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Abstract: in this paper we study a one-dimensional model of bent ferromagnetic nanowire. We establish the existence of static solutions describing either one domain or two domains separated by a wall. We address the stability of these solutions. In particular we exhibit asymptotically-stable wall profiles which are pined at the bent zone even in presence of a small applied magnetic field.

Résumé : dans cet article, on étudie un modèle monodimensionnel de fil ferromagnétique présentant un coude. On explicite toutes les solutions stationnaires décrivant soit un domaine soit deux domaines séparés par un mur. On étudie ensuite la stabilité de ces solutions. On montre en particulier que certains profils de murs sont asymptotiquement stables, l'interprétation physique de ce résultat étant que les murs restent bloqués au niveau du coude, et ce même en présence d'un champ magnétique appliqué.

Keywords: ferromagnetism, Landau-Lifschitz equation, stability, domain walls.

MSC: 35K55, 35Q60.

1 Introduction

Ferromagnetic nanowires are used in a wide range of applications such as microelectronics, paints for radar stealth, transformers and computers. In particular, ferromagnetic nanowires can be used to record and store data in racetrack memories (see [13]). In such devices, the magnetization tends to be aligned in the wire direction, in one sense or in the other sense. As a consequence, one observes in nanowires the formation of domains, large zone in which the magnetic moment is in the wire axis and domain walls, thin zones in which the magnetic moment presents large variations.
In the framework of data storage, the stability of the walls position is crucial. Indeed a non-desired movement of a wall may induce a degradation of the data.
Many papers address the stability of walls in ferromagnetic nanowires (cf. [4, 5, 6, 7, 9, 10, 14]). In [4], the stability of wall profiles is proved in the case of an infinite straight nanowire (*i.e.* without curvature). We remark that we do not have asymptotic stability because of the possible translations and rotations of the wall. In addition, even a small applied magnetic field can produce a displacement of the wall. The situation is even worse in the finite-wire case since the walls profiles are unstable (see [5]).
In this paper, we prove that a bend in a wire attracts the walls, so that profiles for walls located at the bend are asymptotically stable. This property is well known in the Physics literature (see [12, 15] for example) but to our knowledge, this is the first mathematical work concerning the curvature effects on the walls stability in nanowires.

Let us recall the 3d model for ferromagnetic materials (see [3, 11]). We consider a ferromagnetic body occupying the volume $\Omega \subset \mathbb{R}^3$. We denote by $M(t, \mathbf{x})$ the magnetization distribution at the time t and the point $\mathbf{x} \in \Omega$. The material is supposed to be saturated so that the norm of $M(t, \mathbf{x})$, denoted by M_s, does not depend on t and x. The variations of M satisfy the Landau-Lifschitz equation:

$$\partial_t M = -\gamma M \times H_{\mathsf{eff}} - \frac{\alpha\gamma}{M_s} M \times (M \times H_{\mathsf{eff}})$$

where $\times$ is the cross product in $\mathbb{R}^3$, γ is the gyromagnetic ratio, α is the damping coefficient and where the effective field H_{eff} is given by:

$$H_{\text{eff}} = \frac{A}{M_s^2}\Delta M + \mu_0 H_d(M) + \mu_0 H_a. \tag{1.1}$$

In (1.1), A is the exchange constant, μ_0 is the permeability of the vacuum, H_a is the applied magnetic field and $H_d(M)$ is the demagnetizing field.
This last field is deduced from M by the law of Faraday: $\operatorname{div} B = 0$ (B is the magnetic induction), the constitutive relation: $B = H + \bar{M}$ ($\bar{M}$ is the extension of M by zero outside Ω), and by the static Maxwell equation: $\operatorname{curl} H = 0$. So, $H_d(M)$ is obtained from M by solving the following system:

$$\operatorname{curl} H_d(M) = 0 \quad \text{and} \quad \operatorname{div}(H_d(M) + \bar{M}) = 0. \tag{1.2}$$

Rewriting M as

$$M(t,x) = M_s \mathbf{m}(\gamma\mu_0 t, \sqrt{\frac{A}{M_s\mu_0}}\mathbf{x}),$$

we obtain the following rescaled model:

$$\partial_t \mathbf{m} = -\mathbf{m} \times \mathcal{H} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathcal{H}) \tag{1.3}$$

with

$$\mathcal{H} = \mathbf{\Delta m} + H_d(\mathbf{m}) + H_a.$$

In this paper we are interested in an infinitely long wire with one bend. We consider the following one-dimensional model justified by asymptotic process in [2, 4, 5, 16].
The wire is parametrized by:

$$x \longmapsto \begin{cases} x\vec{u} \ \text{ if } x \leq 0, \\ x\vec{e_1} \ \text{ if } x \geq 0, \end{cases} \tag{1.4}$$

where $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is the canonical basis of $\mathbb{R}^3$ and $\vec{u} = \begin{pmatrix} \cos\beta \\ -\sin\beta \\ 0 \end{pmatrix}$ is a unitary vector in the plane $(\vec{e_1}, \vec{e_2})$. The angle $\beta = (\vec{u}, \vec{e_1})$ is supposeed to be in $]0, \pi[$.

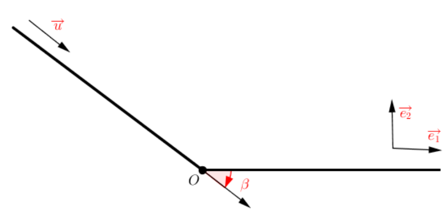

Figure 1: bent nanowire

Using the parametrization (1.4), the magnetic moment m is defined on $\mathbb{R}_+^* \times \mathbb{R}$ with values in $\mathbb{R}^3$ and satisfies the saturation constraint $|m(t,x)| = 1$. As it is proved in [4], [5], [8], the equivalent 1d demagnetizing field reduces to the following local operator:

$$h_d(m)(x) = \frac{1}{2}\left(-m(x) + (m(x)|\vec{\tau}(x))\vec{\tau}(x)\right),$$

where $(.|.)$ is the usual scalar product in $\mathbb{R}^3$, and $\vec{\tau}(x)$ is the direction of the wire at the point x (with $|\vec{\tau}| = 1$). In our case, $\vec{\tau}$ is given by

$$\vec{\tau}(x) = \begin{cases} \vec{u} \text{ for } x < 0, \\ \vec{e_1} \ \text{ for } x > 0. \end{cases} \tag{1.5}$$

We remark that since $m \times h_d(m) = \dfrac{1}{2} m \times ((m|\vec{\tau})\vec{\tau})$, we can replace $h_d(m)$ by $\dfrac{1}{2}((m|\vec{\tau})\vec{\tau})$ in the Landau-Lischitz equation. In addition, by rescaling in space and time, we get rid of the coefficient $\dfrac{1}{2}$ in front of the demagnetizing field so that we obtain the following model for our bent wire:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_e(m) - \alpha m \times (m \times H_e(m)) \quad \text{for } t \geq 0 \text{ and } x \in \mathbb{R}, \\ H_e(m) = \partial_{xx} m + (m|\vec{\tau}(x))\vec{\tau}(x) + H_a(x), \end{cases} \tag{1.6}$$

where $\vec{\tau}$ is defined by (1.5) and $H_a : \mathbb{R} \longrightarrow \mathbb{R}^3$ is the applied field.

Remark 1.1. *In our model, there is no jump for m and $\partial_x m$ at the bendt:*

$$[|m|] := m(t, 0^+) - m(t, 0^-) = 0 \text{ and } [|\partial_x m|] = \partial_x m(t, 0^+) - \partial_x m(t, 0^-) = 0. \tag{1.7}$$

For vanishing applied field, we deal with stationary solutions separating a $\pm\vec{u}$-domain in $\mathbb{R}^-\vec{u}$ and a $\pm\vec{e_1}$-domain in $\mathbb{R}^+\vec{e_1}$. So that we look for solutions satisfying:

$$m(x) \xrightarrow[x \to -\infty]{} \pm \vec{u} \quad \text{and} \quad m(x) \xrightarrow[x \to +\infty]{} \pm \vec{e_1}. \tag{1.8}$$

We first exhibit all the solutions for (1.6)-(1.8). We denote $\vec{v} = \begin{pmatrix} \sin\beta \\ \cos\beta \\ 0 \end{pmatrix}$.

Theorem 1.2. *For $\beta \in]0, \pi[$, there are eight stationary solutions for* (1.6) *with $H_a = 0$ satisfying the limit conditions* (1.8).
The solutions satisfying the limit condition $m(x) \longrightarrow -\vec{u}$ when $x \longrightarrow -\infty$ are given by:

$$\mathbf{m}_1(x) = \begin{cases} \tanh(x-c)\vec{u} + \dfrac{1}{\cosh(x-c)}\vec{v} & \text{if} \quad x \leq 0, \\ \tanh(x+c)\vec{e_1} + \dfrac{1}{\cosh(x+c)}\vec{e_2} & \text{if} \quad x \geq 0, \end{cases} \qquad \textit{with } c = \operatorname{artanh}(\sin\frac{\beta}{2}), \tag{1.9}$$

$$\mathbf{m}_2(x) = \begin{cases} \tanh(x+c)\vec{u} + \dfrac{1}{\cosh(x+c)}\vec{v} & \text{if} \quad x \leq 0, \\ -\tanh(x-c)\vec{e_1} - \dfrac{1}{\cosh(x-c)}\vec{e_2} & \text{if} \quad x \geq 0, \end{cases} \qquad \textit{with } c = \operatorname{artanh}(\cos\frac{\beta}{2}), \tag{1.10}$$

$$\mathbf{m}_3(x) = \begin{cases} \tanh(x+c)\vec{u} - \dfrac{1}{\cosh(x+c)}\vec{v} & \text{if} \quad x \leq 0, \\ \tanh(x-c)\vec{e_1} - \dfrac{1}{\cosh(x-c)}\vec{e_2} & \text{if} \quad x \geq 0, \end{cases} \qquad \textit{with } c = \operatorname{artanh}(\sin\frac{\beta}{2}), \tag{1.11}$$

$$\mathbf{m}_4(x) = \begin{cases} \tanh(x-c)\vec{u} - \dfrac{1}{\cosh(x-c)}\vec{v} & \text{if} \quad x \leq 0, \\ -\tanh(x+c)\vec{e_1} + \dfrac{1}{\cosh(x+c)}\vec{e_2} & \text{if} \quad x \geq 0. \end{cases} \qquad \textit{with } c = \operatorname{artanh}(\cos\frac{\beta}{2}). \tag{1.12}$$

The solutions satisfying the limit condition $m(x) \longrightarrow \vec{u}$ when $x \longrightarrow -\infty$ are given by $-\mathbf{m}_1$, $-\mathbf{m}_2$, $-\mathbf{m}_3$ and $-\mathbf{m}_4$.

It is worth noting that the solutions $\mathbf{m}_1$ and $\mathbf{m}_3$ correspond to a wall profile in the case of a straight wire (case $\beta = 0$). The solution $\mathbf{m}_4$ corresponds to a $+\vec{e_1}$-domain in a straight wire. We remark also that the solution $\mathbf{m}_2$ is specific to the bent-wire case and has no equivalent in the case of a straight wire. Theorem 1.2 is proved in Section 2.

We address now the stability of these solutions. We recall that in the case of straight wire, a $+\vec{e_1}$-domain and $-\vec{e_1}$-domain are asymptotically stable while a wall profile is stable but not asymptotically stable because of the invariance of the system by translation in the x-variable and rotation around the wire axis. We obtain the following result concerning the asymptotic stability of $\mathbf{m}_1$ and $\mathbf{m}_4$.

Theorem 1.3. *Let $\beta \neq 0 \, mod \, \pi$. Then $\mathbf{m}_1$ given by Theorem 1.2 is asymptotically stable for Equation* (1.6)*, that is: for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all initial data m_0 such that*

$$m_0 - \mathbf{m}_1 \in H^1(\mathbb{R}) \text{ and } |m_0(x)| = 1 \text{ for all } x \in \mathbb{R},$$

if $\|m_0 - \mathbf{m}_1\|_{H^1(\mathbb{R})} \leq \eta$, then the solution of (1.6) *with initial data $m(0,x) = m_0(x)$ satisfies:*

- $\forall \, t > 0, \quad \|m(t) - \mathbf{m}_1\|_{H^1(\mathbb{R})} \leq \varepsilon,$
- $\|m(t) - \mathbf{m}_1\|_{H^1(\mathbb{R})}$ *tends to zero when t tends to $+\infty$.*

The same result holds for $-\mathbf{m}_1$, $\mathbf{m}_4$ and $-\mathbf{m}_4$ given by Theorem 1.2.

The other solutions are unstable:

Theorem 1.4. *For $\beta \in]0, \pi[$, the solutions $\mathbf{m}_2$, $-\mathbf{m}_2$, $\mathbf{m}_3$ and $-\mathbf{m}_3$ given by Theorem* 1.2 *are linearly unstable for Equation* (1.6) *with limit conditions* (1.8).

Contrary to the straight-wire case, the wall is pined at the bend, so that the profile $\mathbf{m}_1$ is asymptotically stable. In addition, we loose the invariance by rotation around the wire axis so that only one chirality of the wall profile is relevant. This is the reason why $\mathbf{m}_3$ is unstable.
In order to consider only perturbations satisfying the saturation constraint, we use the mobile frame method developed in [4], [5], [6]. Part 3 is devoted to the obtention of the equivalent system in this mobile frame. This step is followed by a careful study of the linearized equation which ensures the asymptotic stability for $\mathbf{m}_1$ and $\mathbf{m}_4$ (see Part 4) and the linear instability for $\mathbf{m}_2$ and $\mathbf{m}_3$ (see Part 5).

After that we study the behavior of asymptotically-stable configurations when the wire is submitted to an applied magnetic field. In the case of a straight nanowire, a non-vanishing applied field in the direction of the wire induces a displacement of the wall (see [4] and [7]). In our bent-wire case we only assume that the applied field is along the wire far from the origin: let $\xi \in \mathbf{C}^0(\mathbb{R}; \mathbb{R}^3)$ such that

$$\exists \, A > 0, \forall \, x \in \mathbb{R}, \quad x > A \Rightarrow \xi(x) = \vec{e_1} \text{ and } \xi(-x) = \vec{u}. \tag{1.13}$$

We assume that the applied field H_a is given by

$$H_a(x) = \lambda \xi(x), \quad \lambda \in \mathbb{R}. \tag{1.14}$$

We establish that a small non-vanishing applied field defined by (1.14) does not induce wall motion: the wall remains pined at the bend.

Theorem 1.5. *Let $\beta \in]0, \pi[$, let $\mathbf{m}_1$ given by Theorem 1.2. There exist $h_{max} > 0$ and a one parameter family $\lambda \mapsto \mathbf{m}(\lambda)$ satisfying:*

- $\mathbf{m}(0) = \mathbf{m}_1$,
- $\mathbf{m}(\lambda)$ *is defined for $|\lambda| \leq h_{max}$ and is a stationary solution for* (1.6)
- $\lambda \mapsto \mathbf{m}(\lambda) - \mathbf{m}_1$ *is in $\mathcal{C}^1([-h_{max}, h_{max}]; H^2(\mathbb{R}))$.*

In addition, for all $\lambda \in [-h_{max}, h_{max}]$, $\mathbf{m}(\lambda)$ is asymptotically stable for (1.6).

The same result holds for Solutions $-\mathbf{m}_1$, $\mathbf{m}_4$ and $-\mathbf{m}_4$ given by Theorem 1.2.

Part 6 of the present paper is devoted to the proof of this Theorem using the implicit function theorem.

2 Stationary solution

We consider M_0 a stationary solution for (1.6) with $H_a = 0$ satisfying the limit condition (1.8). Writing M_0 as:

$$M_0(x) = \begin{cases} M_0^-(x) = \sin\theta^-(x)\vec{u} + \cos\theta^-(x)\cos\varphi^-(x)\vec{v} + \cos\theta^-(x)\sin\varphi^-(x)\vec{e_3} \text{ for } x \text{ in } \mathbb{R}^-, \\ \\ M_0^+(x) = \sin\theta^+(x)\vec{e_1} + \cos\theta^+(x)\cos\varphi^+(x)\vec{e_2} + \cos\theta^+(x)\sin\varphi^+(x)\vec{e_3} \text{ for } x \text{ in } \mathbb{R}^+, \end{cases} \tag{2.1}$$

where

$$\vec{u} = \begin{pmatrix} \cos\beta \\ -\sin\beta \\ 0 \end{pmatrix} \quad \text{and } \vec{v} = \begin{pmatrix} \sin\beta \\ \cos\beta \\ 0 \end{pmatrix},$$

we obtain that M_0 is a stationary solution for (1.6)-(1.8) if and only if the following four assertions are satisfied:

(i) $M_0^- \times \left(\partial_{xx}M_0^- + (M_0^-|\vec{u})\vec{u}\right)$,

(ii) $M_0^+ \times \left(\partial_{xx}M_0^+ + (M_0^+|\vec{e_1})\vec{e_1}\right)$,

(iii) $M_0^-(0) = M_0^+(0)$ and $\dfrac{dM_0^-(0)}{dx} = \dfrac{dM_0^+(0)}{dx}$ (jump conditions (1.7)),

(iv) $M_0^- \xrightarrow[x\to-\infty]{} \pm\vec{u}, \quad M_0^+ \xrightarrow[x\to+\infty]{} \pm\vec{e_1}$ (limit condition (1.8)).

Plugging (2.1) in the first equation (i), we obtain that:

$$\begin{cases} \dfrac{d^2\theta^-}{dx^2} + \left|\dfrac{d\varphi^-}{dx}\right|^2 \sin\theta^-\cos\theta^- + \sin\theta^-\cos\theta^- = 0 \quad \text{for} \quad x \in \mathbb{R}^-, \\ \\ -\dfrac{d^2\varphi^-}{dx^2}\cos\theta^- + 2\dfrac{d\theta^-}{dx}\dfrac{d\varphi^-}{dx}\sin\theta^- = 0 \qquad\qquad \text{for} \quad x \in \mathbb{R}^-. \end{cases} \tag{2.2}$$

The second equation yields $\dfrac{d}{dx}(\dfrac{d\varphi^-}{dx}\cos^2\theta^-) = 0$, so that

$$\frac{d\varphi^-}{dx}\cos^2\theta^- = cst. \tag{2.3}$$

From (iv), we obtain that $\theta^-(x) \xrightarrow[x\to-\infty]{} \dfrac{\pi}{2} \bmod \pi$. This implies that the constant in (2.3) is zero, so that:

$$\frac{d\varphi^-}{dx}\cos^2\theta^- = 0. \tag{2.4}$$

Therefore

$$\forall\, x \in \mathbb{R}^-,\ \frac{d\varphi^-}{dx} = 0 \text{ or } \theta^-(x) = \frac{\pi}{2} \bmod \pi. \tag{2.5}$$

Let us prove that we have either $\left(\forall x \in \mathbb{R}^-,\ \theta^-(x) = \dfrac{\pi}{2} \bmod \pi\right)$ or $\left(\forall x \in \mathbb{R}^-,\ \dfrac{d\varphi^-}{dx} = 0\right)$.

Assume that we are not in the second case, that is that there exists x_0 such that $\dfrac{d\varphi^-}{dx}(x_0) \neq 0$. By continuity argument, this is also satisfied in a neighborhood of x_0. Therefore, by (2.5), $\theta^-(x) = \dfrac{\pi}{2} \bmod \pi$ in this neighborhood of zero, and by continuity argument, there exists $k \in \mathbb{Z}$ such that

$\theta^-(x) = \dfrac{\pi}{2} + k\pi$ in this neighborhood of zero (k is the same for all x in this neighborhood). So $\dfrac{d\theta^-}{dx}(x_0) = 0$. Therefore, θ^- is a solution for the Cauchy problem:

$$\begin{cases} \dfrac{d^2\theta^-}{dx^2} + \left|\dfrac{d\varphi^-}{dx}\right|^2 \sin\theta^- \cos\theta^- + \sin\theta^- \cos\theta^- = 0 \quad \text{for} \quad x \in \mathbb{R}^-, \\ \\ \theta^-(x_0) = \dfrac{\pi}{2} + k_0\pi, \quad \dfrac{d\theta^-}{dx}(x_0) = 0. \end{cases}$$

By uniqueness argument, we obtain that

$$\forall x \in \mathbb{R}^-, \; \theta^-(x) = \frac{\pi}{2} + k_0\pi.$$

In the second case, we remark that φ^- is constant, and that θ^- is a solution of the pendulum equation

$$\frac{d^2\theta^-}{dx^2} + \sin\theta^- \cos\theta^- = 0 \quad \text{for} \quad x \in \mathbb{R}^-. \tag{2.6}$$

Since θ^- satisfies the limit condition (iv), either θ^- is constant equal to $\frac{\pi}{2}$ mod π or θ^- is a solution represented by a separatrix on the phase portrait.

Remark 2.1. *By solving* (2.6)*, we obtain that the solutions θ represented by a separatrix are on the form $x \mapsto k\pi + \epsilon \arcsin(\tanh(x + c))$ where c is an arbitrary constant, $\epsilon = \pm 1$ and $k \in \mathbb{Z}$. Thus for these solutions, we have* $\sin(\theta(x)) = a \tanh(x + c)$ *and* $\cos(\theta(x)) = b\frac{1}{\cosh(x+c)}$ *where $a = \pm 1$ and $b = \pm 1$.*

From (ii) and (iv), the same analysis on $\mathbb{R}^+$ yields that we have: either $\theta^+ = \dfrac{\pi}{2}$ mod π or θ^+ is a solution on the separatrix of the phase portrait and in the last case, φ^+ is constant on $\mathbb{R}^+$.

We will now discriminate the different cases by using the transmission condition (iii).

Case 1: if $\theta^-(x) \equiv \dfrac{\pi}{2}$ mod π for $x < 0$ and $\theta^+(x) \equiv \frac{\pi}{2}$ mod π for $x > 0$, then $M_0^-(x) = \pm\vec{u}$ and $M_0^+ = \pm\vec{e_1}$. So by jump conditions (iii) we have $\vec{u} = \pm\vec{e_1}$, which is impossible since $\beta \neq 0$ mod π.

Case 2: if $\theta^-(x) \equiv \dfrac{\pi}{2}$ mod π for $x < 0$ and if on $\mathbb{R}^+$, θ^+ is a solution of (2.6) represented by a separatrix and φ^+ is constant, then $M_0^- \equiv \pm\vec{u}$ so by (iii), $\dfrac{dM_0^+}{dx}(0) = 0$. This last equation implies that $\dfrac{d\theta^+}{dx}(0) = 0$, which is impossible on the separatrix. In the same way, the case $\theta^+ \equiv \dfrac{\pi}{2}$mod π is also impossible.

The analysis of the first two cases yields that θ^- and θ^+ are trajectories on the separatrix of the phase portrait and φ^- and φ^+ are constant.

Case 3: let us assume that $\varphi^- \neq 0$ mod π or $\varphi^+ \neq 0$ mod π. Then, for $x > 0$, M_0^+ is in the plane P^+ given by

$$P^+ = \text{vect}(\vec{e_1}, \cos\varphi^+ \vec{e_2} + \sin\varphi^+ \vec{e_3}),$$

thus, $\dfrac{dM_0^+}{dx} \in P^+$ for $x > 0$.

In the same way, for $x < 0$, M_0 and $\dfrac{dM_0^-}{dx}$ belong to P^- given by

$$P^- = vect(\vec{u}, \cos\varphi^- \vec{v} + \sin\varphi^- \vec{w}).$$

By the jump conditions (iii) we have

$$M_0^+(0^+) = M_0^-(0^-) \in P^+ \cap P^- \quad \text{and} \quad \frac{dM_0^-}{dx}(0^-) = \frac{dM_0^+}{dx}(0^+) \in P^+ \cap P^-.$$

Since one of the angles φ^- or φ^+ is different from $0 \bmod \pi$, $P^+ \cap P^-$ is a straight line, so that $M_0^+(0)$ and $\dfrac{dM_0^+}{dx}(0)$ are colinear. In addition, from the saturation constraint $|M_0^+| = 1$, we obtain that $M_0^+(0) \perp \dfrac{dM_0^+}{dx}(0)$, so that $\dfrac{dM_0^+}{dx}(0) = 0$. This implies that $\dfrac{d\theta^+}{dx}(0) = 0$, which is impossible since θ^+ parametrizes a solution on the separatrix.

Therefore the only possible case is the following:

Case 4: $\varphi^\pm = 0 \bmod \pi$ (so that M_0 takes its values in the plane $\mathbb{R}\vec{e_1} + \mathbb{R}\vec{e_2}$) and θ^- and θ^+ are solutions of the pendulum equation on the separatrix. Therefore:

$$M_0^- = a^- \tanh(x + c^-)\vec{u} + b^- \frac{1}{\cosh(x + c^-)}\vec{v},$$

where $b^- \in \{-1, 1\}$ and $a^- = -1$ (resp. $+1$) if $M_0^-(x)$ tends to $\vec{u}$ (resp. $-\vec{u}$) when x tends to $-\infty$, and

$$M_0^+ = a^+ \tanh(x + c^+)\vec{e_1} + b^+ \frac{1}{\cosh(x + c^+)}\vec{e_2},$$

where $b^+ \in \{-1, 1\}$ and $a^+ = 1$ (resp. -1) if $M_0^-(x)$ tends to $\vec{e_1}$ (resp. $-\vec{e_1}$) when x tends to $+\infty$.

We use now the transmission conditions (iii) in order to fix the constants $a^\pm$, $b^\pm$ and $c^\pm$. At the bend, $M_0^-(0) = M_0^+(0)$ and $\dfrac{dM_0^-}{dx}(0) = \dfrac{M_0^+}{dx}(0)$, so we have:

$$\begin{cases} a^- \tanh(c^-)\vec{u} + b^- \dfrac{1}{\cosh(c^-)}\vec{v} = a^+ \tanh(c^+)\vec{e_1} + b^+ \dfrac{1}{\cosh(c^+)}\vec{e_2}, \\ \text{and} \\ \dfrac{1}{\cosh(c^-)}\left(a^- \dfrac{1}{\cosh(c^-)}\vec{u} - b^- \tanh(c^-)\vec{v}\right) = \dfrac{1}{\cosh(c^+)}\left(a^+ \dfrac{1}{\cosh(c^+)}\vec{e_1} - b^+ \tanh(c^+)\vec{e_2}\right). \end{cases} \tag{2.7}$$

The last equation induces that $\dfrac{1}{\cosh(c^-)} = \dfrac{1}{\cosh(c^+)}$, thus $c^- = \epsilon c^+$ with $\epsilon = \pm 1$. System (2.7) is equivalent to the system:

$$Q_1 X = X \quad \text{and} \quad Q_2 X = X, \tag{2.8}$$

where

$$X = \begin{pmatrix} \tanh(c^-) \\ \\ \dfrac{1}{\cosh(c^-)} \end{pmatrix}$$

and where

$$Q_1 = \begin{pmatrix} \epsilon a^- a^+ \cos\beta & \varepsilon a^+ b^- \sin\beta \\ \\ -a^- b^+ \sin\beta & b^- b^+ \cos\beta \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} \epsilon b^- b^+ \cos\beta & \varepsilon a^- b^+ \sin\beta \\ \\ -a^+ b^- \sin\beta & a^+ a^- \cos\beta \end{pmatrix}.$$

The matrices Q_1 and Q_2 are orthogonal (rotation of orthogonal symmetry). In addition, (2.8) induces that one is an eigenvalue of Q_1 and Q_2. So, Q_1 and Q_2 are matrices of orthogonal symmetries ($Q_1 = Id$ is impossible since $\sin\beta \neq 0$). So the determinant of Q_1 equals -1, *i.e.*

$$\epsilon a^- a^+ b^- b^+ = -1. \tag{2.9}$$

Since both matrices have the same eigenvector X associated to $+1$, since they are orthogonal symmetries, $Q_1 = Q_2$. Therefore

$$a^- b^+ = a^+ b^- \quad \text{and} \quad a^+ a^- = b^+ b^-. \tag{2.10}$$

Coupling (2.9) and (2.10), we obtain that $\epsilon = -1$, *i.e.* $c^+ = -c^-$.
Now, we have four cases with $a^- = +1$:

Case 1: $a^- = 1$, $a^+ = 1$, $b^- = 1$ and $b^+ = 1$.

We have in this case:

$$Q_1 = Q_2 = \begin{pmatrix} -\cos\beta & -\sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

The eigenspace associated to the eigenvalue 1 is generated by $\begin{pmatrix} -\sin\dfrac{\beta}{2} \\ \cos\dfrac{\beta}{2} \end{pmatrix}$ and contains $X = \begin{pmatrix} \tanh(c^-) \\ \dfrac{1}{\cosh(c^-)} \end{pmatrix}$. So there exists $\sigma \in \mathbb{R}$ such that $X = \sigma \begin{pmatrix} -\sin\dfrac{\beta}{2} \\ \cos\dfrac{\beta}{2} \end{pmatrix}$. Since both vectors are unitary, $\sigma = \pm 1$, and since the second coordinate is positive (we recall that $\beta \in]0, \pi[$, $\sigma = 1$. In particular, $\tanh(c^-) = -\sin\frac{\beta}{2}$, so $c^- = -\text{artanh}\,(\sin\frac{\beta}{2})$. Thus, in this case, we obtain the following stationary solution:

$$\mathbf{m}_1(x) = \begin{cases} \tanh(x-c)\vec{u} + \dfrac{1}{\cosh(x-c)}\vec{v} & \text{if} \quad x \leq 0, \\ \tanh(x+c)\vec{e_1} + \dfrac{1}{\cosh(x+c)}\vec{e_2} & \text{if} \quad x \geq 0, \end{cases} \qquad \text{with} \quad c = \text{artanh}\,(\sin\frac{\beta}{2}).$$

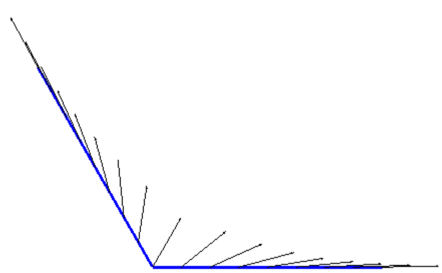

Figure 2: graph of $\mathbf{m}_1$

Case 2: $a^- = 1$, $a^+ = 1$, $b^- = -1$ and $b^+ = -1$.

We have in this case:

$$Q_1 = Q_2 = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}.$$

Considering the eigenspace associated to the eigenvalue 1, we obtain that

$$\begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \tanh(c^-) \\ \dfrac{1}{\cosh(c^-)} \end{pmatrix},$$

so $c^- = \operatorname{artanh}(\cos\frac{\beta}{2})$ and we obtain the following solution:

$$\mathbf{m}_2(x) = \begin{cases} \tanh(x+c)\vec{u} + \dfrac{1}{\cosh(x+c)}\vec{v} & \text{if} \quad x \le 0, \\ -\tanh(x-c)\vec{e_1} - \dfrac{1}{\cosh(x-c)}\vec{e_2} & \text{if} \quad x \ge 0, \end{cases} \qquad \text{with} \quad c = \operatorname{artanh}(\cos\frac{\beta}{2}).$$

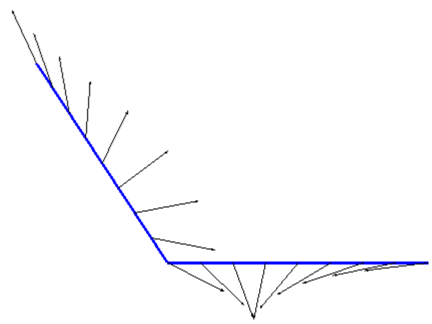

Figure 3: graph of $\mathbf{m}_2$

Case 3: $a^- = 1$, $a^+ = 1$, $b^- = -1$ and $b^+ = -1$.

In this case,

$$Q_1 = Q_2 = \begin{pmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix},$$

so, by considering the eigenspace associated to 1 we obtain that

$$\begin{pmatrix} \sin\frac{\beta}{2} \\ \\ \cos\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \tanh(c^-) \\ \\ \dfrac{1}{\cosh(c^-)} \end{pmatrix}.$$

Thus $c^- = \operatorname{artanh}(\sin\frac{\beta}{2})$, and the associated stationary solution is:

$$\mathbf{m}_3(x) = \begin{cases} \tanh(x+c)\vec{u} - \dfrac{1}{\cosh(x+c)}\vec{v} & \text{if} \quad x \le 0, \\ \tanh(x-c)\vec{e_1} - \dfrac{1}{\cosh(x-c)}\vec{e_2} & \text{if} \quad x \ge 0, \end{cases} \qquad \text{with} \quad c = \operatorname{artanh}(\sin\frac{\beta}{2}).$$

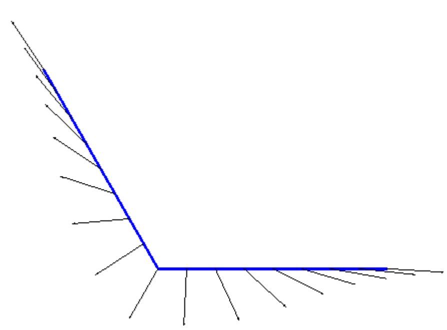

Figure 4: graph of $\mathbf{m}_3$

Case 4: $a^- = 1$, $a^+ = -1$, $b^- = -1$ and $b^+ = 1$.

In this case,

$$Q_1 = Q_2 = \begin{pmatrix} \cos\beta & -\sin\beta \\ -\sin\beta & -\cos\beta \end{pmatrix},$$

so, the eigenspace associated to 1 is generated by:

$$\begin{pmatrix} -\cos(\dfrac{\beta}{2}) \\ \sin(\dfrac{\beta}{2}) \end{pmatrix} = \begin{pmatrix} \tanh(c^-) \\ \\ \dfrac{1}{\cosh(c^-)} \end{pmatrix},$$

so that $c^- = -\mathrm{artanh}\,(\cos\frac{\beta}{2})$. The associated stationary solution is:

$$\mathbf{m}_4(x) = \begin{cases} \tanh(x-c)\vec{u} - \dfrac{1}{\cosh(x-c)}\vec{v} & \text{if} \quad x \le 0, \\ -\tanh(x+c)\vec{e_1} + \dfrac{1}{\cosh(x+c)}\vec{e_2} & \text{if} \quad x \ge 0, \end{cases} \qquad \text{with} \quad c = \mathrm{artanh}\,(\sin\frac{\beta}{2}).$$

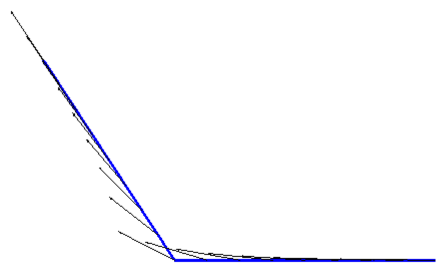

Figure 5: graph of $\mathbf{m}_4$

By symmetry the solutions to (1.6) – (1.8) satisfying the limit conditions $M_0^-(x) \longrightarrow \vec{u}$ when $x \longrightarrow -\infty$ (that is with $a^- = -1$) are $-\mathbf{m}_1$, $-\mathbf{m}_2$, $-\mathbf{m}_3$, and $-\mathbf{m}_4$.

3 Equation for the perturbations

Let M_0 be one of the static solutions for (1.6) with vanishing applied field given by Theorem 1.2:

$$M_0(x) = \begin{cases} M_0^-(x) = \sin\theta^-\vec{u} + \cos\theta^-(x)\vec{v} & \text{if} \quad x \le 0, \\ M_0^+(x) = \sin\theta^+\vec{e_1} + \cos\theta^+(x)\vec{e_2} & \text{if} \quad x \ge 0. \end{cases}$$

We aim to address the stability of M_0 for (1.6). we denote by J the matrix

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As in [4, 5, 7, 9], in order to consider only perturbations m satisfying the saturation constraint $|m| = 1$, we introduce the mobile frame $(M_0(x), M_1(x), M_2)$ with

$$M_1(x) = JM_0(x) \quad \text{and} \quad M_2 = \vec{e_3}$$

and we describe m in this mobile frame writing:

$$m(t,x) = M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2 + \mu(r)M_0(x), \tag{3.1}$$

where $r = (r_1, r_2) \in \mathbb{R}^2$ and $\mu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$, so that the saturation constraint is satisfied.

We remark that $[|M_0|] = [|M_1|] = [|\frac{dM_0}{dx}|] = [|\frac{dM_1}{dx}|] = 0$, then the jump conditions $[|m|] = [|\partial_x m|] = 0$ at $x = 0$ are equivalent to the null-jump condition on r:

$$[|r|] = [|\partial_x r|] = 0 \quad \text{at} \quad x = 0.$$

Now, we plug (3.1) in (1.6). By projection onto $\mathbb{R}M_1$ and $\mathbb{R}M_2$, we obtain that m satisfies (1.6) if and only if r satisfies

$$\partial_t r = \Lambda r + F \text{ on } \mathbb{R}. \tag{3.2}$$

The linear part of (3.2) is given by:

$$\Lambda = \begin{pmatrix} -\alpha L & -L \\ L & -\alpha L \end{pmatrix} \quad \text{with } L = -\partial_{xx} + f_\beta, \tag{3.3}$$

with

$$f_\beta(x) = \sin^2\theta(x) - \cos^2\theta(x),$$

where $\theta(x) = \theta^-(x)$ for $x < 0$ and $\theta(x) = \theta^+(x)$ for $x > 0$.

The nonlinear part $F : \mathbb{R}^+ \times B(0,1) \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of (3.2) is defined by

$$F(x, r, \partial_x r, \partial_{xx}) = A(r)\partial_{xx} r + B(r)(\partial_x r, \partial_x r) + C(x, r)\partial_x r + D(x, r), \tag{3.4}$$

where, for $x \in \mathbb{R}$, $r = (r_1, r_2) \in \mathbb{R}^2$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$,

- $A \in C^\infty(B(0,1), \mathcal{M}_2(\mathbb{R}))$ ($\mathcal{M}_2(\mathbb{R})$ is the set of the real 2×2 matrices) with

$$A(r)\zeta = \begin{pmatrix} -\alpha(r_1)^2 & -\alpha r_1 r_2 + \mu(r) \\ -\alpha r_1 r_2 - \mu(r) & -\alpha(r_2)^2 \end{pmatrix} \zeta - \begin{pmatrix} \alpha r_1(\mu(r)+1) + r_2 \\ \alpha r_2(\mu(r)+1) - r_1 \end{pmatrix} \mu'(r)(\zeta), \tag{3.5}$$

- $B \in C^\infty(B(0,1), \mathcal{L}_2(\mathbb{R}^2))$ ($\mathcal{L}_2(\mathbb{R}^2)$ denotes the set of the bilinear functions defined on $\mathbb{R}^2 \times \mathbb{R}^2$ with values in $\mathbb{R}^2$) given by

$$B(r)(\zeta, \zeta) = -\begin{pmatrix} \alpha r_1(\mu(r)+1) + r_2 \\ \alpha r_2(\mu(r)+1) - r_1 \end{pmatrix} \mu''(r)(\zeta, \zeta). \tag{3.6}$$

- $C \in C^\infty(\mathbb{R} \times B(0,1), \mathcal{M}_2(\mathbb{R}))$ with

$$C(x, r)\zeta = -2\theta'(x) \begin{pmatrix} \alpha r_1(\mu(r)+1) + r_2 \\ \alpha r_2(\mu(r)+1) - r_1 \end{pmatrix} \zeta_1 + 2\theta'(x) \begin{pmatrix} \alpha(r_1^2 - 1) \\ 1 + \mu(r) + \alpha r_1 r_2 \end{pmatrix} \mu'(r)(\zeta) \tag{3.7}$$

- $D \in C^\infty(\mathbb{R} \times B(0,1), \mathbb{R}^2)$ such that

$$D = \Big(2\sin\theta(x)\cos\theta(x) r_1 - \ \mu(r)(\sin^2\theta(x) - \cos^2\theta(x))\Big) \begin{pmatrix} \alpha r_1(\mu(r)+1) + r_2 \\ \alpha r_2(\mu(r)+1) - r_1 \end{pmatrix}. \tag{3.8}$$

We remark that this equation is valid while r takes its values in a neighborhood of zero since μ is singular for $|r| \geq 1$. In order to obtain uniform estimates, we will consider below perturbations such that $\|r\|_{L^\infty} \leq \frac{1}{2}$. We remark also that the asymptotic stability of M_0 for (1.6) is equivalent to the asymptotic stability of zero for (3.2).

4 Proof of Theorem 1.2

4.1 Stability for $\mathbf{m}_1$

For the first solution $\mathbf{m}_1$ given by (1.9), the linear part in (3.2) is given by

$$\Lambda_1 = \begin{pmatrix} -\alpha L_1 & -L_1 \\ L_1 & -\alpha L_1 \end{pmatrix} \quad \text{and} \quad L_1 = -\partial_{xx} + f_{1,\beta}(x), \tag{4.1}$$

where

$$f_{1,\beta}(x) = \begin{cases} f(x+c) \text{ if } x \geq 0 \\ f(x-c) \text{ if } x \leq 0, \end{cases} \qquad \text{with } f(x) = 2\tanh^2(x) - 1 \text{ and } c = \operatorname{artanh}(\sin(\frac{\beta}{2})).$$

Since f is strictly decreasing on $\mathbb{R}^-$ and increasing on $\mathbb{R}^+$, as $c > 0$ since $\beta \in]0, \pi[$, we remark that:

$$\forall\, x \in \mathbb{R}, \quad f_{1,\beta}(x) > f(x). \tag{4.2}$$

4.1.1 First step: Coercivity of L_1

We denote by $\langle .|. \rangle$ the usual L^2-inner product in $L^2(\mathbb{R})$.

We recall that the operator $\mathcal{L} = -\partial_{xx} + f(x)$ is a self-adjoint operator acting on $H^2(\mathbb{R})$, its essential spectrum is $[1, +\infty[$ and zero is its unique eigenvalue which eigenspace is generated by $\dfrac{1}{\cosh x}$ (see [4, 9]). Therefore, for all $w \in H^2(\mathbb{R})$ satisfying $\langle w | \frac{1}{\cosh x} \rangle = 0$, we have

$$\|w\|_{L^2}^2 \leq \langle \mathcal{L}w | w \rangle \leq \|\mathcal{L}w\|_{L^2}^2. \tag{4.3}$$

In addition there exists constants $c_1 > 0$ and $c_2 > 0$ such that for all $w \in \left(\frac{1}{\cosh x}\right)^\perp$,

$$c_1 \|w\|_{H^1}^2 \leq \langle \mathcal{L}w | w \rangle \leq c_2 \|w\|_{H^1}^2. \tag{4.4}$$

We claim the coercivity of L_1 in the following proposition:

Proposition 4.1. *There exists $c > 0$, such that for all $u \in H^2(\mathbb{R})$, we have $\langle L_1 u | u \rangle \geq c \|u\|_{L^2}^2$.*

Proof. Suppose that there exists $(u_n)_n$, such that $\|u_n\|_{L^2} = 1$ and $\langle L_1 u_n | u_n \rangle < \frac{1}{n}$.
We write $u_n = w_n + \dfrac{\sigma_n}{\cosh x}$, where $w_n \in (\dfrac{1}{\cosh x})^\perp$ and $\sigma_n \in \mathbb{R}$. Thus

$$\begin{aligned} \langle L_1 u_n | u_n \rangle = \langle -\partial_{xx} u_n + f_{1,\beta} u_n | u_n \rangle &= \langle -\partial_{xx} u_n + f u_n + (f_{1,\beta} - f) u_n | u_n \rangle \\ &= \langle \mathcal{L} u_n | u_n \rangle + \int_{\mathbb{R}} (f_{1,\beta} - f) |u_n|^2 < \frac{1}{n}. \end{aligned}$$

Moreover $\mathcal{L} u_n = \mathcal{L} w_n$ since $\mathcal{L}(\dfrac{1}{\cosh x}) = 0$. So, since $\mathcal{L}$ is self-adjoint,

$$\langle \mathcal{L} u_n | u_n \rangle = \langle \mathcal{L} w_n | w_n + \sigma_n (\frac{1}{\cosh x}) \rangle = \langle \mathcal{L}(w_n) | w_n \rangle + \langle \mathcal{L}(w_n) | \frac{\sigma_n}{\cosh x} \rangle = \langle \mathcal{L} w_n | w_n \rangle.$$

Therefore we have

$$\langle \mathcal{L} w_n | w_n \rangle + \int_{\mathbb{R}} (f_{1,\beta} - f) |u_n|^2 < \frac{1}{n}.$$

So, with (4.4) we conclude that $\|w_n\|_{H^1}$ tends to zero when n tends to infinity.

On the other hand,

$$\begin{aligned} 1 = \|u_n\|_{L^2}^2 &= \|w_n\|_{L^2}^2 + 2 \langle w_n | \frac{\sigma_n}{\cosh x} \rangle + \| \frac{\sigma_n}{\cosh x} \|_{L^2}^2 \\ &= \|w_n\|_{L^2}^2 + |\sigma_n|^2 \| \frac{1}{\cosh x} \|_{L^2}^2. \end{aligned}$$

Thus, we have $\lim_{n\to\infty} |\sigma_n|^2 = \frac{1}{2}$. Consider now the second term

$$\int_{\mathbb{R}} (f_{1,\beta} - f)|u_n|^2 dx = I_1 + I_2 + I_3, \tag{4.5}$$

with

- $I_1 = \int_{\mathbb{R}} (f_{1,\beta} - f)|w_n|^2 dx \ \leq\ \|f_{1,\beta} - f\|_{L^\infty} \|w_n\|^2_{L^2} \longrightarrow 0$ as $n \longrightarrow \infty$,
- $I_2 = 2\int_{\mathbb{R}} (f_{1,\beta} - f) w_n \frac{\sigma_n}{\cosh x} dx \ \leq\ 2\|f_{1,\beta} - f\|_{L^\infty} \|w_n\|_{L^2} \|\frac{\sigma_n}{\cosh x}\|_{L^2} \longrightarrow 0$ as $n \longrightarrow \infty$,
- $I_3 = |\sigma_n|^2 \int_{\mathbb{R}} (f_{1,\beta} - f) \frac{1}{\cosh^2 x} dx \longrightarrow \frac{1}{2} \int_{\mathbb{R}} (f_{1,\beta} - f) \frac{1}{\cosh^2 x} dx.$

In view of the above analysis, we obtain

$$\int_{\mathbb{R}} (f_{1,\beta} - f)|u_n|^2 dx \longrightarrow \frac{1}{2} \int_{\mathbb{R}} (f_{1,\beta} - f) \frac{1}{\cosh^2 x} dx > 0 \text{ by (4.2).}$$

On the other hand, since $\int_{\mathbb{R}} (f_{1,\beta} - f)|u_n|^2 dx < \frac{1}{n}$, we obtain that

$$\int_{\mathbb{R}} (f_{1,\beta} - f)|u_n|^2 dx \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty,$$

which leads to a contradiction. So the assumption is false and therefore there exists $c > 0$ such that for all $u \in H^2(\mathbb{R})$, we have

$$\langle L_1 u | u \rangle \geq c\|u\|^2_{L^2}.$$

□

Corollary 4.1. *There exist two constants K_1 and K_2 such that for every $u \in H^2(\mathbb{R})$ we have*

$$K_1 \|u\|^2_{H^1} \leq \langle L_1 u | u \rangle \leq K_2 \|u\|^2_{H^1},$$

$$K_1 \|u\|_{H^2} \leq \|L_1 u\|_{L^2} \leq K_2 \|u\|_{H^2}.$$

Proof. Thanks to Proposition 4.1, we have

$$\|\partial_x u\|^2_{L^2} = \langle \partial_x u | \partial_x u \rangle = \langle -\partial_{xx} u | u \rangle = \langle L_1 u - f_{1,\beta} u | u \rangle \leq \langle L_1 u | u \rangle + \|f_{1,\beta}\|_\infty \|u\|^2_{L^2}.$$

Since $\|f_{1,\beta}\|_{L^\infty} = 1$, we obtain by Proposition 4.1 that

$$\|\partial_x u\|^2_{L^2} \leq (1 + \frac{1}{c}) \langle L_1 u | u \rangle.$$

Therefore,

$$\|u\|^2_{H^1} \leq (2 + \frac{1}{c}) \langle L_1 u | u \rangle.$$

In addition, we have:

$$\langle L_1 u | u \rangle = \int_{\mathbb{R}} |\partial_x u|^2 + \int_{\mathbb{R}} f_{1,\beta} |u|^2 \leq \|u\|^2_{H^1} \quad \text{since } \|f_{1,\beta}\|_\infty = 1.$$

This proves the equivalence of norms in $H^1(\mathbb{R})$.

From Proposition 4.1 we have also

$$\|u\|_{L^2} \leq \frac{1}{c} \|L_1 u\|_{L^2}.$$

Furthermore,

$$\|\partial_{xx}u\|_{L^2} = \| -\partial_{xx}u + f_{1,\beta}u - f_{1,\beta}u\| \le \|L_1u\|_{L^2} + \|f_{1,\beta}\|_\infty \|u\|_{L^2} \le (1+\frac{1}{c})\|L_1u\|_{L^2}.$$

Thus, we conclude that there exists a constant K such that

$$\|u\|_{H^2} \le K\|L_1u\|_{L^2}.$$

In addition,

$$\|L_1u\|_{L^2} \le \|\partial_{xx}u\|_{L^2} + \|f_{1,\beta}\|_{L^\infty}\|u\|_{L^2} \le 2\|u\|_{H^2}.$$

This concludes the proof of Corollary 4.1.

□

4.1.2 Second step: estimate for the nonlinear term

In this section we estimate the $L^2(\mathbb{R})$-norm of the nonlinear term F given by :

$$F(x, r, \partial_x r, \partial_{xx}) = A(r)\partial_{xx}r + B(r)(\partial_x r, \partial_x r) + C(x, r)\partial_x r + D(x, r),$$

where the right-hand-side terms are defined by (3.5)-(3.8).

Using the Sobolev injection of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$ and the equivalence of norms claimed in Corollary 4.1, we introduce $\eta_0 > 0$ such that for all $r \in H^2(\mathbb{R})$, $\langle L_1 r|r\rangle^{\frac{1}{2}} \le \eta_0 \Rightarrow \|r\|_{L^\infty(\mathbb{R})} \le \dfrac{1}{2}$. We prove the following estimate for the nonlinear part F:

Proposition 4.2. *There exists a constant k_0 such that for all $r \in H^2(\mathbb{R})$ with $\langle L_1 r|r\rangle^{\frac{1}{2}} \le \eta_0$ then:*

$$\|F\|_{L^2} \le k_0 \langle L_1 r|r\rangle^{\frac{1}{2}} \|L_1 r\|_{L^2}. \tag{4.6}$$

Proof. We estimate each term of F separately. The same notation K is used for different constants independent of β and $r \in H^2(\mathbb{R})$ satisfying that $\|r\|_{L^\infty} \le \frac{1}{2}$.

We first remark that for $r \in \mathbb{R}^2$ in a neighborhood of zero, $\mu(r) = \mathcal{O}(|r|^2)$, $\mu'(r) = \mathcal{O}(|r|)$ and $\mu''(r) = \mathcal{O}(1)$.

By (3.5) we remark that $A(r) = \mathcal{O}(|r|^2)$ so that:

$$|A(r)\partial_{xx}r| \le C|r|^2|\partial_{xx}r|.$$

So using classical Sobolev embedding, we obtain that

$$\|A(r)\partial_{xx}r\|_{L^2} \le K\|r\|_{L^\infty}\|\partial_{xx}r\|_{L^2} \le K\|r\|_{H^1}\|r\|_{H^2}.$$

Concerning the second term defined by (3.6), since $B(r)$ is bounded for $r \in B(0, \frac{1}{2})$, we have

$$|B(r)(\partial_x r, \partial_x r)| \le K|r||\partial_x r|^2.$$

So,

$$\|B(r)(\partial_x r, \partial_x r)\|_{L^2} \le K\|\partial_x r\|_{L^2}\|\partial_x r\|_{L^\infty},$$

thus, by Sobolev embeddings,

$$\|B(r)(\partial_x r, \partial_x r)\|_{L^2} \le K\|r\|_{H^1}\|r\|_{H^2}.$$

By (3.7), since $C(x, r) = \mathcal{O}(|r|)$, we have

$$|C(x, r)\partial_x r| \le K|r|\,|\partial_x r|,$$

so we obtain:

$$\|C(x, r)\partial_x r\|_{L^2} \le K\|r\|_{L^\infty}\|\partial_x r\|_{L^2},$$

thus

$$\|C(x,r)\partial_x r\|_{L^2} \le K\|r\|_{H^1}^2.$$

Concerning the last term, we have $D(x,r) = \mathcal{O}(|r|^2)$, thus

$$\|D(x,r)\|_{L^2} \le \|r\|_{L^2}\|r\|_{L^\infty} \le K\|r\|_{H^1}^2.$$

Finally, there exists a constant K such that if $r \in H^2(\mathbb{R})$ satisfies $\|r\|_{L^\infty} \le \frac{1}{2}$, we have

$$\|F\|_{L^2} \le K\|r\|_{H^1}\|r\|_{H^2}. \tag{4.7}$$

Using Corollary 4.1 and the definition of η_0, we obtain that there exists k_0 such that for all $r \in H^2(\mathbb{R})$ with $\langle L_1 r|r\rangle^{\frac{1}{2}} \le \eta_0$ then:

$$\|F\|_{L^2} \le k_0\langle L_1 r|r\rangle^{\frac{1}{2}}\|L_1 r\|_{L^2}.$$

□

4.1.3 Stability Proof

We recall that the asymptotic stability of $\mathbf{m}_1$ for Equation (1.6) as claimed in Theorem 1.2 is equivalent to the asymptotic stability of 0 for Equation (3.2). We consider an initial data $r_0 \in H^2(\mathbb{R})$ such that $\langle L_1 r_0|r_0\rangle^{\frac{1}{2}} \le \eta_0$ and we denote by r the solution of (3.2) with initial data r_0.
We take the L^2-inner product of (3.2) with $L_1 r$, and we obtain that:

$$\frac{1}{2}\frac{d}{dt}\langle L_1 r|r\rangle = -\alpha\langle L_1 r|L_1 r\rangle + \langle F|L_1 r\rangle.$$

Therefore using (4.6), while $\langle L_1 r(t)|r(t)\rangle^{\frac{1}{2}} \le \eta_0$, we have

$$\frac{1}{2}\frac{d}{dt}\langle L_1 r|r\rangle + \alpha\|L_1 r\|_{L^2}^2 \le k_0\langle L_1 r|r\rangle^{\frac{1}{2}}\|L_1 r\|_{L^2}^2,$$

then,

$$\frac{1}{2}\frac{d}{dt}\langle L_1 r|r\rangle + (\alpha - k_0\langle L_1 r|r\rangle^{\frac{1}{2}})\|L_1 r\|_{L^2}^2 \le 0. \tag{4.8}$$

Thus, while $\langle L_1 r|r\rangle^{\frac{1}{2}}(t) \le \min\{\eta_0, \dfrac{\alpha}{2k_0}\}$, we obtain that $\alpha - k_0\langle L_1 r|r\rangle^{\frac{1}{2}} \ge \dfrac{\alpha}{2}$. Then,

$$\frac{1}{2}\frac{d}{dt}\langle L_1 r|r\rangle + \frac{\alpha}{2}\|L_1 r\|_{L^2}^2 \le 0.$$

By Corollary 4.1, there exists a constant $c_0 > 0$ such that

$$\|L_1 r\|_{L^2}^2 \ge c_0\langle L_1 r|r\rangle.$$

Therefore we obtain that while $\langle L_1 r|r\rangle^{\frac{1}{2}}(t) \le \min\{\eta_0, \dfrac{\alpha}{2k_0}\}$,

$$\frac{d}{dt}\langle L_1 r|r\rangle + c_0\alpha\langle L_1 r|r\rangle \le 0$$

which implies by comparison lemma that

$$\langle L_1 r(t)|r(t)\rangle \le \langle L_1 r_0|r_0\rangle \exp(-\alpha c_0 t). \tag{4.9}$$

We set $\eta_1 = \frac{1}{2}\inf\{\eta_0, \frac{\alpha}{2k_0}\}$. We assume that the initial data r_0 satisfies

$$\langle L_1 r_0|r_0\rangle^{\frac{1}{2}} \le \eta_1. \tag{4.10}$$

Let us show that for all $t \geq 0$,

$$\langle L_1 r(t) | r(t) \rangle^{\frac{1}{2}} < \min\{\eta_0, \frac{\alpha}{2k_0}\}. \tag{4.11}$$

If it is not the case, let t_1 the first time in which (4.11) is false. Since (4.11) is true for small t by continuity reason, then $t_1 > 0$, the property is true for $t \in [0, t_1[$ and at the time t_1, we have:

$$\langle L_1 r(t_1) | r(t_1) \rangle^{\frac{1}{2}} = \min\{\eta_0, \frac{\alpha}{2k_0}\}. \tag{4.12}$$

Now, for $t \in [0, t_1[$, we can apply (4.9) so that $\langle L_1 r(t) | r \rangle(t) \leq \langle L_1 r_0 | r_0 \rangle \exp(-\alpha c_0 t) \leq \eta_1$. By continuity reasons, $\langle L_1 r(t_1) | r \rangle(t_1) \leq \eta_1$ which is in contradiction with (4.12).

So if $\langle L_1 r_0 | r_0 \rangle^{\frac{1}{2}} \leq \eta_1$, then for all $t \geq 0$, Inequality (4.9) is true. This implies that under assumption (4.10), we obtain that $\|r\|_{H^1(\mathbb{R})}$ remains small for all times and $\|r\|_{H^1} \longrightarrow 0$ as t tends to $+\infty$.

This concludes the proof of the asymptotic stability of $\mathbf{m}_1$. $\square$

4.2 Stability for $\mathbf{m}_4$

For the forth solution $\mathbf{m}_4$ defined by (1.12), the linear part in (3.2) is given by

$$\Lambda_4 = \begin{pmatrix} -\alpha L_4 & -L_4 \\ L_4 & -\alpha L_4 \end{pmatrix}, \quad \text{with} \quad L_4 = -\partial_{xx} + f_{4,\beta}$$

where

$$f_{4,\beta} = \begin{cases} f(x+c) \text{ if } x \geq 0, \\ f(x-c) \text{ if } x \leq 0, \end{cases} \quad \text{with } f(x) = 2\tanh^2(x) - 1 \text{ and } c = \operatorname{artanh}(\sin(\frac{\beta}{2})) > 0.$$

So we obtain the same linear part as in Subsection 4.1 for $\mathbf{m}_1$, and we prove the asymptotic stability of $\mathbf{m}_4$ for (1.6) as for $\mathbf{m}_1$.

5 Linear instability of $\mathbf{m}_2$ and $\mathbf{m}_3$

For $\mathbf{m}_2$, the linear part of (3.2) is given by

$$\Lambda_2 = \begin{pmatrix} -\alpha L_2 & -L_2 \\ L_2 & -\alpha L_2 \end{pmatrix}, \quad \text{with} \quad L_2 = -\partial_{xx} + f_{2,\beta}$$

where

$$f_{2,\beta} = \begin{cases} f(x-c) \text{ if } x \geq 0, \\ f(x+c) \text{ if } x \leq 0, \end{cases} \quad \text{with } c = \operatorname{artanh}(\cos\frac{\beta}{2}) > 0.$$

Let us show that the linear operator Λ_2 admits at least one unstable direction.

We have

$$L_2(\frac{1}{\cosh(x-c)}) = (-\partial_{xx} + f(x-c))\,\frac{1}{\cosh(x-c)} + (f_{2,\beta} - f(x-c))\,\frac{1}{\cosh(x-c)}.$$

Therefore, we have

$$L_2(\frac{1}{\cosh(x-c)}) = (f_{2,\beta} - f(x-c))\frac{1}{\cosh(x-c)},$$

since

$$(-\partial_{xx} + f(x-c))(\frac{1}{\cosh(x-c)}) = 0.$$

We have also,

$$\begin{aligned}\langle L_2(\frac{1}{\cosh(x-c)})|\frac{1}{\cosh(x-c)}\rangle &= \int_{\mathbb{R}^+}(f(x-c)-f(x-c))\frac{1}{\cosh^2(x-c)}dx\\ &+\int_{\mathbb{R}^-}(f(x+c)-f(x-c))\frac{1}{\cosh^2(x-c)}dx.\end{aligned}$$

Hence, $\langle L_2(\frac{1}{\cosh(x-c)})|\frac{1}{\cosh(x-c)}\rangle < 0$, we conclude that L_2 has one strictly negative eigenvalue. Therefore the solution is linearly unstable.

Concerning $\mathbf{m}_3$ we obtain that the linear part arising in the stability proof can be written as:

$$\Lambda_3 = \begin{pmatrix} -\alpha L_3 & -L_3 \\ L_3 & -\alpha L_3 \end{pmatrix} \quad \text{with} \quad L_3 = -\partial_{xx} + f_{3,\beta},$$

where

$$f_{3,\beta} = \begin{cases} f(x-c) \text{ if } x \geq 0 \\ f(x+c) \text{ if } x \leq 0, \end{cases} \qquad c = \operatorname{artanh}(\sin\frac{\beta}{2}) > 0.$$

Proceeding in the same way as for $\mathbf{m}_2$, we obtain the linear instability of $\mathbf{m}_3$.

6 Perturbation of stable profiles by small applied fields

Let $\mathbf{m}_1$ be the static solution of (1.6) with vanishing applied field given by (1.9). We denote by $(M_0(x), M_1(x), M_2)$ the mobile frame associated to $\mathbf{m}_1$ defined at the beginning of Section 3:

$$M_0(x) = \mathbf{m}_1(x), \quad M_1(x) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{m}_1(x), \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this part we consider solutions m of (1.6) with applied field $H_a = \lambda\xi$ remaining in the neighborhood of $\mathbf{m}_1$. We describe m in the mobile frame $(M_0(x), M_1(x), M_2)$ writing

$$m(t,x) = M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2 + \mu(r(t,x))M_0(x) \tag{6.1}$$

with $\mu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$, so that the saturation constraint is satisfied. We denote by (ξ_0, ξ_1, ξ_2) the coordinates of ξ in the mobile frame: $\xi(x) = \xi_0(x)M_0(x) + \xi_1(x)M_1(x) + \xi_2(x)M_2$.

As in Section 3, plugging (6.1) in (1.6), we obtain that if m given by (6.1) remains in a neighborhood of $\mathbf{m}_1$, then m satisfies (1.6) if and only if $r = (r_1, r_2)$ satisfies

$$\frac{\partial r}{\partial t} = \mathcal{F}(\lambda, r) := \Lambda_1 r + F(x, r, \partial_x r, \partial_{xx}) + \lambda\kappa(x) + \lambda G(x, r). \tag{6.2}$$

The first two right-hand-side terms are defined in Section 3 by (3.3) and (3.4). We recall that

$$\Lambda_1 = \begin{pmatrix} -\alpha L_1 & -L_1 \\ L_1 & -\alpha L_1 \end{pmatrix}, \quad \text{with } L = -\partial_{xx} + f_{1,\beta}, \tag{6.3}$$

with $f_{1,\beta}(x) = 2\tanh^2(|x| + c) - 1$, $c = \operatorname{artanh}(\sin(\frac{\beta}{2})) > 0$, and that F writes

$$F(x, r, \partial_x r, \partial_{xx}) = A(r)\partial_{xx} r + B(r)(\partial_x r, \partial_x r) + C(x,r)\partial_x r + D(x,r),$$

where A, B, C, D are smooth in the variable r and are defined respectively by (3.5), (3.6), (3.7) and (3.8).

The additional terms coming from the applied field $\lambda\xi$ are given by:

$$\kappa(x) = \begin{pmatrix} \xi_2 + \alpha\xi_1 \\ -\xi_1 + \alpha\xi_2 \end{pmatrix}$$

and

$$G(x,r)=\xi_0(x)\begin{pmatrix}-r_2-\alpha r_1-\alpha r_1\mu(r)\\ r_1-\alpha r_2-\alpha r_2\mu(r)\end{pmatrix}+\xi_1(x)\begin{pmatrix}-\alpha r_1^2\\ -\mu(r)-\alpha r_1r_2\end{pmatrix}+\xi_2(x)\begin{pmatrix}-\alpha r_1r_2+\mu(r)\\ -\alpha r_2^2\end{pmatrix}.$$

We recall that Equation (6.2) remains valid for $|r|<1$ (since $\mu'(r)$ is singular for $|r|=1$).

6.1 Static solution

A small perturbation $\mathbf{m}$ of $\mathbf{m}_1$ is a static solution for (1.6) with $H_a=\lambda\xi$ if and only if $\mathbf{r}=(\mathbf{r}_1,\mathbf{r}_2)$ given by

$$\mathbf{m}(x)=M_0(x)+\mathbf{r}_1(x)M_1(x)+\mathbf{r}_2(x)M_2+\mu(\mathbf{r}(x))M_0(x)$$

satisfies

$$\mathcal{F}(\lambda,\mathbf{r})=0. \tag{6.4}$$

We will prove the existence of a static solution for (6.4) by using the following implicit function theorem in Banach spaces (see [1]):

Theorem 6.1 (Implicit Function Theorem)**.** *Let B_0, B_1, B_2 three Banach spaces, $\mathcal{U}$ a neighborhood of $(x_0,y_0)\in B_0\times B_1$, and $f:\mathcal{U}\longrightarrow B_2$ continuously differentiable. Suppose that $f(x_0,y_0)=0$ and that there exists a continuous linear mapping $A:B_2\longrightarrow B_1$, such that $f'_y(x_0,y_0)A=id_{B_2}$. Then there exists $g\in\mathcal{C}^1$, from neighborhood of x_0 in B_0, such that $f(x,g(x))=0$. If, in addition, $f'_y(x_0,y_0)$ is bijective, then g is unique and $f(x,y)=0$ is equivalent to $y=g(x)$ for (x,y) near to (x_0,y_0).*

In our case, $B_0=\mathbb{R}$ with $x_0=0\in\mathbb{R}$, $B_1=H^2(\mathbb{R};\mathbb{R}^2)$ with $y_0=0\in H^2(\mathbb{R};\mathbb{R}^2)$, and $B_2=L^2(\mathbb{R};\mathbb{R}^2)$.

By Assumption 1.13, we remark that the constant term $\lambda\begin{pmatrix}\xi_2+\alpha\xi_1\\ -\xi_1+\alpha\xi_2\end{pmatrix}$ is in $L^2(\mathbb{R};\mathbb{R}^2)$, so that $\mathcal{F}$ is defined in a neighborhood of zero in $\mathbb{R}\times H^2(\mathbb{R};\mathbb{R}^2)$ and takes its values in $L^2(\mathbb{R};\mathbb{R}^2)$.

Now, we have $\partial_r\mathcal{F}(0,0)=\Lambda_1$ given by (6.3). We know that Λ_1 is strictly negative (see Section 4). So we can apply the implicit function theorem to the operator $\mathcal{F}$. Thus there is a neighborhood $]-\tilde{\eta},\tilde{\eta}[$ of 0 in $\mathbb{R}$, with $\tilde{\eta}>0$, there exists a neighborhood ω of 0 in $H^2(\mathbb{R};\mathbb{R}^2)$ and $\mathbf{R}:]-\tilde{\eta},\tilde{\eta}[\longrightarrow\omega$, such that for all $(\lambda,\mathbf{r})\in]-\tilde{\eta},\tilde{\eta}[\times\omega$,

$$\mathcal{F}(\lambda,\mathbf{r})=0\quad\Longleftrightarrow\quad\mathbf{r}=\mathbf{R}(\lambda).$$

For $\lambda\in]-\tilde{\eta},\tilde{\eta}[$, we write:

$$\mathbf{m}(\lambda)(x)=M_0(x)+\mathbf{R}_1(\lambda)(x)M_1(x)+\mathbf{R}_2(\lambda)(x)M_2+\mu(\mathbf{R}(\lambda)(x))M_0(x).$$

The map $\lambda\mapsto\mathbf{m}(\lambda)$ is at least $\mathcal{C}^1$, $\mathbf{m}(0)=M_0=\mathbf{m}_1$ and for all $\lambda\in]-\tilde{\eta},\tilde{\eta}[$, $\mathbf{m}(\lambda)$ satisfies (1.6).

6.2 Stability

We assume that $|\lambda|<\tilde{\eta}$. The asymptotic stability of $\mathbf{m}(\lambda)$ for Equation (1.6) with applied field $\lambda\xi$ is equivalent to the asymptotic stability of $\mathbf{R}(\lambda)$ for Equation (6.2).

Writing a small perturbation of $\mathbf{R}(\lambda)$ on the form $\mathbf{R}(\lambda)+w$, we have to prove the asymptotic stability of zero for the equation:

$$\frac{\partial w}{\partial t}=\mathcal{F}(\lambda,\mathbf{R}(\lambda)+w). \tag{6.5}$$

Using the Taylor expansion of $\mathcal{F}$ at the neighborhood of $\mathbf{R}(\lambda)$ and the fact that $\mathcal{F}(\lambda,\mathbf{R}(\lambda))=0$, we have

$$\mathcal{F}(\lambda,\mathbf{R}(\lambda)+w)=\Lambda_1 w+\mathcal{N}^\lambda(x,w,\partial_x w,\partial_{xx}w)$$

where

$$\mathcal{N}^\lambda(x,w,\partial_x w,\partial_{xx}w) = \quad A(\mathbf{R}(\lambda)+w)\partial_{xx}w + B(\mathbf{R}(\lambda)+w)(\partial_x w,\partial_x w) + \mathbf{C}(\lambda,x,w)\partial_x w$$
$$+\mathbf{D}(\lambda,x,w) + \lambda\mathbf{G}(\lambda,x,w).$$

The term $\mathbf{C}(\lambda,x,w)$ is defined for λ in a neighborhood of zero, $x\in\mathbb{R}$ and $w\in B(0,\frac{1}{2})$ and takes its values in $\mathcal{M}^2(\mathbb{R})$:

$$\mathbf{C}(\lambda,x,w)\zeta = 2B(\mathbf{R}(\lambda)(x)+w))(\partial_x\mathbf{R}(\lambda)(x),\zeta) + C(x,\mathbf{R}(\lambda)(x)+w)\zeta.$$

The term $\mathbf{D}(\lambda,x,w)$ is defined for λ in a neighborhood of zero, $x\in\mathbb{R}$ and $w\in B(0,\frac{1}{2})$ and takes its values in $\mathbb{R}^2$:

$$\mathbf{D}(\lambda,x,w) = A^\lambda(x,w)(\partial_{xx}\mathbf{R}(\lambda)) + B^\lambda(x,w)(\partial_x\mathbf{R}(\lambda),\partial_x\mathbf{R}(\lambda)) + C^\lambda(x,w)(\partial_x\mathbf{R}(\lambda)) + D^\lambda(x,w)$$

with:

- $A^\lambda(x,w) = \displaystyle\int_0^1 \partial_r A(\mathbf{R}(\lambda)(x)+sw)(w)ds,$
- $B^\lambda(x,w) = \displaystyle\int_0^1 \partial_r B(\mathbf{R}(\lambda)(x)+sw)(w)ds,$
- $C^\lambda(x,w) = \displaystyle\int_0^1 \partial_r C(x,\mathbf{R}(\lambda)(x)+sw)(w)ds,$
- $D^\lambda(x,w) = \displaystyle\int_0^1 \partial_r D(x,\mathbf{R}(\lambda)(x)+sw)(w)ds.$

The last term $\mathbf{G}(\lambda,x,w)$ is given by

$$\mathbf{G}(\lambda,x,w) = \int_0^1 \partial_r G(x,\mathbf{R}(\lambda)(x)+sw)(w)ds.$$

On the one hand, we recall that, in Section 4.1.2, we introduced $\eta_0>0$ such that:

$$\forall\, w\in H^2(\mathbb{R}),\quad \langle L_1 w|w\rangle^{\frac{1}{2}} \le \eta_0 \Rightarrow \|w\|_{L^\infty(\mathbb{R})} \le \frac{1}{2}.$$

In addition, there exists a constant K_3 such that for all $w\in H^2(\mathbb{R})$,

$$\|w\|_{H^1} + \|w\|_{L^\infty} \le K_3\langle L_1 w|w\rangle^{\frac{1}{2}} \quad\text{and}\quad \|w\|_{H^2} \le K_3\|L_1 w\|_{L^2}. \tag{6.6}$$

On the other hand, $\lambda\mapsto\mathbf{R}(\lambda)$ is $\mathcal{C}^1$ from $]-\tilde\eta,\tilde\eta[$ to $H^2(\mathbb{R})$ with $\mathbf{R}(0)=0$, so that there exists a constant $\eta_1>0$ with $\eta_1<\tilde\eta$ and a constant K_4 such that:

$$\forall\lambda\in[-\eta_1,\eta_1],\quad \|\mathbf{R}(\lambda)\|_{L^\infty} \le \frac{1}{4} \quad\text{and}\quad \|\mathbf{R}(\lambda)\|_{L^\infty} + \|\mathbf{R}(\lambda)\|_{H^2} \le K_4|\lambda|. \tag{6.7}$$

Hereafter, we assume that $|\lambda|\le\eta_1$. Since Equation (6.2) is valid for $|r|<1$, we will consider sufficient small initial data w_0 such that $\langle L_1 w(0)|w(0)\rangle^{\frac{1}{2}} \le \eta_0$. While $\langle L_1 w(t)|w(t)\rangle^{\frac{1}{2}} \le \eta_0$, $\|\mathbf{R}(\lambda)+w(t)\|_{L^\infty} \le \frac{3}{4}$ so that (6.5) remains valid.
As in Section 3 we take the L^2-inner product of (6.5) with $L_1 w$ and we obtain that

$$\frac{1}{2}\frac{d}{dt}\langle L_1 w|w\rangle + \alpha\|L_1 w\|_{L^2}^2 = \langle\mathcal{N}^\lambda(x,w,\partial_x w,\partial_{xx}w)|L_1 w\rangle. \tag{6.8}$$

The right-hand-side term is estimated as follows:

Proposition 6.1. *There exists a constant k_1 such that for all $w \in H^2(\mathbb{R})$ with $\langle L_1 w|w\rangle^{\frac{1}{2}} \le \eta_0$ and for all $\lambda \in [-\eta_1, \eta_1]$ then:*

$$\|\mathcal{N}^\lambda(x, w, \partial_x w, \partial_{xx} w)\|_{L^2} \le k_1 \left(\langle L_1 w|w\rangle^{\frac{1}{2}} + |\lambda|\right) \|L_1 w\|_{L^2}. \tag{6.9}$$

Proof. We recall that there exists a constant K_5 such that for all $r \in \mathbb{R}^2$, with $|r| \le \frac{3}{4}$, for all $x \in \mathbb{R}$, we have:

$$\begin{aligned}
&|D(x,r)| \le K_5 |r|^2,\\
&|A(r)| + |\partial_r A(r)| + |C(x,r)| + |\partial_r D(x,r)| + |G(x,r)| \le K_5 |r|,\\
&|B(r)| + |\partial_r B(r)| + |\partial_r C(x,r)| + |\partial_r D(x,r)| + |\partial_r G(x,r)| \le K_5.
\end{aligned} \tag{6.10}$$

Let us estimate each term of $\mathcal{N}^\lambda$. We assume that $\langle L_1 w|w\rangle^{\frac{1}{2}} \le \eta_0$ and that $|\lambda| \le \eta_1$, so that $\|w\|_{L^\infty} \le \frac{1}{2}$ and $\|\mathbf{R}(\lambda)\|_{L^\infty} \le \frac{1}{4}$. Thus $\|\mathbf{R}(\lambda) + w\|_{L^\infty} \le \frac{3}{4}$ and Estimates (6.10) remain valid for $r = \mathbf{R}(\lambda)(x) + w(t,x)$. Therefore, we have:

$$\|A(\mathbf{R}(\lambda)+w)\partial_{xx} w\|_{L^2} \le \quad \|A(\mathbf{R}(\lambda)+w)\|_{L^\infty}\|\partial_{xx} w\|_{L^2} \le K_5 \|\mathbf{R}(\lambda)+w\|_{L^\infty}\|\partial_{xx} w\|_{L^2},$$

$$\|B(\mathbf{R}(\lambda)+w)(\partial_x w, \partial_x w)\|_{L^2} \le \quad \|B(\mathbf{R}(\lambda)+w)\|_{L^\infty}\|\partial_x w\|^2_{L^4} \le K_5 \|w\|_{L^\infty}\|w\|_{H^2},$$

$$\begin{aligned}
\|\mathbf{C}(\lambda, x, w)\partial_x w\|_{L^2} \le\ & 2\|B(\mathbf{R}(\lambda)+w))\|_{L^\infty}\|\partial_x \mathbf{R}(\lambda)\|_{L^\infty}\|\partial_x w\|_{L^2}\\
& + \|C(\cdot, \mathbf{R}(\lambda)+w)\|_{L^\infty}\|\partial_x w\|_{L^2}\\
\le\ & K_5 \left(2\|\partial_x \mathbf{R}(\lambda)\|_{L^\infty} + \|\mathbf{R}(\lambda)+w\|_{L^\infty}\right)\|\partial_x w\|_{L^2}.
\end{aligned}$$

We bound each part of $\mathbf{D}(\lambda, \cdot, w)$ separately:

$$\begin{aligned}
\|A^\lambda(\cdot, w)(\partial_{xx}\mathbf{R}(\lambda))\|_{L^2} \le\ & \|A^\lambda(\cdot, w)\|_{L^\infty}\|\partial_{xx}\mathbf{R}(\lambda)\|_{L^2}\\
\le\ & K_5 \|w\|_{L^\infty}\|\partial_{xx}\mathbf{R}(\lambda)\|_{L^2},
\end{aligned}$$

$$\begin{aligned}
\|B^\lambda(\cdot, w)(\partial_x \mathbf{R}(\lambda), \partial_x \mathbf{R}(\lambda))\|_{L^2} \le\ & \|B^\lambda(\cdot, w)\|_{L^\infty}\|\partial_x \mathbf{R}(\lambda)\|^2_{L^4}\\
\le\ & K_5 \|w\|_{L^\infty}\|\mathbf{R}(\lambda)\|_{L^\infty}\|\mathbf{R}(\lambda)\|_{H^2}
\end{aligned}$$

$$\begin{aligned}
\|C^\lambda(\cdot, w)(\partial_x \mathbf{R}(\lambda))\|_{L^2} \le\ & \|C^\lambda(\cdot, w)\|_{L^\infty}\|\partial_x \mathbf{R}(\lambda)\|_{L^2}\\
\le\ & K_5 \|w\|_{L^\infty}\|\partial_x \mathbf{R}(\lambda)\|_{L^2}
\end{aligned}$$

$$\|D^\lambda(\cdot, w)\|_{L^2} \le \quad K_5(\|\mathbf{R}(\lambda)\|_{L^\infty} + \|w\|_{L^\infty})\|w\|_{L^2}.$$

The last term is estimated as $D^\lambda(x, w)$:

$$\|\mathbf{G}(\lambda, x, w)\|_{L^2} \le K_5 \|w\|_{L^2}$$

Using the previous estimates, (6.6) and (6.7), we conclude the proof of Proposition 6.1.

□

By Proposition 6.1, since we assumed that $|\lambda| \leq \eta_1$, Equation (6.8) yields that while $\langle L_1 w|w\rangle^{\frac{1}{2}} \leq \eta_0$, then

$$\frac{1}{2}\frac{d}{dt}\langle L_1 w|w\rangle + \alpha\|L_1 w\|_{L^2}^2 \leq k_1 \left(\langle L_1 w|w\rangle^{\frac{1}{2}} + |\lambda|\right) \|L_1 w\|_{L^2}^2,$$

that is:

$$\frac{1}{2}\frac{d}{dt}\langle L_1 w|w\rangle + \|L_1 w\|_{L^2}^2 \left(\alpha - k_1 \left(\langle L_1 w|w\rangle^{\frac{1}{2}} + |\lambda|\right)\right) \leq 0.$$

We set

$$h_{max} = \min\{\eta_1, \frac{\alpha}{2k_1}\}.$$

We assume that $|\lambda| \leq h_{max}$. So, while $\langle L_1 w|w\rangle^{\frac{1}{2}} \leq \eta_0$, we have:

$$\frac{1}{2}\frac{d}{dt}\langle L_1 w|w\rangle + \|L_1 w\|_{L^2}^2 \left(\frac{\alpha}{2} - k_1\langle L_1 w|w\rangle^{\frac{1}{2}}\right) \leq 0.$$

Setting $\eta_2 = \min\{\eta_0, \frac{\alpha}{4k_1}\}$, we remark that while $\langle L_1 w|w\rangle^{\frac{1}{2}} \leq \eta_2$, then

$$\frac{1}{2}\frac{d}{dt}\langle L_1 w|w\rangle + \frac{\alpha}{4}\|L_1 w\|_{L^2}^2 \leq 0. \tag{6.11}$$

We prove as in Section 3 that if $\langle L_1 w(0)|w(0)\rangle^{\frac{1}{2}} \leq \frac{\eta_2}{2}$, then for all $t \geq 0$, $\langle L_1 w(t)|w(t)\rangle^{\frac{1}{2}}$ remains less than η_2 so that Equation (6.11) remains valid for all time, and we conclude the proof of stability as in Section 3. □

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