1, 2 and 3 (precisely Corollary 1) and from Theorem 5.

Before stating that theorem, let us recall here that some earlier papers derive weighted continuity results for solutions to (CP). We refer for instance to [10] and [12, Theorem 3.11]. Nevertheless, in these works, we recall that the dynamic f is assumed to satisfy a (quite restrictive) continuous hypothesis of the form $f(\mathbb{C}_\alpha) \subset \mathbb{C}_\alpha$, and thus f is assumed to be continuous with respect to its second variable t (which is not assumed in the present article).

Theorem 6. *Let $(\mathcal{A}_r^{\alpha, \beta})$ be satisfied, with $\beta - \alpha < (1/r)$ if $\beta - \alpha > 0$. If $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{r, \beta})$ are satisfied, then (CP) admits a unique solution. Three cases:*

- (i) *If $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta = 1$, this solution belongs to $\frac{x_a}{\rho_\alpha} + \mathbb{H}^{\alpha - (1/r), 0}$;*
- (ii) *If $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta \in (0, 1)$, this solution belongs to $\frac{x_a}{\rho_\alpha} + \mathbb{H}^{\lambda, 0}$ for all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$;*
- (iii) *If $\alpha - (1/r) + (\beta - 1) \leq 0$, then two cases:*

(a) If $0 < \alpha \leq \frac{r+1}{2r}$, this solution belongs to

$$\frac{x_a}{\rho_\alpha} + \mathbf{H}_\beta^{\eta,0} \cap \mathbf{H}_\gamma^{\alpha-(1/r)+(\beta-\gamma),0}$$

for all $0 < \eta < \alpha - (1/r)$ and all $\beta < \gamma < \alpha - (1/r) + \beta$;

(b) If $\frac{r+1}{2r} < \alpha < 1$, this solution belongs to

$$\frac{x_a}{\rho_\alpha} + \mathbf{H}_{2\alpha-(1/r)+(\beta-1)}^{1-\alpha,0} \cap \mathbf{H}_\gamma^{\alpha-(1/r)+(\beta-\gamma),0}$$

for all $2\alpha - (1/r) + (\beta - 1) < \gamma < \alpha - (1/r) + \beta$.

In all cases, the unique solution x to (CP) belongs to C_α with $(\rho_\alpha x)(a) = x_a$.

Proof. Since $(\mathcal{A}_r^{\alpha,\beta})$ and $(\mathcal{H}_{r,\beta})$ are satisfied, we deduce that $(\mathcal{H}_{1,1})$ is also satisfied. From Theorem 5, we deduce that (CP) admits a unique solution. Let us prove that this solution satisfies the weighted continuity properties stated in Theorem 6. From Proposition 9, we know that the inclusion $f(L_\alpha^\infty) \subset L_\beta^r$ holds true. As a consequence, the inclusion $f(C_\alpha) \subset L_\beta^r$ also holds true. From Theorems 1, 2 and 3, we know that the application

$$\begin{aligned} \mathcal{F}: C_\alpha &\longrightarrow C_\alpha \\ x &\longmapsto \mathcal{F}(x): [a, b] \longrightarrow \mathbb{R}^n \\ t &\longmapsto \mathcal{F}(x)(t) := \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}x_a + \mathbf{I}^\alpha[f(x, \cdot)](t), \end{aligned}$$

is well-defined, in the sense that $\mathcal{F}(x) \in C_\alpha$ for all $x \in C_\alpha$. Then, considering $k \in \mathbb{N}$ sufficiently large for $\frac{2^{2-\alpha}L}{k^\alpha} < 1$, we endow C_α with the Bielecki norm $\|\cdot\|_{C_\alpha, k}$ (see Section 2.3 for details). Similarly to the proof of Theorem 4 we prove that

$$\begin{aligned} &\|\rho_\alpha(t)e_k(t)(\mathcal{F}(y)(t) - \mathcal{F}(x)(t))\|_{\mathbb{R}^n} \\ &\leq \frac{L}{\Gamma(\alpha)}\|y - x\|_{C_\alpha, k}(t-a)^{1-\alpha} \int_a^t (t-\tau)^{\alpha-1}e^{-k(t-\tau)}(\tau-a)^{\alpha-1} d\tau, \end{aligned}$$

for all $t \in [a, b]$ and all $x, y \in C_\alpha$. Denoting by $\theta := \frac{t-a}{2}$, it holds that $a + \theta = t - \theta$ and we get that

$$\begin{aligned} \int_a^t (t-\tau)^{\alpha-1}e^{-k(t-\tau)}(\tau-a)^{\alpha-1} d\tau &\leq \int_a^{a+\theta} (t-\tau)^{\alpha-1}e^{-k(t-\tau)}(\tau-a)^{\alpha-1} d\tau \\ &\quad + \int_{t-\theta}^t (t-\tau)^{\alpha-1}e^{-k(t-\tau)}(\tau-a)^{\alpha-1} d\tau, \end{aligned}$$

and thus

$$\begin{aligned} \int_a^t (t-\tau)^{\alpha-1}e^{-k(t-\tau)}(\tau-a)^{\alpha-1} d\tau &\leq \theta^{\alpha-1} \int_a^{a+\theta} e^{-k(\tau-a)}(\tau-a)^{\alpha-1} d\tau \\ &\quad + \theta^{\alpha-1} \int_{t-\theta}^t (t-\tau)^{\alpha-1}e^{-k(t-\tau)} d\tau, \end{aligned}$$

for all $t \in [a, b]$. Similarly to the proof of Theorem 4, with a simple change of variable, it can be proved that the two last integrals can be bounded by $\frac{\Gamma(\alpha)}{k^\alpha}$. We get that

$$\|\mathcal{F}(y) - \mathcal{F}(x)\|_{C_\alpha, k} \leq \frac{2^{2-\alpha}L}{k^\alpha}\|y - x\|_{C_\alpha, k},$$

for all $x, y \in C_\alpha$. Since $\frac{2^{2-\alpha}L}{k^\alpha} < 1$, we conclude that \mathcal{F} is a contractive map on the Banach space $(C_\alpha, \|\cdot\|_{C_\alpha, k})$ and thus \mathcal{F} admits a unique fixed point. From Proposition 11 and the inclusions $f(C_\alpha) \subset L_\beta^r \subset L^1$, this fixed point coincides with the unique solution to (CP) which thus belongs to C_α . The weighted Hölder continuity results follow from the inclusion $f(C_\alpha) \subset L_\beta^r$ and from Theorems 1, 2 and 3. The last sentence of Theorem 6 is straightforward. \square

Considering the case $(r, \beta) = (\infty, \alpha)$ in Theorem 6 allows to derive the following corollary.

Corollary 3. *If $(\mathcal{H}_{\text{glob}})$ and $(\mathcal{H}_{\infty, \alpha})$ are satisfied, then (CP) admits a unique solution. Three cases:*

- (i) *If $\alpha = 1$, this solution belongs to $x_a + H^{1,0}$;*
- (ii) *If $\frac{1}{2} < \alpha < 1$, this solution belongs to $\frac{x_a}{\rho_\alpha} + H^{\lambda,0}$ for all $0 < \lambda < 2\alpha - 1$;*
- (iii) *If $0 < \alpha \leq \frac{1}{2}$, this solution belongs to*

$$\frac{x_a}{\rho_\alpha} + H_\alpha^{\eta,0} \cap H_\gamma^{2\alpha-\gamma,0}$$

for all $0 < \eta < \alpha$ and all $\alpha < \gamma < 2\alpha$.

In all cases, the unique solution x to (CP) belongs to C_α with $(\rho_\alpha x)(a) = x_a$.

A Technical proofs of Section 4

This appendix is dedicated to the technical proofs of Section 4. All these proofs are based on the following well-known or obvious results.

Lemma 2. *Let $z_1 \geq z_2 \geq 0$ be two nonnegative reals. The inequality*

$$z_1^\xi - z_2^\xi \leq (z_1 - z_2)^\xi$$

holds true for all $0 \leq \xi \leq 1$, while the reverse inequality

$$(z_1 - z_2)^\xi \leq z_1^\xi - z_2^\xi$$

holds true for all $\xi \geq 1$.

Lemma 3 (Hölder inequality). *Let $n = 1$ and let $1 \leq p, q \leq \infty$. If $\frac{1}{p} + \frac{1}{q} \leq 1$, then the product $xy \in L^r$ with*

$$\|xy\|_{L^r} \leq \|x\|_{L^p} \|y\|_{L^q},$$

for all $x \in L^p$ and all $y \in L^q$, where $1 \leq r \leq \infty$ is defined by the equality $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$.

Lemma 4. *If $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied, it holds that*

$$\left(\int_{t_1}^{t_2} (t_2 - \tau)^{r'(\alpha-1)} (\tau - t_1)^{r'(\beta-1)} d\tau \right)^{1/r'} = \frac{\Gamma(\alpha)\Gamma(\beta)}{2} C(\alpha, \beta, r) (t_2 - t_1)^{\alpha - (1/r) + (\beta-1)},$$

for all $t_1, t_2 \in [a, b]$ with $t_1 < t_2$.

A.1 Proof of Proposition 4

Let $x \in L^r_\beta$. From Lemmas 3 and 4, it holds that

$$\begin{aligned} \|\mathbf{I}^\alpha[x](t)\|_{\mathbb{R}^n} &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \|x(\tau)\|_{\mathbb{R}^n} d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-\tau)^{\alpha-1} (\tau-a)^{\beta-1} \|\rho_\beta(\tau)x(\tau)\|_{\mathbb{R}^n} d\tau \\ &\leq \frac{\|\rho_\beta x\|_{L^r}}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^t (t-\tau)^{r'(\alpha-1)} (\tau-a)^{r'(\beta-1)} d\tau \right)^{1/r'} = \frac{1}{2} C(\alpha, \beta, r) \|x\|_{L^r_\beta} (t-a)^{\alpha-(1/r)+(\beta-1)}, \end{aligned}$$

for almost every $t \in [a, b]$.

A.2 Proof of Theorem 1

Theorem 1 (and thus Proposition 3) easily follows from the next proposition.

Proposition 12. *If $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta = 1$, then it holds that*

$$\|\mathbf{I}^\alpha[x](t)\|_{\mathbb{R}^n} \leq \frac{1}{2} C(\alpha, \beta, r) \|x\|_{L^r_\beta} (t-a)^{\alpha-(1/r)+(\beta-1)},$$

for almost every $t \in [a, b]$, and

$$\|\mathbf{I}^\alpha[x](t_2) - \mathbf{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} \leq C(\alpha, \beta, r) \|x\|_{L^r_\beta} (t_2 - t_1)^{\alpha-(1/r)+(\beta-1)},$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, and all $x \in L^r_\beta$.

Proof. Let $x \in L^r_\beta = L^r$. The first inequality comes from Proposition 4. For the second inequality, note that

$$\begin{aligned} \|\mathbf{I}^\alpha[x](t_2) - \mathbf{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} &\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} \|x(\tau)\|_{\mathbb{R}^n} d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right] \|x(\tau)\|_{\mathbb{R}^n} d\tau, \end{aligned}$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Using Lemmas 2 and 3, we get that

$$\begin{aligned} \|\mathbf{I}^\alpha[x](t_2) - \mathbf{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} &\leq \frac{\|x\|_{L^r}}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2 - \tau)^{r'(\alpha-1)} d\tau \right)^{1/r'} \\ &\quad + \frac{\|x\|_{L^r}}{\Gamma(\alpha)} \left(\int_a^{t_1} (t_1 - \tau)^{r'(\alpha-1)} - (t_2 - \tau)^{r'(\alpha-1)} d\tau \right)^{1/r'}, \end{aligned}$$

and, since $(t_1 - a)^{r'(\alpha-1)+1} - (t_2 - a)^{r'(\alpha-1)+1} \leq 0$, we deduce from Lemma 4 that

$$\|\mathbf{I}^\alpha[x](t_2) - \mathbf{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} \leq C(\alpha, 1, r) \|x\|_{L^r} (t_2 - t_1)^{\alpha-(1/r)} = C(\alpha, \beta, r) \|x\|_{L^r_\beta} (t_2 - t_1)^{\alpha+\beta-1-(1/r)},$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. \square

A.3 Proof of Lemma 1

Let $x \in L_\beta^r$ and let $0 < \lambda < \alpha - (1/r) + (\beta - 1)$. Let us denote by $p := \frac{1}{1+(1/r)-(\alpha-\lambda)}$ satisfying $1 < p < \infty$. Note that $0 < 1 - \beta < (\alpha - \lambda) - (1/r)$ and that $p' = \frac{1}{(\alpha-\lambda)-(1/r)}$ which satisfies $1 < p' < \frac{1}{1-\beta}$. Since $x \in L_\beta^r$, we know that $x = \frac{y}{\rho_\beta}$ for some $y \in L^r$ and with $\frac{1}{\rho_\beta} \in L^{p'}$. Since $\frac{1}{r} + \frac{1}{p'} = \alpha - \lambda < \alpha \leq 1$, we know from Lemma 3 that $x \in L^{1/(\alpha-\lambda)}$ with

$$\|x\|_{L^{1/(\alpha-\lambda)}} \leq \left\| \frac{1}{\rho_\beta} \right\|_{L^{p'}} \|y\|_{L^r} = \frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)-\lambda} C(1, \beta, p) \|x\|_{L_\beta^r},$$

from Lemma 4. The proof is complete.

A.4 Proof of Theorem 2

The proof of Theorem 2 is essentially based on the next result.

Proposition 13. *If $(\mathcal{A}_r^{\alpha, \beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta \in (0, 1)$, then it holds that*

$$\|\mathbb{I}^\alpha[x](t)\|_{\mathbb{R}^n} \leq \frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)-\lambda} C(\alpha, \beta, r) \|x\|_{L_\beta^r} (t-a)^\lambda,$$

for almost every $t \in [a, b]$, and

$$\begin{aligned} & \|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} \\ & \leq \frac{1}{2}(b-a)^{\alpha-(1/r)+(\beta-1)-\lambda} C\left(\alpha, 1, \frac{1}{\alpha-\lambda}\right) C\left(1, \beta, \frac{1}{1+(1/r)-(\alpha-\lambda)}\right) \|x\|_{L_\beta^r} (t_2 - t_1)^\lambda, \end{aligned}$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, and for all $x \in L_\beta^r$ and all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$.

Proof. Let $x \in L_\beta^r$ and let $0 < \lambda < \alpha - (1/r) + (\beta - 1)$. The first inequality comes from Proposition 4. For the second inequality, recall from Lemma 1 that $x \in L^q$ where $q := \frac{1}{\alpha-\lambda}$ which satisfies $1 < q < \infty$ and $0 < \frac{1}{q} < \alpha$. Moreover it holds that

$$\begin{aligned} \|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} & \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} \|x(\tau)\|_{\mathbb{R}^n} d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right] \|x(\tau)\|_{\mathbb{R}^n} d\tau, \end{aligned}$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Since $x \in L^q$ and since $1 < q' < \frac{1}{1-\alpha}$, we get from Lemmas 2 and 3 that

$$\begin{aligned} \|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} & \leq \frac{\|x\|_{L^q}}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2 - \tau)^{q'(\alpha-1)} d\tau \right)^{1/q'} \\ & \quad + \frac{\|x\|_{L^q}}{\Gamma(\alpha)} \left(\int_a^{t_1} (t_1 - \tau)^{q'(\alpha-1)} - (t_2 - \tau)^{q'(\alpha-1)} d\tau \right)^{1/q'}, \end{aligned}$$

and, since $(t_1 - a)^{q'(\alpha-1)+1} - (t_2 - a)^{q'(\alpha-1)+1} \leq 0$, we deduce that

$$\|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} \leq C(\alpha, 1, q) \|x\|_{L^q} (t_2 - t_1)^{\alpha-(1/q)} = C\left(\alpha, 1, \frac{1}{\alpha-\lambda}\right) \|x\|_{L^q} (t_2 - t_1)^\lambda,$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. The proof is concluded from Lemma 1. \square

The proof of Theorem 2 is finally concluded with the next result.

Lemma 5. *If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) > 0$ and $\beta \in (0, 1)$, then it holds that*

$$C(\alpha, \beta, r) \leq \frac{1}{2} C\left(\alpha, 1, \frac{1}{\alpha - \lambda}\right) C\left(1, \beta, \frac{1}{1 + (1/r) - (\alpha - \lambda)}\right)$$

for all $0 < \lambda < \alpha - (1/r) + (\beta - 1)$.

Proof. Let $0 < \lambda < \alpha - (1/r) + (\beta - 1)$ and let us denote by $p := \frac{1}{1 + (1/r) - (\alpha - \lambda)}$ and $q := \frac{1}{\alpha - \lambda}$ which satisfy $0 < \frac{1}{p} < \beta < 1$ and $0 < \frac{1}{q} < \alpha \leq 1$. Since $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$, it holds from Lemma 3 that

$$\left(\int_a^b (b - \tau)^{r'(\alpha-1)} (\tau - a)^{r'(\beta-1)} d\tau\right)^{1/r'} \leq \left(\int_a^b (b - \tau)^{q'(\alpha-1)} d\tau\right)^{1/q'} \left(\int_a^b (\tau - a)^{p'(\beta-1)} d\tau\right)^{1/p'}$$

The proof is complete from Lemma 4. \square

A.5 Proof of Theorem 3

In the framework of Theorem 3, recall that necessarily $\alpha, \beta \in (0, 1)$. The proof of Theorem 3 is essentially based on the next proposition.

Proposition 14. *If $(\mathcal{A}_r^{\alpha,\beta})$ is satisfied with $\alpha - (1/r) + (\beta - 1) \leq 0$, then it holds that*

$$\|\mathbb{I}^\alpha[x](t)\|_{\mathbb{R}^n} \leq \frac{1}{2} C(\alpha, \beta, r) \|x\|_{L_\beta^r} (t - a)^{\alpha - (1/r) + (\beta - 1)},$$

for almost every $t \in [a, b]$, and

$$\begin{aligned} & \|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} \\ & \leq \frac{1}{2} \|x\|_{L_\beta^r} \left(\frac{\Gamma(\beta + \mu)}{\Gamma(\beta)} C(\alpha, \beta + \mu, r) (t_1 - a)^{-\mu} (t_2 - t_1)^{\alpha - (1/r) + (\beta - 1) + \mu} \right. \\ & \quad \left. + \frac{\Gamma(\alpha - \eta)}{\Gamma(\alpha)} C(\alpha - \eta, \beta, r) (t_1 - a)^{\alpha - (1/r) + (\beta - 1) - \eta} (t_2 - t_1)^\eta \right), \end{aligned}$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$, and for all $x \in L_\beta^r$, for all $0 \leq \mu \leq 1 - \beta$ and all $0 \leq \eta \leq 1 - \alpha$ such that $0 \leq \eta < \alpha - (1/r)$.

Proof. Let $x \in L_\beta^r$. The first inequality comes from Proposition 4. Let us prove the second inequality. Let $0 \leq \mu \leq 1 - \beta$ and let $0 \leq \eta \leq 1 - \alpha$ such that $0 \leq \eta < \alpha - (1/r)$. It holds that

$$\begin{aligned} \|\mathbb{I}^\alpha[x](t_2) - \mathbb{I}^\alpha[x](t_1)\|_{\mathbb{R}^n} & \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} (\tau - a)^{\beta-1} \|\rho_\beta x(\tau)\|_{\mathbb{R}^n} d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right] (\tau - a)^{\beta-1} \|\rho_\beta x(\tau)\|_{\mathbb{R}^n} d\tau, \quad (3) \end{aligned}$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Since $x \in L_\beta^r$ and from Lemmas 3 and 4, the first

right-hand side term in Inequality (3) can be bounded by

$$\begin{aligned}
& \frac{\|x\|_{L_\beta^r}}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_{t_1}^{t_2} (t_2 - \tau)^{r'(\alpha-1)} (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'} \\
& \leq \frac{\|x\|_{L_\beta^r}}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_{t_1}^{t_2} (t_2 - \tau)^{r'(\alpha-1)} (t_1 - a)^{-r'\mu} (\tau - t_1)^{r'(\beta+\mu-1)} d\tau \right)^{1/r'} \\
& \leq \frac{\|x\|_{L_\beta^r}}{2} \frac{\Gamma(\beta + \mu)}{\Gamma(\beta)} C(\alpha, \beta + \mu, r) (t_1 - a)^{-\mu} (t_2 - t_1)^{\alpha - (1/r) + (\beta-1) + \mu}.
\end{aligned}$$

Similarly, and using moreover Lemma 2, the second right-hand side term in Inequality (3) can be bounded by

$$\begin{aligned}
& \frac{\|x\|_{L_\beta^r}}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^{t_1} \left[(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1} \right]^{r'} (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'} \\
& \leq \frac{\|x\|_{L_\beta^r}}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^{t_1} \left[(t_1 - \tau)^{r'(\alpha-1)} - (t_2 - \tau)^{r'(\alpha-1)} \right] (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'}.
\end{aligned}$$

In the last integral, since $0 < r'(1 - \alpha) < 1$, it holds from Lemma 2 that

$$\begin{aligned}
(t_1 - \tau)^{r'(\alpha-1)} - (t_2 - \tau)^{r'(\alpha-1)} &= \left(\frac{1}{t_1 - \tau} \right)^{r'(1-\alpha)} - \left(\frac{1}{t_2 - \tau} \right)^{r'(1-\alpha)} \\
&\leq \left(\frac{1}{t_1 - \tau} - \frac{1}{t_2 - \tau} \right)^{r'(1-\alpha)} = \left(\frac{1}{t_1 - \tau} - \frac{1}{t_2 - \tau} \right)^{r'(1-\alpha-\eta)} \left(\frac{t_2 - t_1}{(t_1 - \tau)(t_2 - \tau)} \right)^{r'\eta} \\
&\leq (t_1 - \tau)^{r'(\alpha+\eta-1)} (t_2 - t_1)^{r'\eta} (t_1 - \tau)^{-r'\eta} (t_2 - \tau)^{-r'\eta} \\
&\leq (t_2 - t_1)^{r'\eta} (t_1 - \tau)^{r'(\alpha-\eta-1)}.
\end{aligned}$$

Thus, since $0 \leq \frac{1}{r} < \alpha - \eta \leq \alpha \leq 1$, we obtain the bound

$$\begin{aligned}
& \frac{\|x\|_{L_\beta^r}}{\Gamma(\alpha)\Gamma(\beta)} (t_2 - t_1)^\eta \left(\int_a^{t_1} (t_1 - \tau)^{r'(\alpha-\eta-1)} (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'} \\
& = \frac{\|x\|_{L_\beta^r}}{2} \frac{\Gamma(\alpha - \eta)}{\Gamma(\alpha)} C(\alpha - \eta, \beta, r) (t_1 - a)^{\alpha - (1/r) + (\beta-1) - \eta} (t_2 - t_1)^\eta,
\end{aligned}$$

from Lemma 4. The proof is complete. \square

Let us prove Theorem 3. Let $0 < \gamma < \alpha - (1/r) + \beta \leq 1$. Let us define $\mu := \min(1 - \beta, 1 - \gamma)$. Note that

$$0 < \mu \leq 1 - \beta, \quad \alpha - (1/r) + (\beta - \gamma) > 0, \quad 1 - (\mu + \gamma) \geq 0 \quad \text{and} \quad \alpha - (1/r) + (\beta - 1) + \mu > 0.$$

Now let $0 < \eta < \alpha - (1/r)$ such that

$$0 < \eta \leq \min\left(\alpha - (1/r) + (\beta - \gamma), 1 - \alpha\right).$$

From Proposition 14, it holds that

$$\|(\rho_\gamma \mathbf{I}^\alpha[x])(t)\|_{\mathbb{R}^n} \leq \frac{\Gamma(\gamma)}{2} C(\alpha, \beta, r) \|x\|_{L_\beta^r} (t-a)^{\alpha-(1/r)+(\beta-\gamma)},$$

for almost every $t \in [a, b]$. Moreover, it holds that

$$\begin{aligned} \|(\rho_\gamma \mathbf{I}^\alpha[x])(t_2) - (\rho_\gamma \mathbf{I}^\alpha[x])(t_1)\|_{\mathbb{R}^n} &\leq \Gamma(\gamma) \left((t_2 - a)^{1-\gamma} - (t_1 - a)^{1-\gamma} \right) \|\mathbf{I}^\alpha[x](t_2)\|_{\mathbb{R}^n} \\ &\quad + \Gamma(\gamma) (t_1 - a)^{1-\gamma} \|\mathbf{I}^\alpha[x](t_2) - \mathbf{I}^\alpha[x](t_1)\|_{\mathbb{R}^n}, \end{aligned} \quad (4)$$

for almost every $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. From Lemma 2 and Proposition 14 and since $\alpha - (1/r) + (\beta - 1) \leq 0$, the first right-hand side term in Inequality (4) can be bounded by

$$\begin{aligned} \frac{\Gamma(\gamma)}{2} C(\alpha, \beta, r) \|x\|_{L_\beta^r} (t_2 - a)^{\alpha-(1/r)+(\beta-1)} (t_2 - t_1)^{1-\gamma} \\ \leq \frac{\Gamma(\gamma)}{2} C(\alpha, \beta, r) \|x\|_{L_\beta^r} (t_2 - t_1)^{\alpha-(1/r)+(\beta-\gamma)}. \end{aligned}$$

From Proposition 14, the second right-hand side term in Inequality (4) can be bounded by

$$\begin{aligned} \frac{\Gamma(\gamma)}{2} \|x\|_{L_\beta^r} \left(\frac{\Gamma(\beta + \mu)}{\Gamma(\beta)} C(\alpha, \beta + \mu, r) (t_1 - a)^{1-(\mu+\gamma)} (t_2 - t_1)^{\alpha-(1/r)+(\beta-1)+\mu} \right. \\ \left. + \frac{\Gamma(\alpha - \eta)}{\Gamma(\alpha)} C(\alpha - \eta, \beta, r) (t_1 - a)^{\alpha-(1/r)+(\beta-\gamma)-\eta} (t_2 - t_1)^\eta \right). \end{aligned}$$

The proof is complete by noting that

$$\min \left(\alpha - (1/r) + (\beta - \gamma), \alpha - (1/r) + (\beta - 1) + \mu, \eta \right) = \eta$$

and that

$$C(\alpha, \beta, r) \leq \frac{\Gamma(\alpha - \eta)}{\Gamma(\alpha)} C(\alpha - \eta, \beta, r).$$

This last inequality follows from the inequality

$$\left(\int_a^b (b - \tau)^{r'(\alpha-1)} (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'} \leq (b - a)^\eta \left(\int_a^b (b - \tau)^{r'(\alpha-\eta-1)} (\tau - a)^{r'(\beta-1)} d\tau \right)^{1/r'}$$

and from Lemma 4.

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