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Dealing with large numbers: What is important for students and teachers to know?

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National French assessment shows difficulties with writing large numbers at the beginning of the 6th grade. But, what do students need to learn and teachers need to teach? What do they actually learn or teach? We investigate these questions at different levels of the didactical transposition: students' knowledge, teaching practices and reference knowledge. We show a lack of mathematical understanding of large numbers and make a proposal for teaching knowledge which provides justifications for the use of large numbers which could foster a 'number sense' understanding of such numbers.

Keywords: Large numbers, numeration, units, teaching, knowledge.

Introduction

The topic of place value related to whole numbers is a major one in the primary grades, especially because it is the fundament of basic arithmetic. There is an abundant research literature on the issue. Most often it focuses on “small numbers”: from two to four digits. Several authors (Wagner and Davis, 2010; Howe, 2015) consider larger whole numbers, and specific issues on this topic. In this literature, order of magnitude – even more relative magnitudes - appears as a key one. For instance, it is needed for a thorough understanding of economic, political and scientific issues. National French assessment shows difficulties in writing large numbers at the beginning of the 6th grade. The latter can be seen as an anecdotal subject compared to number sense understanding of this kind of numbers. Yet, is there any link between both? How writing and reading large number tasks can be connected to quantity sense or number sense? What do students need to learn and teachers need to teach? What do they actually learn or teach?

Theoretical framework, previous works, and methodology

Theoretical framework

The Theory of Didactic Transposition (TDT) (Chevallard, 1985) (figure 1) considers school mathematics as a reconstruction by the educational institutions from the mathematical knowledge produced by academic scholars. The TDT has been often used for secondary school, more rarely for primary school where scholarly knowledge as a reference is not always taken for granted.

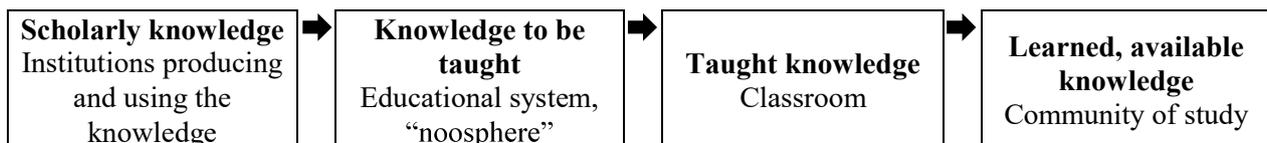


Figure 1: The didactic transposition process (Bosch and Gascon, 2006, p. 56)

The Anthropological Theory of Didactics (ATD) (*ibid.*) extends the TDT. It postulates that practicing math, as any human practice, can be described with the model of *praxeology*. It is

constituted by four pieces: a *type of tasks* -a set of similar problems-, a *technique* -a “way of doing” for all the tasks of the type-, a *technology* justifies the technique and is justified by a *theory*.

Previous works

Chambris (2008, 2015), Tempier (2016) have analyzed teaching and learning of decimal numeration in French context, notably in second and third grades. *Classical* mathematical theory in numeration which embeds units (tens, hundreds, etc.) and relations between them was the reference knowledge up to the New Math. Chambris (2008) proposed the wording “numeration unit” for the units used in numeration (ones, tens, hundreds, etc.). Beginning in the 1980s, classical “scholarly knowledge” has been replaced by transposition of *academic* theory (polynomial decomposition with the exponential notation) within which there is no unit. This might explain why relations between units (e.g. 10 tens = 1 hundred) are little mastered –sometimes not taught at all- in present French context. In turn, Houdement & Chambris (2013), Tempier (2016) designed interventions for reintroducing units for “small numbers” in teaching practices, especially the relations between units, as well as explicit properties of positional notation: 1) The position of each digit in a written number corresponds to a unit (for example hundreds stand in the third place) (“positional principle”); 2) Each unit is equal to ten units of the immediately lower order (e.g. one hundred = ten tens) (“decimal principle”). Ten digits are enough to write any whole number thanks to an iterative process. The names of small numbers present many irregularities: numeration units provide a way to bridge the gap between irregular number names and positional notation. In short: thirty \leftrightarrow three tens \leftrightarrow 3 tens \leftrightarrow 3 in the second place (Houdement & Chambris 2013).

In a broader francophone context (France and Switzerland), there is a range of literature (e.g. Mercier 1997, Ligozat & Leutenegger 2004) on another issue in numeration: students’ and teachers’ difficulties in the topic “how to write large numbers”. Here we present some of their findings. The teachers being observed seem to consider relations between number names and positional notation as a linguistic issue which does not require mathematical knowledge. This generally leads them to teach two rules for writing numbers: 1) replace the words thousand, million by a space¹ (sometimes a dot), 2) put three digits between two spaces. These rules appear to be little powerful to solve the most complex problems with “mute zeros”. Mercier (1997) (related to French context) argues this reflects an institutional problem: the lack of mathematical knowledge on the topic, in the teaching system for several decades. Moreover, only one of the five teachers observed attempts to teach general base ten property of positional notation for large numbers. In all these contexts, it is finally social knowledge which leads to validate (or not) an answer! About “13180” “St.: One-hundred-thirty-one-hundreds and eighty. (...) T.: This would be one of the ways to name this number; but: will everybody understand immediately?” (Ligozat & Leutenegger 2004 p. 15).

These scholars present mathematical knowledge to fill the vacuum: Mercier (1997) indicated the general rule for positional notation using exponential notation algebraically, as well as a brief history of number names. Ligozat and Leutenegger (2004) proposed to distinguish and link two

¹ In France (among other countries), a space is used between the periods for writing large numbers: 34 020 (unlike 34,020 in some other countries). This space is sometimes replaced by a dot (34.020) but never by a comma. The latter is dedicated to decimal numbers: 34,020 is thirty four ones and twenty thousandths. This paper uses “French” notation.

pieces of knowledge: for positional notation (base ten), for number names (base 1000). They formulate this using “powers of ten written with figures” (Chambris 2015, p. 57) notation:

$\bullet 17002058$ $1 \times 10\,000\,000 + 7 \times 1\,000\,000 + 0 \times 100\,000 + 0 \times 10\,000 + 2 \times 1000 + 0 \times 100 + 5 \times 10 + 8 \times 1$ (1)	
\bullet Trente-cinq millions deux cent quarante et un mille neuf cent treize $\bullet (30+5) \times 1\,000\,000 + [(2 \times 100) + 40 + 1] \times 1000 + 9 \times 100 + 10 + 3$ (2) $\bullet 35\,241\,913$	\bullet Dix-sept millions deux mille cinquante-huit $\bullet (10+7) \times 1\,000\,000 + 2 \times 1000 + 50 + 8$ (3) $\bullet 17\,002\,058$

Figure 2: Relations between positional notation and powers of ten (1), between number name and powers of thousand (2 & 3). (Ligozat & Leutenegger 2004, p. 3)

They state: “the point is an institutional “foregone knowledge” phenomenon about how to name the numbers and how this (foregone) knowledge is linked with positional notation” (p. 17, our translation). They suggest tasks like $13180 = 131$ hundreds 80 ones. Years before, Fuson (1990) already indicated this in term of knowledge, the systems of multiunits that are intertwined (named base-ten numeration units, and base-1000 numeration units in the present paper). For large numbers, ten “thousands” make a new unit, a “ten of thousands” which is written in the 5th place; ten “tens of thousands” make a “hundred of thousands” which is written in the 6th place, etc. This reveals that the issue of large number names is mathematically connected with relations between units: in base-ten and in base-1000. That is clearly a first step to “quantity sense”.

Research questions: Finding praxeologies

What are the relations between knowledge learned (by students), knowledge taught (by teachers) and scholarly knowledge? How do they contribute to give sense to large numbers?

Method and data

Within praxeological analysis, exercises to be performed generally indicate the tasks, explanations related to students’ mistakes as well as introduction of new type of tasks often bring the teacher to make explicit the aimed technique and/or technology in classroom episodes, and definitions show technologies. Data will be analyzed in term of praxeologies. A mathematical analysis about reference knowledge is deepened. The data were designed and collected by the second author as follows. A teacher (Soline) was trained on 3-to-4-digit numbers teaching in a collaborative research project aiming at designing a resource for teachers paying specifically attention to the use of base-ten units in relation with written numbers (same vein as Tempier 2016). This grade-4 teacher was later observed during a lesson on another subject: numbers larger than 9999. The lesson was audio-recorded, transcribed, and notes were taken. During an interview (with note-taking by interviewer) just before the lesson, Soline was asked to explain her plan for this lesson. The different tasks of the lesson have been identified then three episodes corresponding to three tasks have been selected as follows. The two first ones are related to students’ mistakes (1- in relation with the introduction of the first 5-digit number, 2- in relation with the first mute zero in a 5-digit number). The third one is the introduction of one million. Finally, a written questionnaire was designed to better identify students’ knowledge and difficulties (n=159, end of grade 6).

Results

A teacher's mathematical praxeology about large numbers (taught praxeology)

Planning the lesson

Reading and writing numbers (in digits) is the only explicit task (“to be taught”) in the French syllabus. The preparation plan shows that Soline chose this writing task with increasingly large numbers from 4 to 8 digits, with various places for “mute zeroes”. She was also planning to introduce the definition of the million as one thousand thousands, but no base-ten relation with the new units (despite the previous study). The teacher wonders whether it is enough to teach how to write large numbers, and that a million is one thousand groups of one thousand.

Implementing the lesson

The first number greater than 9999 to write in digits is "twelve thousand five hundred". A student, Anaïs, is not able to write it. Perhaps she refrains from writing a two digits number in the thousands place in accordance with the technique learned before. The teacher does not identify this cognitive conflict. She shows some confusion by calling out to the researcher; then she tries to help Anaïs.

You see [...] I already have Anaïs who has troubles. She is able to write three thousand but does not know how to write twelve thousand. Does it change something Anaïs? Think about it. Twelllllllve thousand. Twelllllllve thousand five hundred. Twelve thousand it is twelve groups of one thousand.

While the students learn for the first time to write a five-digit number, the teacher seems to consider there is nothing new to know about the *new* 5th place in the written number and about the *old* word (thousand) in the number name. She tries to help the student by emphasizing “twelve”. The technique aimed is to write the number heard before “thousand” (here 12) and then the next number (here 500) with eventually a dot between them.

Later, the teacher asks to write "thirty four thousand and twenty". It is the first time with a “mute zero”. Axel writes “34.20”: it is what he hears: 34 and 20 with a point instead of the word “thousand”.

Soline (T): Thirty four **thousand**. Twenty. It doesn't look strange? After the word one thousand how many digits are there?

Axel: Three

Soline (T): And here you have only two digits. How could you make to have three?

Soline writes on the black board “34.20” and underline 20.

Axel: To put a zero?

Soline (T): Where? (*Axel's answer is inaudible. But Soline writes “34.200” on the board.*)

Soline (T): Look Axel (*and Soline writes on the board: “34.020”*).

Once again, the teacher seems powerless in front of a student's mistake. She gives then the answer without any explanation. With this case the technique to write a number in digits incorporates a new element: after “thousand” there must be three digits. Thus, when there is a two digits number after the thousands it is necessary to write a zero.

Later, after the writing of the number 999 000, Soline intends to introduce the millions. She asks the students how many groups of one thousand there are just after 999 000 and introduces the million thereby:

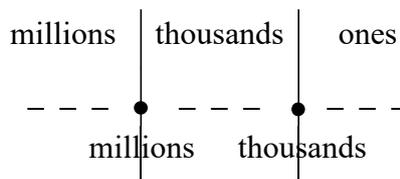
Soline (T): One thousand times one thousand is called otherwise. How it is called?

A pupil: One million.

Soline (T): One million, it is a new word. For the moment we said the word one thousand, now we say too one million.

A pupil: How we are going to write it?

Soline (T): This is what we are trying to discover together. It is necessary to make a kind of small chart (*Soline draws then this place value chart on the black board*).



The million is defined as one thousand thousands, what confirms Soline's plan. She tries to give meaning to large numbers relying on relations between base-thousand units. According to this point of view, the million doesn't appear as ten groups of hundred thousands. In the above table there are the names and places of periods but no reference to the name of base-ten places like the hundred thousands place for example. These technological elements rely only on a period viewpoint of the written number. The corresponding technique for writing numbers is: "write the number heard and points for the words *millions* and *thousand*". This is confirmed during the rest of the lesson. For example for the writing of two million, the teacher tells: "I write my *two* with my little point which means *million*". With this technique, teacher is not able to help students with mute zeros as a whole period or within a period. For example, at the end of the lesson, to write twelve million fifty some students write 12.50.000, 12.000.50 or 12.050.000 (with points or spaces). The teacher continues to explain the three digits in a period without a "base ten/place" viewpoint in addition to this "base 1000/period" viewpoint.

A questionnaire to inform about 6th grade students' knowledge (learned praxeologies)

The first part of our questionnaire concerns the writing of large numbers in digits (table 1).

Numbers to be written in digits	Beginning of gr. 6 (2008)	End of gr. 6 (2016)
Four hundred and seventy-five (475)	94%	
Three thousand and three (3 003)	96%	
Six hundred and twenty seven thousand (627 000)	76%	87 %
One million six hundred thousand (1 600 000)	76%	89 %
Three million fifty thousand three hundred and twenty (3 050 320)		79 %
Seventeen million two thousand and fifty-eight (17 002 058)		69 %
Five hundred and three million thirty-seven (503 000 037)		82 %

Table 1: Results of national assessment (2008) and our assessment (2016) of 6th grade students

To complement the data we also proposed conversions in order to examine the relations between large units and determine which of the relations between base-ten units and base-thousand units are better known. Such tasks are inspired from our prior research, and Ligozat & Leutenegger (2004)'s analysis. Both tasks raise as much difficulties. Approximately half of the students succeed in converting 4 millions into hundreds of thousands (48%) and 3 millions into thousands (50%). Many students did not write any answer. This was not the case for the “writing number” tasks. Perhaps they never performed conversion tasks before.

Complements for reference knowledge for teaching large numbers

We have already recalled some considerations about the specificity of the written and spoken numeration systems and their imbrications in a double system. The above lesson analysis and our previous studies about “small” numbers show that there is a need for an intermediate system between these numeration systems in order to articulate this double system: the numeration units systems for bases ten and 1000. These systems of units enable various decompositions of numbers related to base 10 and base 1000. Up to 9999, number names can be linked with base ten numeration units, whereas beginning of 1000 they are linked with base 1000 numeration units. The transformation from a base-ten decomposition into a base-thousand decomposition is made by conversions.

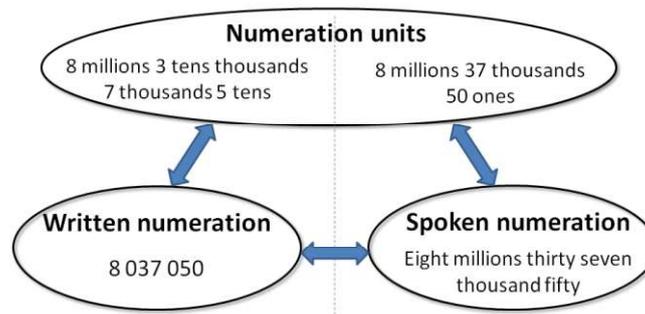


Figure 2: Links between numeration units, written numeration and spoken numeration

The numeration units system supplies a way to justify the writing (in digits) of a number name. It enables to justify the mute zeros. See, for example, the task “writing eight millions thirty seven thousand fifty” (figure 2) in digits. The number can be written or spoken in numeration units “8 millions 37 thousands 50 units”, which can be converted in this register “8 millions 3 tens of thousands 7 thousands 5 tens”. It relies on conversion of 37 thousands in 3 tens of thousands and 7 thousands. The digit of millions is written in the 7th place, that of hundreds of thousands in the 6th place, etc. It is necessary to write a 0 to mark the lack of missing units. The obtained number can then spell 8, 0, 3, 7, 0, 5 and 0 by the positional principle of the written numeration. This example shows how this designation is relevant for reference knowledge for teaching. It enables to make links between the written and spoken system and to explicit the knowledge at stake.

Discussion

The analysis of Soline’s lesson shows that her knowledge of the spoken numeration of large numbers, for one side, and her previous teaching of small numbers with an important concern about relations between base-ten units, on the other side, brought her to a beginning of a teaching of relations between base-thousand units. The latter seems not to be the case in the observations

reported by Mercier (1997) or Ligozat and Leutenegger (2004) where only one teacher focused on the relation between base-ten units, and none on those between base-thousand units. Yet it is not enough to provide explanations for writing large numbers. Indeed, the teacher seems sometimes deprived to help her students during this lesson. In addition, our observation at the end of the lesson shows that it is insufficient to enable the students to avoid some mistakes, particularly those which are related to the mute zeros. The teacher explanations and her place value chart let us think that she assigns a base-thousand system to the written system and ignored its base ten operating. The two first episodes are coherent with our literature review considering large number names as an “institutional foregone knowledge”, and a linguistic issue.

By introducing numeration units in the reference knowledge to make links between base-ten and base-thousand units, we aim to provide a more explicit knowledge for teachers and students. It enables to take into account the double system units for large numbers (base ten and base 1000). It can, on one side, enrich the understanding of the written system by realizing that the system always works in the same way according to the base ten and, on another side, learn the names of large numbers by putting them in connection with the written code. For example, to justify the writing of twelve thousand five hundred, the teacher has to clarify the link between ten thousands, the ten thousands unit and the corresponding places in the written number. In the writing of mute zeros, as in thirty four thousand twenty, the link between the values of 3 and 4 and the corresponding places in the written number can justify the writing of a zero in the hundreds place. Questionnaire results seem to confirm a more effective technique is needed for mute left-hand zeros.

This dual system of units can favor recognition of relative magnitudes of large numbers. For example, understanding the million involves the relation with smaller numbers. For example to understand the million involves the relation with smaller numbers: a million it is “ten times one hundred thousand” as well as “one thousand times one thousand”. Tasks of mental computation on numbers with only one non-zero digit can strengthen these relations. For example, “ten times two hundred thousand” could aim at leaning on the relation between one hundred thousand and one million, relation which does not explicitly appear in our way to speak these numbers. Other tasks aim at the extension of the written numeration. For example it is possible to recover and adapt small numbers tasks as the situations of collection counting and ordering (Tempier 2016). Counting a collection, with representations of large groups, can then be used to introduce new base-ten units. In addition, ordering a collection can be used to produce various decomposition of number under base ten and base 1000 with numeration units. Under this approach the spoken numeration could be secondly brought, in connection with these decompositions.

Conclusion

Obviously students can succeed in writing numbers without knowing relative order of magnitude of numeration units in base ten and in base 1000. Yet, relations between units can provide justification for even writing numbers. This knowledge can be expressed with numeration units. It is missing in French institutional system. It might also contribute to the understanding of quantity sense. In this context, the work on small numbers is not enough to train teachers (even it is only for one case); surely, it helps the teacher to question her usual practices on large numbers. However, it does not

provide tools for justification, neither the specific stakes of large numbers. Further research is needed in order to provide powerful tasks for teaching and learning large numbers.

References

- Bosch M., Gascón J. (2006). 25 years of didactic transposition. *ICMI Bulletin*, 58, 51–64.
- Chambris C. (2008) *Relations entre les grandeurs et les nombres dans les mathématiques de l'école primaire. Évolution de l'enseignement au cours du 20^e siècle. Connaissances des élèves actuels.* Thèse de l'Université Paris Diderot.
- Chambris C. (2015). Mathematical foundations for place value throughout one century of teaching in France. In X. Sun, B. Kaur and J. Novotná. (Eds.), *Conference Proceedings of ICMI Study 23: Primary mathematics study on whole numbers* (pp. 52-59). Macau, China: ICMI.
- Fuson K. (1990). Conceptual Structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction*, 7(4), 343–403.
- Houdement C. & Chambris C. (2013). Why and how to introduce numbers units in 1st and 2nd grades. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 313–322). Ankara, Turkey: ERME.
- Howe R. (2015). The most important thing for your child to learn about arithmetic. In X. Sun, B. Kaur and J. Novotná. (Eds.), *Conference Proceedings of ICMI Study 23: Primary Mathematics Study on Whole Numbers* (pp. 107–114). Macau, China : ICMI
- Ligozat F. & Leutenegger F. (2004). La bivalence mathématique et langagière dans les pratiques d'enseignement sur la numération. *Actes du 9^{ème} colloque de l'AIRDF*. Québec, Canada.
- Mercier A. (1997). La relation didactique et ses effets. In C. Blanchard-Laville (Ed.), *Variations sur une leçon de mathématiques. Analyses d'une séquence : « L'écriture des grands nombres »* (pp. 259-312). Paris: L'Harmattan.
- Tempier F. (2016). New perspectives for didactical engineering. An example for the development of a resource for teaching decimal number system. *Journal of Mathematics Teacher Education*, 19(2), 261-276.
- Wagner D., & Davis B. (2010). Feeling number: Grounding number in a sense of quantity. *Educational Studies in Mathematics*, 74, 39–51.