



**HAL**  
open science

# EXPLICIT EXPRESSIONS RELATED TO DEGENERATE CAUCHY NUMBERS AND THEIR GENERATING FUNCTION

Feng Qi, Dongkyu Lim, Ai-Qi Liu

► **To cite this version:**

Feng Qi, Dongkyu Lim, Ai-Qi Liu. EXPLICIT EXPRESSIONS RELATED TO DEGENERATE CAUCHY NUMBERS AND THEIR GENERATING FUNCTION: Explicit expressions related to degenerate Cauchy numbers. 2018. hal-01725045v1

**HAL Id: hal-01725045**

**<https://hal.science/hal-01725045v1>**

Preprint submitted on 7 Mar 2018 (v1), last revised 19 Jan 2020 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# EXPLICIT EXPRESSIONS RELATED TO DEGENERATE CAUCHY NUMBERS AND THEIR GENERATING FUNCTION

FENG QI, DONGKYU LIM, AND AI-QI LIU

ABSTRACT. In the paper, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, the authors establish an explicit expression for degenerate Cauchy numbers and find explicit, meaningful, and significant expressions for coefficients in a family of nonlinear differential equations for the generating function of degenerate Cauchy numbers.

## 1. MOTIVATIONS AND MAIN RESULTS

It is well known [3, 6, 7, 31, 35] that the Cauchy numbers of the first kind  $C_n$  can be generated by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}.$$

In [2], degenerate Cauchy numbers  $C_n(\lambda)$  were defined by

$$\frac{\lambda[e^{[(1+t)^\lambda-1]/\lambda} - 1]}{(1+t)^\lambda - 1} = \sum_{n=0}^{\infty} C_n(\lambda) \frac{t^n}{n!}$$

which is due to the limit

$$\lim_{\lambda \rightarrow 0} \frac{(1+t)^\lambda - 1}{\lambda} = \ln(1+t) \quad \text{or} \quad \lim_{\lambda \rightarrow 0} e^{[(1+t)^\lambda-1]/\lambda} = 1+t.$$

In [2, Theorem 2.1], it was established inductively and recursively that the family of nonlinear differential equations

$$\begin{aligned} (1+t)^n [(1+t)^\lambda - 1]^n F_\lambda^{(n)}(t) \\ = \sum_{i=1}^{2n} a_i(n, \lambda) (1+t)^{i\lambda} F_\lambda(t) + \sum_{i=1}^{2n-1} b_i(n, \lambda) (1+t)^{i\lambda} \end{aligned} \quad (1)$$

for  $n \in \mathbb{N}$  has the same solution

$$F_\lambda(t) = \frac{e^{[(1+t)^\lambda-1]/\lambda} - 1}{(1+t)^\lambda - 1}, \quad (2)$$

where  $a_i(n, \lambda)$  for  $1 \leq i \leq 2n$  and  $b_i(n, \lambda)$  for  $1 \leq i \leq 2n-1$  are uniquely determined by

$$a_1(n, \lambda) = -\frac{1}{\lambda} \langle n-1-\lambda \rangle_{n+1}, \quad a_2(n, \lambda) = \langle n-1-2\lambda \rangle_{n-1}$$

---

2010 *Mathematics Subject Classification.* Primary 11B68; Secondary 11B83, 33B10, 33B99, 34A05, 34A34.

*Key words and phrases.* explicit expression, degenerate Cauchy number, coefficient, nonlinear differential equation, generating function, Bell polynomial of the second kind, Faà di Bruno formula. This paper was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

$$\begin{aligned}
& -\frac{1}{\lambda} \sum_{i=0}^{n-2} [\lambda - (\lambda + 1)(n - i)] \langle n - i - 2 - \lambda \rangle_{n-i} \langle n - 1 - 2\lambda \rangle_i, \\
& a_i(n, \lambda) = [(i - 1)\lambda - (\lambda + 1)n] a_i(n - 1, \lambda) \\
& + a_{i-2}(n - 1, \lambda) + (n - 1 - i\lambda) a_i(n - 1, \lambda), \quad 3 \leq i \leq 2n - 2, \\
& a_{2n-1}(n, \lambda) = \frac{1}{2} n [(\lambda - 1)(n - 1) - 2(\lambda + 1)], \\
& a_{2n}(n, \lambda) = 1, \quad b_1(n, \lambda) = \langle n - 1 - \lambda \rangle_{n-1}, \\
& b_i(n, \lambda) = [(i - 1)\lambda - (\lambda + 1)(n - 1)] b_{i-1}(n - 1, \lambda) \\
& + a_{i-1}(n - 1, \lambda) + (n - 1 - i\lambda) b_i(n - 1, \lambda), \quad 2 \leq i \leq 2n - 3, \\
& b_{2n-2}(n, \lambda) = (\lambda - 1) \binom{n-1}{2} - 2(n-1) - \lambda, \quad b_{2n-1}(n, \lambda) = 1
\end{aligned}$$

in terms of the falling factorials

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x - k) = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1; \\ 1, & n = 0. \end{cases}$$

It is clear that the generating function  $F_\lambda(t)$  defined by (2) satisfies

$$\lim_{\lambda \rightarrow 0} [\lambda F_\lambda(t)] = \frac{t}{\ln(1+t)}.$$

It is obvious that

- (1) the above expressions for  $a_i(n, \lambda)$  and  $b_i(n, \lambda)$  are too complicated to be remembered, understood, and computed easily;
- (2) the original proof of [2, Theorem 2.1] is very long and tedious,
- (3) there was no any application given in [2].

In this paper, by virtue of the Faà di Bruno formula (7) and two identities (8) and (9) for the Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ , the authors establish an explicit expression (3) for degenerate Cauchy numbers  $C_n(\lambda)$  and find explicit, meaningful, and significant expressions (5) and (6) for coefficients  $a_i(n, \lambda)$  and  $b_i(n, \lambda)$  in the family of nonlinear differential equations (1) for the generating function  $F_\lambda(t)$  of degenerate Cauchy numbers  $C_n(\lambda)$ .

Our main results can be stated as the following theorems.

**Theorem 1.** *For  $n \geq 0$ , degenerate Cauchy numbers  $C_n(\lambda)$  can be explicitly computed by*

$$C_n(\lambda) = \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q), \quad (3)$$

where an empty product is understood to be 1.

**Theorem 2.** *For  $n \in \mathbb{N}$ , the generating function  $F_\lambda(t)$  and its derivatives of degenerate Cauchy numbers  $C_n(\lambda)$  satisfy*

$$(1+t)^n [1 - (1+t)^\lambda]^n F_\lambda^{(n)}(t) = F_\lambda(t) \sum_{i=1}^{2n} \alpha_i(n, \lambda) (1+t)^{i\lambda} + \sum_{i=1}^{2n-1} \beta_i(n, \lambda) (1+t)^{i\lambda} \quad (4)$$

with

$$\alpha_i(n, \lambda) = \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n}} (-1)^m A_k(n, \lambda) \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \quad (5)$$

for  $1 \leq i \leq 2n$  and

$$\beta_i(n, \lambda) = \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n-1}} (-1)^{m+1} A_k(n, \lambda) \sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \quad (6)$$

for  $1 \leq i \leq 2n-1$ , where

$$A_k(n, \lambda) = \frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).$$

## 2. LEMMAS

In order to obtain our main results, we need the following lemmas.

**Lemma 1** ([1, pp. 134 and 139]). *For  $n \geq k \geq 0$ , the Bell polynomials of the second kind, denoted by  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ , are defined by*

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \quad (7)$$

**Lemma 2** ([1, p. 135]). *For  $n \geq k \geq 0$ , we have*

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \quad (8)$$

where  $a$  and  $b$  are any complex numbers.

**Lemma 3.** *For  $n \geq k \geq 0$ , we have*

$$\begin{aligned} B_{n,k} \left( 1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right) \\ = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda). \end{aligned} \quad (9)$$

*Proof.* This explicit formula was first established in [24, Remark 1] and then was applied in [1, p. 135], [17, First proof of Theorem 2], [18, Lemma 2.2], [21, Remark 6.1], [22, Lemma 4], and [29, Lemma 2.6].  $\square$

## 3. PROOFS OF THEOREMS 1 AND 2

We are now in a position to prove our main results.

*Proof of Theorem 1.* For  $n \geq 0$ , applying  $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$  and  $f(u) = \frac{e^u - 1}{u}$  to (7) and making use of (8) and (9) in sequence arrive at

$$\begin{aligned}
\frac{d^n[\lambda F_\lambda(t)]}{d t^n} &= \sum_{k=0}^n \frac{d^k}{d u^k} \left( \frac{e^u - 1}{u} \right) B_{n,k} \left( \frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \right. \\
&\quad \left. \dots, \frac{\lambda(\lambda-1) \cdots [\lambda - (n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda} \right) \\
&= \sum_{k=0}^n \frac{d^k}{d u^k} \left( \sum_{\ell=1}^{\infty} \frac{u^{\ell-1}}{\ell!} \right) (1+t)^{k\lambda-n} B_{n,k}(1, \lambda-1, \dots, (\lambda-1) \cdots [\lambda - (n-k)]) \\
&= \sum_{k=0}^n \frac{d^k}{d u^k} \left( \sum_{\ell=0}^{\infty} \frac{u^\ell}{(\ell+1)!} \right) (1+t)^{k\lambda-n} \\
&\quad \times B_{n,k} \left( 1, \lambda \left( 1 - \frac{1}{\lambda} \right), \dots, \lambda^{n-k} \left( 1 - \frac{1}{\lambda} \right) \cdots \left( 1 - \frac{n-k}{\lambda} \right) \right) \\
&= \sum_{k=0}^n \left[ \sum_{\ell=0}^{\infty} \frac{d^k}{d u^k} \left( \frac{u^\ell}{(\ell+1)!} \right) \right] (1+t)^{k\lambda-n} \\
&\quad \times \lambda^{n-k} B_{n,k} \left( 1, 1 - \frac{1}{\lambda}, \dots, \left( 1 - \frac{1}{\lambda} \right) \cdots \left( 1 - \frac{n-k}{\lambda} \right) \right) \\
&= \sum_{k=0}^n \left[ \sum_{\ell=k}^{\infty} \frac{\ell!}{(\ell-k)! (\ell+1)!} \frac{u^{\ell-k}}{(\ell+1)!} \right] (1+t)^{k\lambda-n} \lambda^{n-k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} \left( \ell - \frac{q}{\lambda} \right) \\
&= \sum_{k=0}^n \left[ \sum_{\ell=k}^{\infty} \frac{1}{(\ell-k)! (\ell+1)} \frac{u^{\ell-k}}{(\ell+1)!} \right] (1+t)^{k\lambda-n} \frac{1}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \\
&\rightarrow \sum_{k=0}^n \frac{1}{k+1} \frac{1}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \\
&= \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q)
\end{aligned}$$

as  $t \rightarrow 0$  and, consequently,  $u \rightarrow 0$ . This implies that

$$C_n(\lambda) = \lambda \lim_{t \rightarrow 0} F_\lambda^{(n)}(t) = \sum_{k=0}^n \frac{(-1)^k}{(k+1)! \lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).$$

The explicit formula (3) is thus proved. The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* For  $n \in \mathbb{N}$ , applying  $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$  and  $f(u) = \frac{e^u - 1}{u}$  to (7) and making use of (8) and (9) in sequence arrive at

$$\frac{d^n[\lambda F_\lambda(t)]}{d t^n} = \sum_{k=1}^n \frac{d^k}{d u^k} \left( \frac{e^u - 1}{u} \right) B_{n,k} \left( \frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \right.$$

$$\begin{aligned}
& \dots, \frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda} \\
& = \sum_{k=1}^n \left[ \sum_{\ell=0}^k \binom{k}{\ell} (e^u - 1)^{(\ell)} \left(\frac{1}{u}\right)^{(k-\ell)} \right] (1+t)^{k\lambda-n} \\
& \quad \times \mathbf{B}_{n,k}(1, \lambda-1, \dots, (\lambda-1)\cdots[\lambda-(n-k)]) \\
& = \sum_{k=1}^n \left[ \frac{(-1)^k k! (e^u - 1)}{u^{k+1}} + \sum_{\ell=1}^k \binom{k}{\ell} e^u \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell+1}} \right] (1+t)^{k\lambda-n} \\
& \quad \times \mathbf{B}_{n,k} \left( 1, \lambda \left(1 - \frac{1}{\lambda}\right), \dots, \lambda^{n-k} \left(1 - \frac{1}{\lambda}\right) \cdots \left(1 - \frac{n-k}{\lambda}\right) \right) \\
& = \sum_{k=1}^n \left[ \lambda F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell}} + \sum_{\ell=1}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} (k-\ell)!}{u^{k-\ell+1}} \right] (1+t)^{k\lambda-n} \\
& \quad \times \lambda^{n-k} \mathbf{B}_{n,k} \left( 1, 1 - \frac{1}{\lambda}, \dots, \left(1 - \frac{1}{\lambda}\right) \cdots \left(1 - \frac{n-k}{\lambda}\right) \right) \\
& = \sum_{k=1}^n \left[ \lambda F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} \lambda^{k-\ell} (k-\ell)!}{[(1+t)^{\lambda} - 1]^{k-\ell}} + \sum_{\ell=1}^k \binom{k}{\ell} \frac{(-1)^{k-\ell} \lambda^{k-\ell+1} (k-\ell)!}{[(1+t)^{\lambda} - 1]^{k-\ell+1}} \right] \\
& \quad \times (1+t)^{k\lambda-n} \lambda^{n-k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} \left(\ell - \frac{q}{\lambda}\right) \\
& = \frac{1}{(1+t)^n} \sum_{k=1}^n \left[ \lambda F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^{\ell} \lambda^{\ell} \ell!}{[(1+t)^{\lambda} - 1]^{\ell}} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \frac{(-1)^{\ell} \lambda^{\ell+1} \ell!}{[(1+t)^{\lambda} - 1]^{\ell+1}} \right] \\
& \quad \times \frac{(1+t)^{k\lambda}}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
(1+t)^n [(1+t)^{\lambda} - 1]^n F_{\lambda}^{(n)}(t) & = \sum_{k=1}^n \left[ \frac{(1+t)^{k\lambda}}{\lambda^k} \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] \\
& \quad \times \left[ F_{\lambda}(t) \sum_{\ell=0}^k \binom{k}{\ell} (-1)^{\ell} \lambda^{\ell} \ell! [(1+t)^{\lambda} - 1]^{n-\ell} \right. \\
& \quad \left. + \sum_{\ell=0}^{k-1} \binom{k}{\ell} (-1)^{\ell} \lambda^{\ell} \ell! [(1+t)^{\lambda} - 1]^{n-\ell-1} \right] \\
& = \sum_{k=1}^n \left[ \frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] (1+t)^{k\lambda} \\
& \quad \times \left[ F_{\lambda}(t) \sum_{\ell=0}^k \frac{(-1)^{\ell} \lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} (1+t)^{m\lambda} (-1)^{n-\ell-m} \right. \\
& \quad \left. + \sum_{\ell=0}^{k-1} \frac{(-1)^{\ell} \lambda^{\ell}}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} \binom{n-\ell-1}{m} (1+t)^{m\lambda} (-1)^{n-\ell-1-m} \right]
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n \sum_{k=1}^n \left[ \frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q) \right] (1+t)^{k\lambda} \\
&\quad \times \left[ F_\lambda(t) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{m\lambda} \right. \\
&\quad \left. + \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{m\lambda} \right] \\
&= (-1)^n F_\lambda(t) \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
&+ (-1)^n \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda}.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell} (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^n (-1)^m \binom{n-\ell}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) (1+t)^{k\lambda} \sum_{m=0}^n (-1)^m \left[ \sum_{\ell=0}^k \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{m\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) (1+t)^{k\lambda} \sum_{m=0}^n (-1)^m \left[ \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{m\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^n (-1)^m \left[ \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{i=1}^{2n} \left[ \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n}} (-1)^m A_k(n, \lambda) \sum_{\ell=0}^{\min\{n-m, k\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell}{m} \right] (1+t)^{i\lambda}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-\ell-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \sum_{m=0}^{n-1} (-1)^{m+1} \binom{n-\ell-1}{m} (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[ \sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^n A_k(n, \lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[ \sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda}
\end{aligned}$$

$$= \sum_{i=1}^{2n-1} \left[ \sum_{\substack{k+m=i \\ 1 \leq k \leq n \\ 0 \leq m \leq n-1}} (-1)^{m+1} A_k(n, \lambda) \sum_{\ell=0}^{\min\{k-1, n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{i\lambda},$$

where an empty sum is understood to be 0 and  $\binom{p}{q} = 0$  for  $q > p \geq 0$ , the equation (4) and the formulas (5) and 6 are thus proved. The proof of Theorem 2 is complete.  $\square$

#### 4. REMARKS

Finally, we list several remarks on our main results and closely related things.

*Remark 1.* It is easy to see that explicit expressions (5) and (6) for  $\alpha_i(n, \lambda)$  and  $\beta_i(n, \lambda)$  are more meaningful and more significant than those in [2, Theorem 2.1] for  $a_i(n, \lambda)$  and  $b_i(n, \lambda)$  mentioned above.

*Remark 2.* The motivations in the papers [4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 36, 37, 38] are same as the one in this paper.

**Acknowledgements.** The authors are thankful to Professor Taekyun Kim for his sending a link for his paper [2] through an e-mail on 28 February 2018.

#### REFERENCES

- [1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.
- [2] T. Kim, D. S. Kim, and G.-W. Jang, *Differential equations associated with degenerate Cauchy numbers*, Iran. J. Sci. Technol. Trans. Sci. **28** (2018), in press; Available online at <https://doi.org/10.1007/s40995-018-0531-y>.
- [3] F. Qi, *A new formula for the Bernoulli numbers of the second kind in terms of the Stirling numbers of the first kind*, Publ. Inst. Math. (Beograd) (N.S.) (2015), in press; Available online at <https://doi.org/10.2298/PIM150501028Q>.
- [4] F. Qi, *A simple form for coefficients in a family of nonlinear ordinary differential equations*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.13392.81928>.
- [5] F. Qi, *A simple form for coefficients in a family of ordinary differential equations related to the generating function of the Legendre polynomials*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.27365.09446>.
- [6] F. Qi, *An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind*, J. Number Theory **144** (2014), 244–255; Available online at <https://doi.org/10.1016/j.jnt.2014.05.009>.
- [7] F. Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28** (2014), no. 2, 319–327; Available online at <https://doi.org/10.2298/FIL14023190>.
- [8] F. Qi, *Explicit formulas for the convolved Fibonacci numbers*, ResearchGate Working Paper (2016), available online at <https://doi.org/10.13140/RG.2.2.36768.17927>.
- [9] F. Qi, *Notes on several families of differential equations related to the generating function for the Bernoulli numbers of the second kind*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.13291.23843>.
- [10] F. Qi, *Simple forms for coefficients in two families of ordinary differential equations*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.10692.73602>.
- [11] F. Qi, *Simplification of coefficients in two families of nonlinear ordinary differential equations*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.30196.24966>.



- [12] F. Qi, *Simplifying coefficients in a family of nonlinear ordinary differential equations*, Acta Comment. Univ. Tartu. Math. (2018), in press; ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.23328.07687>.
- [13] F. Qi, *Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Laguerre polynomials*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.13602.53448>.
- [14] F. Qi, *Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Mittag-Leffler polynomials*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.27758.31049>.
- [15] F. Qi, *Simplifying coefficients in differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials*, ResearchGate Preprint (2017), available online at <https://doi.org/10.13140/RG.2.2.19946.41921>.
- [16] F. Qi, *The inverse of a matrix and several identities related to the Catalan numbers and the Chebyshev polynomials of the second kind*, ResearchGate Presentation (2017), available online at <https://doi.org/10.13140/RG.2.2.22832.46081/1>. (Chinese)
- [17] F. Qi, V. Čerňanová, and Y. S. Semenov, *On tridiagonal determinants and the Cauchy product of central Delannoy numbers*, ResearchGate Working Paper (2016), available online at <https://doi.org/10.13140/RG.2.1.3772.6967>.
- [18] F. Qi, V. Čerňanová, X.-T. Shi, and B.-N. Guo, *Some properties of central Delannoy numbers*, J. Comput. Appl. Math. **328** (2018), 101–115; Available online at <https://doi.org/10.1016/j.cam.2017.07.013>.
- [19] F. Qi and B.-N. Guo, *A diagonal recurrence relation for the Stirling numbers of the first kind*, Appl. Anal. Discrete Math. **12** (2018), no. 1, in press; Available online at <https://doi.org/10.2298/AADM170405004Q>.
- [20] F. Qi and B.-N. Guo, *Explicit formulas and recurrence relations for higher order Eulerian polynomials*, Indag. Math. **28** (2017), no. 4, 884–891; Available online at <https://doi.org/10.1016/j.indag.2017.06.010>.
- [21] F. Qi and B.-N. Guo, *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math. **14** (2017), no. 3, Article 140, 14 pages; Available online at <https://doi.org/10.1007/s00009-017-0939-1>.
- [22] F. Qi and B.-N. Guo, *Several explicit and recursive formulas for the generalized Motzkin numbers*, Preprints **2017**, 2017030200, 11 pages; Available online at <https://doi.org/10.20944/preprints201703.0200.v1>.
- [23] F. Qi and B.-N. Guo, *Some properties of the Hermite polynomials and their squares and generating functions*, Preprints **2016**, 2016110145, 14 pages; Available online at <https://doi.org/10.20944/preprints201611.0145.v1>.
- [24] F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Preprints **2016**, 2016100043, 12 pages; Available online at <https://doi.org/10.20944/preprints201610.0043.v1>.
- [25] F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2018), in press; Available online at <https://doi.org/10.1007/s13398-017-0427-2>.
- [26] F. Qi, D. Lim, and B.-N. Guo, *Some identities relating to Eulerian polynomials and involving Stirling numbers*, Preprints **2017**, 2017080004, 10 pages; Available online at <https://doi.org/10.20944/preprints201708.0004.v1>.
- [27] F. Qi, D.-W. Niu, and B.-N. Guo, *Simplification of coefficients in differential equations associated with higher order Frobenius–Euler numbers*, Preprints **2017**, 2017080017, 7 pages; Available online at <https://doi.org/10.20944/preprints201708.0017.v1>.
- [28] F. Qi, D.-W. Niu, and B.-N. Guo, *Simplifying coefficients in differential equations associated with higher order Bernoulli numbers of the second kind*, Preprints **2017**, 2017080026, 6 pages; Available online at <https://doi.org/10.20944/preprints201708.0026.v1>.
- [29] F. Qi, D.-W. Niu, and B.-N. Guo, *Some identities for a sequence of unnamed polynomials connected with the Bell polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **112** (2018), in press; Available online at <https://doi.org/10.1007/s13398-018-0494-z>.

- [30] F. Qi, X.-L. Qin, and Y.-H. Yao, *The generating function of the Catalan numbers and lower triangular integer matrices*, Preprints **2017**, 2017110120, 12 pages; Available online at <https://doi.org/10.20944/preprints201711.0120.v1>.
- [31] F. Qi, X.-T. Shi, and F.-F. Liu, *Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers*, Acta Univ. Sapientiae Math. **8** (2016), no. 2, 282–297; Available online at <https://doi.org/10.1515/ausm-2016-0019>.
- [32] F. Qi, J.-L. Wang, and B.-N. Guo, *Notes on a family of inhomogeneous linear ordinary differential equations*, Adv. Appl. Math. Sci. **17** (2018), no. 4, 361–368.
- [33] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at <https://doi.org/10.13140/RG.2.2.28855.32166>.
- [34] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying differential equations concerning degenerate Bernoulli and Euler numbers*, Trans. A. Razmadze Math. Inst. **172** (2018), no. 1, 90–94; Available online at <https://doi.org/10.1016/j.trmi.2017.08.001>.
- [35] F. Qi and X.-J. Zhang, *An integral representation, some inequalities, and complete monotonicity of the Bernoulli numbers of the second kind*, Bull. Korean Math. Soc. **52** (2015), no. 3, 987–998; Available online at <https://doi.org/10.4134/BKMS.2015.52.3.987>.
- [36] F. Qi and J.-L. Zhao, *Some properties of the Bernoulli numbers of the second kind and their generating function*, ResearchGate Working Paper (2017), available online at <https://doi.org/10.13140/RG.2.2.13058.27848>.
- [37] F. Qi, Q. Zou, and B.-N. Guo, *Some identities and a matrix inverse related to the Chebyshev polynomials of the second kind and the Catalan numbers*, Preprints **2017**, 2017030209, 25 pages; Available online at <https://doi.org/10.20944/preprints201703.0209.v2>.
- [38] J.-L. Zhao, J.-L. Wang, and F. Qi, *Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1345–1349; Available online at <https://doi.org/10.22436/jnsa.010.04.06>.

(Qi) INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO, HENAN, 454010, CHINA; COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO, INNER MONGOLIA, 028043, CHINA; DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN, 300387, CHINA

*Email address:* [qifeng618@gmail.com](mailto:qifeng618@gmail.com), [qifeng618@hotmail.com](mailto:qifeng618@hotmail.com), [qifeng618@qq.com](mailto:qifeng618@qq.com)  
*URL:* <https://qifeng618.wordpress.com>, <http://orcid.org/0000-0001-6239-2968>

(Lim) DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON, 16419, SOUTH KOREA

*Email address:* [dgrim84@gmail.com](mailto:dgrim84@gmail.com)

(Liu) DEPARTMENT OF MATHEMATICS, SANMENXIA POLYTECHNIC, SANMENXIA, HENAN, 472000, CHINA

*Email address:* [smxptliu@hotmail.com](mailto:smxptliu@hotmail.com)