

# Integrable Toda system as a quantum approximation to the anisotropy of the mixmaster universe

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We present a regularization approach to the study of the quantum dynamics of the mixmaster universe that allows us to approximate the anisotropy potential with the explicitly integrable periodic three-particle Toda potential. This approach is based on a Gaussian weighted Weyl integral quantization. Such a procedure naturally amplifies the dynamical role of the underlying Toda system by smoothing out the three canyons of the anisotropy potential. Since the respective eigenfunctions can be explicitly constructed, our finding paves the way to a perturbative approach to the quantum mixmaster dynamics.

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## I. INTRODUCTION

A widely acknowledged description of the Universe on the approach to the cosmological singularity is given by the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [1,2], generalized and improved in [3,4]. It asserts that on the approach to the singularity the isotropy and homogeneity of space-time break down [5]. However, it also shows that at each spatial point asymptotically towards the singularity the dynamics would approximate to a spatially anisotropic but homogenous model, in particular, the Bianchi type VIII or IX [3,4], which are the most generic ones among the spatially homogeneous models of general relativity. The BKL scenario agrees with the recent numerical studies [6], which assumed no space-time symmetries and used convenient scale-invariant Hubble normalized variables of [3] that reflect the general scale invariance of Einstein equations.

The classical dynamics of the Bianchi IX model, or mixmaster universe, was studied by Misner in the canonical formalism [2]. It involves a Hamiltonian that is formally identical to that of a particle moving in the three-dimensional Minkowski space-time in a time-dependent, exponentially steep, and triangle-like potential. The asymptotic classical dynamics of the model is usually expressed by means of the so-called Kasner map that consists of an infinite sequence of Kasner universes [1–4]. At each transition between two

consecutive Kasner epochs, the Misner particle bounces against the steep potential. Many mathematical studies have been devoted to the classical evolution of the mixmaster universe and have led to some important results on the asymptotic behavior, the nonintegrability, or the chaotic behavior (see, e.g., [3,4,7,8]). On the other hand, the quantum dynamics of mixmaster remains poorly understood despite many interesting studies (see, e.g., [2,9–11] or a review [12]). A recent result [13] shows that the problem can be considerably simplified if one uses the so-called vibronic approach to the coupling between spatial (i.e., anisotropic) and temporal (i.e., isotropic) variables. The difficulty now lies in the elaborate spatial dependence of the anisotropy potential and the fact that solving the quantum dynamics requires the knowledge of the eigenstates and the eigenvalues for the motion in this potential. The solution to the eigenvalue problem seems to have been so far approximated by means of the harmonic and the steep-wall approximation to the anisotropy potential (see, e.g., [14] for their proper derivation). They correspond to the large volume and low anisotropy excitation level or the small volume and high anisotropy excitation level, respectively. However, it was shown [13,15] that the existence of a quantum bounce, a generic prediction of successful quantum models, induces a nonadiabatic evolution to which neither of these approximations is applicable.

It seems therefore crucial to find a more suitable approximation that can be valid for intermediate levels of anisotropy excitations.

In this paper we present an alternative approach to the quantum mixmaster dynamics within which we derive and justify an appealing quantum approximation to the anisotropy potential. We employ the so-called Weyl-Heisenberg (W-H) covariant integral quantization [16–18], which is a kind of weighted Weyl quantization. It is a general procedure that in some circumstances provides conceptually better justified quantization prescriptions than the familiar “canonical prescription,” although it includes the latter as a particular case. While respecting the Weyl-Heisenberg symmetry, our procedure deforms the familiar canonical quantization by introducing a weight function on the classical phase space. In the present case, the weight is a product of Gaussians whose widths label a family of quantum models. The proposed quantization naturally introduces a small perturbation parameter into the model, which emphasizes the role of the Toda potential as a manageable approximation to the anisotropy (mixmaster) potential.

In the simplest case Toda systems consist of a finite chain of  $N$  equal-mass particles with interaction varying exponentially with their separation. They were originally introduced to study analytical solutions for the motion of nonlinear integrable chains [19]. The three-particle Toda lattice is the simplest nontrivial integrable nonlinear chain whose integrability was proved a long time ago [20–22]. On the quantum level the wave functions of the corresponding Schrödinger equations were rigorously studied [23] through the symmetries of the system. In this approach, the Toda system describes slightly perturbed harmonic chains in the small energy limit, whereas in the high energy limit it becomes equilateral triangular billiard [24]. Those low and high energy limits closely resemble the analogous limits of the anisotropic part of the mixmaster Hamiltonian, recently discussed in [14].

On the classical level, the similarity between the anisotropy potential and the Toda potential was used to introduce the so-called disturbed Toda lattices [25] that include the anisotropy potential and the Toda one as special cases. Nevertheless, due to the integrability of the Toda lattices this similarity seems too little to be useful in the context of the classical dynamics (though see [26]). On the other hand, it has never been shown justified and exploited on the quantum level. In the present paper we provide a link between the mixmaster anisotropy and the exact Toda potentials, which is valid for quantum models. Since the eigenvalue problem for the Toda lattice can be solved, the proposed approximation should be very useful in future investigations of the quantum dynamics, in particular, if combined with the mentioned vibronic approach.

Our approach is based on the observation that the mixmaster anisotropy potential is equal to a difference between two three-particle Toda potentials. The weight introduced in

the Weyl quantization procedure amplifies the relative contribution of the positive Toda potential that is shown to dominate over the negative one. Thus, the latter can be neglected in the first approximation and the approximate evolution of the anisotropic variables for a fixed volume of the Universe becomes integrable. The negative Toda potential can be treated as a perturbation to the integrable dynamics. This split opens the door to a better understanding of the quantum dynamics, essentially through perturbative methods.

The outline of the paper is as follows. In Sec. II we consider the Bianchi IX Hamiltonian constraint, the anisotropy potential, its previously known approximations, and the approximation based on the Toda potential. In Sec. III we explain the basics of the Weyl-Heisenberg covariant integral quantization. In Sec. IV we study the W-H integral covariant quantization of the anisotropy potential and show that the proposed approximation is reaffirmed on the quantum level. We conclude in Sec. V.

## II. DECOMPOSITION OF THE CLASSICAL ANISOTROPY POTENTIAL

### A. Definition of the model

Let us first recall the definition and the Hamiltonian formulation of the Bianchi type IX model. We assume the line element,

$$ds^2 = -\mathcal{N}^2 dt^2 + \sum_i a_i^2 (\omega^i)^2, \quad (1)$$

where  $d\omega_i = \frac{1}{2} \mathbf{n} \varepsilon_i^{jk} \omega_j \wedge \omega_k$  and  $\mathcal{N}$ ,  $a_i$  are functions of time. The Hamiltonian constraint in the Misner variables reads [2]

$$\begin{aligned} \mathbf{C} = & \frac{\mathcal{N} e^{-3\Omega}}{24} \left( \frac{2\kappa}{\mathcal{V}_0} \right)^2 \left( -p_+^2 + \mathbf{p}^2 \right. \\ & \left. + 36 \left( \frac{\mathcal{V}_0}{2\kappa} \right)^3 \mathbf{n}^2 e^{4\Omega} [V(\boldsymbol{\beta}) - 1] \right), \\ & (\Omega, p_\Omega, \boldsymbol{\beta}, \mathbf{p}) \in \mathbb{R}^6, \end{aligned} \quad (2)$$

where  $\boldsymbol{\beta} := (\beta_+, \beta_-)$ ,  $\mathbf{p} := (p_+, p_-)$ ,  $\mathcal{V}_0 = \frac{16\pi^2}{\mathbf{n}^3}$  is the fiducial volume,  $\kappa = 8\pi G$  is the gravitational constant, and  $\mathcal{N}$  is the nonvanishing and otherwise arbitrary lapse function. In what follows we set  $\mathbf{n} = 1$  and  $2\kappa = \mathcal{V}_0$ . The gravitational Hamiltonian  $\mathbf{C}$  resembles the Hamiltonian of a particle in the three-dimensional Minkowski space-time in a certain potential. The space-time variables used in Eq. (2) have the cosmological interpretation,

$$\begin{aligned} \Omega = & \frac{1}{3} \ln a_1 a_2 a_3, & \beta_+ = & \frac{1}{6} \ln \frac{a_1 a_2}{a_3^2}, \\ \beta_- = & \frac{1}{2\sqrt{3}} \ln \frac{a_1}{a_2}. \end{aligned} \quad (3)$$

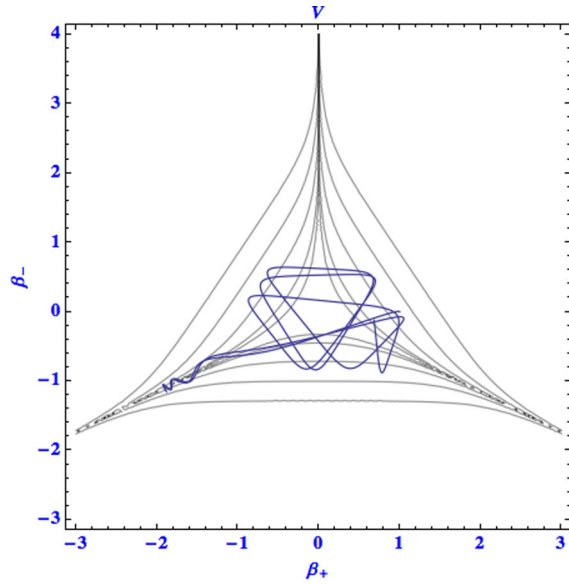


FIG. 1. A sample solution to the motion of a particle in the anisotropy potential. The contours are  $3, 10^1, 10^2, 10^3, 10^4$  and the initial condition for the plotted solution is  $p_+ = 10, p_- = -10, \beta_+ = 0, \beta_- = 1$ .

Thus, the variable  $\Omega$  describes the isotropic geometry, whereas  $\beta_{\pm}$  describe distortions to the isotropic geometry and are referred to as the anisotropic variables. The potential that drives the motion of the geometry represents the spatial curvature  ${}^3R$ .

The Hamiltonian constraint (2) is a sum of the isotropic and anisotropic parts,  $C = -C_{\text{iso}} + C_{\text{ani}}$ , where (up to a factor)

$$C_{\text{iso}} = p_{\Omega}^2 + 36e^{4\Omega}, \quad C_{\text{ani}} = \mathbf{p}^2 + 36e^{4\Omega}V(\boldsymbol{\beta}), \quad (4)$$

and where

$$V(\boldsymbol{\beta}) = \frac{e^{4\beta_+}}{3} [(2 \cosh(2\sqrt{3}\beta_-) - e^{-6\beta_+})^2 - 4] + 1. \quad (5)$$

The potential  $V(\boldsymbol{\beta})$  is plotted in Fig. 1. In the present paper we study the anisotropic Hamiltonian  $C_{\text{ani}}$  in (4) for a fixed value of the isotropic variable  $\Omega$  (whose value we set as  $36e^{4\Omega} = 1$ ). This Hamiltonian now reads

$$C_{\text{ani}} = \mathbf{p}^2 + V(\boldsymbol{\beta}). \quad (6)$$

The issue of the coupling between the anisotropic and isotropic variables (at the quantum level) is addressed elsewhere [15]. The presently known approximations to the anisotropy potential are the harmonic and steep-wall approximation,  $V_h(\boldsymbol{\beta})$  and  $V_{\Delta}(\boldsymbol{\beta})$ ,

$$V_h(\boldsymbol{\beta}) = 8(\beta_+^2 + \beta_-^2), \quad (7)$$

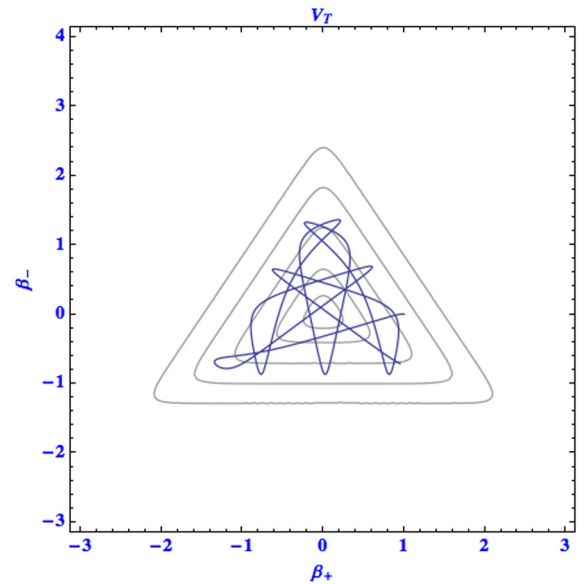


FIG. 2. Contour plot of the Toda potential and a sample solution. The contours are  $3, 10^1, 10^2, 10^3, 10^4$  and the initial condition for the plotted solution is  $p_+ = 10, p_- = -10, \beta_+ = 0, \beta_- = 0.3$ . We notice that the motion of the particle resembles the exact motion outside the narrow canyons, which are smoothed out in the present approximation.

$$V_{\Delta}(\boldsymbol{\beta}) = \lim_{q \rightarrow 0} \frac{1}{3} \exp [4 |\ln q| (\beta_+ + \sqrt{3} |\beta_-|)]. \quad (8)$$

## B. Decomposition into two Toda potentials

We decompose the anisotropy potential as follows:

$$V(\boldsymbol{\beta}) = \frac{1}{3} (e^{4\sqrt{3}\beta_- + 4\beta_+} + e^{-4\sqrt{3}\beta_- + 4\beta_+} + e^{-8\beta_+}) - \frac{2}{3} (e^{-2\sqrt{3}\beta_- - 2\beta_+} + e^{2\sqrt{3}\beta_- - 2\beta_+} + e^{4\beta_+}) + 1. \quad (9)$$

The introduction of new variables  $q_1, q_2, q_3$  such that  $q_1 - q_2 = 4\sqrt{3}\beta_- + 4\beta_+$  and  $q_2 - q_3 = -4\sqrt{3}\beta_- + 4\beta_+$  leads to

$$V(q_1, q_2, q_3) = V_T - V_p + 1, \quad (10)$$

where  $V_T = \frac{1}{3} (e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_1})$ , and  $V_p = \frac{2}{3} (e^{-\frac{1}{2}(q_1 - q_2)} + e^{-\frac{1}{2}(q_2 - q_3)} + e^{-\frac{1}{2}(q_3 - q_1)})$ . This coordinate transformation can be consistently extended by assuming an extra variable, say  $\beta_z$  (with corresponding momentum  $p_z$ ), which is absent in the potential, leading to the following transformation and relation between conjugate momenta,

$$[\beta_+, \beta_-, \beta_z]^T = \mathbf{A} [q_1, q_2, q_3]^T, \quad (11)$$

$$[p_1, p_2, p_3]^T = \mathbf{A}^T [p_+, p_-, p_z]^T, \quad (12)$$

where

$$\mathbf{A} := \begin{bmatrix} \frac{1}{8} & 0 & -\frac{1}{8} \\ \frac{1}{8\sqrt{3}} & -\frac{1}{4\sqrt{3}} & \frac{1}{8\sqrt{3}} \\ a & b & c \end{bmatrix}$$

and coefficients  $a, b, c$  are such that the transformation is invertible. The Hamiltonian

$$H = 32(p_1^2 + p_2^2 + p_3^2) + V_T - V_p + 1 \quad (13)$$

yields the original anisotropic one (6) for the vanishing total momentum  $p_1 + p_2 + p_3 = 0$  (or,  $p_z = 0$ ). Note that the total momentum is a quantity conserved by (13).

### C. Perturbation of the Toda system

Upon rescaling,  $q_i \rightarrow \lambda q_i$ ,  $p_i \rightarrow \lambda^{-1} p_i$  and  $t \rightarrow 3e^{-\lambda} t$  such that  $3e^{-\lambda} = \lambda^2$ , one brings (13) to the following form (up to an irrelevant constant):

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_1} - 3e^{-\frac{1}{2}\lambda} V_p. \quad (14)$$

The above Hamiltonian describes the periodic three-particle Toda system [27] perturbed by the negative three-particle Toda potential  $-3e^{\frac{1}{2}\lambda} V_p$ .

The periodic Toda system is a system of  $N$  equal-mass particles interacting via exponential forces, described by the Hamiltonian,

$$H = \frac{1}{2} \sum_{k=1}^N p_k^2 + \sum_{k=1}^N e^{-(q_k - q_{k+1})}, \quad (15)$$

with periodicity condition  $q_0 \equiv q_N$  and  $q_1 \equiv q_{N+1}$ .

The periodic three-particle Toda system is the simplest nontrivial periodic crystal consisting of three particles. It is known that the Toda systems are integrable [20–22] and solutions can be made explicit. This system, with three degrees of freedom, has also three independent conserved quantities: the total momentum  $P = p_1 + p_2 + p_3$ , the total energy  $H$ , and an additional third invariant,

$$K = -p_1 p_2 p_3 + p_1 e^{-(q_3 - q_2)} + p_2 e^{-(q_1 - q_3)} + p_3 e^{-(q_2 - q_1)}, \quad (16)$$

which gives rise to the integrability of the system and the removal of chaotic behavior. See Fig. 2 for an example of solution in this Toda potential.

## III. WEYL-HEISENBERG COVARIANT INTEGRAL QUANTIZATION

In this section, we present an alternative to the canonical quantization that, apart from being well suited for dealing with singular observables, possesses an attractive probabilistic interpretation. This procedure emphasizes the resemblance of the anisotropy Hamiltonian (14) to the Toda system, which we prove in the next section, and which is another indication of the importance of the classical approximation that we have earlier proposed.

For the sake of simplicity, we restrict the presentation of the method to the phase space of the motion of a particle on the line, namely,  $\mathbb{R}^2 = \{(q, p), q, p \in \mathbb{R}\}$ . We follow [16–18] where justifications and details are given. The idea is to combine symmetry with resources of measure/integral calculus, where we can ignore points or lines to some extent. It can be viewed as an extension of the so-called Wigner-Weyl integral quantization by combining the latter with a regularizing weight function, a kind of coarse-graining, on the classical phase space, say

$$(q, p) \mapsto \Pi(q, p), \quad \text{with } \Pi(0, 0) = 1. \quad (17)$$

While this function is not required to be a probability distribution, this property is required for its symplectic Fourier transform, defined as

$$\mathfrak{F}_s[\Pi](q, p) = \int_{\mathbb{R}^2} e^{-i(qp' - q'p)} \Pi(q', p') \frac{dq' dp'}{2\pi}, \quad (18)$$

for reasons that are explained in [17,18].  $\mathfrak{F}_s$  is involutive,  $\mathfrak{F}_s[\mathfrak{F}_s[f]] = f$  like its *dual* defined as  $\overline{\mathfrak{F}}_s[f](q, p) = \mathfrak{F}_s[f](-q, -p)$ .

Let us now consider the operator defined by the integral

$$\mathfrak{Q}_0 = \int_{\mathbb{R}^2} U(q, p) \Pi(q, p) \frac{dq dp}{2\pi}, \quad (19)$$

where  $U(q, p)$  is the Weyl unitary displacement operator, built from the self-adjoint  $\hat{q}$  and  $\hat{p}$  mentioned above,

$$U(q, p) = e^{i(p\hat{q} - q\hat{p})}. \quad (20)$$

The map  $(q, p) \mapsto U(q, p)$  is a projective unitary irreducible representation of the Abelian group  $\mathbb{R}^2$ . Then, by using Schur's lemma, the following resolution of the identity holds true:

$$\int_{\mathbb{R}^2} \mathfrak{Q}(q, p) \frac{dq dp}{2\pi} = I, \quad \mathfrak{Q}(q, p) = U(q, p) \mathfrak{Q}_0 U^\dagger(q, p). \quad (21)$$

Equipped with one choice of  $\Pi(q, p)$  or equivalently  $\mathfrak{Q}_0$ , one proceeds with the corresponding Weyl-Heisenberg covariant integral quantization,

$$f(q, p) \mapsto A_f \equiv A_f^{\mathfrak{Q}_0} = \int_{\mathbb{R}^2} f(q, p) \mathfrak{Q}(q, p) \frac{dq dp}{2\pi}. \quad (22)$$

The above quantization based on  $\mathfrak{Q}_0$  is possible if  $\mathfrak{Q}_0$  is unit trace. Indeed, we have the relation

$$\Pi(q, p) = \text{Tr}(U(-q, -p)\mathfrak{Q}_0), \quad (23)$$

which can be interpreted as the Weyl-Heisenberg transform of operator  $\mathfrak{Q}_0$ , Eq. (19) being viewed as the inverse Weyl-Heisenberg transform of the distribution  $\Pi(q, p)$ . The unit trace condition is a consequence of (23) since  $1 = \Pi(0, 0) = \text{Tr}(\mathfrak{Q}_0)$ . From  $2\pi\delta(q, p) \mapsto A_\delta = \mathfrak{Q}_0$ , the operator  $\mathfrak{Q}_0$  is the quantum version (up to a constant) of the origin of the phase space. Covariance means that the origin of the phase space is arbitrary. It holds in the sense that the quantization of the translation of  $f$  is unitarily equivalent to the quantization of  $f$  as follows:

$$U(q_0, p_0)A_fU^\dagger(q_0, p_0) = A_{\mathcal{T}(q_0, p_0)f}, \\ (\mathcal{T}(q_0, p_0)f)(q, p) := f(q - q_0, p - p_0). \quad (24)$$

The properties of  $U(q, p)$  allow us to establish an equivalent form of the WH integral quantization, which is more convenient for practical calculations.

$$A_f = \int_{\mathbb{R}^2} U(q, p) \overline{\mathfrak{F}_s}[f](q, p) \Pi(q, p) \frac{dq dp}{2\pi}. \quad (25)$$

There are several features independent of the choice of the quantization ingredients  $\mathfrak{Q}_0$  or  $\Pi(q, p)$ . First, we have the general important result: if  $\mathfrak{Q}_0$  is symmetric, i.e.,  $\overline{\Pi(-q, -p)} = \Pi(q, p)$ , a real function  $f(q, p)$  is mapped to a symmetric operator  $A_f$ . Moreover, if  $\mathfrak{Q}_0$  is a non-negative operator, i.e., is a density operator, then a real semibounded function  $f(q, p)$  is mapped to a self-adjoint operator  $A_f$  through the Friedrich extension of its associated semibounded quadratic form. Next, the canonical commutation rule is preserved,

$$q \mapsto A_q = \hat{q} + c_0, \quad p \mapsto A_p = \hat{p} + d_0, \\ c_0, d_0 \in \mathbb{R}, \rightarrow [A_q, A_p] = iI. \quad (26)$$

For the kinetic energy we have the following formula:

$$p^2 \mapsto A_{p^2} = \hat{p}^2 + e_1 \hat{p} + e_0, \quad e_0, e_1 \in \mathbb{R}. \quad (27)$$

The constants  $c_0, d_0, e_0, e_1$  appearing in the above can be made equal to 0 with a suitable choice of  $\Pi(q, p)$ , and thus, we may reestablish the results of the canonical quantization concerning the variables  $q, p$  and their quadratic combinations.

A potential energy becomes the multiplication operator in the position representation,

$$V(q) \mapsto A_{V(q)} = \mathfrak{B}(\hat{q}), \\ \mathfrak{B}(\hat{q}) = \frac{1}{\sqrt{2\pi}} V * \bar{\mathcal{F}}[\Pi(0, \cdot)](\hat{q}), \quad (28)$$

where  $\bar{\mathcal{F}}$  is the inverse  $1-d$  Fourier transform, and  $f * g(x) = \int_{\mathbb{R}} dt f(x-t)g(t)$ . Such a convolution formula can be of crucial importance when it is needed to smooth classical singularities or modify in a suitable way the strengths of some potentials as is shown in the sequel.

In the next section, we choose separable Gaussian distributions,

$$\Pi(q, p) = e^{-\frac{q^2}{2\sigma^2}} e^{-\frac{p^2}{2\tau^2}}. \quad (29)$$

They yield simple formulas with familiar probabilistic content. Standard coherent state (or Berezin or anti-Wick) quantization corresponds to the particular values  $\sigma = \sqrt{2} = \tau$ . The Weyl quantization holds as the widths  $\sigma$  and  $\tau$  are infinite.

## IV. QUANTIZATION OF THE POTENTIAL

### A. General result

We apply the integral quantization method described in the previous section to the anisotropy potential. For the quantization of the two-dimensional anisotropy potential (5) we employ the separable Gaussian distributions mentioned in the previous section,

$$\Pi(\beta_+, p_+; \beta_-, p_-) = e^{-\frac{\beta_+^2}{2\sigma_+^2}} e^{-\frac{\beta_-^2}{2\sigma_-^2}} e^{-\frac{p_+^2}{2\tau_+^2}} e^{-\frac{p_-^2}{2\tau_-^2}}, \quad (30)$$

where we use the formula

$$e^{\alpha q} \mapsto A_{e^{\alpha q}} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(q-y)^2}{2\sigma^2}} e^{\alpha y} dy = e^{\frac{\alpha^2}{2\sigma^2}} e^{\alpha q}. \quad (31)$$

We obtain

$$V(\beta_+, \beta_-) \mapsto A_{V(\beta_+, \beta_-)} \\ = \frac{1}{3} (2D_+^4 D_-^{12} e^{4\beta_+} \cosh 4\sqrt{3}\beta_- \\ - 4D_+ D_-^3 e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + D_+^{16} e^{-8\beta_+} - 2D_+^4 e^{4\beta_+}) + 1, \quad (32)$$

where  $D_+ = e^{\frac{2}{\sigma_+^2}}$  and  $D_- = e^{\frac{2}{\sigma_-^2}}$ . The classical anisotropy potential  $V(\beta_+, \beta_-)$  is recovered for  $D_+ = D_- = 1$  (or  $\sigma_+, \sigma_- \rightarrow \infty$ ).

Figure 3 (left plot) shows the quantum potential (32) for sample values of  $D_+$  and  $D_-$ . The classical escape canyons are absent and the potential is fully confining. Moreover, it is anisotropic in the variables  $\beta_+$  and  $\beta_-$ , and the position of its minimum is, in comparison with the classical case,

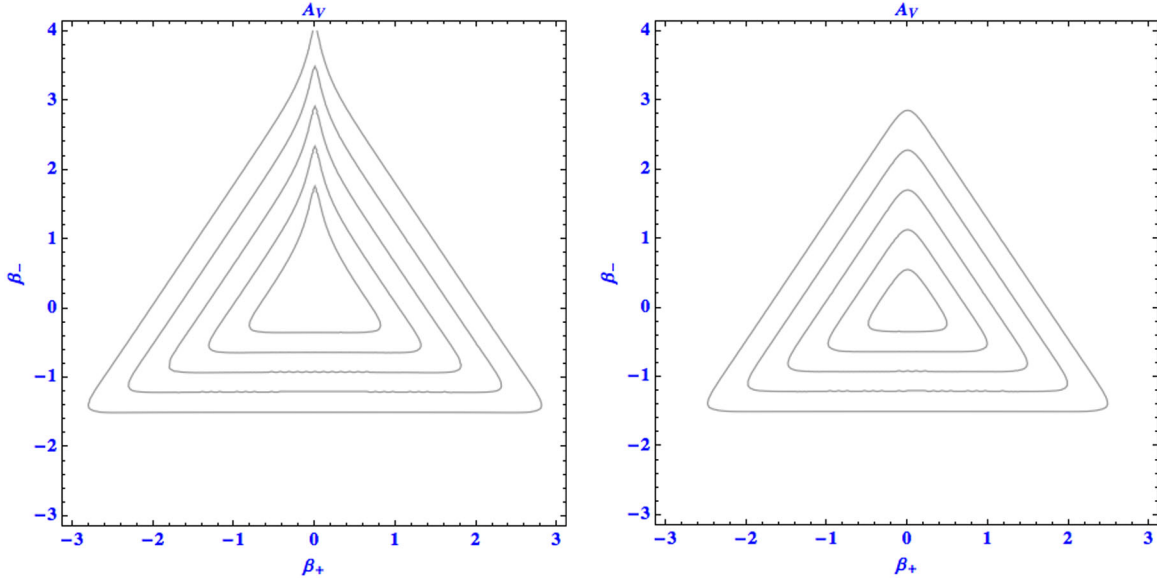


FIG. 3. On the left, contour plot of the quantized Bianchi IX potential near its minimum, for sample values  $D_+ = 1.2$ ,  $D_- = 1.005$ . On the right, contour plot of the symmetric quantized potential near its minimum,  $D_+ = D_- = 1.2$ .

shifted from  $(0,0)$  to  $(\beta_0, 0)$ , where the value  $\beta_0$  satisfies the following equation:

$$-D_+^{16}e^{-8\beta_0} + D_+D_-^3e^{-2\beta_0} - D_+^4e^{4\beta_0} + D_+^4D_-^{12}e^{4\beta_0} = 0, \quad (33)$$

derived from condition  $\partial A_{V(\beta_+, \beta_-)} / \partial \beta_+ = 0$  (condition  $\partial A_{V(\beta_+, \beta_-)} / \partial \beta_- = 0$  is trivially fulfilled). The minimal value of the potential is shifted from 0 to the value

$$A_{V(\beta_+, \beta_-)}(\beta_0, 0) = \frac{1}{3}e^{-8\beta_0}(D_+^{16} - 4D_+D_-^3e^{6\beta_0} + 3e^{8\beta_0} + 2D_+^4(-1 + D_-^{12})e^{12\beta_0}). \quad (34)$$

### B. Minimum and isotropy preservation

We want our quantization procedure to preserve the basic properties of the classical potential by requiring (i) the isotropy around its minimum and (ii) keeping the position of the minimum at the point  $(0,0)$ .

Expansion of the potential (32) around the point  $(0,0)$ <sup>1</sup> reads

$$A_{V(\beta_+, \beta_-)} \approx \frac{1}{3}[D_+^{16} - 2D_+^4 + 2D_+^4D_-^{12} - 4D_+D_-^3 - 8(D_+^4 + D_+^{16} - D_+D_-^3 - D_+^4D_-^{12})\beta_+ + 8(4D_+^{16} - 2D_+^4 - D_+D_-^3 + 2D_+^4D_-^{12}) \times \beta_+^2 + 24(2D_+^4D_-^{12} - D_+D_-^3)\beta_-^2] + 1. \quad (35)$$

<sup>1</sup>Expanding around  $(\beta_0, 0)$  yields similar results but the presentation is far more sophisticated.

The coefficients in front of  $\beta_+^2$  and  $\beta_-^2$  are equal if

$$4D_+^{16} - 2D_+^4 - D_+D_-^3 + 2D_+^4D_-^{12} = 3(2D_+^4D_-^{12} - D_+D_-^3).$$

This condition may be rewritten as

$$(D_+^4 - D_+D_-^3)[2(D_+^4 + D_+D_-^3)(D_+^8 + (D_+D_-^3)^2) - 1] = 0. \quad (36)$$

For values of  $D_+, D_- \geq 1$  (condition fulfilled by exponents) it has only one unique solution,

$$D_+ = D_- := D. \quad (37)$$

The condition (37) also implies the vanishing of the coefficient in front of  $\beta_+$  in Eq. (35). This is consistent with Eq. (33), which for  $\beta_0 = 0$  can be rearranged as

$$(D_+^4 - D_+D_-^3)[(D_+^4 + D_+D_-^3)(D_+^8 + (D_+D_-^3)^2) + 1] = 0, \quad (38)$$

which again for positive values of  $D_+$  and  $D_-$  has only one unique solution  $D_+ = D_-$ , the same as the one for symmetry preservation. Thus, there is no shift in the position of the minimum.

The full quantized Bianchi IX potential reads now

$$A_{V(\beta_+, \beta_-)} = \frac{1}{3}(2D^{16}e^{4\beta_+} \cosh 4\sqrt{3}\beta_- - 4D^4e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + D^{16}e^{-8\beta_+} - 2D^4e^{4\beta_+}) + 1. \quad (39)$$

It is shown in Fig. 3 (right plot). One may verify that it is invariant with respect to the rotations by  $2\pi/3$  and  $4\pi/3$ , and thus, the  $C_{3v}$  symmetry is preserved in the full plane.

### C. Underlying Toda system

The quantized potential (39), in full analogy to the classical potential in Eq. (9), may be written as a difference between two Toda potentials,

$$A_{V(\beta_+, \beta_-)} = D^{16}V_T - D^4V_p + 1, \quad (40)$$

where the coefficients  $D^{16}$  and  $D^4$  strengthen the dynamical role of  $V_T$  as  $D > 1$ . We are interested in verifying whether the following,

$$\left| \frac{V_p}{V_T} \right| = \left| \frac{2XY + 1}{2Y^2 + X^2/2 - 1} \right|, \quad (41)$$

is bounded from above, where

$$X = e^{-6\beta_+} \geq 0, \quad (42)$$

$$Y = \cosh 2\sqrt{3}\beta_- \geq 1. \quad (43)$$

Direct computation shows that in the open domain  $(X, Y) \in ]0, \infty[ \times ]1, \infty[$  the ratio  $|\frac{V_p}{V_T}|(X, Y)$  does not have any critical points. On the boundary  $Y = 1$  it has one maximum at  $X = 1$ , with  $|\frac{V_p}{V_T}|(1, 1) = 2$ , which is therefore a unique global maximum. This proves that for exponential weights  $D > 1$ , the exact quantum potential may be viewed as a perturbed Toda potential  $D^{16}V_T$ .

### V. DISCUSSION

The Bianchi IX model plays an essential role in our understanding of the near-singularity dynamics of the Universe. The classical Hamiltonian for this model involves an elaborate two-dimensional anisotropy potential. Therefore, the quantizations of the model found in the literature often rely on one of the two approximations to the anisotropy potential: the steep-wall or the harmonic approximation. They enable us to analytically determine the respective eigenstates and eigenvalues and apply adiabatic or nonadiabatic approaches to study the quantum dynamics of that model.

In this work, we developed an alternative proposal for approximating the anisotropic part of the quantum

Hamiltonian with the integrable three-particle Toda system. It approximates the three exponential walls while smoothing out the problematic canyons. This result brings additional possibilities to studies of the quantum dynamics of mixmaster. The periodic three-body Toda system has been analyzed both on the classical and quantum levels. In the literature, one may find ways to construct classical solutions [21,28] as well as the corresponding quantum eigenfunctions and eigenvalues [23,29]. Thus, we provide an analytically solvable approximation to the anisotropy potential of the Bianchi IX model, which is valid not only in the large volume and low anisotropy excitation level or in the small volume and high anisotropy excitation level limits (given by the harmonic and steep-wall approximation, respectively), but also in the vast and unexplored in-between region.

In [14] we proved the discreteness of the spectrum originated by the exact Bianchi IX anisotropic potential despite its noncompact character. This result validates all implementations of approximations of the potential, which remove the three nonconfining canyons. Moreover, it was shown on the classical level [3,4] that the canyons (referred to there as “channels”) do not contribute to the chaotic behavior of the mixmaster. Therefore our procedure preserves key properties of the model, specially those regarding energy levels or possible chaotic behavior.

In the ongoing studies, we want to calculate explicitly the eigenstates and energy levels of the quantum mixmaster, starting from the integrable Toda potential. This approximation would serve as an exact model, around which we could use some standard perturbative methods. We also plan to examine the level spacing distribution, which usually points out the chaotic behavior.

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