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An Itô type formula for the additive stochastic heat equation

Carlo Bellingeri *

Abstract

We use the recent theory of regularity structures to develop an Itô formula for u , the stochastic heat equation with space-time white noise in one space dimension with periodic boundary conditions. In particular for any smooth enough function φ we can express the random distribution $(\partial_t - \partial_{xx})\varphi(u)$ and the random field $\varphi(u)$ in terms of the reconstruction of some modelled distributions. The law of the resulting objects is also identified with some classical constructions of stochastic calculus.

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1 Introduction

We consider $\{u(t, x) : t \in [0, T], x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}\}$ the solution of the additive stochastic heat equation with periodic boundary conditions and zero initial value:

$$\begin{cases} \partial_t u = \partial_{xx} u + \xi \\ u(t, 0) = u(t, 1) & t \in [0, T], \\ u(0, x) = 0 & x \in \mathbb{T}, \end{cases} \quad (1.1)$$

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where $\xi = \partial W / \partial t \partial x$ is the space-time white noise over $\mathbb{R} \times \mathbb{T}$ associated to W , the Brownian sheet on $\mathbb{R} \times \mathbb{T}$. This equation was originally formulated to model a random string with values in \mathbb{R} (see [Fun83]) and this is the simplest example of a stochastic PDE driven by space-time white noise. It is well known that (1.1) admits a unique solution which can be written as the explicit continuous gaussian random field:

$$u(t, x) = \int_0^t \int_{\mathbb{T}} P(t-s, x-y) dW_{s,y}, \quad (1.2)$$

where $P: (0, +\infty) \times \mathbb{T} \rightarrow \mathbb{R}$ is the fundamental solution of the heat equation with periodic boundary conditions:

$$P(t, x) = \sum_{m \in \mathbb{Z}} G(t, x+m),$$

G is the Heat kernel on \mathbb{R} and the integral is taken with respect the martingale measure associated to the brownian sheet W (we refer to [Wal84] for the general theory and [Zam17, Chapter 4] for a quick introduction to the Dirichlet case). The classical theory shows also that the process $\{u(t, \cdot) : t \in [0, T]\}$ is a Feller diffusion taking values in $C(\mathbb{T})$ (the space of periodic continuous functions). Its hitting properties were intensively studied in [MT02] by means of the Markov property, potential theory and the theory of gaussian processes. However it is still impossible to study u using stochastic calculus with respect to W mainly because for any fixed $x \in \mathbb{T}$ it has been shown in [Swa07] that the process $t \rightarrow u(t, x)$ has an a.s. infinite quadratic variation, therefore the powerful theory of Itô calculus cannot be applied in this context.

Introduced in 2014 and explained trough the famous “quartet” of articles ([Hai14] [BHZ16] [CH16] [BCCH17]), the theory of regularity structures has provided a very general framework and some robust analytical methods to prove local path-wise existence and uniqueness of a wide family of SPDEs driven by space time white noise. In this paper we will show how these new techniques allow to formulate an Itô formula for the solution of the equation (1.1). This new formula is expressed under a new form, reflecting the new perspective under which SPDEs are analysed. Contrary to the classic literature where other Ito formulae were already treated but they cannot be applied in our context (see e.g. [DPZ92]), we will not look at u as an infinite dimensional diffusion with values on an Hilbert space, but for any fixed smooth function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$, we will decide to study the quantity $(\partial_t - \partial_{xx})\varphi(u)$, interpreted as a space time random distribution. This choice is heuristically motivated by the parabolic form of the equation (1.1) defining u and it is manageable by the regularity structures, where it is possible to manipulate random distributions. Thus we are searching for a random distribution g_φ , depending on higher derivatives of φ , such that the identity

$$\langle (\partial_t - \partial_{xx})\varphi(u), \psi \rangle = \langle g_\varphi, \psi \rangle, \quad (1.3)$$

holds a.s. for any test function $\psi: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$. We will refer to this formula as a *differential Itô formula*, because of the presence of a differential operator on the left hand side of (1.3). By uniqueness of the heat equation, we can identify $\varphi(u)$ with the solution of the heat equation with the distribution g_φ thus obtaining formally

$$\varphi(u(t, x)) = \varphi(0) + \int_0^t \int_{\mathbb{T}} P(t-s, x-y) g_\varphi(s, y) ds dy \quad (1.4)$$

a.s. for every $(t, x) \in [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$. We call a similar kind of identity an *integral Itô formula* because of the double integral on the right hand side of (1.4). This resulting formula may be one possible tool to improve our comprehension of the trajectories of u , even if it is still not clear wheter it will be as effective as it is for finite-dimensional diffusions (see e.g. [RY04]).

In order to explain how to obtain it, we will follow the general philosophy of the regularity structure. Instead of working directly with the process u , we will look at it as the limit of a sequence of smooth processes u_ε which resolve a ‘‘Wong-Zakai’’ approximation of (1.1):

$$\begin{cases} \partial_t u_\varepsilon = \partial_{xx} u_\varepsilon + \xi_\varepsilon \\ u_\varepsilon(t, 0) = u_\varepsilon(t, 1) & t \in [0, T], \\ u_\varepsilon(0, x) = 0 & x \in \mathbb{T}, \end{cases} \quad (1.5)$$

where for any $\varepsilon > 0$, ξ_ε is a smoothed version of ξ converging to it as $\varepsilon \rightarrow 0^+$ (see [HP15] for this approximation procedure on a wider class of equations). More precisely we extend ξ periodically on \mathbb{R}^2 and for any given smooth, compactly supported function $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\int \rho = 1$ we define for $\varepsilon > 0$ the functions:

$$\rho_\varepsilon = \varepsilon^{-3} \rho(\varepsilon^{-2}t, \varepsilon^{-1}x), \quad \xi_\varepsilon(t, x) = (\rho_\varepsilon * \xi)(t, x).$$

The inhomogeneous scaling in the mollification procedure is chosen in accordance with the parabolic nature of the equation (1.1). This regularisation makes ξ_ε an a.s. periodic smooth function and the equation (1.5) admits an a.s. periodic strong solution (in the analytical sense) $u_\varepsilon: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ which is smooth in space and time. Therefore the classical chain rule between u_ε and φ holds, obtaining

$$\partial_t(\varphi(u_\varepsilon)) = \varphi'(u_\varepsilon)\partial_t u_\varepsilon, \quad \partial_x(\varphi(u_\varepsilon)) = \varphi'(u_\varepsilon)\partial_x u_\varepsilon, \quad (1.6)$$

$$\partial_{xx}(\varphi(u_\varepsilon)) = \varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \varphi'(u_\varepsilon)\partial_{xx} u_\varepsilon. \quad (1.7)$$

which yields:

$$\partial_t(\varphi(u_\varepsilon)) - \partial_{xx}(\varphi(u_\varepsilon)) = \varphi'(u_\varepsilon)\xi_\varepsilon - \varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2. \quad (1.8)$$

Let us understand what happens when $\varepsilon \rightarrow 0^+$. Thanks to convergence result in Proposition 3.2, the process u_ε will converge to u in probability for the topology of continuous functions, so then the left hand side of (1.8) must converge in probability to $(\partial_t - \partial_{xx})\varphi(u)$ for the topology of distribution because the derivatives are continuous maps. Therefore the right hand side of (1.8) must converge too and the formula is obtained once we can express this limit in a different way. However, written under this form, it is very hard to study this right hand side because it is possible to show that a.s.

$$\|\varphi'(u_\varepsilon)\xi_\varepsilon\|, \|\varphi''(u_\varepsilon)(\partial_x u_\varepsilon)^2\| \rightarrow +\infty$$

with respect the same distributional norm under which the left hand side converge. These two paradoxical results suggest that the formula (1.8) should hide a *cancellation phenomenon* between two objects whose divergences compensate between each other. By means of the notion of modelled distribution and the reconstruction theorem, we can give a rigorous meaning to such compensation and at same time we will be able to identify the non trivial limit to which the right hand side converges. However, these limits are

only characterised by some analytical properties which cannot allow to understand immediately their probabilistic properties. Therefore the convergence is also linked with some specific *identification theorems* which describe their law. Thanks to the presence in the previous literature of other Itô formulae related to (1.1) in the homogenous Dirichlet boundary case (see in particular [Zam06] and [GNT05], [Lan07] [BS10] as other examples), we were able to fully describe this random distribution with some classical objects. Summing up both these results we can state the main theorem of the paper:

Theorem 1.1 (Integral and differential Itô formula). *Let φ be a function of class $C^4(\mathbb{R})$. Then for any test function $\psi: (0, T) \times \mathbb{T} \rightarrow \mathbb{R}$ one has*

$$\begin{aligned} \langle (\partial_t - \partial_{xx})\varphi(u), \psi \rangle &= \int_0^T \int_{\mathbb{T}} \varphi'(u(s, y))\psi(s, y)dW_{s,y} + \frac{1}{2} \int_0^T \int_{\mathbb{T}} \psi(s, y)\varphi''(u(s, y))C(s)dy ds \\ &- 2 \int_{\Delta_{2,T} \times \mathbb{T}^2} \int_{\mathbf{s}_1}^T \int_{\mathbb{T}} \psi(s, y)\varphi''(u(s, y))P_x(s - \mathbf{s}_1, y - \mathbf{y}_1)P_x(s - \mathbf{s}_2, y - \mathbf{y}_2)dy ds W(ds, d\mathbf{y}), \end{aligned}$$

Moreover for any $(t, x) \in [0, T] \times \mathbb{T}$ we have also

$$\begin{aligned} \varphi(u(t, x)) &= \varphi(0) + \int_0^t \int_{\mathbb{T}} P(t - s, x - y)\varphi'(u(s, y))dW_{s,y} \\ &- 2 \int_{\Delta_{2,t} \times \mathbb{T}^2} \int_{\mathbf{s}_1}^t \int_{\mathbb{T}} P(t - s, x - y)P_x(s - \mathbf{s}_1, y - \mathbf{y}_1)P_x(s - \mathbf{s}_2, y - \mathbf{y}_2)\varphi''(u_s(y))dy ds W(ds, d\mathbf{y}) \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}} P(t - s, x - y)\varphi''(u(s, x))C(s)dy ds. \end{aligned}$$

where in both case $W(ds, d\mathbf{y})$ denotes the Skorohod integral over $\Delta_{2,t} = \{0 \leq \mathbf{s}_1 < \mathbf{s}_2 \leq t\}$ and $C: (0, T) \rightarrow \mathbb{R}$ is the deterministic integrable function

$$C(s) := \|P(s, \cdot)\|_{L^2(\mathbb{T})}^2.$$

Remark 1.2. Looking at this type of result, it is natural to ask if we can lower the regularity of φ to be C^2 or the existence analogous Tanaka Formula where φ is the absolute value or a generic convex function using a classical approximation argument (e.g. [RY04]). Under very low assumptions on φ the techniques of the regularity structures theory cannot be applied, therefore we should understand the limit of each stochastic object in the right hand side classically. Unfortunately even if the convergence of the stochastic integral term should not worry at the limit, it is still quite hard to understand the convergence of the deterministic integral and the Skorohod integral under the hypothesis of unbounded derivatives up to order 4 (see Remark 4.7, where the fourth derivatives of φ appears to bound the second moment). Finally any Tanaka formula requires also a robust theory of local times associated to its driving process and, in case of u , this notion is very ambiguous in the literature: indeed using some general results on gaussian variables (such as [GH80]) we can prove the existence of a local times for the process $\{u(t, x)\}_{t \in [0, T]}$ with respect its occupation measure on $[0, T]$ but at the same time another notion of local time for u is developed by means of distribution on the Wiener space (see [GNT05]). A link between these two notions is still mysterious.

We discuss now the organization of the paper: in the section 2 and 3 we will apply the general theorems of the regularity structure theory to build the analytical and algebraic tools to study the problem: all the construction are mostly self contained, starting from their abstract definition till the associated calculations. In some cases we will also recall in our context some previous results obtained in [HP15] and [Hai16]. Then in section 4 we will apply all these tools to obtain firstly two formulae involving only analytical quantities (we will refer them to rough Itô formulae) and secondly we will identify them to yield Theorem 1.1.

We finally remark that some of the techniques presented here could be also used to formulate an Itô formula on a non-linear perturbation of (1.1), the so called generalised KPZ equation:

$$\begin{cases} \partial_t u = \partial_{xx} u + g(u)(\partial_x u)^2 + h(u)(\partial_x u) + k(u) + f(u)\xi \\ u(t, 0) = u(t, 1) \quad t \in [0, T], \\ u(0, \cdot) = u_0(\cdot) \end{cases} \quad (1.9)$$

where g, f, h, k are generic smooth function and $u_0 \in C(\mathbb{T})$ a generic initial condition. (We refer the reader to [Hai16], [Bru15]). Other possible directions of research may also take into account the Itô formula for SPDEs with Dirichlet boundary conditions (see [GH17]) and, using the reformulation in the regularity structures context of differential equations driven by fractional brownian motion (see [BFG⁺17]), we could recover some classical results in the literature of fractional processes (see e.g. [FR02], [RV93]).

Notations and functional spaces

The functional spaces where the whole theory is set up are slightly different from what we find in the classical literature of elliptic PDE. Therefore we recall here some basic notions and notations.

For any space time variable $z \in \mathbb{R}^2$, $z = (t, x)$, in order to preserve the parabolic *scaling* of equation (1.1) (that is time counts twice in space) we define the parabolic “norm” as

$$\|z\| := \sqrt{|t|} + |x|.$$

we put the word norm in quotes because $\|\cdot\|$ does not define a norm on \mathbb{R}^2 but certainly a metric. We adopt the shorthand notation $z^k = t^{k_1} x^{k_2}$ for any multi-index $k = (k_1, k_2)$. For any continuous function η , any point $z = (t, x) \in \mathbb{R}^2$ and $\lambda > 0$, we define

$$\eta_z^\lambda(\bar{z}) := \lambda^{-3} \eta\left(\frac{\bar{t} - t}{\lambda^2}, \frac{\bar{x} - x}{\lambda}\right)$$

where \bar{t}, \bar{x} are the time and space variable of \bar{z} . Both notations allow us to introduce $\mathcal{C}^\alpha(\mathbb{R}^2)$ spaces, a family of Besov spaces indexed by α , any real non natural number.

- If $0 < \alpha < 1$ it consists of the space of α Hölder functions with respect metric $\|\cdot\|$, namely continuous functions $f: \mathbb{R}^2 \mapsto \mathbb{R}$ such that there exists a constant C satisfying

$$|f(x) - f(y)| \leq C \|x - y\|^\alpha$$

uniformly over x, y over a compact.

- If $\alpha > 1$ then $f: \mathbb{R}^2 \mapsto \mathbb{R}$ has recursively all partial derivatives continuous up to order $[\alpha]$, the biggest integer less or equal than α , and for any multi-index β of order $[\alpha]$, $D^\beta f \in \mathcal{C}^{\alpha-[\alpha]}(\mathbb{R}^2)$.
- If $\alpha < 0$ then $\mathcal{C}^\alpha(\mathbb{R}^2)$ is a space included in \mathcal{S}' , the space of Schwartz distributions. More precisely f belongs to $\mathcal{C}^\alpha(\mathbb{R}^2)$ if for every compact set \mathcal{K} , there exists a constant C such that the bound

$$|\langle f, \eta_x^\lambda \rangle| \leq C\lambda^\alpha$$

holds uniformly over $0 < \lambda \leq 1$, all $x \in \mathcal{K}$ and any test function η whose support is included in $B(0, 1) = \{z \in \mathbb{R}^2: \|z\| \leq 1\}$ and it has any directional derivative up to order $[\alpha]$ bounded in the sup norm.

All these spaces are endowed with a Fréchet structure when we consider the family of norms given by the minimal constants satisfying the bounds on their definition. Moreover for any fixed compact \mathcal{K} of \mathbb{R}^2 , we can easily define also the Banach space $\mathcal{C}^\alpha(\mathcal{K})$ by restriction on \mathcal{K} of the test functions or the evaluation points $\mathcal{C}^\alpha(\mathbb{R}^2)$. In our context we will always put $\mathcal{K} = [-1, T+1] \times \mathbb{T}$ where we assume the functions being periodic in the space variable and the time interval is large enough to strictly contain $[0, T]$. Most of the classical analytical operations apply to \mathcal{C}^α spaces with some slight differences given by the metric $\|\cdot\|$.

- *Derivation* if $f \in \mathcal{C}^\alpha$ and $k = (k_1, k_2)$ is a multi-index then map $f \rightarrow \partial^k f = \partial_x^{k_1} \partial_t^{k_2} f$ is a continuous map from \mathcal{C}^α to \mathcal{C}^β where $\beta = \alpha - 2k_1 - k_2$.
- *Schauder estimates* (see [Sim97]) if P is the Heat kernel on some domain, map the space-time convolution with P , $f \rightarrow P * f$ is a well defined map for every f supported on positive times and it sends continuously \mathcal{C}^α in $\mathcal{C}^{\alpha+2}$ for every real α except for those values such that $\alpha + 2$ is natural.
- *Product* (see [Hai14, Proposition 4.14]) for any non natural β the map $(f, g) \rightarrow f \cdot g$ defined over smooth functions extends continuously to a bilinear map $B: \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^{\alpha \wedge \beta}$ if and only if $\alpha + \beta > 0$.

One of the main properties of \mathcal{C}^α spaces consists in their relation with ξ . For every $\kappa > 0$ we can choose a modification of the periodic extension to \mathbb{R}^2 of ξ belonging to $\mathcal{C}^{-3/2-\kappa}(\mathbb{R} \times \mathbb{T})$. [Hai14, Lemma 10.2]. We recall that for any distribution $g \in \mathcal{C}^\beta(\mathbb{R} \times \mathbb{T})$ with $\beta \in (-2, 0)$ non integer, and $T > 0$ the PDE

$$\begin{cases} \partial_t v - \partial_{xx} v = g \\ v(t, 0) = v(t, 1) \\ v(0, \cdot) = v_0(\cdot). \end{cases}$$

with periodic boundary conditions and $v_0 \in \mathcal{C}(\mathbb{T})$ continuous, admits a unique strong solution $v: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$, $v \in \mathcal{C}^{\beta+2}$ which is given explicitly by the formula

$$v(t, x) = \int_{\mathbb{T}} P(t, x - y) v_0(y) dy + (P * \mathbf{1}_+ g)(t, x), \quad (1.10)$$

where $\mathbf{1}_+g$ denotes the product of g with the indicator function of the interval $(0, +\infty) \times \mathbb{T}$. This product is clearly not defined point-wise but under the condition $\beta \in (-2, 0)$, for any $(s, t) \subset \mathbb{R}$, $s < t \leq +\infty$ we define for any test function

$$\mathbf{1}_{(s,t)}g(\psi) := \lim_N g(\varphi_N\psi), \quad (1.11)$$

where φ_N is a sequence of periodic bounded smooth functions compactly supported in $(s, t) \times \mathbb{T}$ such that for any $z \in (s, t) \times \mathbb{R}$, $\varphi_N(z)$ converges to 1 and for any $z \in \mathbb{R} \setminus [s, t] \times \mathbb{T}$, $\varphi_N(z)$ converges to 0. The condition $\beta > -2$ makes $g(\varphi_N\psi)$ a Cauchy sequence and we can also prove that the limit is still a periodic distribution. Moreover the map $g \rightarrow \mathbf{1}_{(s,t)}g \in \mathcal{C}^\beta(\mathbb{R} \times \mathbb{T})$ is a continuous extension of the pointwise product and $\mathbf{1}_{(s,t)}g(\psi)$ is uniquely characterised as the only distribution of $\mathcal{C}^\beta(\mathbb{R} \times \mathbb{T})$ such that

$$\mathbf{1}_{(s,t)}g(\psi) = g(\psi)$$

for any smooth test function ψ such that $\text{supp}(\psi) \subset (s, t) \times \mathbb{T}$ and $\mathbf{1}_{(s,t)}g(\psi) = 0$ if $\text{supp}(\psi) \cap [s, t] \times \mathbb{T} = \emptyset$. (for many references to this multiplication see [HP15, Lemma 6.1], [Hai14, Proposition 6.9] and [GH17, Proposition 2.15]). Applying this deterministic construction with $g = \xi$, $\kappa < 1/2$ and $v_0 = 0$, since strong solution are also mild solution for any realisation of ξ , we obtain immediately

$$u(t, x) = (P * \mathbf{1}_+\xi)(t, x) \quad (1.12)$$

a.s. and therefore u belongs to $\mathcal{C}^{1/2-\kappa}$. This regularity implies immediately there is no classical analytical way to define distributions like $(\partial_x u)^2$ for instance. As always we suppose that the space time white noise ξ on $\mathbb{R} \times \mathbb{T}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ enhanced with its natural filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$

$$\mathcal{F}_t := \sigma(\{\xi(\psi) : \psi|_{(t, +\infty) \times \mathbb{T}} = 0, \quad \psi \in L^2(\mathbb{R} \times \mathbb{T})\}).$$

The gaussian nature of space time white noise ξ allows us to use also all the classical tools of Malliavin Calculus on the Hilbert space $H = L^2(\mathbb{R} \times \mathbb{T})$. Here we recall some notations and we refer the reader to [Nua95] for the general theory. In general for any random variable $X \in L^2(\Omega, \mathbb{P})$ we denote by $DX = \{D_{s,y}X : s \in \mathbb{R}, y \in \mathbb{T}\}$ its Malliavin derivative whenever is well defined on X and by $\mathbb{D}^{2,k} \subset L^2(\Omega, \mathbb{P})$ the space of random variables X such that we can derivate k times and the derivative is square integrable. On the other hand, for any $v \in L^2(\Omega \times \mathbb{R} \times \mathbb{T})$ whenever it is well defined we use the notation

$$\int_{\mathbb{R}} \int_{\mathbb{T}} v(s, y) W(ds, dy)$$

to write the adjoint of D , $\delta: \text{Dom}(\delta) \subset L^2(\Omega \times \mathbb{R} \times \mathbb{T}) \rightarrow L^2(\Omega, \mathbb{P})$, which is referred in the literature as the Skorokhod Integral. The operator δ is an extension of the classical stochastic integral and we are allowed to apply it over a class of non adapted integrands. Extending periodically the brownian sheet W to \mathbb{R}^2 we can transfer the Walsh integral as well the Skorokhod integral to stochastic processes $H: \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ through the definition:

$$\int_{\mathbb{R}^2} H(s, y) d\tilde{W}_{s,y} := \int_{\mathbb{R}} \int_{\mathbb{T}} \sum_{m \in \mathbb{Z}} H(s, y + m) dW_{s,y},$$

When the right hand side is well defined.

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2 Abstract regularity structures

We begin the article with the construction of an explicit *regularity structure* and a *model* such that they are capable to take in account u and its compositions with smooth functions. All objects are built using elementary constructions and they come as a simplification of the algebraic structure explained in [Hai16]. For a synthetic description of the general theory we refer the reader to [FH14].

2.1 Algebraic construction

The key starting idea behind the whole theory focuses around the notion of *regularity structures* (A, T, G) , a triple of the following elements:

- A discrete lower bounded real subset A containing 0.
- A graduated vectorial space $T = \bigoplus_{\alpha \in A} T_\alpha$ such that each space T_α is a Banach space with norm $\|\cdot\|_\alpha$ and $\dim(T_0) = 1$.
- A group of linear operators G preserving T_0 such that for each $\alpha \in A$, a in T_α and Γ in G , one has

$$\Gamma a - a \in \bigoplus_{\beta < \alpha} T_\beta. \quad (2.1)$$

This definition comes as generalisation of the space of polynomials and since they approximate smooth functions, these abstract objects should be able to approximate functions or even distributions with very low regularity as u .

It is reasonable to start including in our structure $\mathbb{R}[X_1, X_2]$, the real polynomials on 2 indeterminates. For any multi-index $k \in \mathbb{N}^2$, $k = (k_1, k_2)$ we will write \mathbf{X}^k as a shorthand for the monomial $X_1^{k_1} X_2^{k_2}$ (we imagine X_1 as a time variable) while the unit will be denoted by $\mathbf{1}$. At the same time, we introduce an additional abstract symbol Ξ to represent the space-time white noise ξ which is a.s. a distribution. To take in account convolution of ξ with the heat kernel as in formula (1.12), for any symbol σ and $k \in \mathbb{N}^2$ we introduce a family of symbols $\mathcal{J}_k(\sigma)$ ($\mathcal{J}_{(0,0)}(\sigma)$ is noted with $\mathcal{J}(\sigma)$), that represent the convolution of the k -th derivative of the heat kernel with the function associated to the symbol σ . Since $\mathcal{J}_k(\mathbf{X}^m)$ should be identified with a smooth function, we simply put it to 0 to avoid repetitions. Finally for any two symbols τ_1, τ_2 we consider also their products $\tau_1 \tau_2$ by juxtaposition of τ_1 and τ_2 up to classical identification, namely the juxtaposition

with $\mathbf{1}$ does not change the symbol, $\mathbf{X}^l \mathbf{X}^k = \mathbf{X}^{l+m}$ and the iterated juxtaposition of the same symbol is noted with an integer power. Adding all these formal rules, we note with F the smallest set of symbols satisfying

- $\{\mathbf{X}^k\}_{k \in \mathbb{N}^2} \cup \{\Xi\} \subset F$.
- For any $\tau_1, \tau_2 \in F$, then $\tau_1 \tau_2 \in F$.
- For any $\sigma \in F$ and $m \in \mathbb{N}^2$, $\mathcal{J}_m(\sigma) \in F$.

We write \mathcal{F} for the free vector space generated by all formal symbols of F . The interpretation of symbols as functions allows us to define a *homogeneity* map $|\cdot|: F \rightarrow \mathbb{R}$ that associates to any symbol a kind of “regularity” of the function associated to it. We define recursively $|\cdot|$: In case of polynomials and Ξ , we fix a parameter $\kappa > 0$, that we imagine sufficiently small, and for any multi-index $k = (k_1, k_2) \in \mathbb{N}^2$ we set

$$|\mathbf{X}^k| := 2k_1 + k_2, \quad |\Xi| := -\frac{3}{2} - \kappa;$$

Moreover for any $\tau, \tau' \in F$, we set

$$|\mathcal{J}_k(\tau)| := \tau + 2 - 2k_1 - k_2, \quad |\tau\tau'| := |\tau| + |\tau'|,$$

To imitate Schauder estimates and the multiplicative property of polynomial degree. Starting from the linear space \mathcal{F} and the set $|F|$ we introduce a subset of symbols where we choose all reasonable products that we will need in our calculations. We write $\mathcal{J}_1(\Xi)$ as shorthand of $\mathcal{J}_{(0,1)}(\Xi)$.

Definition 2.1. We define the sets of symbols $T, U, U' \subset F$ as the smallest triple of sets satisfying:

- $\{\Xi\} \subset T$, $\{\mathcal{J}(\Xi)\} \cup \{\mathbf{X}^k\}_{k \in \mathbb{N}^2} \subset U$, $\{\mathcal{J}_1(\Xi)\} \cup \{\mathbf{X}^k\}_{k \in \mathbb{N}^2} \subset U'$;
- for every $k \geq 0$ and any finite family of elements $\tau_1, \dots, \tau_k \in U$ and any couple of elements $\sigma_1, \sigma_2 \in U'$ then $\{\tau, \tau\Xi, \tau\sigma_1, \tau\sigma_1\sigma_2\} \subset T$ and $\tau \in U$, where $\tau = \tau_1 \cdots \tau_n$.

We denote also by \mathcal{T} and \mathcal{U} respectively the free vectorial space upon T and U .

The definition of T allows also an alternative description of all its symbols. We define V as the set of all symbol of the form $\mathcal{J}(\Xi)^m \mathbf{X}^l$ with $m \in \mathbb{N}, l \in \mathbb{N}^2$ and for any $\sigma \in \{\Xi, \mathcal{J}_1(\Xi), \mathcal{J}_1(\Xi)^2\}$ we define the set $V_\sigma := \sigma V$, namely the set of all symbols of the form σ times an element of V . Using these definitions, it is straightforward to show the identities

$$U = V, \quad T = V_\Xi \sqcup V_{\mathcal{J}_1(\Xi)^2} \sqcup V_{\mathcal{J}_1(\Xi)} \sqcup V. \quad (2.2)$$

Moreover, noting with \mathcal{V}_σ the free vectorial space generated upon V_σ , we obtain from (2.2) the decomposition

$$\mathcal{F} = \mathcal{V}_\Xi \oplus \mathcal{V}_{\mathcal{J}_1(\Xi)^2} \oplus \mathcal{V}_{\mathcal{J}_1(\Xi)} \oplus \mathcal{U}. \quad (2.3)$$

\mathcal{F} and the homogeneity $|\cdot|$ are a reasonable “candidate” to build the first two components of a regularity structure. Let us give the construction of the structure group associated

to \mathcal{T} . Similarly to polynomials, for any $h \in \mathbb{R}^3$, $h = (h_1, h_2, h_3)$ we define the linear map $\Gamma_h: \mathcal{T} \rightarrow \mathcal{T}$, defined as

$$\Gamma_h(\sigma \mathcal{F}(\Xi)^m \mathbf{X}^l) = \sigma[(X_1 + h_1 \mathbf{1})^{l_1} (X_2 + h_2 \mathbf{1})^{l_2} (\mathcal{F}(\Xi) + h_3 \mathbf{1})^m], \quad (2.4)$$

for any $\sigma \in \{\Xi, \mathcal{F}_1(\Xi), \mathcal{F}_1(\Xi)^2, \mathbf{1}\}$, $m \in \mathbb{N}$, $l \in \mathbb{N}^2$. Thanks to (2.2) all symbols under the form $\sigma \mathcal{F}(\Xi)^m \mathbf{X}^l$ form a basis and the map Γ_h is well defined. Using also the explicit definition (2.4) it is straightforward to show

$$\Gamma_h \Gamma_k = \Gamma_{h+k} \quad (2.5)$$

for any $h, k \in \mathbb{R}^3$. This property implies that the map $h \rightarrow \Gamma_h$ is an injective homomorphism from $(\mathbb{R}^3, +)$ to a subgroup of the invertible maps acting on \mathcal{T} . Therefore $\mathcal{G} = \{\Gamma_h: h \in \mathbb{R}^3\}$ is a group.

Proposition 2.2. *For any $\kappa < 1/2$, the triple $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ where $A = \{|\tau|: \tau \in T\}$ defines a regularity structure.*

Proof. To prove that \mathcal{A} is bounded from below, we show that for any $\beta \in \mathbb{R}$ the set $\{\tau \in T: |\tau| \leq \beta\}$ is finite. Let $\tau \in T$ satisfying $|\tau| \leq \beta$. Using the decomposition (2.2) there exist $m \in \mathbb{N}$, $n \in \mathbb{N}^2$ and $\sigma \in \{\Xi, \mathcal{F}_1(\Xi), \mathcal{F}_1(\Xi)^2\}$ such that $\tau = \sigma I(\Xi)^m \mathbf{X}^n$. The property of τ implies

$$n_1 + 2n_2 + (1/2 - \kappa)m \leq \gamma - |\sigma|. \quad (2.6)$$

Under the condition $\kappa < 1/2$, the left hand side of the inequality is strictly bigger or equal than 0. Therefore, for any choice of $\gamma \in \mathbb{R}$ there exists only a finite number of indexes n_1, n_2, m satisfying (2.6). This result implies also for any $\gamma \in \mathcal{A}$, denoting with \mathcal{T}_γ the linear space generated upon all symbol with homogeneity equal to γ , these spaces satisfy the identity $\mathcal{T} = \bigoplus_{\gamma \in \mathcal{A}} \mathcal{T}_\gamma$ and each one of them is finite dimensional, therefore there is no need to specify a norm on each space. Property (2.1), comes directly from Newton's binomial formula and the positive homogeneity of the symbol $\mathcal{F}(\Xi)$. \square

Remark 2.3. As a matter of fact in what follows we can restrict our considerations to a subspace of \mathcal{T} generated by all symbols with homogeneity less than some parameter $\zeta > 0$, that we will chose all along the article. In general for any $\beta \in A$, $\tau \in \mathcal{T}$ we denote by $|\tau|_\beta$ the euclidean norm on \mathcal{T}_β of the β component of τ and with $\mathcal{Q}_\beta \tau$ the projection of τ on $\bigotimes_{\alpha < \beta} \mathcal{T}_\alpha$ (We decided to take euclidean norm to be coherent with [HP15] but there is no "canonical" choice of this norm since \mathcal{T}_β is finite dimensional). We stress also that under this choice of \mathcal{G} one has $\Gamma_h \tau \tau' = \Gamma_h \tau \Gamma_h \tau'$ for every symbol $\tau, \tau' \in T$ such that also their product $\tau \tau'$ belongs to T . Therefore, whenever it is well defined, the juxtaposition product is a *regular product* on \mathcal{T} accordingly to [Hai14, Definition 4.6]. By the definition of Γ_h , the subspace \mathcal{U} is invariant by the action of the group \mathcal{G} . This property implies \mathcal{U} is a *function-like sector*, following the terminology of [Hai14, Definition 2.4].

Remark 2.4. We may express our symbols in terms of trees. We consider labelled, rooted trees τ (LR tree), that is τ consists of a combinatorial rooted tree (finite connected simple graph with a non-empty set of nodes N_τ and a set of edges E_τ without cycles and not planar), enhanced with a *typing map* $\tau': E_\tau \rightarrow \{\Xi, \mathcal{F}\}$, two abstract symbols related

This notion plays a fundamental role in the whole theory because it gives a concrete family of distributions to approximate function with low regularity. In order to compare different models on the same structure, we endow \mathcal{M} , the set of all models on $(\mathcal{T}, \mathcal{G})$, with the topology associated to the corresponding system of semi-distances induced by conditions (2.7) and (2.8):

$$\|(\Pi, \Gamma); (\bar{\Pi}, \bar{\Gamma})\|_{\mathcal{X}} := \|\Pi - \bar{\Pi}\|_{\mathcal{X}} + \|\Gamma - \bar{\Gamma}\|_{\mathcal{X}} . \quad (2.9)$$

Since we fixed our analysis on a compact space-time domain \mathcal{X} including $[0, T] \times [0, 1]$ we will restrict all elements of Definition (2.5) on \mathcal{X} and we will avoid any reference of it in the notation. In this way $(\mathcal{M}, \|\cdot\|)$ becomes also a complete space (\mathcal{M} is not a Banach space because the sum of models is not necessarily a model!). In particular if a sequence (Π^n, Γ^n) converge to (Π, Γ) , then $\Pi_z^n \tau$ converges to $\Pi_z \tau$ in the sense of tempered distributions for any z, τ . To define correctly a model over symbols under the form $\mathcal{F}(\sigma)$, we need a technical lemma related to a suitable decomposition of G , the heat kernel on \mathbb{R} . For its proof see [Hai14, Lemma 5.5], [Hai14, Lemma 7.7].

Lemma 2.6. *For any fixed $T > 0$, there exist two functions $K: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$, $R: \mathbb{R}^2 \rightarrow \mathbb{R}$ with the following properties:*

- For every periodic distribution u supported in $\mathbb{R}_+ \times \mathbb{R}$ and every $z \in (-\infty, T+1] \times \mathbb{R}$ one has

$$G * u(z) = (K * u)(z) + (R * u)(z) . \quad (2.10)$$

- K is smooth, supported on $\{(t, x) \in \mathbb{R}^2: x^2 + |t| \leq 1\}$ and equal to G on $\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}: x^2 + t < 1/2, t > 0\}$.
- $K(t, x) = 0$ for $t \leq 0, x \neq 0$ and $K(t, -x) = K(t, x)$.
- For every polynomial $Q: \mathbb{R}^2 \mapsto \mathbb{R}$ of degree $|\cdot|$ less than ζ , one has

$$\int_{\mathbb{R}^2} K(t, x) Q(t, x) dx dt = 0 . \quad (2.11)$$

- R is smooth, $R(t, x) = 0$ for $t \leq 0$ and it is compactly supported.

From this lemma it is also possible to show [Hai14, Theorem 5.12] that the map $v \rightarrow K * v$ sends continuously \mathcal{C}^α in $\mathcal{C}^{\alpha+2}$ for any non natural $\alpha \in \mathbb{R}$ and any distribution v not necessarily compactly supported. The choice of constants 1 and 1/2 in Lemma 2.6 is completely arbitrary.

In what follows for any given realisation of ξ^ε we will provide the construction of $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ a sequence of models associated to ξ^ε and converging to a model $(\hat{\Pi}, \hat{\Gamma})$ related to ξ . As a further simplification, we parametrise all possible models with a couple $(\mathbf{\Pi}, f)$ where $\mathbf{\Pi}: \mathcal{T} \rightarrow \mathcal{S}'$ and $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Indeed it is straightforward to check that the operators

$$\Pi_z = \mathbf{\Pi} \Gamma_{f(z)}, \quad \Gamma_{z\bar{z}} = \Gamma_{f(z)-f(\bar{z})} . \quad (2.12)$$

satisfy trivially the algebraic relationships in Definition 2.5, because of the identity (2.5). We will divide the construction in two parts: Since any realisation of ξ_ε is smooth, we firstly build a model $(\mathbf{\Pi}^c, f^c)$ upon any deterministic smooth function $\xi: \mathbb{R}^2 \rightarrow \mathbb{R}$ adding randomness as well as low regularity in a second time.

Proposition 2.7. *Let ξ be a smooth periodic function. Then the conditions*

$$\mathbf{\Pi}^c \mathbf{1} = 1, \quad \mathbf{\Pi}^c \mathbf{X}^k \tau = z^k \mathbf{\Pi}^c \tau, \quad (2.13)$$

$$\mathbf{\Pi}^c \mathcal{J}_k(\sigma) = D^k(K * \mathbf{\Pi}^c(\sigma)), \quad \mathbf{\Pi}^c \Xi = \xi, \quad (2.14)$$

$$\mathbf{\Pi}^c \bar{\tau} \tau = \mathbf{\Pi}^c \bar{\tau} \mathbf{\Pi}^c \tau. \quad (2.15)$$

for any $k \in \mathbb{N}^d$ and $\tau, \bar{\tau} \in T$ such that $\tau \mathbf{X}^k \in T$, $\tau \bar{\tau} \in T$, identify uniquely a model $(\mathbf{\Pi}^c, \Gamma^c)$ on \mathcal{T} . We call it canonical model.

Proof. Using the hypothesis on ξ and conditions (2.13) as well as (2.14), it is straightforward to show $\mathbf{\Pi}^c \tau$ is a smooth function for any $\tau \in T$ which is not a product of symbols. Therefore in this case point-wise product on the right hand side of the second equation of (2.15) is well defined and $\mathbf{\Pi}^c$ is a well posed operator with values on smooth functions. In order to choose f we compute explicitly

$$\mathbf{\Pi}_z^c(\sigma \mathcal{J}(\Xi)^m \mathbf{X}^k)(\bar{z}) = \mathbf{\Pi}^c(\sigma)(\bar{z})(\bar{z} + (f^c(z))_{1,2})^k [(K * \xi)(\bar{z}) + (f^c(z))_3]^m. \quad (2.16)$$

for any $z, \bar{z} \in \mathbb{R}^2$, $\sigma \in \{\mathcal{J}_1(\Xi), \mathcal{J}_1(\Xi)^2, \Xi, \mathbf{1}\}$ and k, m as before. Therefore if we want to satisfy bound (2.7) when $\sigma = \mathbf{1}$ we need to impose

$$f^c(z)_i = -z_i, \quad (f^c(z))_3 = -(K * \mathbf{\Pi} \Xi)(z). \quad (2.17)$$

Hence (2.7) follows from simple calculations. On the other hand, the multiplicative property of Γ implies that (2.8) holds if and only if this property is verified on $\tau = \{\mathcal{J}(\Xi), X_1, X_2\}$, which is trivial. \square

Remark 2.8. The choice of f^c given by (2.17) does not depend on the condition (2.15). Indeed, if ξ were a distribution belonging to $\mathcal{C}^\beta(\mathbb{R}^2)$ with $\beta \geq -3/2 - \kappa$ with $\kappa < 1/2$, Formulae (2.13), (2.14) and (2.17) would still have a meaning in this context and operators $\Gamma_{z\bar{z}}$ given by (2.12) would satisfy (2.8) again. Therefore for any map $\mathbf{\Pi}$ satisfying (2.13), (2.14) we can uniquely associate to it a couple $\mathcal{L}(\mathbf{\Pi}) := (\mathbf{\Pi}, \Gamma)$ through the new identities

$$\mathbf{\Pi}_z = \mathbf{\Pi} \Gamma_{f^c(z)}, \quad \Gamma_{z\bar{z}} = \Gamma_{f^c(z) - f^c(\bar{z})}. \quad (2.18)$$

which satisfies almost all properties of a model. We call these maps *admissible maps* (this terminology slightly differs from the usual concept but we adopt this modification for sake of brevity). The choice of kernel K satisfying (2.11) is also compatible with our previous assumption on symbols $\mathcal{S}(\mathbf{X}^k)$.

Remark 2.9. If ξ is also periodic in the space variable (which is true in case of ξ^ε), it is straightforward to prove that

$$\mathbf{\Pi}_{(t,x+m)}^c \tau(t', x' + m) = \mathbf{\Pi}_{(t,x)}^c \tau(t', x'), \quad \Gamma_{(t,x+m)(t',x'+m)}^c \tau = \Gamma_{(t,x)(t',x')}^c \tau \quad (2.19)$$

for any couple of space time points $z = (t, x)$, $z' = (t', x')$ and for $m \in \mathbb{Z}$, $\tau \in T$. Thus the canonical model is also *adapted* to the action of translation on \mathbb{R} (for this definition see [Hai14, Definition 3.33]). Roughly speaking this property allows a model $(\mathbf{\Pi}, \Gamma)$ to approximate distributions which are periodic in space.

Let us complete the existence of a non-trivial random model associated to space-time white noise. We denote alternatively by $(\Pi^\varepsilon, \Gamma^\varepsilon)$, $\mathcal{L}(\mathbf{\Pi}^\varepsilon)$ the random model associated to any realisation of ξ_ε via Theorem 2.7. Since ξ_ε converges to ξ , we would like to define a model by sending ε to 0 in each term $\mathbf{\Pi}^\varepsilon\tau$ and then in $(\Pi^\varepsilon, \Gamma^\varepsilon)$. Unfortunately, it is well known that the sequence $\mathbf{\Pi}^\varepsilon\tau$ does not converge as a random distribution for any τ . Thus $\mathcal{L}(\mathbf{\Pi}^\varepsilon)$ is not a good approximating model. However, the convergence of $\mathbf{\Pi}^\varepsilon$ on other symbols (e.g. on Ξ) tells us we should not change it completely. A natural way to get rid of this partial ill-posedness will be to consider for any fixed $\varepsilon > 0$ a new family of operator $\mathbf{\Pi}^\varepsilon M_\varepsilon$ where $M_\varepsilon: \mathcal{T} \rightarrow \mathcal{T}$ is a linear map. In particular if we impose

$$\begin{aligned} M_\varepsilon \mathbf{1} &= \mathbf{1}, & M_\varepsilon \mathcal{J}_k(\tau) &= \mathcal{J}_k(M_\varepsilon \tau), \\ M_\varepsilon \mathbf{X}^k \tau &= \mathbf{X}^k M_\varepsilon \tau, & M_\varepsilon \Xi &= \Xi. \end{aligned} \quad (2.20)$$

Then $\mathbf{\Pi}^\varepsilon M_\varepsilon$ is still an admissible map and we can consider the convergence of the couple $\mathcal{L}(\mathbf{\Pi}^\varepsilon M_\varepsilon) = (\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$. A big contribution of [BHZ16] [CH16] was to identify a deterministic procedure to define for any $\varepsilon > 0$ a ‘‘canonical’’ map $\tilde{M}_\varepsilon: \mathcal{T} \rightarrow \mathcal{T}$ satisfying (2.20) such that $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon) \in \mathcal{M}$ converges in probability to some random limiting model $(\hat{\Pi}, \hat{\Gamma})$; the model $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ is called *the BPHZ renormalisation* of $(\Pi^\varepsilon, \Gamma^\varepsilon)$. To recover such operator in this context we introduce the constants

$$C_\varepsilon^1 = \mathbb{E}[\mathbf{\Pi}^\varepsilon \Xi \mathcal{J}(\Xi)(0)] = \int_{\mathbb{R}^2} \rho_\varepsilon(z) K * \rho_\varepsilon(z) dz \quad (2.21)$$

$$C_\varepsilon^2 = \mathbb{E}[\mathbf{\Pi}^\varepsilon \mathcal{J}_1(\Xi)^2(0)] = \int_{\mathbb{R}^2} (K_x * \rho_\varepsilon)^2(z) dz \quad (2.22)$$

and for any $m \in \mathbb{N}$ we define

$$\begin{aligned} M_\varepsilon \Xi \mathcal{J}(\Xi)^m &= \Xi \mathcal{J}(\Xi)^m - m C_\varepsilon^1 \mathcal{J}(\Xi)^{m-1}, \\ M_\varepsilon \mathcal{J}_1(\Xi)^2 \mathcal{J}(\Xi)^m &= \mathcal{J}_1(\Xi)^2 \mathcal{J}(\Xi)^m - C_\varepsilon^2 \mathcal{J}(\Xi)^m. \\ M_\varepsilon \tau &= \tau \quad \text{for any } \tau \in V_{\mathcal{J}_1(\Xi)} \cup U. \end{aligned} \quad (2.23)$$

Decomposition (2.2) implies that the conditions (2.20) (2.23) are sufficient to define a family of linear maps $M_\varepsilon: \mathcal{T} \rightarrow \mathcal{T}$. In the following theorem we will calculate the BPHZ renormalisation following the formalism and the algebraic structures introduced in [Hai16] and we will relate them to these previous conditions.

We represent each symbol of T as a decorated tree as explained in Remark 2.4. Using this graphical notation we can easily define two new algebraic structures associated to \mathcal{T} . We firstly consider $\hat{\mathcal{T}}_-$ as the free commutative algebra generated upon T with respect the product given by the disjoint union of graphs (we refer to this operation as the forest product and we will write using \cdot when we use the formalism of abstract symbols). We also define $\mathcal{F}_- := \hat{\mathcal{T}}_- / \mathcal{J}$, the quotient of $\hat{\mathcal{T}}_-$ with respect \mathcal{J} , the ideal of $\hat{\mathcal{T}}_-$ generated by the set

$$J := \{ \tau \in T : |\tau| \geq 0 \}; \quad (2.24)$$

Under this notation the map \tilde{M}_ε is given by

$$\tilde{M}_\varepsilon \tau_\varepsilon^n = (g^{\mathbf{\Pi}^\varepsilon} \hat{A}_- \otimes id) \Delta_- \tau_\varepsilon^n, \quad (2.25)$$

where $\Delta_- : \mathcal{T} \rightarrow \mathcal{T}_- \otimes \mathcal{T}$ is the coaction given by

$$\Delta_- \tau_\epsilon^n := \sum_{\gamma \subset \tau, \gamma \in \hat{T}_-} \sum_{\epsilon_\gamma, \mathbf{n}_\gamma \leq \mathbf{n}} \frac{1}{\epsilon_\gamma!} \binom{\mathbf{n}}{\mathbf{n}_\gamma} p(\gamma, \mathbf{n}_\gamma + \pi \epsilon_\gamma, \epsilon |_\gamma) \otimes (\mathcal{K}_\gamma \tau, \mathcal{K}_\gamma(\mathbf{n} - \mathbf{n}_\gamma), \mathcal{K}_\gamma \epsilon + \epsilon_\gamma) \quad (2.26)$$

where γ is a generic subforest of τ contained in \hat{T}_- with no isolated points, p is the projection operator on \mathcal{T}_- and \mathcal{K}_γ the contraction of tree τ on γ . We specify in this context that for any tree $\gamma \subset \tau_\epsilon^n$ under the form $\gamma = \sigma_1 \dots \sigma_n$ the *contraction tree* of τ_ϵ^n over γ , which is noted with $\mathcal{K}_\gamma \tau_\epsilon^n = (\mathcal{K}_\gamma \tau, \mathcal{K}_\gamma \mathbf{n}, \mathcal{K}_\gamma \epsilon)$, satisfies the following conditions:

- $\mathcal{K}_\gamma \tau$ is the tree obtained from τ replacing each σ_i with a node.
- Denoting with $\bullet_1, \dots, \bullet_n$ each node associated to the contraction of tree σ_i , function $\mathcal{K}_\gamma \mathbf{n} : N_{\mathcal{K}_\gamma} \rightarrow \mathbb{N}^2$ is equal to \mathbf{n} on every non contracted node of $\mathcal{K}_\gamma \tau$ and for every i , $\mathbf{n}(\bullet_i) = \sum_{y \in N_{\sigma_i}} \mathbf{n}(y)$.
- $\mathcal{K}_\gamma \epsilon : E_{\mathcal{K}_\gamma} \rightarrow \mathbb{N}^2$ is equal to ϵ on every non contracted edge of $\mathcal{K}_\gamma \tau$.

Since this contraction operation may produce some trees $\mathcal{K}_\gamma \tau$ that do not belong to F , in this case we impose $\mathcal{K}_\gamma \tau_\epsilon^n = 0$. The map $\hat{A}_- : \mathcal{T}_- \rightarrow \hat{\mathcal{T}}_-$ is called the twisted-Antipode and it is the only homomorphism between these algebras such that on any non empty tree τ_ϵ^n

$$\hat{A}_- \tau_\epsilon^n = -M^*(\hat{A}_- \otimes id)(\Delta_- \tau_\epsilon^n - \tau_\epsilon^n \otimes \mathbf{1})$$

where M^* is the forest product and $g^{\mathbf{I}^\epsilon} : \hat{\mathcal{T}}_- \rightarrow \mathbb{R}$ is the only character on the algebra $\hat{\mathcal{T}}_-$ such that if $\sigma = \tau_1 \dots \tau_n \in \hat{\mathcal{T}}_-$ then

$$g^{\mathbf{I}^\epsilon}(\sigma) := \mathbb{E} \mathbf{I}^\epsilon \tau_1(0) \dots \mathbb{E} \mathbf{I}^\epsilon \tau_n(0).$$

Theorem 2.10. *Let $\varepsilon > 0$. By fixing κ sufficiently small, the map $M_\varepsilon : \mathcal{T} \rightarrow \mathcal{T}$ is equal to \tilde{M}_ε , therefore the resulting model $\mathcal{L}(\mathbf{I}^\varepsilon M_\varepsilon)$ corresponds to the BPHZ renormalisation of $\mathcal{L}(\mathbf{I}^\varepsilon)$ as explained in [CH16], [BHZ16].*

Proof. To prove the theorem it is sufficient to show that also \tilde{M}_ε satisfies conditions (2.20) and (2.23). Thanks to [BHZ16, Theorem 6.17] the map \tilde{M}_ε does satisfy the first line of (2.20) automatically. Let us prove (2.23). We fix $m \in \mathbb{N}$ and we calculate $\tilde{M}_\varepsilon \tau_m$ when

$$\tau_m = \underbrace{\begin{array}{c} \bullet \dots \bullet \\ \vdots \\ \underbrace{}_m \end{array}}_m,$$

where \dots denotes the iterated product with $\mathcal{S}(\Xi)$. Under the condition $\kappa < 1/10$, the only values m such that $\tau_m \in \mathcal{T}_-$ are given by $m \leq 3$. We calculate \tilde{M}_ε firstly in this case. By definition (2.26) one has

$$\begin{aligned} \Delta_- \dot{\bullet} &= \emptyset \otimes \dot{\bullet} + \dot{\bullet} \otimes \mathbf{1}, & \Delta_- \dot{\vee} &= \emptyset \otimes \dot{\vee} + \dot{\bullet} \otimes \dot{\bullet} + {}^{(0,1)} \dot{\bullet} \otimes \dot{\bullet} + \dot{\vee} \otimes \mathbf{1}, \\ \Delta_- \dot{\vee} \dot{\vee} &= \emptyset \otimes \dot{\vee} \dot{\vee} + \dot{\bullet} \otimes \dot{\vee} \dot{\vee} + 2 {}^{(0,1)} \dot{\bullet} \otimes \dot{\vee} \dot{\vee} + 2 \dot{\vee} \otimes \dot{\bullet} + 2 {}^{(0,1)} \dot{\vee} \otimes \dot{\bullet} + \dot{\vee} \dot{\vee} \otimes \mathbf{1}, \end{aligned}$$

$$\Delta_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = \emptyset \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + 3 \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + 3 \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + 6 \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \\ + \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \otimes \mathbf{1}.$$

Hence we obtain by definition of \hat{A}_-

$$\begin{aligned} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= -\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, \quad \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = -\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - \hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, \\ \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= -\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - 2\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - 2\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - 2\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, \\ \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= -\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - 3\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} - 3\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} + 6\hat{A}_- \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}. \end{aligned}$$

As we will see, the calculation of \hat{A}_- is recursive but in this case the map g^{Π^ε} is zero on many trees, so we do not need more algebraic calculations. Indeed $g^{\Pi^\varepsilon} \Xi \mathcal{F}(\Xi) = C_\varepsilon^1$ by definition and to complete the calculation it is sufficient to calculate the constants

$$g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}, g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}.$$

The first five constant from the left are 0 because we are taking the expectations over a product of an *odd* number of centered gaussian variables, on the other hand denoting with $K^\varepsilon = K * \rho_\varepsilon$ and $K_x^\varepsilon = K_x * \rho_\varepsilon$ we have

$$\begin{aligned} g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= \mathbb{E} \left[\int_{\mathbb{R}^2} K^\varepsilon(-z_1) d\tilde{W}_{z_1} \int_{\mathbb{R}^2} K^\varepsilon(-z_2) d\tilde{W}_{z_2} \right] = \int_{\mathbb{R}^2} (K_\varepsilon(z))^2 dz, \\ g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= \mathbb{E} \left[\int_{\mathbb{R}^2} K^\varepsilon(-z_1) d\tilde{W}_{z_1} \int_{\mathbb{R}^2} K_x^\varepsilon(-z_2) d\tilde{W}_{z_2} \right] = \int_{\mathbb{R}^2} K^\varepsilon(z) K_x^\varepsilon(z) dz = 0, \end{aligned}$$

because the function $z \rightarrow K^\varepsilon(z) K_x^\varepsilon(z)$ is odd in the space variable and the integral on the space variable is taken over all \mathbb{R} . For the last integral we reduce to the product of 4 gaussian random variable using Wick's formula

$$\begin{aligned} g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} &= \int_{\mathbb{R}^8} K(-z_1) K(-z_2) K(-z_3) \mathbb{E} [\xi_\varepsilon(z_1) \xi_\varepsilon(z_2) \xi_\varepsilon(z_3) \xi_\varepsilon(z_4)] dz_1 dz_2 dz_3 dz_4 \\ &= 3 \int_{\mathbb{R}^8} K(-z_1) K(-z_2) K(-z_3) \mathbb{E} [\xi_\varepsilon(z_1) \xi_\varepsilon(z_2)] \mathbb{E} [\xi_\varepsilon(z_3) \xi_\varepsilon(z_4)] dz_1 dz_2 dz_3 dz_4 \\ &= 3C_\varepsilon^1 g^{\Pi^\varepsilon} \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \end{aligned}$$

By replacing these values in the above calculations one has

$$g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = 0, \quad g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = -C_\varepsilon^1, \quad g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = 0, \quad g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = 0. \quad (2.27)$$

We can also prove with similar arguments

$$g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = 0, \quad g^{\Pi^\varepsilon} \hat{A}_- \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} = 0. \quad (2.28)$$

Proof. Let us check this identity by a direct computation. Since Γ_h leaves invariant the subspaces $\mathcal{V}_{\mathcal{F}_1(\Xi)}$ and \mathcal{U} , the identity (2.33) is trivially satisfied. On the other hand the commutative property of M_ε with the polynomials reduces to verify (2.33) over the symbols $\mathcal{F}_1(\Xi)^2 \mathcal{F}(\Xi)^m$ and $\Xi \mathcal{F}(\Xi)^m$ for any $m \geq 1$. In both cases we have

$$\begin{aligned}
M_\varepsilon \Gamma_h \mathcal{F}_1(\Xi)^2 \mathcal{F}(\Xi)^m &= \sum_{n=0}^m \binom{m}{n} M_\varepsilon (\mathcal{F}_1(\Xi)^2 \mathcal{F}(\Xi)^n) h_1^{m-n} \\
&= \sum_{n=0}^m \binom{m}{n} (\mathcal{F}_1(\Xi)^2 - C_\varepsilon^2) \mathcal{F}(\Xi)^n h_1^{m-n} \\
&= (\mathcal{F}_1(\Xi)^2 - C_\varepsilon^2) (\mathcal{F}(\Xi) + h_1 \mathbf{1})^m = \Gamma_h M_\varepsilon \mathcal{F}_1(\Xi)^2 \mathcal{F}(\Xi)^m; \\
M_\varepsilon \Gamma_h \Xi \mathcal{F}(\Xi)^m &= \sum_{n=0}^m \binom{m}{n} M_\varepsilon (\Xi \mathcal{F}(\Xi)^n) h_1^{m-n} \\
&= \sum_{n=0}^m \binom{m}{n} \Xi \mathcal{F}(\Xi)^n h_1^{m-n} - \sum_{n=0}^m \binom{m}{n} n C_\varepsilon^1 \mathcal{F}(\Xi)^{n-1} h_1^{m-n} \\
&= \sum_{n=0}^m \binom{m}{n} \Xi \mathcal{F}(\Xi)^n h_1^{m-n} - m C_\varepsilon^1 \sum_{n'=0}^m \binom{m-1}{n'} \mathcal{F}(\Xi)^{n'} h_1^{m-1-n'} \\
&= \Xi (\mathcal{F}(\Xi) + h_1 \mathbf{1})^m - m C_\varepsilon^1 (\mathcal{F}(\Xi) + h_1 \mathbf{1})^{m-1} = \Gamma_h M_\varepsilon \Xi \mathcal{F}(\Xi)^m;
\end{aligned}$$

and the Lemma is proved. \square

The identity 2.32 allows a practical way to calculate $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ and it implies immediately that $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ satisfies again the conditions (2.19) related to the periodic behaviour of distributions.

The second results has already been obtained in [HP15] and it explores the law of the of the random distributions associated to the model $(\hat{\Pi}, \hat{\Gamma})$. In general, it is straightforward to show for any compactly supported test function $\psi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

$$\langle \xi_\varepsilon, \psi \rangle \xrightarrow{L^2(\mathbb{P})} \int_{\mathbb{R}^2} \psi(s, y) d\tilde{W}_{s,y} \quad (\varepsilon \rightarrow 0^+),$$

where the integral is a classical Wiener integral because the function ψ does not depend on Ω . Since the convergence of models implies the convergence as distributions we have by uniqueness of the limit

$$\hat{\Pi}_z \Xi(\psi) = \int_{\mathbb{R}^2} \psi(s, y) d\tilde{W}_{s,y}.$$

Starting from this simple identity we could ask ourselves if there is an explicit description of the law random distributions $\hat{\Pi}_z \tau$. In case of the BPHZ renormalisation there is an explicit formula in [CH16, Proposition 4.22] where the law of $\hat{\Pi}_z \tau$ is expressed in terms of a finite sum of Wiener integrals with respect some kernels associated to the tree identification of the symbol τ . However for our purposes it is sufficient to recall a partial result which was obtained previously in [HP15, Theorem 4.5]. As we mention before, it is possible to check that both \mathcal{V}_Ξ and \mathcal{U} may be embedded in the algebraic structure they build; moreover $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ restricted to these subspaces coincides with the

so called renormalised version of the limiting ‘‘Itô Model’’ developed in that article. This identification allows to obtain the same a probabilistic description which was developed there.

Theorem 2.14 (Hairer Pardoux). *For every $\tau \in U$ and every $z = (t, x) \in \mathbb{R}^2$, the process $\{\hat{\Pi}_z \tau(s, \cdot)\}_{s \in \mathbb{R}}$ is \mathcal{F}_s adapted for any $s > t$ and, for every smooth test function ψ such that for any $s \leq t$ $\psi(s, y) = 0$.*

$$\hat{\Pi}_z \Xi \tau(\psi) = \int_t^\infty \int_{\mathbb{R}} \Pi_z \tau(s, y) \psi(s, y) d\tilde{W}_{s,y} \quad (2.34)$$

Remark 2.15. The identity (2.34) holds only when the test function is supported in the future. Otherwise the right hand side integrand will not be adapted and the identity is reformulated with other terms. In principle we could have also applied [CH16, Proposition 4.22] to deduce the explicit law of $\hat{\Pi}_z$ on $\mathcal{V}_{\mathcal{F}_1(\Xi)} \cup \mathcal{V}_{\mathcal{F}_1(\Xi)_2}$ but these formulae turn out to be very messy and they do not provide any additional information as Theorem 2.14 does.

3 Calculus on regularity structures

In this section we will show how the existence of model $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ as well as $(\hat{\Pi}, \hat{\Gamma})$ are linked respectively with process u_ε and u and, more generally, which kind of new operations are allowed to do in this context differently from the case of Hölder spaces.

3.1 Modelled distributions

If a given space \mathcal{T} as well as a model (Π, Γ) generalise the notion of polynomials to perform a Taylor expansion, we now consider the equivalent version of \mathcal{C}^γ spaces in this context. As a matter of fact the theory will consider the vector of coordinates which behaves as if they were the coefficients of a Taylor polynomial related to a \mathcal{C}^γ function. This leads us to definition:

Definition 3.1. For any given $\gamma > 0$, $\eta \in (-2, \gamma]$, a function $U: \mathbb{R}^2 \mapsto \bigoplus_{\alpha < \gamma} \mathcal{T}_\alpha$ belongs to $\mathcal{D}^{\gamma, \eta}$ if for every compact space-time domain \mathcal{K} , one has

$$\|U\|_{\gamma, \eta} := \sup_{z \in \mathcal{K}} \sup_{\alpha < \gamma} \frac{|U(z)|_\alpha}{|t|^{(\frac{\eta-\alpha}{2}) \wedge 0}} + \sup_{(z, \bar{z}) \in \mathcal{K}^{(2)}} \sup_{\alpha < \gamma} \frac{|U(z) - \Gamma_{z\bar{z}} U(\bar{z})|_\alpha}{(|t| \wedge |\bar{t}|)^{\frac{\eta-\gamma}{2}} |z - \bar{z}|^{\gamma-\alpha}} < +\infty. \quad (3.1)$$

Where $\mathcal{K}^{(2)}$ denotes the set of pairs of points $(z, \bar{z}) \in \mathcal{K}^2$ such that $|z - \bar{z}| \leq 1/2 \sqrt{|t| \wedge |\bar{t}|}$. We call U a modelled distributions.

Similarly to the models, the norm $\|U\|_{\gamma, \eta}$ should depend also on the compact set \mathcal{K} , but we avoid it in the notation because we work in a finite time horizon. Looking at their definition, spaces $\mathcal{D}^{\gamma, \eta}$ have one parameter more than classical spaces (it allows the the coordinates of U to blow at rate η near $\{(t, x) \in \mathbb{R}^2: t = 0\}$ with the integrability condition $\eta > -2$) and they possess a natural linear structure. In particular they do depend in a crucial way on the underlying model (Π, Γ) (to remark this dependency we will adopt the shorthand notation $\mathcal{D}^{\gamma, \eta}(\Pi)$) and it is not obvious to compare elements

belonging to two space associated on two different models. For this purpose we define for any couple of modelled distributions U, \bar{U} associated to two different models (Π, Γ) and $(\bar{\Pi}, \bar{\Gamma})$ the quantity

$$\|U; \bar{U}\|_{\gamma, \eta} := \sup_{z, w, \alpha} \frac{|U(z) - \bar{U}(z) - \Gamma_{zw}U(w) + \bar{\Gamma}_{zw}\bar{U}(w)|_{\alpha}}{(|t| \wedge |\bar{t}|)^{\frac{\eta-\gamma}{2}} |z - \bar{z}|^{\gamma-\alpha}} + \sup_{z, \alpha} \frac{|U(z) - \bar{U}(z)|_{\alpha}}{|t|^{(\frac{\eta-\alpha}{2}) \wedge 0}}.$$

This function together with the norm $\|\cdot\|$ on models endows the fibred space

$$\mathcal{M} \times \mathcal{D}^{\gamma, \eta} := \{((\Pi, \Gamma), U) : (\Pi, \Gamma) \in \mathcal{M}, U \in \mathcal{D}^{\gamma, \eta}(\Pi)\}$$

of a complete metric structure. At the same time if $\Pi = \bar{\Pi}$, the quantity $\|U; \bar{U}\|_{\gamma, \eta}$ becomes the natural metric associated to the linear structure of $\mathcal{D}^{\gamma, \eta}(\Pi)$. Modelled distribution are also intimately related to Hölder spaces. A fundamental theorem proved in [Hai14] shows there exists a unique locally Lipschitz continuous map $\mathcal{R} : \mathcal{M} \times \mathcal{D}^{\gamma, \eta} \rightarrow \mathcal{S}'(\mathbb{R}^2)$ with the property that

$$|(\mathcal{R}U - \Pi_z U(z))(\eta_z^\lambda)| \leq C\lambda^\gamma \quad (3.2)$$

uniformly over $\eta \in \mathcal{B}$, $\lambda \in (0, 1]$ and z locally. The map \mathcal{R} is called the reconstruction map because it “reconstructs” a distribution starting from the local description provided by Π_x and (3.2) is a generalisation of Taylor remainder formula for distributions. In case when $\Pi_z \tau$ represents a continuous function for every $\tau \in T$ belonging to the decomposition of U , $\mathcal{R}U$ is a function and it satisfies the key identity:

$$\mathcal{R}(U)(z) = \Pi_z(U(z))(z). \quad (3.3)$$

In general if the model is distributional valued and $U \in \mathcal{D}^{\gamma, \eta}$ then $\mathcal{R}U \in \mathcal{C}^\alpha$ with α is the lowest strictly negative homogeneity in the symbols of U otherwise, if $\mathcal{R}U$ is a function, then $\mathcal{R} \in \mathcal{C}^\chi((0, T) \times \mathbb{R})$ where χ is the lowest non integer homogeneity in the symbols associated to U , which can be uniquely extended to the space of bounded functions with the sup norm on $[0, T] \times \mathbb{R}$ (see [Hai14, Remark 6.11]). Moreover for any value $\gamma \geq \gamma' > 0$ the projection $\mathcal{Q}_{\gamma'} U \in \mathcal{D}^{\gamma', \eta}$ and $\mathcal{R}\mathcal{Q}_{\gamma'} U = \mathcal{R}U$ thus the reconstruction is obtained uniquely using the knowledge of $\Pi_z U$ on symbols with negative homogeneity. We will consider also the subspace $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta} \subset \mathcal{D}^{\gamma, \eta}$ where its elements $V \in \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}$ take value only in the subspace \mathcal{U} . Since we are interested in periodic functions, the additional conditions (2.19) on Π allow to pass the Definition 3.1 and the operator \mathcal{R} when the functions U is defined on $\mathbb{R} \times \mathbb{T}$ \mathcal{R} but we decide to express all general result in this section as if they were formulated on \mathbb{R}^2 for sake of concision (we can always recovering them by periodical extension). A trivial example to check all of these properties is given by the function $\mathbf{1}_+ \Xi : \mathbb{R}^2 \rightarrow \mathcal{T}$ which is simply defined by:

$$(\mathbf{1}_+ \Xi)(z) = \mathbf{1}_{(0, +\infty) \times \mathbb{R}}(z) \Xi = \begin{cases} \Xi & \text{if } t > 0, \\ 0 & \text{Otherwise.} \end{cases}$$

Applying the definitions of $\hat{\Gamma}, \hat{\Gamma}^\varepsilon$ and Γ^ε , one has immediately $\mathbf{1}_+ \Xi \in \mathcal{D}^{\gamma, \eta}(\hat{\Pi}^\varepsilon)$, $\mathbf{1}_+ \Xi \in \mathcal{D}^{\gamma, \eta}(\Pi^\varepsilon)$ and $\mathbf{1}_+ \Xi \in \mathcal{D}^{\gamma, \eta}(\hat{\Pi})$ for any $\gamma > 0$ and $-2 < \eta < \gamma$ a.s. ($\mathcal{D}^{\gamma, \eta}$ depends on the model, which is random, then it is a random space). Moreover since in the first two cases

the associated model takes values in the space of function for any $\varepsilon > 0$ we can easily calculate $\hat{\mathcal{R}}^\varepsilon$ and \mathcal{R}^ε , the reconstruction operator associated respectively to $\mathcal{D}^{\gamma,\eta}(\hat{\Pi}^\varepsilon)$ and $\mathcal{D}^{\gamma,\eta}(\Pi^\varepsilon)$ using (3.3). Indeed we have

$$\hat{\mathcal{R}}^\varepsilon \mathbf{1}_+ \Xi(z) = \mathbf{1}_{(0,+\infty) \times \mathbb{R}}(z) \hat{\Pi}_z^\varepsilon \Xi(z) = \mathbf{1}_{(0,+\infty) \times \mathbb{R}}(z) \xi_\varepsilon(z), \quad (3.4)$$

and similarly with \mathcal{R}^ε by construction of the canonical model. Nevertheless we can only send the parameter ε to zero only in the case of the renormalised model because only this one converges; in this case, using the continuity of the reconstruction map with respect the convergence of models the random distribution $\hat{\mathcal{R}}^\varepsilon \mathbf{1}_+ \Xi(z)$ converges in the $\mathcal{C}^{-3/2-\kappa}$ topology in probability to the random distribution $\hat{\mathcal{R}} \mathbf{1}_+ \Xi$ which is equal in law to

$$\hat{\mathcal{R}} \mathbf{1}_+ \Xi = \mathbf{1}_+ \xi,$$

where the distribution $\mathbf{1}_+ \xi$ was introduced in (1.10).

3.2 Operations with SHE

Although modelled distributions look very unusual, the reconstruction theorem associates to them a classical function. Under this identification it is possible to lift up some classical operations on \mathcal{C}^γ spaces directly at the $\mathcal{D}^{\gamma,\eta}$ level as it was explained in detail in [Hai14]. We briefly recall their definitions to put them in relation with the stochastic heat equation.

Convolution

The first operation is the convolution with the heat kernel on \mathbb{R} , $G: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$. In other terms, for any $V \in \mathcal{D}^{\gamma,\eta}(\Pi)$ there exists $\mathcal{P}(V) \in \mathcal{D}^{\bar{\gamma},\bar{\eta}}(\Pi)$ depending on Π and the parameters $\bar{\gamma} = \gamma + 2$, $\bar{\eta} = \alpha \wedge \eta + 2$ such that the map $\mathcal{P}: \mathcal{M} \times \mathcal{D}^{\gamma,\eta} \rightarrow \mathcal{M} \times \mathcal{D}^{\bar{\gamma},\bar{\eta}}$ is locally Lipschitz and

$$\mathcal{R}\mathcal{P}(V) = G * \mathcal{R}V. \quad (3.5)$$

(see [Hai14, Proposition 6.16, Lemma 7.3]). In general the convolution on the right hand side may not be well-defined because G is not compactly supported. However, looking at a finite time horizon and under the hypothesis that $\mathcal{R}V$ is supported in $\mathbb{R}_+ \times \mathbb{R}$, we can apply the Lemma 2.6 and the right hand sides of (3.5) writes as $(K + R) * \mathcal{R}V$. Therefore it is sufficient to express \mathcal{P} under the form $\mathcal{P} := \mathcal{K} + R_{\bar{\gamma}}$ where the operators \mathcal{K} , $R_{\bar{\gamma}}: \mathcal{M} \times \mathcal{D}^{\gamma,\eta} \rightarrow \mathcal{M} \times \mathcal{D}^{\bar{\gamma},\bar{\eta}}$ are also locally Lipschitz and they satisfy

$$\mathcal{R}\mathcal{K}(V) = K * \mathcal{R}V, \quad \mathcal{R}R_{\bar{\gamma}}V = R * \mathcal{R}V. \quad (3.6)$$

How these operators are defined? the function $(R * \mathcal{R}V)$ is always smooth by hypothesis on R , therefore we can always associate to it for any $\bar{\gamma} \in A$, $\bar{\gamma} > 0$ the lifting of its Taylor polynomial:

$$R_{\bar{\gamma}}(V)(z) := \mathcal{Q}_{\bar{\gamma}} \sum_k (\partial^k R * (\mathcal{R}V))(z) \frac{\mathbf{X}^k}{k!}, \quad (3.7)$$

which belongs to $\mathcal{D}^{\bar{\gamma},\bar{\eta}}$ for any $\bar{\eta} \leq \bar{\gamma}$ by a simple check. On the other hand the operator \mathcal{K} is more complex and it is given in general by this abstract formula

$$\mathcal{K}V(z) := I(V)(z) + \mathcal{Q}_{\bar{\gamma}} \sum_k \partial^k K * (\mathcal{R}V - \Pi_x \mathcal{Q}_{(2k_1+k_2-2)} V(z))(z) \frac{\mathbf{X}^k}{k!} \quad (3.8)$$

where I is deterministic linear map such that $I(\Xi) = \mathcal{J}(\Xi)$ (for more information on I see [Hai14, Chapter 5]). Noting by $\hat{\mathcal{K}}_\varepsilon \hat{R}_\varepsilon^\gamma \hat{\mathcal{P}}_\varepsilon$ the operators with respect the model $\hat{\Pi}^\varepsilon$ and by $\hat{\mathcal{K}} \hat{R}^\gamma \hat{\mathcal{P}}$, the operator associated to $(\hat{\Pi}, \hat{\Gamma})$, if we apply both them to the function $\mathbf{1}_+\Xi \in \mathcal{D}^{\gamma, \eta}$ for some choice of $\gamma > 0$, $0 < \eta < \gamma$ and $\varepsilon > 0$, the formula (3.8) becomes:

$$\hat{\mathcal{K}}_\varepsilon(\mathbf{1}_+\Xi)(z) = K * \hat{\mathcal{R}}_\varepsilon(\mathbf{1}_+\Xi)(z) \mathbf{1} + \mathbf{1}_+\dot{\downarrow}, \quad \hat{\mathcal{K}}(\mathbf{1}_+\Xi)(z) = K * \hat{\mathcal{R}}(\mathbf{1}_+\Xi)(z) \mathbf{1} + \mathbf{1}_+\dot{\downarrow}. \quad (3.9)$$

Then the modelled distributions $U_\varepsilon = \mathcal{P}_\varepsilon(\mathbf{1}_+\Xi)$, $U = \hat{\mathcal{P}}(\mathbf{1}_+\Xi)$ both belong to $\mathcal{D}^{\bar{\gamma}, 1/2-\kappa}$ and they are written as follows:

$$U_\varepsilon(z) = u_\varepsilon(z) \mathbf{1} + \mathbf{1}_+\dot{\downarrow} + \mathcal{Q}_{\bar{\gamma}} \sum_{k > (0,0)} u_\varepsilon^{(k)}(z) \frac{\mathbf{X}^k}{k!}, \quad (3.10)$$

$$U(z) = u(z) \mathbf{1} + \mathbf{1}_+\dot{\downarrow} + \mathcal{Q}_{\bar{\gamma}} \sum_{k > (0,0)} u^{(k)}(z) \frac{\mathbf{X}^k}{k!}, \quad (3.11)$$

where $u_\varepsilon^{(k)} = \partial_k R * \mathbf{1}_+\xi^\varepsilon$, $u^{(k)} = \partial_k R * \mathbf{1}_+\xi$ (we note the projection $\mathcal{Q}_{\gamma+2}$ makes the series a finite sum). As a straightforward application of the identities (3.3) we obtain immediately $\hat{\mathcal{R}}^\varepsilon U_\varepsilon = u_\varepsilon$ and $\hat{\mathcal{R}}U = u$ a.s. For this reason we call the function U_ε (respectively U) the lifting of u_ε (u), because they “lift” these random fields to the level of the $\mathcal{D}^{\gamma, \eta}$ spaces. Finally the convergence of $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ to $(\hat{\Pi}, \hat{\Gamma})$ together with the continuity of \mathcal{P} allows to deduce a convergence result of the approximated solution u_ε towards to u with respect the Hölder topology [Hai14, Theorem 3.10, Proposition 6.9]. This result comes also as a trivial application of [HP15, Theorem 1.1].

Proposition 3.2. *Let u_ε be the solution of (1.5) and u the solution of (1.1). Then*

$$\sup_{(t,x) \in [0,T] \times \mathbb{T}} |u_\varepsilon(t, x) - u(t, x)| \rightarrow 0 \quad (\varepsilon \rightarrow 0^+)$$

in probability. Moreover, $u_\varepsilon \rightarrow u$ in probability for the topology of $\mathcal{C}^{1/2-\kappa}((0, T) \times \mathbb{T})$.

Remark 3.3. For any fixed $\varepsilon > 0$ we can consider also \mathcal{P}_ε , the convolution operator associated the canonical model $(\Pi^\varepsilon, \Gamma^\varepsilon)$. In this particular case the following identities hold:

$$\Pi_z^\varepsilon \Xi = \hat{\Pi}_z^\varepsilon \Xi = \xi_\varepsilon, \quad \Gamma_{z\bar{z}}^\varepsilon \Xi = \hat{\Gamma}_{z\bar{z}}^\varepsilon \Xi = \Xi.$$

Therefore this implies that $\mathbf{1}_+\Xi \in \mathcal{D}^{\gamma, \eta}(\Pi^\varepsilon)$ and $\mathcal{P}_\varepsilon(\mathbf{1}_+\Xi) = \hat{\mathcal{P}}_\varepsilon(\mathbf{1}_+\Xi)$. Then we could have also used the classical model to lift u_ε . However the absence of a limiting model prevents us to send the limit as $\varepsilon \rightarrow 0^+$ at the level of modelled distributions.

Composition

Given any $V \in \mathcal{D}_\mathcal{U}^{\gamma, \eta}$, the algebraic structure of \mathcal{U} allows us to decompose uniquely V as follows

$$V(z) = v(z) \mathbf{1} + \tilde{V}(z),$$

for some function $v: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\tilde{V}: \mathbb{R}^2 \rightarrow \bigotimes_{\alpha \geq \chi} \mathcal{T}_\alpha$ for some $\chi > 0$. Under these conditions for any smooth function $h: \mathbb{R} \rightarrow \mathbb{R}$ we define the function $H(V): \mathbb{R}^2 \rightarrow \mathcal{U}$ as follows:

$$H(V)(z) := \mathcal{Q}_\gamma \sum_{k \geq 0} \frac{h^{(k)}(v(z))}{k!} (\tilde{V}(z))^k = \mathcal{Q}_\gamma \sum_{k \geq 0} \frac{h^{(k)}(v(z))}{k!} (V(z) - v(z)\mathbf{1})^k. \quad (3.12)$$

Following [Hai14, Theorem 6.13] and [HP15, Proposition 3.11], $H(V)$ is a well defined modelled distribution belonging to $\mathcal{D}^{\gamma, \eta}$ for any $h \in C^\beta(\mathbb{R}, \mathbb{R})$ with β integer $\geq (1 \vee \gamma/\zeta)$, $\eta \in [0, \gamma]$, the map $V \rightarrow H(V)$ is locally Lipschitz if $\beta \geq (1 \vee \gamma/\chi) + 1$, $\eta \in [0, \gamma]$ and

$$\mathcal{R}H(V) = h \circ (\mathcal{R}V). \quad (3.13)$$

Let us apply these theorems when $V = U_\varepsilon$ and $h = \varphi', \varphi''$ (the associated operator will be denoted by Φ' and Φ''). In our case the parameter χ is equal to $1/2 - \kappa$, therefore for any fixed parameter $\bar{\gamma}$ in the definition of U_ε from (3.10) the modelled distribution $\Phi'(U_\varepsilon)$ and $\Phi''(U_\varepsilon)$ are both well defined modelled distribution of $\mathcal{D}^{\bar{\gamma}, \bar{\eta}}(\hat{\Pi}^\varepsilon)$ converging respectively to $\Phi'(U)$ and $\Phi''(U)$ for any φ of class C^β with $\beta \geq 2\bar{\gamma}/(1 - 2\kappa) + 3$. Since this operation depends only on the algebraic structure one has also the same result on $\mathcal{D}^{\bar{\gamma}, \bar{\eta}}(\Pi^\varepsilon)$ but only for finite $\varepsilon > 0$.

Space derivative

Thanks to its definition, the regularity structure \mathcal{T} allows us to define also a linear map $D_x: \mathcal{U} \rightarrow \mathcal{T}$ which behaves like a space derivative on abstract symbols. Indeed it is sufficient to set:

$$\begin{aligned} D_x \mathbf{1} &= 0, & D_x X_1 &= 0, & D_x X_2 &= \mathbf{1} & D_x \mathcal{J}(\Xi) &= \mathcal{J}_1(\Xi), \\ D_x(\tau\sigma) &= D_x(\tau)\sigma + D_x(\sigma)\tau, \end{aligned}$$

for any couple τ, σ such that $\sigma\tau \in U$. Extending these conditions linearly we identify uniquely a map $D_x: \mathcal{U} \rightarrow \mathcal{U} \cup \mathcal{V}_{\mathcal{J}_1(\Xi)}$. By taking the space derivative on both sides in (2.16) it is straightforward to show also for any $z \in \mathbb{R}^2$ and any $\tau \in \mathcal{U}$

$$\partial_x \Pi_z^\varepsilon \tau = \Pi_z^\varepsilon D_x \tau.$$

These property make the operator D_x an abstract gradient which is compatible with the canonical model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ (see [Hai14, Definition 5.25, Definition 5.26]). Moreover, since the canonical model equals its BPHZ renormalisation over the set $\mathcal{U} \cup \mathcal{V}_{\mathcal{J}_1(\Xi)}$, one has also

$$\partial_x \hat{\Pi}_z^\varepsilon \tau = \hat{\Pi}_z^\varepsilon D_x \tau.$$

Thanks to these properties, the operator D_x can be applied at the level of modelled distributions and the map $V \rightarrow D_x V$ becomes a continuous operator from $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta}$ to $\mathcal{D}^{\gamma-1, \eta-1}$ such that for any $\gamma > 1$

$$\hat{\mathcal{R}}^\varepsilon D_x V = \partial_x \hat{\mathcal{R}}^\varepsilon V, \quad \mathcal{R}^\varepsilon D_x V = \partial_x \mathcal{R}^\varepsilon V \quad (3.14)$$

and as $\varepsilon \rightarrow 0^+$ we have

$$\hat{\mathcal{R}} D_x V = \partial_x \hat{\mathcal{R}} V,$$

where the derivative in the right hand side is interpreted in the distributional sense when $\hat{\mathcal{R}} V$ is not smooth enough.

Product

We conclude the list of operation on modelled distributions with the product. Even if \mathcal{T} is not an algebra with respect the juxtaposition product, by construction we are allowed to consider some products between symbols living in \mathcal{T} . For example we can multiply an element in \mathcal{U} with Ξ obtaining an element of \mathcal{V}_Ξ or by multiplication of two elements living in $\mathcal{V}_{\mathcal{F}_1(\Xi)}$ we have an element of $\mathcal{V}_{\mathcal{F}_1(\Xi)^2}$. In particular for any couple of modelled distributions $U_1 \in \mathcal{D}^{\gamma_1, \eta_1}$, $U_2 \in \mathcal{D}^{\gamma_2, \eta_2}$ such that $\hat{\mathcal{R}}U_1 \in \mathcal{C}^{\alpha_1}$ and $\hat{\mathcal{R}}U_2 \in \mathcal{C}^{\alpha_1}$, if it is possible to multiply any symbol appearing in U_1 with any symbol in U_2 , we can define a new function $U_1U_2: \mathbb{R}^2 \rightarrow \mathcal{T}$. As a consequence of [Hai14, Proposition 6.12] it is possible to fit this function in the modelled distribution formalism, indeed one has $\mathcal{Q}_\gamma U_1U_2 \in \mathcal{D}^{\gamma, \eta}$ where

$$\gamma = (\gamma_1 + (\alpha_2 \wedge 0)) \wedge (\gamma_2 + (\alpha_1 \wedge 0)), \quad \eta = (\eta_1 + \eta_2) \wedge (\eta_1 + (\alpha_2 \wedge 0)) \wedge (\eta_2 + (\alpha_1 \wedge 0)).$$

In our case we can apply this result together with other previous ones to identify a range of parameter $\bar{\gamma}$ such that, starting from U given in (3.11), the functions $\Phi'(U)\Xi$ and $\Phi''(U)(D_xU)^2$ are well defined modelled distributions. In the first case $\Phi'(U)\Xi$ belongs to $\mathcal{D}^{\bar{\gamma}-3/2-\kappa, -1-\kappa}$. On the other hand one has $D_xU \in \mathcal{D}^{\bar{\gamma}-1, -1/2-\kappa}$, $\hat{\mathcal{R}}D_xU \in \mathcal{C}^{-1/2-\kappa}$ and its square $(D_xU)^2 \in \mathcal{D}^{\bar{\gamma}-3/2-\kappa, -1-2\kappa}$ is a modelled distribution satisfying $\hat{\mathcal{R}}(D_xU)^2 \in \mathcal{C}^{-1-2\kappa}$. Therefore $\Phi''(U)(D_xU)^2 \in \mathcal{D}^{\bar{\gamma}-3/2-\kappa, -1-2\kappa}$ and we will need to fix $\bar{\gamma} \in (3/2 + \kappa, \zeta)$. Under this hypothesis on $\bar{\gamma}$, recalling the results on the composition of function, the function φ must belong to C^β with $\beta \geq 6$.

Contrary to the previous operations, there is no classical interpretation of the distribution $\hat{\mathcal{R}}(U_1U_2)$ as a product of functions because in some cases (like e.g. when $D_xU = U_1 = U_2$) there is no classical way to multiply the distributions $\hat{\mathcal{R}}U_1$ $\hat{\mathcal{R}}U_2$ and this resulting object does depend in strong way on the underlying model and the algebraic structure. However in case of the canonical model $\Pi_x^\varepsilon \tau$ is the point-wise product of continuous functions symbol τ belonging to the decomposition of U_1U_2 , then we deduce

$$\mathcal{R}^\varepsilon(U_1U_2) = \mathcal{R}^\varepsilon U_1 \mathcal{R}^\varepsilon U_2.$$

As a consequence of the identity (3.3).

4 Itô formula

We sum up the convergence result of $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ together with the calculus on $\mathcal{D}^{\gamma, \eta}$ spaces to show an explicit example of how this new paradigm of Itô formulae work in the case of the additive stochastic heat equation.

4.1 Rough Itô formulae

We begin with a general differential identity involving only objects defined from the regularity structures setting.

Theorem 4.1 (Rough differential Itô Formula). *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^6 . Then one has*

$$(\partial_t - \partial_{xx})(\varphi(u)) = \hat{\mathcal{R}}(\Phi'(U)\Xi) - \hat{\mathcal{R}}(\Phi''(U)(D_xU)^2) \quad (4.1)$$

a.s. as distributions in $\mathcal{E}^{-3/2-\kappa}((0, T) \times \mathbb{T})$.

Proof. The identity will be obtained by means of a rearrangement of the calculations in the introduction. we consider u_ε , the solution of (1.5) and we rewrite the identity (1.8) using $U_\varepsilon \in \mathcal{D}^{\bar{\gamma}, 1/2-\kappa}$ from (3.10). We decide to choose $\bar{\gamma} = 3/2 + 2\kappa$ so then, together with the hypothesis on φ , the functions $\Phi'(U_\varepsilon)\Xi$ and $\Phi''(U_\varepsilon)D_x U_\varepsilon^2$ will be two modelled distributions converging to $\Phi'(U_\varepsilon)\Xi$ and $\Phi''(U_\varepsilon)D_x U_\varepsilon^2$. Using the operations on $\mathcal{D}^{\gamma, \eta}$ spaces we rewrite (1.8) as

$$(\partial_t - \partial_{xx})\varphi(u_\varepsilon) = (\hat{\mathcal{R}}_\varepsilon \Phi'(U_\varepsilon))(\hat{\mathcal{R}}_\varepsilon \Xi) - (\hat{\mathcal{R}}_\varepsilon \Phi''(U_\varepsilon))(\hat{\mathcal{R}}_\varepsilon D_x U_\varepsilon^2).$$

Since all the modelled distribution under the form $\hat{\mathcal{R}}^\varepsilon V$ converge to $\hat{\mathcal{R}}^\varepsilon V$ in probability, we rewrite this identity putting the products inside the operator $\hat{\mathcal{R}}^\varepsilon$. In order to do that we calculate $\hat{\mathcal{R}}_\varepsilon \Phi'(U_\varepsilon)\Xi$ and $\Phi''(U_\varepsilon)D_x U_\varepsilon^2$. Firstly we write down

$$\begin{aligned} \Phi'(U_\varepsilon)\Xi &= \varphi'(u_\varepsilon)\dot{\cdot} + \varphi''(u_\varepsilon)\dot{\vee} + \varphi''(u_\varepsilon)u'_\varepsilon{}^{(0,1)}\dot{\cdot} + \frac{\varphi'''(u_\varepsilon)}{2}\dot{\cdot}\dot{\cdot} \\ &\quad + \varphi'''(u_\varepsilon)u'_\varepsilon{}^{(0,1)}\dot{\vee} + \frac{\varphi^{(4)}(u_\varepsilon)}{6}\dot{\cdot}\dot{\cdot}\dot{\cdot}, \\ \Phi''(U_\varepsilon)(D_x U_\varepsilon)^2 &= \varphi''(u_\varepsilon)\left(\dot{\vee} + 2(u'_\varepsilon)\dot{\cdot} + (u'_\varepsilon)^2\mathbf{1}\right) + \varphi'''(u_\varepsilon)\dot{\vee}\dot{\vee} + 2\varphi'''(u_\varepsilon)u'_\varepsilon\dot{\vee} \\ &\quad + \varphi'''(u_\varepsilon)u'_\varepsilon{}^{(0,1)}\dot{\vee}\dot{\vee} + \varphi^{(iv)}(u_\varepsilon)\dot{\vee}\dot{\vee}\dot{\vee}. \end{aligned}$$

(We omit the dependency on z and $\mathbf{1}_+$ in the coefficients as a shorthand in the notation). Then we use (2.20), (2.23) to calculate

$$\begin{aligned} M_\varepsilon \Phi'(U_\varepsilon)\Xi &= \varphi'(u_\varepsilon)\dot{\cdot} + \varphi''(u_\varepsilon)\left(\dot{\vee} - C_\varepsilon^1\mathbf{1}\right) + \varphi''(u_\varepsilon)u'_\varepsilon{}^{(0,1)}\dot{\cdot} + \frac{\varphi'''(u_\varepsilon)}{2}\left(\dot{\cdot}\dot{\cdot} - 2C_\varepsilon^1\dot{\cdot}\right) \\ &\quad + \varphi'''(u_\varepsilon)u'_\varepsilon\left(\dot{\vee} - C_\varepsilon^1\bullet_{(0,1)}\right) + \frac{\varphi^{(4)}(u_\varepsilon)}{6}\left(\dot{\cdot}\dot{\cdot}\dot{\cdot} - 3C_\varepsilon^1\dot{\vee}\right), \\ M_\varepsilon \Phi''(U_\varepsilon)(D_x U_\varepsilon)^2 &= \varphi''(u_\varepsilon)(D_x U_\varepsilon)^2 - \varphi''(u_\varepsilon)C_\varepsilon^2\mathbf{1} + 2\varphi'''(u_\varepsilon)u'_\varepsilon\dot{\vee} + \varphi'''(u_\varepsilon)\left(\dot{\vee}\dot{\vee} - C_\varepsilon^2\dot{\cdot}\right) \\ &\quad + \varphi'''(u_\varepsilon)u'_\varepsilon\left(\dot{\vee}\dot{\vee} - C_\varepsilon^2\bullet_{(0,1)}\right) + \frac{\varphi^{(iv)}(u_\varepsilon)}{2}\left(\dot{\vee}\dot{\vee}\dot{\vee} - C_\varepsilon^2\dot{\vee}\dot{\vee}\right). \end{aligned}$$

Applying operator $\Pi_z^\varepsilon \cdot (z)$ on both sides, we obtain a drastic simplification of many terms because if $\tau \in T$ is under the form $\sigma_1\sigma_2$ with $|\sigma_1| > 0$, then the multiplicative property of Π_z^ε together with the model assumptions implies $\Pi_z^\varepsilon \tau(z) = 0$. Hence we stay with the identities

$$\begin{aligned} \hat{\mathcal{R}}_\varepsilon \Phi'(U_\varepsilon)\Xi &= \Pi_z^\varepsilon M_\varepsilon \Phi'(U_\varepsilon)\Xi(z) = \varphi'(u_\varepsilon)\xi_\varepsilon - \varphi''(u_\varepsilon)C_\varepsilon^1, \\ \hat{\mathcal{R}}_\varepsilon (\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) &= \Pi_z^\varepsilon M_\varepsilon \Phi''(U_\varepsilon)(D_x U_\varepsilon)^2(z) = \varphi''(u_\varepsilon) [\Pi_z^\varepsilon [(D_x U_\varepsilon)^2](z) - C_\varepsilon^2]. \end{aligned}$$

Using again (3.3) and the properties of the reconstruction with respect the canonical model one has

$$\Pi_z^\varepsilon [(D_x U_\varepsilon)^2](z) = \mathcal{R}_\varepsilon(D_x U_\varepsilon)^2 = (\mathcal{R}_\varepsilon(D_x U_\varepsilon))^2 = (\partial_x u_\varepsilon)^2.$$

Therefore the left hand side of (1.8) becomes

$$\partial_t(\varphi(u_\varepsilon)) - \partial_x^2(\varphi(u_\varepsilon)) = \hat{\mathcal{R}}_\varepsilon(\Phi'(U_\varepsilon)\Xi) - \hat{\mathcal{R}}_\varepsilon(\Phi''(U_\varepsilon)(D_x U_\varepsilon)^2) + \varphi''(u_\varepsilon) (C_\varepsilon^1 - C_\varepsilon^2) . \quad (4.2)$$

By Sending $\varepsilon \rightarrow 0^+$ the left hand side converges to $(\partial_t - \partial_{xx})(\varphi(u))$ thanks to the Proposition 3.2. On the other hand, the continuity of reconstruction map $\mathcal{R}: \mathcal{M} \times \mathcal{D}^{\gamma,\eta} \rightarrow \mathcal{E}^{-3/2-\kappa}(\mathbb{R} \times S^1)$ and Lemma A.2 implies that the right hand side of (4.2) converge in probability to the right hand side of (4.1). \square

From the formula (4.1) we can derive an explicit expression of $\varphi(u)$ by convolution with the heat kernel P .

Corollary 4.2 (Rough integral Itô Formula). *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^6 Then a.s.*

$$\varphi(u(t, x)) = \varphi(0) + P * \mathbf{1}_+ \hat{\mathcal{R}}(\Phi'(U)\Xi)(t, x) - P * \mathbf{1}_+ \hat{\mathcal{R}}((\Phi''(U)(D_x U)^2))(t, x). \quad (4.3)$$

Proof. We consider the random PDE

$$\begin{cases} \partial_t v - \partial_{xx} v = \hat{\mathcal{R}}(\Phi'(U)\Xi) - \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2) \\ v(0, \cdot) = \varphi(0). \end{cases} \quad (4.4)$$

where $v: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ has periodic boundary conditions on the space variable. The right hand side of equation (4.4) is a.s. a distribution living in $\mathcal{E}^{-3/2-\kappa}(\mathbb{R} \times \mathbb{T})$ and solution $v \in \mathcal{E}^{1/2-\kappa}([0, T] \times \mathbb{T})$ is given by the right hand side of (4.3). Since $\varphi(u)$ satisfies the equation too, we conclude by uniqueness. \square

4.2 Identification of the laws

Formula (4.3) expresses $\phi(u)$ in term of a convolution with two random distributions. In this section we identify their laws with some classical objects of stochastic analysis. Since the kernel P is non anticipative (that is for any $s > t$, $P(t-s, x) = 0$ for any $x \in \mathbb{T}$) one has the following identities for any fixed value $(t, x) \in (0, T) \times \mathbb{T}$

$$P * \mathbf{1}_+ \hat{\mathcal{R}}(\Phi'(U)\Xi)(t, x) = P * \mathbf{1}_{(0,t)} \hat{\mathcal{R}}(\Phi'(U)\Xi)(t, x)$$

$$P * \mathbf{1}_+ \hat{\mathcal{R}}((\Phi''(U)(D_x U)^2))(t, x) = P * \mathbf{1}_{(0,t)} \hat{\mathcal{R}}((\Phi''(U)(D_x U)^2))(t, x)$$

Therefore we express firstly the law of the reconstructions and then we will convolve them with P . In the first case a first result was already known in the literature.

Proposition 4.3. *Let $(\hat{\Pi}, \hat{\Gamma})$ be the BPHZ model and $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^6 . Then for any test function $\psi: \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$ with $\text{supp}(\psi) \subset (0, +\infty) \times \mathbb{T}$, one has*

$$(\mathbf{1}_{(0,t)} \hat{\mathcal{R}}(\Phi'(U)\Xi))(\psi) = \int_0^t \int_{\mathbb{T}} \varphi'(u(s, y)) \psi(s, y) dW_{s,y} . \quad (4.5)$$

Proof. Thanks to the identification of \mathcal{V}_Ξ and \mathcal{U} into the algebraic structure of [HP15], this result is a consequence of [HP15, Theorem 6.2] applied to the modelled distribution $\Phi'(U) \in \mathcal{D}_u^{\bar{\gamma}, 1/2-\kappa}$, with $\bar{\gamma} = 3/2 + 2\kappa$, which is given by

$$\Phi'(U) = \varphi'(u)\mathbf{1} + \varphi''(u)\dot{\mathbf{1}} + \varphi''(u)u'X_1 + \varphi'''(u)\dot{\mathbf{V}} + 2\varphi'''(u)X_1\dot{\mathbf{1}} + \varphi^{(iv)}(u)\dot{\mathbf{V}}\dot{\mathbf{1}}.$$

Clearly $(t, x) \rightarrow \Phi'(U)(t, x)$ is an adapted process with respect the filtration \mathcal{F}_t and $\|\Phi'(U)\|_{\gamma, \eta}$ is a.s. bounded because u and u' are a.s. continuous (then bounded on compact domains). Thus we can apply directly the result. \square

We pass to the identification of the other stochastic object in (4.1).

Theorem 4.4. *Let $(\hat{\Pi}, \hat{\Gamma})$ be the BPHZ model and $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^6 . Then, for any test function $\psi: \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$ with $\text{supp}(\psi) \subset (0, +\infty) \times \mathbb{T}$, one has*

$$\mathbf{1}_{(0,t)} \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)(\psi) = -\frac{1}{2} \int_0^t \int_{\mathbb{T}} \psi(s, y) \varphi''(u(s, y)) C(s) dy ds \quad (4.6)$$

$$+ 2 \int_{\Delta_{2,t} \times \mathbb{T}^2} \int_{\mathbf{s}_1}^t \int_{\mathbb{T}} \psi(s, y) \varphi''(u(s, y)) P_x(s - \mathbf{s}_1, y - \mathbf{y}_1) P_x(s - \mathbf{s}_2, y - \mathbf{y}_2) dy ds W(d\mathbf{s}, d\mathbf{y}),$$

where $W(d\mathbf{s}, d\mathbf{y})$ denotes the Skorohod integral, $\Delta_{2,t} = \{0 \leq \mathbf{s}_1 < \mathbf{s}_2 \leq t\}$ and $C: (0, T) \rightarrow \mathbb{R}$ is the deterministic integrable function

$$C(s) := \|P(s, \cdot)\|_{L^2(\mathbb{T})}^2.$$

Proof. We prove firstly the result when $\psi = h \otimes l$ where $h: [0, t] \rightarrow \mathbb{R}$ is a compactly supported smooth function and $l: \mathbb{T} \rightarrow \mathbb{R}$. Moreover, since u is a.s. a continuous function and we are looking all the stochastic objects in a finite time horizon we can suppose that φ has all bounded derivatives up to order 6. Under these hypothesis we can forget about multiplicative factor $\mathbf{1}_{(0,t)}$ obtaining

$$\begin{aligned} \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)(\psi) &= \left(-\partial_t(\varphi(u)) + \partial_x^2(\varphi(u)) + \hat{\mathcal{R}}(\Phi'(U)\Xi) \right) (\psi) = \\ &= \int_0^t \langle \varphi(u(s, \cdot)) h'(s), l(\cdot) \rangle + \langle \varphi(u(s, \cdot)) h(s), l''(\cdot) \rangle ds + \int_0^t \int_{\mathbb{T}} \varphi'(u(s, y)) h(s) l(y) dW_{s,y}, \end{aligned} \quad (4.7)$$

where the quantity under brackets $\langle \cdot, \cdot \rangle$ is the L^2 product on \mathbb{T} . In order to obtain the equality between the right hand side of (4.6) and the above random variable we will adapt the same argument of [Zam06] in this context. for any $\varepsilon > 0$, $x \in \mathbb{T}$ we introduce the process

$$u_t^\varepsilon(x) =: \int_0^t \int_{\mathbb{T}} P(\varepsilon + t - s, x - y) dW_{s,y}.$$

u^ε is a semimartingale converging in $L^2(\mathbb{P})$ to u with respect the sup norm such that for any t , the function $x \rightarrow u_t^\varepsilon(x)$ is a.s. smooth. Moreover for any $x \in \mathbb{T}$ the process $(u_t^\varepsilon(x))_{t \in [0, T]}$ satisfies

$$du_t^\varepsilon(x) = \partial_{xx}(u_t^\varepsilon)(x) dt + dW_t^\varepsilon(x)$$

where $W_t^\varepsilon(x)$ is the \mathcal{F}_t martingale

$$W_t^\varepsilon(x) = \int_{\mathbb{T}} P(\varepsilon, x - y) W(t, dy), \quad \langle W^\varepsilon(x) \rangle_t = t \int_{\mathbb{T}} P(\varepsilon, x - y)^2 dy =: tC_\varepsilon(x).$$

Applying the classical Itô formula to the process $h(s)\varphi(u_s^\varepsilon(x))$ one has

$$\begin{aligned} h(t)\varphi(u_t^\varepsilon(x)) - h(0)\varphi(u_0^\varepsilon(x)) &= \int_0^t h'(s)\varphi(u_s^\varepsilon(x)) ds + \int_0^t h(s)\partial_{xx}(u_s^\varepsilon)(x)\varphi'(u_s^\varepsilon(x)) ds \\ &\quad + \int_0^t h(s)\varphi'(u_s^\varepsilon(x)) dW_s^\varepsilon(x) + \frac{1}{2}C_\varepsilon(x) \int_0^t h(s)\varphi''(u_s^\varepsilon(x)) ds. \end{aligned} \quad (4.8)$$

The left hand side of (4.8) is a.s. equal to zero by hypothesis on h and we can still apply formula (1.7) with u^ε instead of u_ε . Therefore we can arrange the term under this form:

$$\begin{aligned} \int_0^t h'(s)\varphi(u_s^\varepsilon(x)) + h(s)\partial_{xx}(\varphi(u_s^\varepsilon(x))) ds + \int_0^t h(s)\varphi'(u_s^\varepsilon(x)) dW_s^\varepsilon(x) \\ = ((\partial_x u^\varepsilon)^2(x) - \frac{C_\varepsilon(x)}{2}) \int_0^t h(s)\varphi''(u_s^\varepsilon(x)) ds. \end{aligned} \quad (4.9)$$

By multiplying both sides of (4.9) with l and integrating over \mathbb{T} this identity becomes

$$\begin{aligned} \int_0^t \langle h'(s)\varphi(u_s^\varepsilon(\cdot)), l \rangle h(s) \langle \varphi(u_s^\varepsilon(\cdot)), l'' \rangle ds + \langle \int_0^t h(s)\varphi'(u_s^\varepsilon(\cdot)) dW_s^\varepsilon(\cdot), l \rangle \\ = \int_0^t \int_{\mathbb{T}} l(x)h(s)\varphi''(u_s^\varepsilon(x))((\partial_x u_s^\varepsilon)^2(x) - \frac{C_\varepsilon(x)}{2}) ds dx. \end{aligned} \quad (4.10)$$

Using a simple Fubini theorem with respect the stochastic integral

$$\langle \int_0^t h(s)\varphi'(u_s^\varepsilon(\cdot)) dW_s^\varepsilon(\cdot), l \rangle = \int_0^t \int_{\mathbb{T}} h(s) \left(\int_{\mathbb{T}} P(\varepsilon, y - x)\varphi'(u_s^\varepsilon(x))l(x) dx \right) dW_{s,y}$$

and it is easy to see that the left hand side of (4.10) converges in $L^2(\mathbb{P})$ to the right hand side of (4.7). Then to conclude the theorem it is sufficient to study the convergence of the right hand side of (4.10) to obtain an a.s. identity. For any $x \in \mathbb{T}$, $(\partial_x u_s^\varepsilon(x))_{s \geq 0}$ is the \mathcal{F}_s martingale

$$N_s(x) := \int_0^s \int_{\mathbb{T}} \partial_x P(\varepsilon + s - r, x - y) dW_{r,y}$$

therefore we can apply again Itô formula on $N_s(x)$

$$N_s(x)^2 = 2 \int_0^s N_r(x) dN_r(x) + \langle N(\cdot)(x) \rangle_s.$$

and the right hand side of (4.10) becomes

$$\begin{aligned} \int_0^t \int_{\mathbb{T}} l(x)h(s)\varphi''(u_s^\varepsilon(x)) \left(\langle N(\cdot)(x) \rangle_s - \frac{C_\varepsilon(x)}{2} \right) ds dx \\ + 2 \int_0^t \int_{\mathbb{T}} l(x)h(s)\varphi''(u_s^\varepsilon(x)) \int_0^s N_r(x) dN_r(x) ds dx =: A_1^\varepsilon + A_2^\varepsilon \end{aligned}$$

We treat both terms separately. In case of A_1^ε for any $s \geq 0$ and x we have

$$\langle N_\cdot(x) \rangle_s - \frac{C_\varepsilon(x)}{2} = \int_\varepsilon^{s+\varepsilon} \int_{\mathbb{T}} (\partial_x P(r, x-y))^2 dr dy - \frac{1}{2} \int_{\mathbb{T}} P(\varepsilon, x-y)^2 dy. \quad (4.11)$$

Since $P(\cdot, x)$ is smooth outside the origin and \mathbb{T} does not have a boundary one has

$$\begin{aligned} & \int_{\mathbb{T}} \frac{P(s+\varepsilon, x-y)^2}{2} dy - \int_{\mathbb{T}} \frac{P(\varepsilon, x-y)^2}{2} dy = \int_\varepsilon^{s+\varepsilon} \partial_t \int_{\mathbb{T}} \frac{P(r, x-y)^2}{2} dy dr \\ & = \int_\varepsilon^{s+\varepsilon} \int_{\mathbb{T}} P(r, x-y) \partial_{xx} P(r, x-y) dy dr = - \int_\varepsilon^{s+\varepsilon} \int_{\mathbb{T}} (\partial_x P(r, x-y))^2 dy dr. \end{aligned}$$

Thus A_1^ε equals to

$$-\frac{1}{2} \int_0^t \int_{\mathbb{T}} l(x) h(s) \varphi''(u_s^\varepsilon(x)) \left(\int_{\mathbb{T}} P(s+\varepsilon, x-y)^2 dy \right) ds dx,$$

which converges in $L^2(\mathbb{P})$ to the first term on the right hand side of (4.6) because the function $\zeta \rightarrow \int_{\mathbb{T}} P(s, \zeta-y)^2 dy$ is constant by invariance of translations. On the other hand

$$A_2^\varepsilon = 2 \int_0^t \int_{\mathbb{T}} l(x) h(s) \varphi''(u_s^\varepsilon(x)) \left(\int_0^s \int_{\mathbb{T}} N_r(x) \partial_x P(\varepsilon+s-r, x-y) dW_{r,y} \right) ds dx.$$

In order to identify their limit we want to commute the deterministic integral in ds with the iterated stochastic Itô integral. However, the resulting integrand $\varphi''(u_s^\varepsilon(x)) N_r(x)$ will not be adapted to \mathcal{F}_r because $s > r$. Therefore we interpret the stochastic integrals as Skorokhod integrals and for any s, x $\varphi''(u_s^\varepsilon(x))$ is interpreted as a random variable with square integrable Malliavin derivative (we refer the reader to [Nua95, Chapter 1] for further explications). Under this new point of view, by means of the shorthand notation

$$p_{s-r}^\varepsilon(x-y) := \partial_x P(\varepsilon+s-r, x-y), \quad P_{s-r}^\varepsilon(x-y) := P(\varepsilon+s-r, x-y)$$

we can use the product formula for the Malliavin divergence [Nua95, Proposition 1.3.3] and the chain formula of the Malliavin derivative [Nua95, Proposition 1.2.3] (the hypothesis are satisfied because u lives a.s. on a compact and φ is smooth enough) obtaining

$$\begin{aligned} \varphi''(u_s^\varepsilon(x)) \int_0^s N_r(x) dN_r(x) &= \int_0^s \int_{\mathbb{T}} \varphi''(u_s^\varepsilon(x)) N_{\mathbf{s}_2}(x) p_{s-s_2}^\varepsilon(x-\mathbf{y}_2) W(d\mathbf{s}_2, d\mathbf{y}_2) \\ &+ \int_0^s \varphi'''(u_s^\varepsilon(x)) N_{\mathbf{s}_2}(x) \int_{\mathbb{T}} P_{s-s_2}^\varepsilon(x-\mathbf{y}_2) p_{s-s_2}^\varepsilon(x-\mathbf{y}_2) d\mathbf{y}_2 ds_2. \end{aligned} \quad (4.12)$$

The absence of boundary on \mathbb{T} implies the identity

$$\int_{\mathbb{T}} P_{s-s_2}^\varepsilon(x-\mathbf{y}_2) p_{s-s_2}^\varepsilon(x-\mathbf{y}_2) d\mathbf{y}_2 = \int_{\mathbb{T}} \partial_x \frac{(P_{s-s_2}^\varepsilon(x-\mathbf{y}_2))^2}{2} d\mathbf{y}_2 = 0.$$

And the deterministic integral on the right hand side of (4.12) is zero. Iterating again this calculation with $\varphi''(u_s^\varepsilon(x)) N_{\mathbf{s}_2}(x)$, we can pass the product inside the stochastic integral and we have

$$A_2^\varepsilon = 2 \int_{\Delta_{2,t} \times \mathbb{T}^2} \int_{\mathbf{s}_1} \int_{\mathbb{T}} h(s) l(y) \varphi''(u_s^\varepsilon(y)) p_{s-s_1}^\varepsilon(y-\mathbf{y}_1) p_{s-s_2}^\varepsilon(y-\mathbf{y}_2) dy ds W(d\mathbf{s}, d\mathbf{y}).$$

The iterated Skorokhod integral is a closed operator and all the previous term converge in $L^2(\mathbb{P})$, then A_2^ε must converge too to some limit in $L^2(\mathbb{P})$ and the theorem is obtained once we know the convergence on $L^2(\Omega \times ([0, t] \times \mathbb{T})^2)$ of the integrand function

$$F^\varepsilon(\mathbf{s}, \mathbf{y}) = \mathbf{1}_{\Delta_{2,t}}(\mathbf{s}) \int_{\mathbb{T}} \int_{\mathbf{s}_1+\varepsilon}^{t+\varepsilon} h(s-\varepsilon)l(y)\varphi''(u_{s-\varepsilon}^\varepsilon(x))p_{s-\mathbf{s}_1}^0(x-\mathbf{y}_1)p_{s-\mathbf{s}_2}^0(x-\mathbf{y}_2)dy ds. \quad (4.13)$$

which belongs to the domain of the operator for every $\varepsilon > 0$. We estimate the norm of F^ε in $L^2([0, t] \times \mathbb{T})^2$:

$$\begin{aligned} \|F^\varepsilon\|^2 &= \int \left(\int_{\mathbb{T}} \int_{\mathbf{s}_1+\varepsilon}^{t+\varepsilon} h(s-\varepsilon)l(y)\varphi''(u_{s-\varepsilon}^\varepsilon(y))p_{s-\mathbf{s}_1}^0(y-\mathbf{y}_1)p_{s-\mathbf{s}_2}^0(y-\mathbf{y}_2)dy ds \right)^2 ds dy \\ &= 2 \int_\varepsilon^{t+\varepsilon} h(s-\varepsilon) ds \int_\varepsilon^{t+\varepsilon} h(r-\varepsilon) dr \int_{\mathbb{T}} l(y)dy \int_{\mathbb{T}} l(x)dx \varphi''(u_{s-\varepsilon}^\varepsilon(x))\varphi''(u_{r-\varepsilon}^\varepsilon(y)) \\ &\quad \times \int_0^{s \wedge r} \int_{\mathbb{T}} \int_0^{\mathbf{s}_2} \int_{\mathbb{T}} p_{s-\mathbf{s}_1}^0(x-\mathbf{y}_1)p_{r-\mathbf{s}_1}^0(x-\mathbf{y}_2)p_{s-\mathbf{s}_2}^0(y-\mathbf{y}_1)p_{r-\mathbf{s}_2}^0(y-\mathbf{y}_2)ds dy. \end{aligned}$$

Integrating by parts with respect to \mathbf{y}_1 and \mathbf{y}_2 and applying the semigroup property of P we obtain

$$\begin{aligned} &\int_0^{s \wedge r} \int_{\mathbb{T}} \int_0^{\mathbf{s}_2} \int_{\mathbb{T}} p_{s-\mathbf{s}_1}^0(x-\mathbf{y}_1)p_{r-\mathbf{s}_1}^0(x-\mathbf{y}_2)p_{s-\mathbf{s}_2}^0(y-\mathbf{y}_1)p_{r-\mathbf{s}_2}^0(y-\mathbf{y}_2)ds dy \\ &= \int_0^{s \wedge r} \int_{\mathbb{T}} \int_0^{\mathbf{s}_2} \int_{\mathbb{T}} \partial_t P_{s-\mathbf{s}_1}^0(x-\mathbf{y}_1)P_{s-\mathbf{s}_2}^0(y-\mathbf{y}_1)P_{r-\mathbf{s}_1}^0(x-\mathbf{y}_2)\partial_t P_{s-\mathbf{s}_2}^0(y-\mathbf{y}_2)ds dy \\ &= \int_0^{s \wedge r} \int_0^{\mathbf{s}_2} \partial_t P_{s+r-2\mathbf{s}_1}^0(x-y)\partial_t P_{s+r-2\mathbf{s}_2}^0(y-x)d\mathbf{s}_1 d\mathbf{s}_2 = \frac{1}{2}(P_{|s-r|}^0(y-x) - P_{s+r}^0(y-x))^2 \end{aligned} \quad (4.14)$$

Using the hypothesis on φ , h and l , there exists a constant $C > 0$ depending only on these tree objects such that for any $\varepsilon > 0$ and a.s.

$$\|F^\varepsilon\|^2 \leq C \int_{\mathbb{T}^2} \int_0^{4t} (P_s^0(y-x))^2 ds dy dx < +\infty.$$

Finally since u_ε converges to u in $L^2(\mathbb{P})$ there is a subsequence converging a.s. and under this subsequence F^ε converges a.s. and a.e. with respect the variables \mathbf{s} and \mathbf{y} to the second term on the right hand side of (4.6), we conclude by dominated convergence. To conclude the result when ψ is a generic smooth function supported on $(0, t) \times \mathbb{T}$ we apply the identity (4.6) with a sequence of test functions $h_N \otimes l_N: (0, t) \times \mathbb{T} \rightarrow \mathbb{R}$ converging to ψ as compactly supported smooth functions. Thanks to the equation (4.7) and the convergence of $h_N \otimes l_N$, the reconstruction term and the deterministic integral converge in $L^2(\mathbb{P})$ to their natural limit, then the Skorokhod integral must also converge in $L^2(\mathbb{P})$ and we can repeat almost the same calculations to show that the sequence

$$F^N(\mathbf{s}, \mathbf{y}) = \mathbf{1}_{\Delta_{2,t}}(\mathbf{s}) \int_{\mathbb{T}} \int_{\mathbf{s}_1}^t h_N(s)l_N(y)\varphi''(u_s(x))p_{s-\mathbf{s}_1}^0(x-\mathbf{y}_1)p_{s-\mathbf{s}_2}^0(x-\mathbf{y}_2)dy ds. \quad (4.15)$$

converges in $L^2(\Omega \times ([0, t] \times \mathbb{T})^2)$ to the integrand in the right hand side of (4.6). The same extension procedure applies when ψ is supported on $(0, +\infty) \times \mathbb{T}$. In this case we repeat the calculation with the sequence of tests function $\varphi_N \psi$ where φ_N is introduced in (1.11) and converges a.e. to the indicator function of $\mathbf{1}_{(0,t) \times \mathbb{T}}$. \square

Remark 4.5. The approximating procedure we used to prove this result is very different compared to the proof of [HP15, Theorem 6.2] that implies Proposition 4.3. In that case, the theorem is more general and it uses deeply the definition of $\hat{\mathcal{R}}\Phi'(U)\Xi$ as the limit in the $\mathcal{C}^{-3/2-\kappa}$ topology of a sequence of smooth space-time functions $\hat{\mathcal{R}}^n\Phi'(U)\Xi$ associated to a wavelet basis. This sequence can be written explicitly even in case of $\Phi''(U)(D_x U)^2$ but it seems very hard to recover the same result using this approximation. In particular, the splitting of the heat kernel G as a sum $K + R$ as explained in Lemma 2.6 make all calculations very indirect and it does not allow to use directly some nice property of P such as the semigroup property or its explicit definition. On the other hand the approximations we use to prove the result are very peculiar and they do not seem appropriate in other contexts. A general methodology to describe the law of some reconstructed modelled distributions with respect the BPHZ model is still missing.

The stochastic object we found to have also the advantage to pass the convolution into them.

Theorem 4.6 (Identification Theorem). *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^6 . Then one has*

$$P * (\mathbf{1}_+ \hat{\mathcal{R}}(\Phi'(U)\Xi))(t, x) = \int_0^t \int_{\mathbb{T}} P(t-s, x-y) \varphi'(u(s, y)) dW_{s, y} \quad (4.16)$$

$$P * (\mathbf{1}_+ \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2))(t, x) = -\frac{1}{2} \int_0^t \int_{\mathbb{T}} P(t-s, x-y) \varphi''(u(s, y)) C(s) dy ds$$

$$+ 2 \int_{\Delta_{2, t} \times \mathbb{T}^2} \int_{\mathbf{s}_1}^t \int_{\mathbb{T}} P(t-s, x-y) P_x(s-\mathbf{s}_1, y-\mathbf{y}_1) P_x(s-\mathbf{s}_2, y-\mathbf{y}_2) \varphi''(u_s(y)) dy ds W(ds, dy) \quad (4.17)$$

Proof. In principle the result comes from Proposition 4.3 and Proposition 4.4 using the test function $\psi: \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}$ given by $\psi(s, y) = P(t-s, x-y)$ but ψ is not smooth because ψ has a singularity at (t, x) and its support is not included inside $(0, \infty) \times \mathbb{T}$. In order to overcome these constraints, we consider the periodic extensions of $g = \hat{\mathcal{R}}(\Phi'(U)\Xi)$, $\hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)$ on the space variable. Then $(P * \mathbf{1}_+ g)(t, x)$ becomes the periodic function $(\tilde{G} * \mathbf{1}_+ g)(t, x)$ defined for $(t, x) \in (0, T) \times \mathbb{R}$ (we use a tilde to denote the periodic extension of a distribution). Using [Hai14, Lemma 5.5] and [Hai14, Lemma 5.24], we can find a couple of functions $\tilde{K}: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ and $\tilde{R}: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

- $G(z - \bar{z}) = \tilde{K}(z - \bar{z}) + \tilde{R}(z - \bar{z})$ for any $z, \bar{z} \in \mathbb{R}^2, z \neq \bar{z}$.
- $\tilde{K}(t, x) = 0$ for $t \leq 0, x \neq 0$ and $\tilde{K}(t, -x) = K(t, x)$.
- \tilde{R} is smooth, $\tilde{R}(t, x) = 0$ for $t \leq 0$.
- for any $z, \bar{z} \in \mathbb{R}^2, z \neq \bar{z}$, \tilde{K} can be decomposed as

$$\tilde{K}(z - \bar{z}) = \sum_{n \geq 0} K_n(z - \bar{z}).$$

where all the functions $K_n: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ are smooth and they satisfy

$$\text{supp}(K_n) = \{z = (t, x) \in \mathbb{R}^2: 2^{-n-1} \leq \|z\| \leq 2^{1-n}, t > 0\}.$$

Since the value (t, x) is fixed, there exists a value $n_0(t, x)$ such that $K_n(t, x) = 0$ for any $n \geq n_0$. Then we write

$$G * (\mathbf{1}_+g)(t, x) = \sum_{n \geq 0} K_n * (\tilde{\mathbf{1}}_+g)(t, x) + \tilde{R} * (\tilde{\mathbf{1}}_+g)(t, x), \quad (4.18)$$

In order to take in account also the product indicator function $\mathbf{1}_+$ in both terms we fix $\{\varphi_N\}_{N \geq 0}$ the sequence of bounded smooth functions converging pointwise to the indicator function of $(0, +\infty) \times \mathbb{T}$ as in the definition (1.11). Thanks to the decomposition (4.18) we obtain that the left hand side of (4.16) (4.17) is the a.s. limit of the sequence

$$\sum_{n=0}^N K_n * (\varphi_N g)(t, x) + \tilde{R} * (\varphi_N g)(t, x). \quad (4.19)$$

Written under this form, for any fixed N the law of (4.19) can be identified with the stochastic object of the Proposition 4.3 and Proposition 4.4 applied to the sequence of test functions

$$\eta_N(s, y) := \sum_{n=0}^N \tilde{\varphi}_N(s, y) K_n(t - s, x - y) + \tilde{\varphi}_N(s, y) K^{(2)}(t - s, x - y).$$

which converges for any $s, y \in [0, t] \times \mathbb{R}$ to the function $G(t - s, x - y)$. In case $g = \hat{\mathcal{R}}(\Phi'(U)\Xi)$ the sequence (4.19) equals to

$$\int_0^t \int_{\mathbb{R}} \eta_N(s, y) \varphi'(\tilde{u}(s, y)) d\tilde{W}_{s,y}.$$

Since both $\tilde{\varphi}_N(s, y)$ and φ' are bounded (\tilde{u} will stay in a compact set because it is continuous), there exists a constant an a.s. bounded positive random variable M such that for any $(s, y) \in [0, t] \times \mathbb{R}$

$$|\eta_N(s, y) \varphi'(\tilde{u}(s, y))| \leq MG(t - s, x - y).$$

The function $(s, y) \rightarrow G(t - s, x - y)$ is L^2 integrable on $[0, t] \times \mathbb{R}$, therefore using the Itô we can apply the dominated convergence theorem of stochastic integrals and (4.19) converges in probability to

$$\int_0^t \int_{\mathbb{R}} G(t - s, x - y) \varphi'(\tilde{u}(s, y)) d\tilde{W}_{s,y} = \int_0^t \int_{\mathbb{T}} P(t - s, x - y) \varphi'(u(s, y)) dW_{s,y}.$$

On the other hand when $g = \hat{\mathcal{R}}(\Phi''(U)(D_x U)^2)$ the sequence (4.19) equals to

$$\begin{aligned} &+ 2 \int_{\Delta_{2,t} \times \mathbb{R}^2} \int_{\mathbf{s}_1}^t \int_{\mathbb{R}} \eta_N(s, y) \varphi''(\tilde{u}(s, y)) G_x(s - \mathbf{s}_1, y - \mathbf{y}_1) G_x(s - \mathbf{s}_2, y - \mathbf{y}_2) dy ds \tilde{W}(ds, dy) \\ &- \frac{1}{2} \int_0^t \int_{\mathbb{R}} \eta_N(s, y) \varphi''(\tilde{u}(s, y)) C(s) dy ds =: A_N^1 + A_N^2. \end{aligned}$$

the sequence of integral A_N^2 satisfies a.s.

$$|A_N^2| \leq C \int_0^t \int_{\mathbb{R}} G(t-s, x-y) C(s) dy ds = C \int_0^t C(s) ds < \infty$$

for some constant fixed constant $C > 0$ and then A_N^2 converges a.s. by dominated convergence. Let us prove the convergence of A_N^1 . Using the shorthand notation

$$g_{s-r}(x-y) := \partial_x G(s-r, x-y), \quad G_{s-r}(x-y) := G(s-r, x-y),$$

and $[0, t] \times \mathbb{R} = O_t$ we study the convergence of the integrands defining A_N^1 :

$$F^N(\mathbf{s}, \mathbf{y}) = \mathbf{1}_{\Delta_{2,t}}(\mathbf{s}) \int_{\mathbf{s}_1}^t \int_{\mathbb{R}} \eta_N(s, y) \varphi''(\tilde{u}(s, y)) g_{s-s_1}(y - \mathbf{y}_1) g_{s-s_2}(y - \mathbf{y}_2) dy ds.$$

Firstly, we can repeat in this context an identical calculation of (4.14) with G_x instead of P_x to show the convergence in $L^2(\Omega \times O_t \times O_t)$ to the right integral. This yields:

$$\begin{aligned} \int_{([0,t] \times \mathbb{R})^2} (F^N(\mathbf{s}, \mathbf{y}))^2 ds d\mathbf{y} &= \int_{\Delta_{2,t} \times \mathbb{R}^2} (F^N(\mathbf{s}, \mathbf{y}))^2 ds d\mathbf{y} = \\ &\int \int \eta_N(s, y) \eta_N(r, z) \varphi''(\tilde{u}(s, z)) \varphi''(\tilde{u}(r, y)) (G_{|s-r|}(z-y) - G_{s+r}(z-y))^2 ds dz dr dy \end{aligned}$$

(we avoid to write down the integration set which is always equal to $O_t \times O_t$ to shorten the notation) by hypothesis the absolute value of this integral is bounded a.s. by

$$\begin{aligned} &\tilde{C} \int \int G_{t-s}(x-y) G_{t-r}(x-z) (G_{|s-r|}(z-y) - G_{s+r}(z-y))^2 \\ &\leq 2\tilde{C} \int \int G_{t-s}(x-y) G_{t-r}(x-z) (G_{|s-r|}(z-y)^2 + G_{s+r}(z-y)^2) \end{aligned} \quad (4.20)$$

for some deterministic constant $\tilde{C} > 0$ depending on φ'' . We show now this quantity is finite. By definition of G

$$G_{|s-r|}(z-y)^2 + G_{s+r}(z-y)^2 = \frac{G_{|s-r|/2}(z-y)}{\sqrt{4\pi|s-r|}} + \frac{G_{(s+r)/2}(z-y)}{\sqrt{4\pi(s+r)}}$$

Therefore we apply the semi-group property of G

$$\begin{aligned} &\int \int G_{t-s}(x-y) G_{t-r}(x-z) (G_{|s-r|}(z-y)^2 + G_{s+r}(z-y)^2) \\ &= \int \int G_{t-s}(x-y) G_{t-r}(x-z) \left(\frac{G_{|s-r|/2}(z-y)}{\sqrt{4\pi|s-r|}} + \frac{G_{(s+r)/2}(z-y)}{\sqrt{4\pi(s+r)}} \right) \\ &= \int \int G_{t-s}(x-y) \left(\frac{G_{t-s+|s-r|/2}(x-y)}{\sqrt{4\pi|s-r|}} + \frac{G_{t+(r-s)/2}(x-y)}{\sqrt{4\pi(s+r)}} \right) ds dr dy \end{aligned}$$

$$\leq \int_0^t \int_0^t \left(\frac{1}{\sqrt{|s-r|}} \frac{1}{\sqrt{t-r}} + \frac{1}{\sqrt{(s+r)}} \frac{1}{\sqrt{(t-r)}} \right) ds dr < +\infty.$$

and we conclude. Since all the remaining terms converge a.s., in order to obtain the convergence it is sufficient to show that second moment of the iterated integral is uniformly bounded. Looking at the iterated integral as a single Skorohod integral on $L^2(\Omega \times O_t \times O_t)$ we can calculate its variance by means of the isometry formula.

$$\begin{aligned} \mathbb{E} \left(\int_{O_t \times O_t} F^N(\mathbf{s}, \mathbf{y}) \tilde{W}(ds, d\mathbf{y}) \right)^2 &= \mathbb{E} \int_{\Delta_{2,t} \times \mathbb{R}^2} (F^N(\mathbf{s}, \mathbf{y}))^2 ds d\mathbf{y} \\ &+ \mathbb{E} \int_{O_t \times O_t} \int_{O_t \times O_t} D_\zeta F^N(u) D_u F^N(\zeta) d\zeta du. \end{aligned} \quad (4.21)$$

Where u and ζ are double space time variables. One term is bounded thanks to the convergence, on the other hand for any $\zeta \in O_t \times O_t$, $u \in O_t \times O_t$ $u = (s_1, y_1, s_2, y_2)$, $\zeta = (s'_1, y'_1, s'_2, y'_2)$ the function $D_\zeta F^N(u)$ is given by

$$\mathbf{1}_{\Delta_{2,t}}(s_1, s_2) \int_{s_1}^t \int_{\mathbb{R}} \eta_N(s, y) \varphi^{(iv)}(\tilde{u}) G_{s-s'_1}(y-y'_1) G_{s-s'_2}(y-y'_2) g_{s-s_1}(y-y_1) g_{s-s_2}(y-y_2) dy ds$$

and the second integral in (4.21) becomes

$$\begin{aligned} &\int_{O_t} \int_{O_t} \eta_N(s, y) \eta_N(r, z) \varphi^{(iv)}(\tilde{u}(s, y)) \varphi^{(iv)}(\tilde{u}(r, z)) \\ &\times \left(\int_0^{s \wedge r} ds_2 \int g_{s-s_2}(y-y_2) G_{r-s_2}(z-y_2) \int_0^{s_2} ds_1 \int G_{r-s_1}(z-y_1) g_{s-s_1}(y-y_1) \right)^2 \end{aligned} \quad (4.22)$$

(we shorten the space integrals with a simple symbol of integral). Let us look at the integral inside the square. Thanks to the semigroup property and the estimation

$$|g_t(y)| \leq \sup_{u \in \mathbb{R}} \left(\frac{u}{\sqrt{4\pi}} e^{-u^2} \right) \frac{1}{t}$$

one has

$$\begin{aligned} &\left| \int_0^{s \wedge r} ds_2 \int g_{s-s_2}(y-y_2) G_{r-s_2}(y_2-z) \int_0^{s_2} ds_1 \int g_{s-s_1}(y-y_1) G_{r-s_1}(y_1-z) \right| \\ &= \left| \int_0^{s \wedge r} g_{r+s-2s_2}(y-z) \left(\int_0^{s_2} g_{r+s-2s_1}(y-z) ds_1 \right) ds_2 \right| \lesssim \int_0^{s \wedge r} \frac{1}{(r+s-2s_2)} \\ &\times \int_0^{s_2} \frac{1}{(r+s-2s_1)} ds_1 = \int_0^{s \wedge r} \frac{1}{2(r+s-2s_2)} (\ln(r+s) - \ln(r+s-2s_2)) ds_2 = \\ &= -\frac{1}{4} \ln(r+s)^2 - \frac{1}{4} \ln(|r-s|) \ln(r+s) + \frac{1}{2} \ln(|r-s|)^2 \lesssim \ln(|r-s|)^2 + \ln(r+s)^2 =: B(s, r). \end{aligned}$$

(The symbol \lesssim means less or equal up to a universal contest). Therefore for every N the continuity of the derivatives can allow to bound a.s. the integral (4.22) with the quantity

$$\int_{O_t} \int_{O_t} G_{t-s}(x-y) G_{t-r}(x-z) B(s, r)^2 dz dy ds dr = \int_0^t \int_0^t B(s, r)^2 ds dr < \infty$$

and the convergence applies. \square

Remark 4.7. Looking at the proof of Theorem 4.6 we realise that in order to bound the second moment of the iterated integrals the only hypothesis we should carry is the boundedness of the derivatives of φ up to order 4. Thus if φ_δ is a sequence of C^∞ functions converging to φ in the topology of C^4 , we can apply the Theorem 1.1 with φ_δ and using the same strategy of the previous proof with their natural approximations, we can pass to the the limit into all the objects.

A Behaviour of the constants

We put in the appendix the explicit estimation on renormalisations constants. This identity lies on a remarkable identity on G , the heat kernel on \mathbb{R} .

Lemma A.1. *For any $z \in \mathbb{R}^2 \setminus \{0\}$ one has*

$$2 \int G_x(z - \bar{z})G_x(-\bar{z})d\bar{z} = G(z) + G(-z) \quad (\text{A.1})$$

Proof. We verify this identity by calculating the space time fourier transform of both sides. In order to do that we recall the formula

$$\widehat{G}(\xi) = \frac{1}{2\pi i \xi_1 + 4\pi^2 \xi_2^2}.$$

Rewriting the LHS as $2G_x * \overline{G_x}(z)$, it yields

$$2\widehat{G_x * \overline{G_x}}(\xi) = 2\widehat{G_x}(\xi)\widehat{\overline{G_x}}(\xi) = (2\pi i \xi_2 \widehat{G}(\xi))(-2\pi i \xi_2 \widehat{G}(-\xi)) = \frac{8\pi^2 \xi_2^2}{4\pi^2 \xi_1^2 + (4\pi^2 \xi_2^2)^2}.$$

On the other hand

$$\widehat{G}(\xi) + \widehat{G}(-\xi) = \frac{8\pi^2 \xi_2^2}{(2\pi i \xi_1 + 4\pi^2 \xi_2^2)(-2\pi i \xi_1 + 4\pi^2 \xi_2^2)} = \frac{8\pi^2 \xi_2^2}{4\pi^2 \xi_1^2 + (4\pi^2 \xi_2^2)^2}.$$

□

Identity (A.1) allows to prove

Lemma A.2. *Let $C_\varepsilon^1, C_\varepsilon^2$ be the sequences introduced in (2.21), (2.22). The the following estimations hold as $\varepsilon \rightarrow 0^+$*

$$C_\varepsilon^1 = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} G(s, y) \rho^{*2}(s, y) ds dy + o(1); \quad (\text{A.2})$$

$$C_\varepsilon^2 = \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (G_x * \rho)^2(s, t) ds dy + o(1); \quad (\text{A.3})$$

$$C_\varepsilon^1 = C_\varepsilon^2 + o(1). \quad (\text{A.4})$$

Proof. In what follows all integrals will be taken on the whole space \mathbb{R}^2 and all integration variable lives in space-time. For any function $K: \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ we will adopt also the notation

$$S_\varepsilon^m K(t, x) := \varepsilon^m K(\varepsilon^2 t, \varepsilon x)$$

for any integer m and any $\varepsilon > 0$. By definition

$$\begin{aligned} C_\varepsilon^1 &= \int \int K(w) \rho_\varepsilon(z) \rho_\varepsilon(z-w) dw dz = \\ &= \int K(w) \int \rho_\varepsilon(z) \rho_\varepsilon(w-z) dz dw = \int K(w) (\rho_\varepsilon)^{*2}(w) dw. \end{aligned}$$

Using a simple change of variable formula, one has that $(\rho_\varepsilon)^{*2}(w) = (\rho^{*2})_\varepsilon(w)$ and the last integral is equal to

$$\begin{aligned} \int K(s, y) \varepsilon^{-3} \rho^{*2}\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon}\right) ds dy &= \int K(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) ds dy = \\ &= \frac{1}{\varepsilon} \int \varepsilon K(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) ds dy. \end{aligned}$$

Since K is locally equal to G and $\varepsilon G(\varepsilon^2 s, \varepsilon y) = G$ one has

$$\varepsilon K(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) \rightarrow G(s, y) \rho^{*2}(s, y) \quad \text{a.e.}$$

Moreover since $G \rho^{*2}$ is integrable, dominated convergence theorem implies that

$$\int \varepsilon K(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) ds dy \rightarrow \int G(s, y) \rho^{*2}(s, y) ds dy.$$

Using decomposition $G = K + R$ with under the convolution with ρ^{*2} one has

$$\frac{1}{\varepsilon} \left[\int (G(s, y) - \varepsilon K(\varepsilon^2 s, \varepsilon y)) \rho^{*2}(s, y) ds dy \right] = \int R(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) ds,$$

and properties of R imply straightforwardly

$$\int R(\varepsilon^2 s, \varepsilon y) \rho^{*2}(s, y) ds \rightarrow R(0, 0) \int \rho^{*2}(s, y) ds = 0.$$

We pass then to the other constant

$$C_\varepsilon^2 = \int_{\mathbb{R}^2} (K_x * \rho_\varepsilon)^2(z) dz = \frac{1}{\varepsilon} \int \varepsilon (K_x * \rho_\varepsilon)^2(z) dz$$

for any $z = (t, x)$

$$(K_x * \rho_\varepsilon)(\varepsilon^2 t, \varepsilon x) = \int K_x(\varepsilon^2 t - \varepsilon^2 s, \varepsilon x - \varepsilon y) \rho(s, y) ds dy = (S_\varepsilon(K_x) * \rho)(z)$$

then

$$(K_x * \rho_\varepsilon)(t, x) = (S_\varepsilon(K_x) * \rho)\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right).$$

and

$$\int \varepsilon (K_x * \rho_\varepsilon)^2(z) dz = \varepsilon \int (S_\varepsilon(K_x) * \rho)^2\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) dt dx = \int (S_\varepsilon^2(K_x) * \rho)^2(t, x) dt dx$$

for almost any (t, x) (s, y) with $s \leq t$

$$S_\varepsilon^2(K_x)(t-s, x-y)\rho(s, y) \rightarrow S_\varepsilon^2(G_x)(t-s, x-y)\rho(s, y) = G_x(t-s, x-y)\rho(s, y) \in L^1$$

then for a.e. (t, x)

$$(S_\varepsilon^2(K_x) * \rho)(t, x) \rightarrow (G_x * \rho)(t, x) \in L^2$$

and we obtain that

$$\int \varepsilon(K_x * \rho_\varepsilon)^2(z)dz \rightarrow \int (G_x * \rho)^2(z)dz.$$

Again the decomposition $G = K + R$ implies that

$$\begin{aligned} & \frac{1}{\varepsilon} \int (G_x * \rho)^2(z)dz - \frac{1}{\varepsilon} \int \varepsilon(K_x * \rho_\varepsilon)^2(z)dz = \\ & \frac{1}{\varepsilon} \int (S_\varepsilon^2(G_x) * \rho)^2(z)dz - \frac{1}{\varepsilon} \int (S_\varepsilon^2(K_x) * \rho)^2(z)dz = \\ & \frac{1}{\varepsilon} \int (S_\varepsilon^2(R_x) * \rho)^2(z)dz = \varepsilon^3 \int (S_\varepsilon(R_x) * \rho)^2(z)dz = \int ((R_x) * \rho_\varepsilon)^2(z)dz. \end{aligned}$$

Since R_x is C^∞ and ρ_ε is an approximation of the identity, the above quantity converges in L^2 to $R_x(0, 0) = 0$ and the second estimation holds. To finally prove the third one it is sufficient to show

$$\int G(s, y)\rho^{*2}(s, y)dsdy = \int (G_x * \rho)^2(s, y)dsdy.$$

Starting from identity (A.1) we convolve both sides with the function ρ^{*2} and for any space time variable u that left hand side is equal to

$$2(G_x * \overline{G_x}) * \rho^{*2}(u) = \int \int \int 2G_x(u-v-w)G_x(-w)\rho(v-x)\rho(x)dxvdwdw$$

where x, v, w are space time variables. We impose the following change of space-time variables

$$\begin{cases} v' = v - x \\ x' = x \\ w' = w + x \end{cases} \quad \begin{cases} v = v' + x' \\ x = x' \\ w = w' - x' \end{cases} \quad dvdxdw = dv'dw'dw'$$

then the integral becomes

$$\int \int \int 2G_x(u-v'-w')G_x(-w'+x')\rho(v')\rho(x')dx'dv'dw'$$

ρ is even then this integral equals to

$$\begin{aligned} & 2 \int \int \int G_x(u-v'-w')G_x(-w'-x')\rho(v')\rho(x')dx'dv'dw' = \\ & 2 \int (G_x * \rho)(u-w')(G_x * \rho)(-w')dw'. \end{aligned}$$

At the same time the right hand side is equal to

$$\begin{aligned} & \int G(u-w)\rho^{*2}(w)dw + \int G(w-u)\rho^{*2}(w)dw = \\ & \int G(u-w)\rho^{*2}(w)dw + \int G(-u-w)\rho^{*2}(w)dw = (G * \rho^{*2})(u) + \overline{(G * \rho^{*2})}(u). \end{aligned}$$

Evaluating both sides in $u = 0$ we obtain the desired equality. \square

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