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EXPLICIT EXPRESSIONS RELATED TO DEGENERATE CAUCHY NUMBERS AND THEIR GENERATING FUNCTION

FENG QI, AI-QI LIU, AND DONGKYU LIM

ABSTRACT. In the paper, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, the authors establish an explicit expression for degenerate Cauchy numbers and find explicit, meaningful, and significant expressions for coefficients in a family of nonlinear differential equations for the generating function of degenerate Cauchy numbers.

1. MOTIVATIONS AND MAIN RESULTS

It is well known [4, 7, 8, 34, 39] that the Cauchy numbers of the first kind C_n can be generated by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}.$$

In [2], degenerate Cauchy numbers $C_n(\lambda)$ were defined by

$$\frac{\lambda[e^{[(1+t)^\lambda-1]/\lambda} - 1]}{(1+t)^\lambda - 1} = \sum_{n=0}^{\infty} C_n(\lambda) \frac{t^n}{n!}.$$

Because

$$\lim_{\lambda \rightarrow 0} \frac{(1+t)^\lambda - 1}{\lambda} = \ln(1+t) \quad \text{or} \quad \lim_{\lambda \rightarrow 0} e^{[(1+t)^\lambda-1]/\lambda} = 1+t,$$

it follows that

$$\lim_{\lambda \rightarrow 0} C_n(\lambda) = C_n, \quad n \geq 0. \tag{1}$$

In [2, Theorem 2.1], it was established that the family of nonlinear differential equations

$$(1+t)^n [(1+t)^\lambda - 1]^n F_\lambda^{(n)}(t) = F_\lambda(t) \sum_{i=1}^{2n} a_i(n, \lambda) (1+t)^{i\lambda} + \sum_{i=1}^{2n-1} b_i(n, \lambda) (1+t)^{i\lambda} \tag{2}$$

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for $n \in \mathbb{N}$ has the same solution

$$F_\lambda(t) = \frac{e^{[(1+t)^\lambda - 1]/\lambda} - 1}{(1+t)^\lambda - 1}, \quad (3)$$

where $a_i(n, \lambda)$ for $1 \leq i \leq 2n$ and $b_i(n, \lambda)$ for $1 \leq i \leq 2n-1$ are uniquely determined by

$$\begin{aligned} a_1(n, \lambda) &= -\frac{1}{\lambda} \langle n-1-\lambda \rangle_{n+1}, & a_2(n, \lambda) &= \langle n-1-2\lambda \rangle_{n-1} \\ &- \frac{1}{\lambda} \sum_{i=0}^{n-2} [\lambda - (\lambda+1)(n-i)] \langle n-i-2-\lambda \rangle_{n-i} \langle n-1-2\lambda \rangle_i, \\ a_i(n, \lambda) &= [(i-1)\lambda - (\lambda+1)n] a_i(n-1, \lambda) \\ &+ a_{i-2}(n-1, \lambda) + (n-1-i\lambda) a_i(n-1, \lambda), \quad 3 \leq i \leq 2n-2, \\ a_{2n-1}(n, \lambda) &= \frac{1}{2} n [(\lambda-1)(n-1) - 2(\lambda+1)], \\ a_{2n}(n, \lambda) &= 1, \quad b_1(n, \lambda) = \langle n-1-\lambda \rangle_{n-1}, \\ b_i(n, \lambda) &= [(i-1)\lambda - (\lambda+1)(n-1)] b_{i-1}(n-1, \lambda) \\ &+ a_{i-1}(n-1, \lambda) + (n-1-i\lambda) b_i(n-1, \lambda), \quad 2 \leq i \leq 2n-3, \\ b_{2n-2}(n, \lambda) &= (\lambda-1) \binom{n-1}{2} - 2(n-1) - \lambda, \quad b_{2n-1}(n, \lambda) = 1 \end{aligned}$$

in terms of the falling factorials

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1; \\ 1, & n = 0. \end{cases}$$

It is clear that the generating function $F_\lambda(t)$ defined by (3) satisfies

$$\lim_{\lambda \rightarrow 0} [\lambda F_\lambda(t)] = \frac{t}{\ln(1+t)}.$$

It is obvious that

- (1) the above expressions for $a_i(n, \lambda)$ and $b_i(n, \lambda)$ are recursive and can not be computed easily;
- (2) the original proof of [2, Theorem 2.1] is inductive, recursive, and long;
- (3) there was no any application given in [2].

In this paper, by virtue of the Faà di Bruno formula (9) and two identities (10) and (11) for the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, we will establish an explicit expression (4) for degenerate Cauchy numbers $C_n(\lambda)$ and find explicit, meaningful, and significant expressions (7) and (8) for coefficients $a_i(n, \lambda)$ and $b_i(n, \lambda)$ in the family of nonlinear differential equations (2) related to the generating function $F_\lambda(t)$ of degenerate Cauchy numbers $C_n(\lambda)$.

Our main results can be stated as the following theorems.

Theorem 1. *For $n \geq 0$, degenerate Cauchy numbers $C_n(\lambda)$ and the Cauchy numbers C_n can be explicitly and respectively computed by*

$$C_n(\lambda) = \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \ell \lambda \rangle_n \quad (4)$$

and

$$C_n = \sum_{k=0}^n \frac{s(n, k)}{k+1}. \quad (5)$$

Theorem 2. For $n \in \mathbb{N}$, the generating function $F_\lambda(t)$ and its derivatives of degenerate Cauchy numbers $C_n(\lambda)$ satisfy

$$F_\lambda^{(n)}(t) = \frac{F_\lambda(t)}{(1+t)^n [1 - (1+t)^\lambda]^n} \sum_{i=1}^{2n} \alpha_i(n, \lambda) (1+t)^{i\lambda} + \sum_{i=1}^{2n-1} \beta_i(n, \lambda) (1+t)^{i\lambda} \quad (6)$$

with

$$\alpha_i(n, \lambda) = \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n}} (-1)^m A_k(n, \lambda) \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \quad (7)$$

for $1 \leq i \leq 2n$ and

$$\beta_i(n, \lambda) = \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n-1}} (-1)^{m+1} A_k(n, \lambda) \sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \quad (8)$$

for $1 \leq i \leq 2n-1$, where

$$A_k(n, \lambda) = \frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).$$

2. LEMMAS

In order to obtain our main results, we need the following lemmas.

Lemma 1 ([1, pp. 134 and 139]). For $n \geq k \geq 0$, the Bell polynomials of the second kind, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \quad (9)$$

Lemma 2 ([1, p. 135]). For $n \geq k \geq 0$, we have

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \quad (10)$$

where a and b are any complex numbers.

Lemma 3. For $n \geq k \geq 0$ and $\lambda, \alpha \in \mathbb{C}$, we have

$$B_{n,k} \left(1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \quad (11)$$

or, equivalently,

$$B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \dots, \langle \alpha \rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \alpha \ell \rangle_n. \quad (12)$$

Proof. This explicit formula (11) was first established in [25, Remark 1] and then was applied in [16, Section 2], [17, First proof of Theorem 2], [19, Lemma 2.2], [22, Remark 6.1], [23, Lemma 4], and [32, Lemma 2.6]. The formula (12) and the equivalence were presented in [33, Theorems 2.1 and 4.1]. \square

3. PROOFS OF THEOREMS 1 AND 2

We are now in a position to prove our main results.

Proof of Theorem 1. For $n \geq 0$, applying $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$ and $f(u) = \frac{e^u - 1}{u}$ to (9) and making use of (10) and (11) in sequence arrive at

$$\begin{aligned} \frac{d^n[\lambda F_\lambda(t)]}{dt^n} &= \sum_{k=0}^n \frac{d^k}{du^k} \left(\frac{e^u - 1}{u} \right) B_{n,k} \left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \right. \\ &\quad \left. \dots, \frac{\lambda(\lambda-1) \cdots [\lambda - (n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda} \right) \\ &= \sum_{k=0}^n \frac{d^k}{du^k} \left(\sum_{\ell=1}^{\infty} \frac{u^{\ell-1}}{\ell!} \right) (1+t)^{k\lambda-n} B_{n,k}(1, \lambda-1, \dots, (\lambda-1) \cdots [\lambda - (n-k)]) \\ &= \sum_{k=0}^n \frac{d^k}{du^k} \left(\sum_{\ell=0}^{\infty} \frac{u^\ell}{(\ell+1)!} \right) (1+t)^{k\lambda-n} \\ &\quad \times B_{n,k} \left(1, \lambda \left(1 - \frac{1}{\lambda} \right), \dots, \lambda^{n-k} \left(1 - \frac{1}{\lambda} \right) \cdots \left(1 - \frac{n-k}{\lambda} \right) \right) \\ &= \sum_{k=0}^n \left[\sum_{\ell=0}^{\infty} \frac{d^k}{du^k} \left(\frac{u^\ell}{(\ell+1)!} \right) \right] (1+t)^{k\lambda-n} \\ &\quad \times \lambda^{n-k} B_{n,k} \left(1, 1 - \frac{1}{\lambda}, \dots, \left(1 - \frac{1}{\lambda} \right) \cdots \left(1 - \frac{n-k}{\lambda} \right) \right) \\ &= \sum_{k=0}^n \left[\sum_{\ell=k}^{\infty} \frac{\ell!}{(\ell-k)!} \frac{u^{\ell-k}}{(\ell+1)!} \right] (1+t)^{k\lambda-n} \lambda^{n-k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} \left(\ell - \frac{q}{\lambda} \right) \\ &= \sum_{k=0}^n \left[\sum_{\ell=k}^{\infty} \frac{1}{(\ell-k)!} \frac{u^{\ell-k}}{\ell+1} \right] (1+t)^{k\lambda-n} \frac{1}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \\ &\rightarrow \sum_{k=0}^n \frac{1}{k+1} \frac{1}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \end{aligned}$$

$$= \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \ell \lambda \rangle_n$$

as $t \rightarrow 0$ and, consequently, $u \rightarrow 0$. This implies that

$$C_n(\lambda) = \lambda \lim_{t \rightarrow 0} F_\lambda^{(n)}(t) = \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \ell \lambda \rangle_n.$$

The explicit formula (4) is thus proved.

It is well known [1, Theorem A] that the Stirling numbers of the first kind $s(n, k)$ can be generated by

$$\langle x \rangle_n = \sum_{k=0}^n s(n, k) x^k.$$

Hence, by the L'Hôspital rule, we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \ell \lambda \rangle_n &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{m=0}^n s(n, m) (\ell \lambda)^m \\ &= \frac{1}{k!} \lim_{\lambda \rightarrow 0} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{m=0}^n s(n, m) \langle m \rangle_k \ell^k (\ell \lambda)^{m-k} \\ &= \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} s(n, k) \langle k \rangle_k \ell^k = s(n, k) \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \ell^k = (-1)^k k! s(n, k). \end{aligned}$$

Combining this with (1) and (4) gives

$$C_n = \lim_{\lambda \rightarrow 0} C_n(\lambda) = \sum_{k=0}^n \frac{(-1)^k}{(k+1)!} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \ell \lambda \rangle_n = \sum_{k=0}^n \frac{s(n, k)}{k+1}.$$

The explicit formula (5) follows. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. For $n \in \mathbb{N}$, applying $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$ and $f(u) = \frac{e^u - 1}{u}$ to (9) and making use of (10) and (11) in sequence arrive at

$$\begin{aligned} \frac{d^n [\lambda F_\lambda(t)]}{d t^n} &= \sum_{k=1}^n \frac{d^k}{d u^k} \left(\frac{e^u - 1}{u} \right) B_{n,k} \left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \right. \\ &\quad \left. \dots, \frac{\lambda(\lambda-1) \cdots [\lambda - (n-k)] (1+t)^{\lambda-(n-k+1)}}{\lambda} \right) \\ &= \sum_{k=1}^n \left[\sum_{\ell=0}^k \binom{k}{\ell} (e^u - 1)^{(\ell)} \left(\frac{1}{u} \right)^{(k-\ell)} \right] (1+t)^{k\lambda-n} \\ &\quad \times B_{n,k}(1, \lambda-1, \dots, (\lambda-1) \cdots [\lambda - (n-k)]) \\ &= \sum_{k=1}^n \left[\frac{(-1)^k k! (e^u - 1)}{u^{k+1}} + \sum_{\ell=1}^k \binom{k}{\ell} e^u \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell+1}} \right] (1+t)^{k\lambda-n} \\ &\quad \times B_{n,k} \left(1, \lambda \left(1 - \frac{1}{\lambda} \right), \dots, \lambda^{n-k} \left(1 - \frac{1}{\lambda} \right) \cdots \left(1 - \frac{n-k}{\lambda} \right) \right) \\ &= \sum_{k=1}^n \left[\lambda F_\lambda(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell}} + \sum_{\ell=1}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell+1}} \right] (1+t)^{k\lambda-n} \end{aligned}$$

$$\begin{aligned}
& \times \lambda^{n-k} B_{n,k} \left(1, 1 - \frac{1}{\lambda}, \dots, \left(1 - \frac{1}{\lambda} \right) \cdots \left(1 - \frac{n-k}{\lambda} \right) \right) \\
& = \sum_{k=1}^n \left[\lambda F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} \lambda^{k-\ell} (k-\ell)!}{[(1+t)^{\lambda} - 1]^{k-\ell}} + \sum_{\ell=1}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} \lambda^{k-\ell+1} (k-\ell)!}{[(1+t)^{\lambda} - 1]^{k-\ell+1}} \right] \\
& \quad \times (1+t)^{k\lambda-n} \lambda^{n-k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} \left(\ell - \frac{q}{\lambda} \right) \\
& = \frac{1}{(1+t)^n} \sum_{k=1}^n \left[\lambda F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{\ell} \lambda^{\ell} \ell!}{[(1+t)^{\lambda} - 1]^{\ell}} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \frac{(-1)^{\ell} \lambda^{\ell+1} \ell!}{[(1+t)^{\lambda} - 1]^{\ell+1}} \right] \\
& \quad \times \frac{(1+t)^{k\lambda}}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
(1+t)^n [(1+t)^{\lambda} - 1]^n F_{\lambda}^{(n)}(t) & = \sum_{k=1}^n \left[\frac{(1+t)^{k\lambda}}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] \\
& \times \left[F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} (-1)^{\ell} \lambda^{\ell} \ell! [(1+t)^{\lambda} - 1]^{n-\ell} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} (-1)^{\ell} \lambda^{\ell} \ell! [(1+t)^{\lambda} - 1]^{n-\ell-1} \right] \\
& = \sum_{k=1}^n \left[\frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] (1+t)^{k\lambda} \\
& \quad \times \left[F_{\lambda}(t) \sum_{\ell=0}^k \frac{(-1)^{\ell} \lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} (1+t)^{m\lambda} (-1)^{n-\ell-m} \right. \\
& \quad \left. + \sum_{\ell=0}^{k-1} \frac{(-1)^{\ell} \lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} \binom{n-\ell-1}{m} (1+t)^{m\lambda} (-1)^{n-\ell-1-m} \right] \\
& = (-1)^n \sum_{k=1}^n \left[\frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] (1+t)^{k\lambda} \\
& \quad \times \left[F_{\lambda}(t) \sum_{\ell=0}^k \frac{\lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{m\lambda} \right. \\
& \quad \left. + \sum_{\ell=0}^{k-1} \frac{\lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{m\lambda} \right] \\
& = (-1)^n F_{\lambda}(t) \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
& \quad + (-1)^n \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda}.
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^n (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) (1+t)^{k\lambda} \sum_{m=0}^n (-1)^m \left[\sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{m\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) (1+t)^{k\lambda} \sum_{m=0}^n (-1)^m \left[\sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{m\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^n (-1)^m \left[\sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{i=1}^{2n} \left[\sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n}} (-1)^m A_k(n, \lambda) \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{i\lambda}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[\sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[\sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{i=1}^{2n-1} \left[\sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n-1}} (-1)^{m+1} A_k(n, \lambda) \sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{i\lambda},
\end{aligned}$$

where an empty sum is understood to be 0 and $\binom{p}{q} = 0$ for $q > p \geq 0$, the equation (6) and the formulas (7) and 8 are thus proved. The proof of Theorem 2 is complete. \square

4. REMARKS

Finally, we list several remarks on our main results and closely related things.

Remark 1. Comparing (6) with (2) reveals that

$$a_i(n, \lambda) = (-1)^n \alpha_i(n, \lambda) \quad \text{and} \quad b_i(n, \lambda) = (-1)^n \beta_i(n, \lambda).$$

Remark 2. It is easy to see that explicit expressions (7) and (8) for $\alpha_i(n, \lambda)$ and $\beta_i(n, \lambda)$ are more meaningful and more significant than those in [2, Theorem 2.1] for $a_i(n, \lambda)$ and $b_i(n, \lambda)$ mentioned above.

Remark 3. The formula (5) was derived in [3] and mentioned in [4, 8].

Remark 4. Per requests of anonymous referees, the preprint [17] is split into and simplified as two formally published papers [18, 29].

Remark 5. The motivations in the papers [5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 24, 25, 26, 27, 30, 31, 32, 38, 35, 36, 37, 40, 41, 42] are same as the one in this paper.

Remark 6. This paper is a slightly revised version of the preprint [28].

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