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► **To cite this version:**

| Anne Beaulieu. The kernel of the linearized Ginzburg-Landau operator. 2018. hal-01724551

HAL Id: hal-01724551

<https://hal.science/hal-01724551v1>

Preprint submitted on 6 Mar 2018

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The kernel of the linearized Ginzburg-Landau operator.

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March 6, 2018

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Abstract.

We consider a linear system of ordinary differential equations from the two dimensional Ginzburg-Landau equation. We prove that this system doesn't admit globally bounded solutions, except those that come from invariance of the Ginzburg-Landau equation under the action of the group of the translations and rotations.

AMS classification : 34B40: Ordinary Differential Equations, Boundary value problems on infinite intervals. 35J60: Nonlinear PDE of elliptic type.

1 Introduction

Let n and d be given integers, $n \geq 1$, $d \geq 1$. We define the following system

$$\begin{cases} a'' + \frac{a'}{r} - \frac{(n-d)^2}{r^2}a - f_d^2 b & = -(1 - 2f_d^2)a \\ b'' + \frac{b'}{r} - \frac{(n+d)^2}{r^2}b - f_d^2 a & = -(1 - 2f_d^2)b \end{cases} \quad (1.1)$$

and for $n = 0$, we define the following equation

$$\begin{cases} a'' + \frac{a'}{r} - \frac{d^2}{r^2}a & = -(1 - f_d^2)a \end{cases} \quad (1.2)$$

with the variable $r > 0$.

Both problems come from the Ginzburg-Landau Theory. Here f_d is the only solution of the differential equation

$$f_d'' + \frac{f_d'}{r} - \frac{d^2}{r^2}f_d = -f_d(1 - f_d^2). \quad (1.3)$$

with the conditions $f_d(0) = 0$ and $\lim_{+\infty} f_d = 1$. The equation (1.3) is entirely studied by Hervé and Hervé in [4].

Let us consider the Ginzburg-Landau equation

$$-\Delta u = u(1 - |u|^2) \text{ in } \mathbb{R}^2 \quad (1.4)$$

where u takes its values in \mathbb{C} . The system (1.1) and the equation (1.2) appear when we linearize the Ginzburg-Landau operator $\mathcal{N}(u) = \Delta u + u(1 - |u|^2)$ around the solutions of the form $f_d(r)e^{id\theta}$, $d \in \mathbb{N}^*$. The linearized operator has been studied by several authors, amongst them [5], [8], [6] and [7]. In the third chapter of the book [9], Pacard and Rivière study the system (1.1) for $d = 1$. The aim of these authors is the construction of some solutions for the Ginzburg-Landau equation on a bounded connected domain Ω ,

$$\begin{aligned} -\Delta u &= \frac{1}{\varepsilon^2}u(1 - |u|^2) \text{ in } \Omega \\ u &= g \text{ in } \partial\Omega \end{aligned} \quad (1.5)$$

where $\varepsilon > 0$ is a small parameter, u and g having complex values and degree $(g, \partial\Omega) \geq 1$. The study of the minimizing solutions of equation (1.1) is in the book of Bethuel, Brezis Hélein, [2].

Let us call a bounded solution of (1.1) any solution (a, b) which is defined at $r = 0$ and which has a finite limit as $r \rightarrow +\infty$. Concerning the bounded solutions of (1.1) or (1.2), the following theorem is known

Theorem 1.1 *For all $d \geq 1$ and for $n = 0$, the real vector space of the bounded solutions of (1.2) is one-dimensional, spanned by f_d . For $n = 1$, the vector space of the bounded solutions of (1.1) is also a one dimensional vector space, spanned by $(f'_d + \frac{d}{r}f_d, f'_d - \frac{d}{r}f_d)$. For $d = 1$ and $n \geq 2$, there are no bounded solutions. For $d > 1$ and for $n \geq 2d - 1$, there are no bounded solutions.*

For all $d \geq 1$, the known bounded solutions, for $n = 0$ and $n = 1$, come from the invariance of the Ginzburg-Landau equation with respect to the translations and the rotations.

The aim of the present paper is to prove the following

Theorem 1.2 *For all $d \geq 1$ and for all $n > 1$, the system (1.1) has no bounded solution.*

We will have to allow n to be a real parameter. To begin with, let us consider the system

$$\begin{cases} a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2}a - f_d^2b - f_d^2a &= -(1 - f_d^2)a \\ b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2}b - f_d^2a - f_d^2b &= -(1 - f_d^2)b \end{cases} \quad (1.6)$$

where γ_1 and γ_2 are real parameters verifying

$$\gamma_2 > \gamma_1 \geq 0.$$

Letting $x = a + b$ and $y = a - b$, we will have to consider also the system verified by (x, y) , that is

$$\begin{cases} x'' + \frac{x'}{r} - \frac{\gamma_2^2}{r^2}x + \frac{\xi^2}{r^2}y - 2f_d^2x &= -(1 - f_d^2)x \\ y'' + \frac{y'}{r} - \frac{\gamma_1^2}{r^2}y + \frac{\xi^2}{r^2}y &= -(1 - f_d^2)y \end{cases} \quad (1.7)$$

with

$$\gamma^2 = \frac{\gamma_1^2 + \gamma_2^2}{2} \quad \text{and} \quad \xi^2 = \frac{\gamma_2^2 - \gamma_1^2}{2}.$$

Let us give a precise description of two basis of solutions for the system (1.6), one base being defined near 0, and one other base being defined near $+\infty$. Let us give the following definition

Definition 1.1 *We say that*

1. $a = O(f)$ at 0 if there exists $R > 0$ and $C > 0$ such that

$$\forall r \in]0, R], \quad |a(r)| \leq C|f(r)|.$$

2. a has the behavior f at 0, and we denote $a \sim_0 f$, if there exists a map g , such that

$$\lim_0 g = 0, \quad |a - f| = O(fg).$$

3. $a = o(f)$ at 0 if there exists a map g , such that

$$\lim_0 g = 0, \quad a = fg.$$

We will use the same convention at $+\infty$.

We will consider that (d, γ_1, γ_2) is allowed to move into the set

$$\mathcal{D} = \{(d, \gamma_1, \gamma_2) \in (\mathbb{R}_+)^3; d \geq 1; \gamma_2 > 1; 0 \leq \gamma_1 \leq \gamma_2 < \gamma_1 + 2d + 2\}.$$

The condition $\gamma_1 \leq \gamma_2 < \gamma_1 + 2d + 2$ and $\gamma_2 > 1$ is satisfied for $\gamma_1 = |n - d|$ and $\gamma_2 = n + d$, whenever $d \geq 1$ and $n \geq 1$. Moreover, we don't need to use more general (γ_1, γ_2) in the course of the paper. We will need the following subsets of \mathcal{D} .

$$\mathcal{D}_1 = \{(d, \gamma_1, \gamma_2) \in \mathcal{D}; \gamma_1 > 0\}, \quad \text{that is } n \neq d$$

and

$$\mathcal{D}_2 = \{(d, \gamma_1, \gamma_2) \in \mathcal{D}; 0 \leq \gamma_1 < \frac{1}{4}; -\gamma_1 - \gamma_2 + 2d + 2 > 0; -\gamma_2 + 2d + 1 > 0\},$$

$$\text{that is } |n - d| < \frac{1}{4}. \tag{1.8}$$

Let us recall the following expansion for f_d (see [4])

$$f_d(r) = 1 - \frac{d^2}{2r^2} + O\left(\frac{1}{r^4}\right) \text{ near } +\infty \tag{1.9}$$

and

$$f_d(r) = ar^d - \frac{a}{4(d+1)}r^{d+2} + O(r^{d+4}) \text{ near } 0, \tag{1.10}$$

for some $a > 0$.

Then, we can state the following theorem, about a base of solutions defined near 0.

Theorem 1.3 *For all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, there exist four independent solutions (a, b) of (1.6) verifying the following conditions*

1.

$$(a_1(r), b_1(r)) \sim_0 (O(r^{\gamma_2+2d+2}), r^{\gamma_2}) \text{ and } (a'_1(r), b'_1(r)) \sim_0 (O(r^{\gamma_2+2d+1}), \gamma_2 r^{\gamma_2-1}).$$

2.

$$(a_2(r), b_2(r)) \sim_0 \begin{cases} (O(r^2\theta(r)), r^{-\gamma_2}) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (O(r^{-\gamma_2+2d+2}), r^{-\gamma_2}) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$$

$$(a'_2(r), b'_2(r)) \sim_0 \begin{cases} (O(r\theta(r)), -\gamma_2 r^{-\gamma_2-1}) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (O(r^{-\gamma_2+2d+1}), -\gamma_2 r^{-\gamma_2-1}) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$$

where

$$\theta(r) = \begin{cases} \frac{-r^{\gamma_1-2} + r^{-\gamma_2+2d}}{\gamma_1 + \gamma_2 - 2d - 2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ -r^{\gamma_1-2} \log r & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0. \end{cases}$$

3.

$(a_3(r), b_3(r)) \sim_0 (r^{\gamma_1}, O(r^{\gamma_1+2d+2}))$ and, if $\gamma_1 \neq 0$ $(a'_3(r), b'_3(r)) \sim_0 (\gamma_1 r^{\gamma_1-1}, O(r^{\gamma_1+2d+1}))$ while, if $\gamma_1 = 0$, $(a'_3(r), b'_3(r)) = (O(r), O(r^{2d+1}))$.

4.

$$(a_4(r), b_4(r)) \sim_0 \begin{cases} (r^{-\gamma_1}, O(r^2\tilde{\theta}(r))) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (\tau(r), O(\tau(r)r^{2d+2})) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$$

and

$$(a'_4(r), b'_4(r)) \sim_0 \begin{cases} (r^{-\gamma_1-1}, O(r\tilde{\theta}(r))) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \\ (\tau'(r), O(\tau'(r)r^{2d+2})) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$$

where

$$\tilde{\theta}(r) = \begin{cases} \frac{-r^{\gamma_2-2} + r^{-\gamma_1+2d}}{\gamma_1 + \gamma_2 - 2d - 2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ -r^{\gamma_2-2} \log r & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \end{cases}$$

and

$$\tau(r) = \begin{cases} \frac{r^{-\gamma_1} - r^{\gamma_1}}{2\gamma_1} & \text{if } \gamma_1 \neq 0 \\ -\log r & \text{if } \gamma_1 = 0. \end{cases}$$

5. For $j = 1$ and for $j = 3$, for all $r > 0$, the maps

$$(d, \gamma_1, \gamma_2) \mapsto (a_j(r), a'_j(r), b_j(r), b'_j(r)) \text{ are continuous in } \mathcal{D}.$$

6. For $j = 1$ and for $j = 3$, and for all $r > 0$, $(a_j(r), a'_j(r), b_j(r), b'_j(r))$ is derivable wrt to γ_1 and wrt γ_2 , whenever $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, and $\gamma_2 > \gamma_1$.

Moreover the map $(d, \gamma_1, \gamma_2) \mapsto \frac{\partial}{\partial \gamma_i} (a_j(r), a'_j(r), b_j(r), b'_j(r))$ is continous, for $i = 1$ and $i = 2$. And we have

$$\left(\frac{\partial a_1}{\partial \gamma_i}, \frac{\partial a'_1}{\partial \gamma_i}, \frac{\partial b_1}{\partial \gamma_i}, \frac{\partial b'_1}{\partial \gamma_i} \right)(r) \sim_0 \log r (O(r^{\gamma_2+2d+2}), O(r^{\gamma_2+2d+1}), r^{\gamma_2}, \gamma_2 r^{\gamma_2-1}) \quad (1.11)$$

and, if $\gamma_1 \neq 0$

$$\begin{aligned} & \left(\frac{\partial a_3}{\partial \gamma_i}, \frac{\partial a'_3}{\partial \gamma_i}, \frac{\partial b_3}{\partial \gamma_i}, \frac{\partial b'_3}{\partial \gamma_i} \right)(r) \\ & \sim_0 \log r (r^{\gamma_1}, \gamma_1 r^{\gamma_1-1} + O(r^{\gamma_1+1}), O(r^{\gamma_1+2d+2}), O(r^{\gamma_1+2d+1})) \end{aligned} \quad (1.12)$$

7. For $j = 2$ or for $j = 4$, the same notation (a_j, b_j) is used for two solutions, one of them being defined for $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$, the other one being defined for $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$.

Moreover, for each domain \mathcal{D}_i , $i = 1, 2$ and for all $r > 0$ the maps $(d, \gamma_1, \gamma_2) \mapsto (a_j(r), a'_j(r), b_j(r), b'_j(r))$ are continuous in \mathcal{D}_i . For each $r > 0$, the partial derivability of $(a_j(r), a'_j(r), b_j(r), b'_j(r))$ wrt γ_1 or wrt γ_2 is also true separately in each domain \mathcal{D}_i , $i = 1, 2$.

Let us remark that our method of construction near 0 doesn't permit to obtain smooth solutions wrt the parameter $(d, \gamma_1, \gamma_2) \in \mathcal{D}$ and keeping the behavior of (a_2, b_2) or the behavior of (a_4, b_4) at 0 for all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$. It is not a problem for us, since in our final proof of Theorem 1.2, we only need two independent smooth solutions wrt (d, γ_1, γ_2) and having bounded behaviors at 0. Also, we don't have to use the derivability of (a_2, b_2) and (a_4, b_4) wrt γ_1 and γ_2 .

The second theorem is about a base of solutions defined near $+\infty$.

Theorem 1.4 *We suppose that $\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 > 0$. Let us denote*

$$n = \sqrt{\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2}.$$

1. *We have a base of four solutions (a, b) of (1.6), with given behaviors at $+\infty$. In order to distinguish these solutions from the solutions defined in Theorem 1.3, we use the notation (u_i, v_i) , $i = 1, \dots, 4$, for these solutions. We have*

$$(u_1(r), v_1(r)) \sim_{r \rightarrow +\infty} \left(\frac{e^{\sqrt{2}r}}{\sqrt{r}}, \frac{e^{\sqrt{2}r}}{\sqrt{r}} \right) (1 + O(r^{-2}));$$

$$(u_2(r), v_2(r)) \sim_{r \rightarrow +\infty} \left(\frac{e^{-\sqrt{2}r}}{\sqrt{r}}, \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \right) (1 + O(r^{-2}));$$

and

$$(u_3(r), v_3(r)) \sim_{r \rightarrow +\infty} (r^{-n}, -r^{-n}) (1 + O(r^{-2}));$$

$$(u_4(r), v_4(r)) \sim_{r \rightarrow +\infty} (r^n, -r^n) (1 + O(r^{-2})).$$

2. *Except for $j = 2$, the construction of (u_j, v_j) is done separately for each compact subset \mathcal{K} of \mathcal{D} . For each of the four solutions and for all $r > 0$ the map $(d, \gamma_1, \gamma_2) \mapsto (u_j(r), u'_j(r), v_j(r), v'_j(r))$ is continuous on \mathcal{K} . There partial derivatives wrt γ_1 and wrt γ_2 exist whenever $\gamma_1 < \gamma_2$ and are continuous. We have*

$$\begin{aligned} & \left(\frac{\partial u_1}{\partial \gamma_i}, \frac{\partial u'_1}{\partial \gamma_i}, \frac{\partial v_1}{\partial \gamma_i}, \frac{\partial v'_1}{\partial \gamma_i} \right) (r) \\ & \sim_{r \rightarrow +\infty} \frac{e^{\sqrt{2}r}}{\sqrt{r}} \log r (O(r^{-2}), O(r^{-3}), O(r^{-2}), O(r^{-3})) \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\partial u_2}{\partial \gamma_i}, \frac{\partial u'_2}{\partial \gamma_i}, \frac{\partial v_2}{\partial \gamma_i}, \frac{\partial v'_2}{\partial \gamma_i} \right)(r) \\
& \sim_{r \rightarrow +\infty} \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \log r (O(r^{-2}), O(r^{-3}), O(r^{-2}), O(r^{-3})) \\
& \left(\frac{\partial u_3}{\partial \gamma_i}, \frac{\partial u'_3}{\partial \gamma_i}, \frac{\partial v_3}{\partial \gamma_i}, \frac{\partial v'_3}{\partial \gamma_i} \right)(r) \\
& \sim_{r \rightarrow +\infty} \log r (r^n, O(r^{n-1}), -r^n, O(r^{n-1})) (1 + O(r^{-2})) \\
& \left(\frac{\partial u_4}{\partial \gamma_i}, \frac{\partial u'_4}{\partial \gamma_i}, \frac{\partial v_4}{\partial \gamma_i}, \frac{\partial v'_4}{\partial \gamma_i} \right)(r) \\
& \sim_{r \rightarrow +\infty} \log r (r^{-n}, O(r^{-n-1}), -r^{-n}, O(r^{-n-1})) (1 + O(r^{-2})).
\end{aligned}$$

Let us remark that, by our construction, the solution (u_j, v_j) depends on the given compact set \mathcal{K} , except for $j = 2$. But, for $j = 1$, we can say that this difficulty disappears after the proof of Theorem 1.3, since the definition of (a_1, b_1) is the same for all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$. For the other solutions, called (u_3, v_3) and (u_4, v_4) , we will have to make sure that the parameter (d, γ_1, γ_2) stays in a compact set, as soon as we want and use the continuity and the derivability of these solutions wrt the parameters.

In [1] we have already give the behaviors of a base of solutions at 0 and at $+\infty$. But the smooth dependence of the solutions wrt the parameters, announced in Theorem 1.3 and Theorem 1.4, was not taken into account in this previous paper. In the present paper, the continuity wrt to (d, γ_1, γ_2) , specially of the five solutions (a_3, b_3) and (a_1, b_1) (defined at 0) and (u_1, v_1) , (u_2, v_2) , (u_3, v_3) , (u_4, v_4) (defined at $+\infty$) and there derivability wrt γ_1 and γ_2 , are essential and are not entirely trivial facts. Indeed, although it is clear by the ODE theory that for any given Cauchy data $(a_0, a'_0, b_0, b'_0) \in \mathbb{R}^4$ at some $r_0 > 0$, there exists a solution of (1.6) that is continuous wrt (d, γ_1, γ_2) and derivable wrt γ_1 and γ_2 , it is not clear that this solution keeps the same behavior at 0 and at $+\infty$ for all the values of $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, and this is generally false.

Now, let us rely the problem (1.6) to an eigenvalue problem.

Let $0 \leq \gamma_1 < \gamma_2$, $\mu \in \mathbb{R}$ and $\varepsilon > 0$ be given and let us consider the following system

$$\begin{cases} a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2} a - \frac{1}{\varepsilon^2} f^2 a - \frac{1}{\varepsilon^2} f^2 b & = -\frac{1}{\varepsilon^2} \mu (1 - f^2) a \\ b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2} b - \frac{1}{\varepsilon^2} f^2 b - \frac{1}{\varepsilon^2} f^2 a & = -\frac{1}{\varepsilon^2} \mu (1 - f^2) b \end{cases} \quad (1.13)$$

for $r \in]0, 1]$, with the notation

$$f(r) = f_d\left(\frac{r}{\varepsilon}\right)$$

and the condition

$$a(1) = b(1) = 0.$$

Let us explain in which sense this can be considered as an eigenvalue problem.

We define, for a given $\gamma_1 \geq 0$

$$\mathcal{H}_{\gamma_1} = \{r \mapsto (a(r), b(r)); (ae^{i\gamma_1\theta}, be^{i\theta}) \in H_0^1(B(0, 1)) \times H_0^1(B(0, 1))\},$$

where (r, θ) are the polar coordinates in \mathbb{R}^2 .

We endow \mathcal{H}_{γ_1} with the scalar product

$$\langle (a, b) | (u, v) \rangle = \frac{1}{2\pi} \int_{B(0,1)} \nabla(ae^{i\gamma_1\theta}) \cdot \nabla(be^{i\theta}) dx = \int_0^1 (ra'u' + rb'v' + \frac{\gamma_1^2}{r}au + \frac{1}{r}bv) dr$$

and then \mathcal{H}_{γ_1} is a Hilbert space.

Let \mathcal{H}'_{γ_1} be the topological dual space of \mathcal{H}_{γ_1} .

We consider the following operator $\mathcal{T}_{\gamma_1, \gamma_2} : \mathcal{H}_{\gamma_1} \rightarrow \mathcal{H}'_{\gamma_1}$

$$\mathcal{T}_{\gamma_1, \gamma_2}(a, b) = \left(\begin{array}{l} -e^{-i\gamma_1\theta} \Delta e^{i\gamma_1\theta} a + \frac{1}{\varepsilon^2} f^2 a + \frac{1}{\varepsilon^2} f^2 b \\ -e^{-i\gamma_2\theta} \Delta e^{i\gamma_2\theta} b + \frac{1}{\varepsilon^2} f^2 b + \frac{1}{\varepsilon^2} f^2 a. \end{array} \right) \quad (1.14)$$

Then we have

$$\begin{aligned} & \langle \mathcal{T}_{\gamma_1, \gamma_2}(a, b), (u, v) \rangle_{\mathcal{H}', \mathcal{H}} \\ &= \frac{1}{2\pi} \int_{B(0,1)} (\nabla(e^{i\gamma_1\theta} a) \cdot \nabla(e^{-i\gamma_1\theta} \bar{u}) + \nabla(e^{i\gamma_2\theta} b) \cdot \nabla(e^{-i\gamma_2\theta} \bar{v}) + \frac{r}{\varepsilon^2} f^2 (a+b)(u+v)) dx \\ &= \int_0^1 (ra'u' + rb'v' + \frac{\gamma_1^2}{r} au + \frac{\gamma_2^2}{r} bv + \frac{r}{\varepsilon^2} f^2 (a+b)(u+v)) dr. \end{aligned}$$

We remark that

$$\begin{aligned} ((a, b), (u, v)) &\mapsto \int_{B(0,1)} (\nabla(e^{i\gamma_1\theta} a) \cdot \nabla(e^{-i\gamma_1\theta} u) + \nabla(e^{i\gamma_2\theta} b) \cdot \nabla(e^{-i\gamma_2\theta} v) \\ &\quad + \frac{1}{\varepsilon^2} f^2 (a+b)(u+v)) dx \end{aligned}$$

is a scalar product on \mathcal{H}_{γ_1} . So, $\mathcal{T}_{\gamma_1, \gamma_2}$ is an isomorphism, by the Riesz Theorem.

Last, let us define the embedding

$$\begin{aligned} I : \mathcal{H}_{\gamma_1} &\rightarrow \mathcal{H}'_{\gamma_1} \\ (a, b) &\mapsto ((u, v) \mapsto \int_0^1 r(au + bv) dr) \end{aligned}$$

Since the embedding $H_0^1(B(0, 1)) \times H_0^1(B(0, 1)) \subset L^2(B(0, 1)) \times L^2(B(0, 1))$ is compact, then I is compact.

For $\mu \in \mathbb{R}$, we define the operator

$$\Phi = \mathcal{T}_{\gamma_1, \gamma_2} - \frac{\mu}{\varepsilon^2} (1 - f^2) I.$$

Then

$$\mathcal{T}_{\gamma_1, \gamma_2}^{-1} \Phi = id_{\mathcal{H}_{\gamma_1}} - \mu \mathcal{T}_{\gamma_1, \gamma_2}^{-1} \mathcal{C},$$

where we define

$$\mathcal{C} = \frac{1}{\varepsilon^2} (1 - f^2) I. \quad (1.15)$$

Since \mathcal{C} is a compact operator and thanks to the continuity of $\mathcal{T}_{\gamma_1, \gamma_2}^{-1}$, then $\mathcal{T}_{\gamma_1, \gamma_2}^{-1} \mathcal{C}$ is a compact operator from \mathcal{H}_{γ_1} into itself. By the standard theory of self adjoint compact

operators, we can deduce that the kernel $\mathcal{N}(\mathcal{T}_{\gamma_1, \gamma_2} - \mu\mathcal{C})$ has a finite dimension in \mathcal{H}_{γ_1} and that the range $\mathcal{R}(\mathcal{T}_{\gamma_1, \gamma_2} - \mu\mathcal{C})$ is closed in \mathcal{H}'_{γ_1} and that

$$\mathcal{R}(\mathcal{T}_{\gamma_1, \gamma_2} - \mu\mathcal{C}) = \mathcal{N}(\mathcal{T}_{\gamma_1, \gamma_2} - \mu\mathcal{C})^\perp.$$

When $\mathcal{N}(\mathcal{T}_{\gamma_1, \gamma_2} - \mu\mathcal{C}) \neq \emptyset$, we say that μ is a \mathcal{C} -eigenvalue of $\mathcal{T}_{\gamma_1, \gamma_2}$.

There exists a Hilbertian base of \mathcal{H}_{γ_1} formed of eigenvectors of $\mathcal{T}_{\gamma_1, \gamma_2}^{-1}\mathcal{C}$. Let $x \in \mathcal{H}_{\gamma_1}$ be an eigenvector associated to an eigenvalue γ of $\mathcal{T}_{\gamma_1, \gamma_2}^{-1}\mathcal{C}$. Then $\gamma \neq 0$ and we have

$$\mathcal{T}_{\gamma_1, \gamma_2}(x) - \frac{1}{\gamma}\mathcal{C}x = 0.$$

Then $\frac{1}{\gamma}$ is a \mathcal{C} -eigenvalue of $\mathcal{T}_{\gamma_1, \gamma_2}$. In what follows, we simply call μ an eigenvalue. Because of the dependence on ε , we denote it by $\mu(\varepsilon)$.

Now let us define $m_{\gamma_1, \gamma_2}(\varepsilon)$ as the first eigenvalue for the above eigenvalue problem in \mathcal{H}_{γ_1} , that is

$$m_{\gamma_1, \gamma_2}(\varepsilon) = \inf_{(a, b) \in \mathcal{H}_{\gamma_1} \times \mathcal{H}_{\gamma_1} / \{(0, 0)\}} \frac{\int_0^1 (ra'^2 + rb^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + \frac{r}{\varepsilon^2}f_d^2(\frac{r}{\varepsilon})(a+b)^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f_d^2(\frac{r}{\varepsilon}))(a^2 + b^2)dr} \quad (1.16)$$

and let us define

$$m_0(\varepsilon) = \inf_{a \in \mathcal{H}_d / \{0\}} \frac{\int_0^1 (ra'^2 + \frac{d^2}{r}a^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f_d^2(\frac{r}{\varepsilon}))a^2dr} \quad (1.17)$$

It is classical that these infimum are attained. Considering the rescaling $(\tilde{a}, \tilde{b})(r) = (a(\varepsilon r), b(\varepsilon r))$ and an extension by 0 outside $[0, 1/\varepsilon]$, we see that $\varepsilon \mapsto m_{\gamma_1, \gamma_2}(\varepsilon)$ decreases when ε decreases. Then $\lim_{\varepsilon \rightarrow 0} m_{\gamma_1, \gamma_2}(\varepsilon)$ exists.

Moreover, $m_{\gamma_1, \gamma_2}(\varepsilon)$ is a simple eigenvalue and there exists an eigenvector (a, b) verifying

$$a(r) \geq -b(r) \geq 0 \text{ for all } r > 0.$$

Also, $m_0(\varepsilon)$ is realized by some function $a(r) \geq 0$.

We consider that $d > 0$, that $\gamma_2 > \gamma_1 \geq 0$ are given and we suppose that

$$\frac{\gamma_2^2 + \gamma_1^2}{2} > d^2.$$

Let $\mu(\varepsilon)$ be a bounded eigenvalue. Then, we can suppose that

$$\mu(\varepsilon) \rightarrow \mu \text{ as } \varepsilon \rightarrow 0$$

where $\mu \geq 0$. Let

$$\omega_\varepsilon = (a_\varepsilon, b_\varepsilon)$$

be an eigenvector associated to $\mu(\varepsilon)$. We define

$$\tilde{\omega}_\varepsilon(r) = \omega_\varepsilon(\varepsilon r), \quad \text{for } r \in [0, \frac{1}{\varepsilon}].$$

An examination of the proof of Theorem 1.3 gives, for some constants A_ε and B_ε ,

$$\tilde{\omega}_\varepsilon \sim_{r \rightarrow 0} A_\varepsilon(r^{\gamma_1}, o(r^{\gamma_1})) + B_\varepsilon(o(r^{\gamma_2}), r^{\gamma_2}).$$

We may suppose that $\max\{|A_\varepsilon|, |B_\varepsilon|\} = 1$. Then by the ODE theory

$$\tilde{\omega}_\varepsilon \rightarrow \omega_0, \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly on each compact subset of $[0, +\infty]$, where $\omega_0 = (a_0, b_0)$ verifies

$$\begin{cases} a_0'' + \frac{a_0'}{r} - \frac{\gamma_1^2}{r^2} a_0 - f_d^2 a_0 - f_d^2 b_0 &= -\mu(1 - f_d^2) a_0 \\ b_0'' + \frac{b_0'}{r} - \frac{\gamma_2^2}{r^2} b_0 - f_d^2 b_0 - f_d^2 a_0 &= -\mu(1 - f_d^2) b_0 \end{cases} \quad (1.18)$$

It seems to us that this eigenvalue problem is better suited to our purpose than that used in previous work. Nevertheless, the following theorem can be deduced from previous work on the subject [5], [8], [6] and [7].

Theorem 1.5 *For all $d \geq 1$,*

(i) *there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $\frac{m_0(\varepsilon)-1}{\varepsilon^2} \geq C$; $m_0(\varepsilon) \rightarrow 1$ and there exists an associated eigenvector a such that $\tilde{a}_\varepsilon \rightarrow f_d$, uniformly on each $[0, R]$, $R > 0$.*

(ii) *$m_{d-1, d+1}(\varepsilon) > 1$ and $\frac{m_{d-1, d+1}(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$.*

(iii) *for $d > 1$ and $n \geq 2d - 1$, there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $\frac{m_{|d-n|, d+n}(\varepsilon)-1}{\varepsilon^2} \geq C$.*

(iv) *There exists an eigenvector ω_ε associated to the eigenvalue $m_{d-1, d+1}(\varepsilon)$ such that $\|(1 - f_d^2)^{\frac{1}{2}}(\tilde{\omega}_\varepsilon - F_d)\|_{L^2(B(0, \frac{1}{\varepsilon}))} \rightarrow 0$, as $\varepsilon \rightarrow 0$, where $F_d = (f_d' + \frac{d}{r} f_d, f_d' - \frac{d}{r} f_d)$ appears in Theorem 1.1.*

Let us remark that the function f used here ($f(r) = f_d(\frac{r}{\varepsilon})$) is not the same as the one used in the previous works [8], [6] and [7]. For this reason, we will give a direct proof of (i) in the appendix and we will give a proof of (iv) in the course of the paper. The norm L^∞ , used in [7] is a nonsense here, since $F_d(\frac{1}{\varepsilon}) \neq 0$ and $\tilde{\omega}_\varepsilon(\frac{1}{\varepsilon}) = 0$.

We have

Proposition 1.1 (i) *With the notation above, if $\mu(\varepsilon) \rightarrow \mu$, if $\tilde{\omega}_\varepsilon \rightarrow \omega_0$, if $\frac{\gamma_2^2 + \gamma_1^2}{2} - \mu d^2 > 0$ and if ω_0 blows up at $+\infty$, then $\frac{\mu(\varepsilon)-1}{\varepsilon^2} \geq C$, where C is a given positive number, independent of ε .*

(ii) *If there exists some bounded solution (a, b) of (1.6), then there exists an eigenvalue $\mu(\varepsilon)$ verifying $\mu(\varepsilon) - 1 \rightarrow 0$.*

With the additional condition $\frac{\gamma_2^2 + \gamma_1^2}{2} - d^2 \geq 1$ and $m_{\gamma_1, \gamma_2}(\varepsilon) \geq 1$, there exists an eigenvalue $\mu(\varepsilon)$ verifying $\frac{\mu(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$.

And we have

Proposition 1.2 *Let $d > 1$ be given. For all $n \in]1, d + 1[$, there exists $C_n > 0$ independent of ε such that*

$$m_{|d-n|, d+n}(\varepsilon) \leq 1 - C_n.$$

None of the propositions above are very new, because we have already given the proof of Proposition 1.2, in [1]. Also, Proposition 1.1 can be found there, but for a slightly different eigenvalue problem.

There are two new results in this paper, that will allow us to reach our goal, that is to prove Theorem 1.2. The first one is that the solution having the least behavior at 0 (ie (a_1, b_1) , that tends the faster to 0 as $r \rightarrow 0$) blows up exponentially at $+\infty$ and that the solution having the least behavior at $+\infty$ (ie (u_2, v_2) , that tends exponentially to 0 as $r \rightarrow +\infty$) has the greater blowing up behavior at 0. In other words

Proposition 1.3 *When $d > 0$ and when $\gamma_2 \geq \gamma_1 \geq 0$, $(\gamma_2^2 + \gamma_1^2)/2 \geq d^2$, then the behavior of (a_1, b_1) at $+\infty$ is the behavior of (u_1, v_1) and the behavior of (u_2, v_2) at 0 is the behavior of (a_2, b_2) .*

The second result is the following

Proposition 1.4 *When $\frac{\gamma_1^2 + \gamma_2^2}{\varepsilon^2} - d^2 > 0$, if there exists a bounded solution $\omega = (a, b)$ of (1.6), then we have $m_{\gamma_1, \gamma_2}(\varepsilon) - 1 \rightarrow 0$ and there exists an eigenvector $\omega_\varepsilon = (a_\varepsilon, b_\varepsilon)$ associated to $m_{\gamma_1, \gamma_2}(\varepsilon)$ and such that $\tilde{\omega}_\varepsilon$ tends to ω , uniformly on each $[0, R]$, $R > 0$.*

Propositions 1.3 and 1.4 allow us to achieve the proof of Theorem 1.2. More, we can also enonce

Theorem 1.6 *For $d \geq 1$, $n > 1$, $\gamma_1 = |n - d|$ and $\gamma_2 = n + d$, there is no eigenvalue $\mu(\varepsilon)$, with eigenvector in $\mathcal{H}_{|n-d|}$, such that $\frac{\mu(\varepsilon) - 1}{\varepsilon^2} \rightarrow 0$, as $\varepsilon \rightarrow 0$.*

Let us remark that the Hilbert space \mathcal{H}_{γ_1} does depend on γ_1 . In other words, the notation $m_{\gamma_1, \gamma_2}(\varepsilon)$ doesn't mean the continuity on this simple eigenvalue wrt the parameter (γ_1, γ_2) . The theorem on this subject, in [3], doesn't work here.

In Part II and Part III, we give detailed proves of Theorem 1.3 and of Theorem 1.4, although the proves are altogether technical and classical. But these theorems play a crucial role in our final proof.

In Part IV, we prove Proposition 1.3. In Part V, in order to make the paper as self contained as possible, we give the proof of Proposition 1.1 and of Proposition 1.2. In Part VI, we give the proof of Proposition 1.4 and also the proof of Theorem 1.5 (iv). We chose to give a direct proof of this claim, since the eigenvalue problem is not exactly the same as in the previous works on the subject and the function f_d is not exactly the same, too. In Part VII, we conclude the proof of Theorem 1.2. Last, in the appendix, we give a direct proof of Theorem 1.5 (i).

We will use Theorem 1.2 in a separated paper.

2 The possible behaviors at zero and the dependence of the solutions wrt the parameters

Let us explain the way to prove the existence and the continuity wrt the parameter $(d, \gamma_1, \gamma_2) \in \mathcal{D}$ of a solution having a given behavior at 0, eg the solution (a_1, b_1) . We construct some solution (a_1, b_1) such that for all compact subset \mathcal{K} of \mathcal{D} , there exists some $R > 0$, depending only on \mathcal{K} and some $C > 0$, also depending only on \mathcal{K} , such that for all $r \in]0, R]$ and all $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, we have

$$|a_1(r)| + |b_1(r) - r^{\gamma_2}| \leq Cr^{\gamma_2+2d+1}$$

and such that, for all $r \in]0, R]$, $(d, \gamma_1, \gamma_2) \mapsto (a_1(r), a_1'(r), b_1(r), b_1'(r))$ is continuous on \mathcal{K} , and derivable wrt γ_1 and wrt γ_2 . First, the construction is done for $r \in]0, R]$. Then the definition of this solution in $[0, +\infty[$ and the continuity wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, for all $r > 0$, follows from the ODE Theory.

We use a constructive method, similar to the proof of the Banach fixed point Theorem. For each solution, we define a fixed point problem of the form

$$(a, b) = \Phi(a, b)$$

whose solutions verify the differential system that we have to solve. Then we define two maps $r \mapsto \zeta_1(r)$ and $r \mapsto \zeta_2(r)$. In order to construct a solution (a, b) , verifying, for each compact subset \mathcal{K} of \mathcal{D} ,

$$|a(r)\zeta_1^{-1}(r)| + |b(r)\zeta_2^{-1}(r) - 1| \leq Cr^2$$

for all $r < R$ and with R and C depending only on \mathcal{K} , we define two sequences

$$\begin{cases} \alpha_0 = 0 & \beta_0 = \zeta_2 \\ (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \end{cases} \quad (2.19)$$

Then, for each compact subset \mathcal{K} of \mathcal{D} , we prove that for all $0 < r < 1$ and for all $(d, \gamma_1, \gamma_2) \in \mathcal{K}$ we have

$$|\alpha_{k+1} - \alpha_k|(r) \leq C\zeta_1(r)r^2(\|\zeta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])}), \quad (2.20)$$

$$|\beta_{k+1} - \beta_k|(r) \leq C\zeta_2(r)r^2(\|\zeta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])}) \quad (2.21)$$

and

$$|\alpha_1 - \alpha_0|(r) \leq Cr^2\zeta_1(r), \quad |\beta_1 - \beta_0|(r) \leq Cr^2\zeta_2(r) \quad (2.22)$$

where C depends only on \mathcal{K} .

Then we deduce that

$$\begin{aligned} & \|\zeta_1^{-1}(\alpha_{k+1} - \alpha_k)\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\beta_{k+1} - \beta_k)\|_{L^\infty([0,r])} \\ & \leq (Cr)^{2k}(\|\zeta_1^{-1}(\alpha_1 - \alpha_0)\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\beta_1 - \beta_0)\|_{L^\infty([0,r])}) \end{aligned}$$

We choose R such that $CR < 1$ and we define,

$$\text{for all } 0 < r < R \quad a(r) = \sum_{k=0}^{k=+\infty} (\alpha_{k+1} - \alpha_k) + \alpha_0, \quad b(r) = \sum_{k=0}^{k=+\infty} (\beta_{k+1} - \beta_k) + \beta_0. \quad (2.23)$$

Then we have $(a, b) = \Phi(a, b)$ and the continuity of $(a(r), b(r))$ wrt (d, γ_1, γ_2) follows from the continuity of (α_k, β_k) for all k and from the convergence of the sums uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Then we have to prove the uniform convergence wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$ of the sums

$$\sum_{k=0}^{k=+\infty} (\alpha'_{k+1} - \alpha'_k), \quad \text{and} \quad \sum_{k=0}^{k=+\infty} (\beta'_{k+1} - \beta'_k),$$

in order to prove the continuity of $(a'(r), b'(r))$ wrt (d, γ_1, γ_2) .

Then, since the derivability of (a, a', b, b') wrt γ_1 and wrt γ_2 is needed only for the solutions (a_1, b_1) and (a_3, b_3) , we will prove it only for the solution (a_1, b_1) , but the proof can be adapted for the other solutions.

In what follows, we will use the following forms of the first equation of (1.6)

$$(r^{2\gamma_1+1}(ar^{-\gamma_1})')' = r^{\gamma_1+1}(f_d^2 b - (1 - f_d^2)a) \quad (2.24)$$

or

$$(r^{-2\gamma_1+1}(ar^{\gamma_1})')' = r^{-\gamma_1+1}(f_d^2 b - (1 - f_d^2)a) \quad (2.25)$$

or, when γ_1 may reach 0,

$$(r\tau^2(\tau^{-1}a)')' = r\tau(f_d^2 b - (1 - 2f_d^2)a) \quad (2.26)$$

where

$$\tau(r) = \begin{cases} \frac{r^{-\gamma_1} - r^{\gamma_1}}{2\gamma_1} & \text{if } \gamma_1 > 0 \\ -\log r & \text{if } \gamma_1 = 0. \end{cases}$$

and the following form of the second equation of (1.6)

$$(r^{2\gamma_2+1}(br^{-\gamma_2})')' = r^{\gamma_2+1}(f_d^2 a - (1 - 2f_d^2)b) \quad (2.27)$$

or

$$(r^{-2\gamma_2+1}(br^{\gamma_2})')' = r^{-\gamma_2+1}(f_d^2 a - (1 - 2f_d^2)b). \quad (2.28)$$

We denote

$$\nu : r \mapsto r.$$

2.1 The solution (a_1, b_1) .

Let us consider the integral system

$$\begin{cases} a &= r^{\gamma_1} + r^{\gamma_1} \int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1} (f_d^2 b - (1 - 2f_d^2)a) ds dt \\ b &= r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2 a - (1 - 2f_d^2)b) ds dt. \end{cases} \quad (2.29)$$

By the reformulation given at the begining of the section, it is clear that any solution of this system is a solution of (1.6).

Let us denote by

$$\Phi(a, b)$$

the rsm of (2.29).

Following the method described just above, we prove

Proposition 2.5 *There exists a solution (a_1, b_1) of (1.6) such that, for any compact subset \mathcal{K} of \mathcal{D} , there exist some real numbers R and C verifying*

$$\text{for all } r \leq R, \quad |a_1(r)r^{-2d}| + |b_1(r) - r^{\gamma_2}| \leq Cr^{\gamma_2+2}$$

where C and R remain the same for all $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, and $(d, \gamma_1, \gamma_2) \rightarrow (a_1(r), a'_1(r), b_1(r), b'_1(r))$ is continuous. Moreover

$$|a'_1(r)r^{-2d}| + |b'_1(r) - \gamma_2 r^{\gamma_2-1}| \leq Cr^{\gamma_2+1}$$

for all $r < R$ and for some C depending only on \mathcal{K} .

For all $r > 0$, $(a_1(r), a'_1(r), b_1(r), b'_1(r))$ is derivable wrt γ_1 and with respect to γ_2 , as soon as $\gamma_2 > \gamma_1$ and $(d, \gamma_1, \gamma_2) \in \mathcal{D}$ and, for $i = 1, 2$

$$\left| \frac{\partial a_1}{\partial \gamma_i} \right|(r) \leq Cr^{\gamma_2+2d+2} |\log r|, \quad \left| \frac{\partial b_1}{\partial \gamma_i} \right|(r) - r^{\gamma_2} \log r \leq Cr^{\gamma_2+2} |\log r|, \quad (2.30)$$

and

$$\left| \frac{\partial a'_1}{\partial \gamma_i} \right|(r) \leq Cr^{\gamma_2+2d+2} |\log r|, \quad \left| \frac{\partial b'_1}{\partial \gamma_i} \right|(r) - \gamma_2 r^{\gamma_2-1} \log r \leq Cr^{\gamma_2+1} |\log r|. \quad (2.31)$$

with the same property for C and R as above.

Proof We define $\zeta_1(r) = r^{\gamma_2+2d}$ and $\zeta_2(r) = r^{\gamma_2}$ and we define (α_k, β_k) by (2.19). For $k \geq 1$, assuming that $\alpha_k - \alpha_{k-1}$ and $\beta_k - \beta_{k-1}$ are continuous wrt (d, γ_1, γ_2) , we prove the continuity of $\alpha_{k+1} - \alpha_k$ and $\beta_{k+1} - \beta_k$ in \mathcal{K} by use of the Lebesgue Theorem. Then, involving the estimate $f_d^2(t) \leq Mt^{2d}$ and $|1 - 2f_d^2(t)| \leq M$, the desired estimate (2.20) remains to the estimation for all $r > 0$, $r < 1$,

$$r^{\gamma_1} \int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1+\gamma_2+2d} ds dt = \frac{r^{\gamma_2+2d+2}}{(-\gamma_1 + \gamma_2 + 2d + 2)(\gamma_1 + \gamma_2 + 2d + 2)} \leq Cr^{\gamma_2+2d+2}$$

where C depends only on \mathcal{K} .

This gives (2.20) and also the estimate of $|\alpha_1 - \alpha_0|$.

Now the desired estimate (2.21) follows from both estimations

$$r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{2\gamma_2+1} ds dt = \frac{r^{\gamma_2+2}}{2(2\gamma_2+2)} \leq Cr^{2+\gamma_2}$$

and

$$r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{2\gamma_2+1+4d} ds dt \leq Cr^{\gamma_2+2+4d}$$

where C depends only on \mathcal{K} . This gives the proof of (2.21) and also the estimate of $|\beta_1 - \beta_0|$.

This terminates the proof of the existence of (a_1, b_1) , the continuity wrt (d, γ_1, γ_2) and the desired behavior at 0.

To prove the continuity of $(a'(r), b'(r))$ wrt to (d, γ_1, γ_2) , we compute

$$(\alpha'_{k+1} - \alpha'_k)(r) = \gamma_1 r^{-1} (\alpha_{k+1} - \alpha_k)(r) + r^{-\gamma_1-1} \int_0^r s^{\gamma_1+1} (f_d^2(\beta_k - \beta_{k-1}) + (1 - 2f_d^2)(\alpha_k - \alpha_{k-1})) ds dt$$

that gives, for $k \geq 1$

$$r^{-\gamma_2-2d}|(\alpha'_{k+1}-\alpha'_k)(r)| \leq (Cr^2)^k \gamma_1 r^{-1} (\|(\alpha_1-\alpha_0)\nu^{-\gamma_2-2d}\|_{L^\infty([0,r])} + \|(\beta_1-\beta_0)\nu^{-\gamma_2}\|_{L^\infty([0,r])}) \\ + \tilde{C}(Cr^2)^{k-1} r^{2d+\gamma_2+1} (\|(\alpha_1-\alpha_0)\nu^{-\gamma_2-2d}\|_{L^\infty([0,r])} + \|(\beta_1-\beta_0)\nu^{-\gamma_2}\|_{L^\infty([0,r])})$$

with \tilde{C} depending only on \mathcal{K} , and consequently

$$r^{-\gamma_2-2d}|(\alpha'_{k+1}-\alpha'_k)(r)| \leq \tilde{C}(Cr^2)^{k-1} r^2$$

with another \tilde{C} depending only on \mathcal{K} . Then the sum

$$\sum_{k=1}^{+\infty} (\alpha'_{k+1}-\alpha'_k)\nu^{-\gamma_2-2d}$$

converges for all $r < R$, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Thus $a'(r)$ is continuous wrt (d, γ_1, γ_2) . Moreover, a direct estimate gives

$$|\alpha'_1(r)| \leq Cr^{\gamma_2+2d+1}$$

for all $r < R$. We deduce that

$$|a'(r)| \leq Cr^{\gamma_2+2d+1}$$

with C depending only on \mathcal{K} .

Now we compute, for $k \geq 1$

$$(\beta'_{k+1}-\beta'_k)(r) = \gamma_2 r^{-1} (\beta_{k+1}-\beta_k)(r) + r^{-\gamma_2-1} \int_0^r s^{\gamma_2+1} (f_d^2(\alpha_k-\alpha_{k-1}) + (1-2f_d^2)(\beta_k-\beta_{k-1})) ds dt$$

that gives

$$r^{-\gamma_2}|(\beta'_{k+1}-\beta'_k)(r)| \leq ((Cr^2)^k \gamma_2 r^{-1} + \tilde{C}(Cr^2)^{k-1}) \\ (\|(\beta_1-\beta_0)\nu^{-\gamma_2}\|_{L^\infty([0,r])} + \|(\alpha_1-\alpha_0)\nu^{-\gamma_2-2d}\|_{L^\infty([0,r])})$$

and consequently

$$r^{-\gamma_2+1}|(\beta'_{k+1}-\beta'_k)(r)| \leq \tilde{C}(Cr^2)^{k-1} r$$

with \tilde{C} depending only on \mathcal{K} . Recalling

$$b'(r) = \sum_{k=1}^{+\infty} (\beta'_{k+1}-\beta'_k)(r) + \beta'_1,$$

a direct calculation gives

$$|\beta'_1 - \beta'_0| \leq Cr^{\gamma_2+1}.$$

This gives that $b'(r)$ is continuous wrt (d, γ_1, γ_2) and that for all $r < R$

$$|b'(r) - \gamma_2 r^{\gamma_2-1}| \leq Cr^{\gamma_2+1}$$

with C depending only on \mathcal{K} .

Now, let us prove the derivability, wrt γ_2 , of $(a_1(r), b_1(r))$, for $0 < r < R$ and for $\gamma_2 > \gamma_1$, $\gamma_1 \geq 0$ and $d > 0$ being given, and the continuity of the derivative function wrt (d, γ_1, γ_2) . First, we use the Lebesgue Theorem to prove by induction that $\alpha_{k+1} - \alpha_k$ and $\beta_{k+1} - \beta_k$ are derivable wrt γ_1 and wrt γ_2 . Then, since a and b are defined for $r \in [0, R]$ by (2.23), it is sufficient to prove that the sums

$$\sum_{k \geq 0} \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2} \quad \text{and} \quad \sum_{j \geq 0} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2}$$

are convergent, for all $r \in [0, R]$, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, for any compact subset \mathcal{K} of $\{(d, \gamma_1, \gamma_2) \in \mathcal{D}, d > 0, \gamma_2 > \gamma_1\}$. In fact, we are going to prove that

$$\sum_{j \geq 0} \|\zeta_1^{-1} |\log \nu|^{-1} \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2}\|_{L^\infty([0, R])} \quad \text{and} \quad \sum_{j \geq 0} \|\zeta_2^{-1} |\log \nu|^{-1} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2}\|_{L^\infty([0, R])}$$

are convergent, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. This will give the desired derivability result, and also the estimate (3.80).

We have

$$\frac{\partial \alpha_0}{\partial \gamma_2} = 0 \quad \text{and} \quad \frac{\partial \beta_0}{\partial \gamma_2} = r^{\gamma_2} \log r.$$

Let $k \geq 0$ be given. We easily verify that

$$\frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2} = r^{\gamma_1} \int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1} (f_d^2 \frac{\partial(\beta_k - \beta_{k-1})}{\partial \gamma_2} - (1 - 2f_d^2) \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial \gamma_2}) ds dt$$

and that

$$\begin{aligned} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2} &= r^{\gamma_2} \log r \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2(\alpha_k - \alpha_{k-1}) - (1 - 2f_d^2)(\beta_k - \beta_{k-1})) ds dt \\ &+ r^{\gamma_2} \int_0^r (-2\gamma_2 - 1) t^{-2\gamma_2-2} \int_0^t s^{\gamma_2+1} (f_d^2(\alpha_k - \alpha_{k-1}) - (1 - 2f_d^2)(\beta_k - \beta_{k-1})) ds dt \\ &+ r^{\gamma_2} \int_0^r -2t^{-2\gamma_2-1} \log t \int_0^t s^{\gamma_2+1} \log s (f_d^2(\alpha_k - \alpha_{k-1}) - (1 - 2f_d^2)(\beta_k - \beta_{k-1})) ds dt \\ &+ r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2 \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial \gamma_2} - (1 - 2f_d^2) \frac{\partial(\beta_k - \beta_{k-1})}{\partial \gamma_2}) ds dt. \end{aligned}$$

Now we estimate, for all $r > 0$

$$\begin{aligned} r^{-\gamma_2-2d} (\log r)^{-1} \left| \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2} \right|(r) &\leq \frac{Mr^2 \|\nu^{-\gamma_2} (\log \nu)^{-1} \frac{\partial(\beta_k - \beta_{k-1})}{\partial \gamma_2}\|_{L^\infty([0, r])}}{(\gamma_2 - \gamma_1 + 2d + 2)(\gamma_2 + \gamma_1 + 2d + 2)} \\ &+ \frac{2r^2 \|\nu^{-\gamma_2-2d} (\log \nu)^{-1} \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial \gamma_2}\|_{L^\infty([0, r])}}{(\gamma_2 + \gamma_1 + 2d + 4)(\gamma_2 - \gamma_1 + 2d + 4)}. \end{aligned} \quad (2.32)$$

Now, let us estimate the first term for $r^{-\gamma_2} (\log r)^{-1} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2}(r)$

$$\begin{aligned}
& \left| \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2(\alpha_k - \alpha_{k-1}) - (1 - 2f_d^2)(\beta_k - \beta_{k-1})) ds dt \right| \\
& \leq \frac{Mr^{4d+2} \|\nu^{-\gamma_2-2d}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,R])}}{(4d+2)(2\gamma_2+4d+2)} + \frac{r^2 \|\nu^{-\gamma_2}(\beta_k - \beta_{k-1})\|_{L^\infty([0,R])}}{2(2\gamma_2+2)}.
\end{aligned}$$

We can estimate the second term as follows

$$\begin{aligned}
& |(\log r)^{-1} \int_0^r (-2\gamma_2 - 1)t^{-2\gamma_2-2} \int_0^t s^{\gamma_2+1} (f_d^2(\alpha_k - \alpha_{k-1}) - (1 - 2f_d^2)(\beta_k - \beta_{k-1})) ds dt| \\
& \leq \frac{Mr^{4d+2} \|\nu^{-\gamma_2-2d}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,R])}}{(4d+2)(2\gamma_2+4d+2)} + \frac{r^2 \|\nu^{-\gamma_2}(\beta_k - \beta_{k-1})\|_{L^\infty([0,R])}}{2(2\gamma_2+2)}.
\end{aligned}$$

We have a similar estimate for the third term. The fourth term gives

$$\begin{aligned}
& |(\log r)^{-1} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2 \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_2} - (1 - 2f_d^2) \frac{\partial(\beta_k - \beta_{k-1})}{\partial\gamma_2}) ds dt| \\
& \leq \frac{Mr^{4d+2} \|\nu^{-\gamma_2-2d}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,R])}}{(4d+2)(2\gamma_2+4d+2)} + \frac{r^2 \|\nu^{-\gamma_2}(\beta_k - \beta_{k-1})\|_{L^\infty([0,R])}}{2(2\gamma_2+2)}.
\end{aligned}$$

Finally, we can find some constant C , independent of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, such that, for all $0 < r < 1$,

$$\begin{aligned}
r^{-\gamma_2-2d} |\log r|^{-1} \left| \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial\gamma_2} \right|(r) & \leq Cr^2 (\|\nu^{-\gamma_2}(\log \nu)^{-1} \frac{\partial(\beta_k - \beta_{k-1})}{\partial\gamma_2}\|_{L^\infty([0,r])} \\
& \quad + \|\nu^{-\gamma_2-2d}(\log \nu)^{-1} \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_2}\|_{L^\infty([0,r])})
\end{aligned}$$

and

$$\begin{aligned}
r^{-\gamma_2} |\log r|^{-1} \left| \frac{\partial(\beta_{k+1} - \beta_k)}{\partial\gamma_2} \right|(r) & \leq Cr^2 (\|\nu^{-\gamma_2-2d}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} \\
& \quad + \|\nu^{-\gamma_2}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])} + \|\nu^{-\gamma_2-2d}(\log \nu)^{-1} \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_2}\|_{L^\infty([0,r])} \\
& \quad + \|\nu^{-\gamma_2}(\log \nu)^{-1} \frac{\partial(\beta_k - \beta_{k-1})}{\partial\gamma_2}\|_{L^\infty([0,r])})
\end{aligned}$$

Summing the both inequalities just above and (2.20) and (2.21) (with $\zeta_1 = \nu^{\gamma_2+2d}$ and $\zeta_2 = \nu^{\gamma_2}$), we get

$$\begin{aligned}
& r^{-\gamma_2-2d} |\log r|^{-1} \left| \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial\gamma_2} \right|(r) + r^{-\gamma_2} |\log r|^{-1} \left| \frac{\partial(\beta_{k+1} - \beta_k)}{\partial\gamma_2} \right|(r) \\
& \leq (Cr^2)^k (\|\nu^{-\gamma_2}(\log \nu)^{-1} \frac{\partial(\beta_1 - \beta_0)}{\partial\gamma_2}\|_{L^\infty([0,r])} + \|\nu^{-\gamma_2-2d}(\alpha_1 - \alpha_0)\|_{L^\infty([0,r])} \\
& \quad + \|\nu^{-\gamma_2-2d}(\log \nu)^{-1} \frac{\partial(\alpha_1 - \alpha_0)}{\partial\gamma_2}\|_{L^\infty([0,r])} + \|\nu^{-\gamma_2}(\log \nu)^{-1} \frac{\partial(\beta_1 - \beta_0)}{\partial\gamma_2}\|_{L^\infty([0,r])}).
\end{aligned}$$

Then, we directly estimate

$$r^{-\gamma_2} |(\log r)^{-1} \frac{\partial(\beta_1 - \beta_0)}{\partial \gamma_2}|(r) \leq \frac{r^2}{\gamma_2 + 1} \quad (2.33)$$

and

$$r^{-\gamma_2 - 2d} |(\log r)^{-1} \frac{\partial(\alpha_1 - \alpha_0)}{\partial \gamma_2}|(r) \leq \frac{Mr^2}{(\gamma_2 - \gamma_1 + 2d + 2)(\gamma_2 + \gamma_1 + 2d + 2)} \quad (2.34)$$

and we deduce that, choosing $r < R$, where R depends only on \mathcal{K} , the sum

$$\sum_{k=0}^{+\infty} \|\nu^{-\gamma_2 - 2d} (\log \nu)^{-1} \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2}\|_{L^\infty([0, R])} + \sum_{k=0}^{+\infty} \|\nu^{-\gamma_2} (\log \nu)^{-1} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2}\|_{L^\infty([0, R])} \quad (2.35)$$

converges, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Consequently, the same claim is true for

$$\sum_{k=0}^{+\infty} \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_2} \quad \text{and} \quad \sum_{k=0}^{+\infty} \frac{\partial(\beta_{k+1} - \beta_k)}{\partial \gamma_2}.$$

We can deduce that a and b , defined by (2.23), are differentiable wrt γ_2 and that the partial differential is continuous wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Moreover, we get the behavior of the derivatives near 0.

Now let us prove the differentiability of $(a_1(r), b_1(r))$ wrt γ_1 . We have

$$\frac{\partial \alpha_0}{\partial \gamma_1} = \frac{\partial \beta_0}{\partial \gamma_1} = 0.$$

By induction, we have that $\frac{\partial \alpha_k}{\partial \gamma_1}$ and $\frac{\partial \beta_k}{\partial \gamma_1}$ exist for all k .

Then we write

$$\begin{aligned} \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_1} &= r^{\gamma_1} \log r \int_0^r t^{-2\gamma_1 - 1} \int_0^t s^{\gamma_1 + 1} (f_d^2(\beta_k - \beta_{k-1}) - (1 - 2f_d^2)(\alpha_k - \alpha_{k-1})) ds dt \\ &+ r^{\gamma_1} \int_0^r -2t^{-2\gamma_1 - 1} \log t \int_0^t s^{\gamma_1 + 1} (f_d^2(\beta_k - \beta_{k-1}) - (1 - 2f_d^2)(\alpha_k - \alpha_{k-1})) ds dt \\ &+ r^{\gamma_1} \int_0^r t^{-2\gamma_1 - 1} \int_0^t s^{\gamma_1 + 1} \log s (f_d^2(\beta_k - \beta_{k-1}) - (1 - 2f_d^2)(\alpha_k - \alpha_{k-1})) ds dt \\ &+ r^{\gamma_1} \int_0^r t^{-2\gamma_1 - 1} \int_0^t s^{\gamma_1 + 1} (f_d^2 \frac{\partial(\beta_k - \beta_{k-1})}{\partial \gamma_1} - (1 - 2f_d^2) \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial \gamma_1}) ds dt \end{aligned}$$

We can estimate the first three terms of $r^{-2\gamma_2 - 2d} |\log r|^{-1} |\frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial \gamma_1}|(r)$ by

$$Cr^2 (\|(\beta_k - \beta_{k-1})\nu^{-\gamma_2}\|_{L^\infty([0, r])} + \|(\alpha_k - \alpha_{k-1})\nu^{-\gamma_2 - 2d}\|_{L^\infty([0, r])})$$

where C is independent of r and of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

The estimate of the fourth term gives

$$|r^{-\gamma_2 - 2d + \gamma_1} (\log r)^{-1} \int_0^r t^{-2\gamma_1 - 1} \int_0^t s^{\gamma_1 + 1} (f_d^2 \frac{\partial(\beta_k - \beta_{k-1})}{\partial \gamma_1} - (1 - 2f_d^2) \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial \gamma_1}) ds dt|$$

$$\leq Cr^2 \left(\left\| \frac{\partial(\beta_k - \beta_{k-1})}{\partial\gamma_1} \nu^{-\gamma_2} (\log \nu)^{-1} \right\|_{L^\infty([0,r])} + \left\| \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_1} \nu^{-\gamma_2-2d} (\log \nu)^{-1} \right\|_{L^\infty([0,r])} \right)$$

where C is independent of r and of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Finally

$$\begin{aligned} & \left| \frac{\partial(\alpha_{k+1} - \alpha_k)}{\partial\gamma_1}(r) r^{-\gamma_2-2d} (\log r)^{-1} \right| \\ & \leq Cr^2 \left(\left\| (\beta_k - \beta_{k-1}) \nu^{-\gamma_2} \right\|_{L^\infty([0,r])} + \left\| (\alpha_k - \alpha_{k-1}) \nu^{-\gamma_2-2d} \right\|_{L^\infty([0,r])} \right) \quad (2.36) \\ & + \left\| \frac{\partial(\beta_k - \beta_{k-1})}{\partial\gamma_1} \nu^{-\gamma_2} (\log \nu)^{-1} \right\|_{L^\infty([0,r])} + \left\| \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_1} \nu^{-\gamma_2-2d} (\log \nu)^{-1} \right\|_{L^\infty([0,r])}. \end{aligned}$$

Now,

$$\frac{\partial\beta_{k+1}}{\partial\gamma_1} = r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2 \frac{\partial\alpha_k}{\partial\gamma_1} - (1-2f_d^2) \frac{\partial\beta_k}{\partial\gamma_1}) ds dt.$$

Then, we get

$$\begin{aligned} r^{-\gamma_2} |\log r|^{-1} \left| \frac{\partial(\beta_{k+1} - \beta_k)}{\partial\gamma_1} \right|(r) & \leq Cr^2 \left(\left\| \frac{\partial(\alpha_k - \alpha_{k-1})}{\partial\gamma_1} \nu^{-\gamma_2-2d} (\log \nu)^{-1} \right\|_{L^\infty([0,r])} \right. \\ & \left. + \left\| \frac{\partial(b_j - b_{j-1})}{\partial\gamma_1} \nu^{-\gamma_2} (\log \nu)^{-1} \right\|_{L^\infty([0,r])} \right). \quad (2.37) \end{aligned}$$

Recalling (2.37), (2.36), (2.20) and (2.21), we can conclude as in the proof of the derivability wrt γ_2 . Now, the same proof as for a' and b' permit to prove that the sums

$$\sum_{k=0}^{+\infty} \frac{\partial(\alpha'_{k+1} - \alpha'_k)}{\partial\gamma_i} \quad \text{and} \quad \sum_{k=0}^{+\infty} \frac{\partial(\beta'_{k+1} - \beta'_k)}{\partial\gamma_i}$$

converge, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$ and to get (2.31).

2.2 The solution (a_3, b_3) .

Proposition 2.6 *There exist a solution (a_3, b_3) of (1.6) and, for any compact subset $\mathcal{K} \in \mathcal{D}$, some real numbers R and C verifying*

$$\begin{aligned} & \text{for all } 0 < r < R, \quad |a_3(r) - r^{\gamma_1}| \leq Cr^{\gamma_1+2}, \quad |b_3(r)| \leq Cr^{\gamma_1+2d+2}, \\ & |a'_3(r) - \gamma_1 r^{\gamma_1-1}| \leq Cr^{\gamma_1+1} \quad \text{and} \quad |b'_3(r)| \leq Cr^{\gamma_1+2d+1}, \end{aligned}$$

where C and R remain the same for all $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Moreover, for all $r > 0$ $(d, \gamma_1, \gamma_2) \mapsto (a_3(r), a'_3(r), b_3(r), b'_3(r))$ is continuous on \mathcal{D} and is derivable wrt γ_1 and γ_2 whenever $\gamma_1 < \gamma_2$ and we have for all $0 < r < R$ and for $i = 1, 2$,

$$\left| \frac{\partial a_3}{\partial\gamma_i}(r) - (\log r) r^{\gamma_1} \right| \leq Cr^{\gamma_1+2} |\log r|, \quad \left| \frac{\partial b_3}{\partial\gamma_i}(r) \right| \leq Cr^{\gamma_1+2d+2} |\log r| \quad (2.38)$$

and

$$\left| \frac{\partial a'_3}{\partial\gamma_i}(r) - (\log r) \gamma_1 r^{\gamma_1-1} \right| \leq Cr^{\gamma_1+1} |\log r|, \quad \left| \frac{\partial b'_3}{\partial\gamma_i}(r) \right| \leq Cr^{\gamma_1+2d+1} |\log r| \quad (2.39)$$

Proof We consider the fixed point problem

$$\begin{cases} a &= r^{\gamma_1} \int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1} (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b &= r^{\gamma_2} + r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1} (f_d^2 a - (1 - 2f_d^2) b) ds dt. \end{cases} \quad (2.40)$$

We define by induction

$$(\alpha_0, \beta_0) = (r^{\gamma_1}, 0) \quad \text{and, for all } j \in \mathbb{N}, \quad (\alpha_{j+1}, \beta_{j+1}) = \Phi(\alpha_j, \beta_j). \quad (2.41)$$

and we define $\zeta_1(r) = r^{\gamma_1}$ and $\zeta_2(r) = r^{\gamma_1+2d}$.

We have to verify (2.20). For this purpose, we estimate

$$r^{\gamma_1} \int_0^r t^{-\gamma_1-1} \int_0^t s^{\gamma_1+1} s^{\gamma_1} ds dt = \frac{r^{2+\gamma_1}}{2(2+2\gamma_1)} \leq Cr^{\gamma_1+2}.$$

where C depends only on \mathcal{K} , and

$$r^{\gamma_1} \int_0^r t^{-\gamma_1-1} \int_0^t s^{\gamma_1+1} s^{\gamma_1+4d+2} ds dt = \frac{r^{2+\gamma_1}}{2(2+2\gamma_1)} \leq Cr^{\gamma_1+2+4d}.$$

The both inequalities give (2.20), for all $0 < r < 1$, and give also

$$|\alpha_1 - \alpha_0|(r) \leq Cr^{\gamma_1+1}.$$

In order to verify (2.21), we compute

$$r^{\gamma_2} \int_0^r t^{-2\gamma_2-1} \int_0^t s^{\gamma_2+1+2d+\gamma_1} ds dt = \frac{r^{2d+\gamma_1+2}}{(-\gamma_2+2d+2)(\gamma_2+2d+2)} \leq Cr^{\gamma_1+2d+2}$$

where C depends only on \mathcal{K} . This gives (2.21), and gives also

$$|\beta_1 - \beta_0|(r) \leq Cr^{\gamma_1+2d+2}.$$

Then, as explained at the beginning of the chapter, we can deduce that

$$\text{for all } r \leq R, \quad |a(r) - r^{\gamma_1}| + |b(r)| \leq Cr^{\gamma_1+2d+2}$$

with R and C depending only on \mathcal{K} , and we have the continuity of $(d, \gamma_1, \gamma_2) \mapsto (a_3(r), b_3(r))$. Now, the continuity of $(d, \gamma_1, \gamma_2) \mapsto (a'_3(r), b'_3(r))$ and the estimate near 0 of (a'_3, b'_3) can be proved exactly by the same proof as the continuity of $(d, \gamma_1, \gamma_2) \mapsto (a'_1(r), b'_1(r))$, and we obtain

$$\text{for all } 0 < r < R \quad |a'_3(r) - \gamma_1 r^{\gamma_1-1}| \leq Cr^{\gamma_1+1} \quad \text{and} \quad |b'_3(r)| \leq Cr^{\gamma_1+2d+1}.$$

Now, when $\gamma_1 \neq 0$, the proof of (2.38) and of (2.39) are similar to the corresponding property of (a_1, b_1) and are left to the reader.

When $\gamma_1 = 0$, we write, for $k \geq 1$

$$|\alpha'_{k+1} - \alpha'_k|(r) \leq r^{-1} \int_0^r s |f_d^2(\beta_k - \beta_{k-1}) - (1 - 2f_d^2)(\alpha_k - \alpha_{k-1})| ds$$

$$\begin{aligned}
&\leq C \left(\frac{r^{4d+1}}{4d+2} \|(\beta_k - \beta_{k-1})\nu^{-2d}\|_{L^\infty([0,r])} + \frac{r}{2} \|\alpha_k - \alpha_{k-1}\|_{L^\infty([0,r])} \right) \\
&\leq Cr(Cr^2)^{k-1} (\|(\beta_1 - \beta_0)\nu^{-2d}\|_{L^\infty([0,r])} + \|\alpha_1 - \alpha_0\|_{L^\infty([0,r])}).
\end{aligned}$$

Then the sum $\sum_{k \geq 0} (\alpha'_{k+1} - \alpha'_k)$ converges, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. We estimate directly

$$|\alpha'_1 - \alpha'_0|(r) \leq Cr$$

that gives

$$|a'_3(r)| \leq Cr.$$

The estimate of $|b'_3(r)|$ is left to the reader. The proof of the derivability wrt γ_1 and γ_2 and the behaviors of the derivatives works as for (a_1, b_1) and is left to the reader, too.

2.3 The solution (a_2, b_2) .

We distinguish the construction of (a_2, b_2) when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and the construction of (a_2, b_2) when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$.

2.3.1 (a_2, b_2) , for $\gamma_1 \neq 0$.

First, we construct a solution, for $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1 = \{(d, \gamma_1, \gamma_2) \in \mathcal{D}; \gamma_1 > 0\}$.

Proposition 2.7 *For all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, such that $\gamma_1 \neq 0$, there exists a solution (a_2, b_2) of (1.6) having the following property : for all compact set $\mathcal{K} \subset \mathcal{D}_1$, there exists R and C depending only on \mathcal{K} , for which we have*

$$\text{for all } 0 < r \leq R \quad |b_2(r) - r^{-\gamma_2}| \leq Cr^{-\gamma_2+2} \quad (2.42)$$

and

$$\text{for all } 0 < r \leq R \quad |a_2(r)| \leq C(r^2\theta(r) + r^{\gamma_1}) \quad (2.43)$$

where

$$\theta(r) = \begin{cases} \frac{-r^{\gamma_1-2} + r^{-\gamma_2+2d}}{\gamma_2 + \gamma_1 - 2d - 2} & \text{if } -\gamma_2 - \gamma_1 + 2d + 2 \neq 0 \\ -r^{\gamma_1-2} \log r & \text{if } -\gamma_2 - \gamma_1 + 2d + 2 = 0 \end{cases}$$

(We have $\theta(r) \geq 0$ for all $0 < r < 1$).

Moreover, for all $0 < r < R$,

$$|a'_2(r)| \leq C(r\theta(r) + r^{\gamma_1-1}) \quad \text{and} \quad |b'_2(r) + \gamma_2 r^{-\gamma_2-1}| \leq Cr^{-\gamma_2-1} \quad (2.44)$$

where C and R depend only on \mathcal{K} , and, for all $r > 0$

$$(d, \gamma_1, \gamma_2) \mapsto (a_2(r), a'_2(r), b_2(r), b'_2(r))$$

is continuous on \mathcal{D}_1 .

Proof

We consider the following fixed point problem

$$\begin{cases} a &= r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_1^t s^{-\gamma_1+1} (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b &= r^{-\gamma_2} + r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_1^t s^{-\gamma_2+1} (f_d^2 a - (1 - 2f_d^2) b) ds dt. \end{cases} \quad (2.45)$$

We define

$$\zeta_1(r) = \theta(r) + r^{\gamma_1-2} \quad \text{and} \quad \zeta_2(r) = r^{-\gamma_2}$$

and

$$(\alpha_0, \beta_0) = (0, r^{-\gamma_2}) \quad \text{and, for all } k \in \mathbb{N}, \quad (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \quad (2.46)$$

We give $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, a compact subset of \mathcal{D}_1 .

Then we have $\gamma_1 \geq c$, for some $c > 0$, depending only on \mathcal{K} .

In order to prove (2.20), we estimate, for $r < 1$

$$\begin{aligned} & r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_t^1 s^{-\gamma_1+1} (\theta(s) + s^{\gamma_1-2}) ds dt \\ &= r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \left(\begin{cases} \frac{-\log t - \frac{1-t^{-\gamma_2-\gamma_1+2d+2}}{-\gamma_2-\gamma_1+2d+2}}{\gamma_1+\gamma_2-2d-2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ (\log t)^2 & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \end{cases} - \log t \right) dt \quad (2.47) \\ &= \begin{cases} \frac{-\frac{r^{\gamma_1}}{2\gamma_1} \log r + \frac{r^{\gamma_1}}{(2\gamma_1)^2} - \frac{\frac{r^{\gamma_1}}{2\gamma_1} - \frac{r^{-\gamma_2+\gamma_1+2d+2}}{-\gamma_2+\gamma_1+2d+2}}{\gamma_1+\gamma_2+2d+2}}{\gamma_1+\gamma_2+2d+2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ \frac{r^{\gamma_1}}{2\gamma_1} (\log r)^2 - 2 \frac{r^{\gamma_1}}{(2\gamma_1)^2} \log r + 2 \frac{r^{\gamma_1}}{(2\gamma_1)^3} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \end{cases} - \frac{r^{\gamma_1}}{2\gamma_1} \log r + \frac{r^{\gamma_1}}{(2\gamma_1)^2} \\ & \leq Cr^2 (-\log r) (\theta(r) + r^{\gamma_1-2}) \end{aligned}$$

for some C depending only on \mathcal{K} .

Then we compute

$$\begin{aligned} & r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_t^1 s^{-\gamma_1+1+2d-\gamma_2} ds dt \\ & r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \left(\begin{cases} \frac{1-t^{-\gamma_1-\gamma_2+2d+2}}{-\gamma_1-\gamma_2+2d+2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ -\log t & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \end{cases} \right) dt \quad (2.48) \\ &= \frac{r^{\gamma_1}}{(\gamma_1 - \gamma_2 + 2d + 2)2\gamma_1} + r^2 \frac{\theta(r)}{(\gamma_1 - \gamma_2 + 2d + 2)} \leq Cr^2 (\theta(r) + r^{\gamma_1-2}) \end{aligned}$$

where C depends only on \mathcal{K} .

Now (2.47) and (2.48) give, for all $k \geq 2$ and in place of (2.20)

$$|\alpha_{k+1} - \alpha_k|(r) \leq C \zeta_1(r) r^2 (-\log r) (\|\zeta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])}),$$

and gives also

$$|\alpha_1 - \alpha_0|(r) \leq Cr^2(\theta(r) + r^{\gamma_1-2}).$$

To obtain (2.21), we compute

$$\begin{aligned}
& r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-\gamma_2+1+2d}(\theta(s) + s^{\gamma_1-2}) ds dt \leq \\
& r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \left(\begin{aligned} & \frac{\frac{1-t^{-\gamma_1-\gamma_2+2d+2}}{-\gamma_1-\gamma_2+2d} + \frac{1-t^{-2\gamma_2+4d}}{-2\gamma_2+4d}}{\gamma_1+\gamma_2-2d-2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0, \gamma_1 - \gamma_2 + 2d \neq 0, \gamma_2 \neq 2d \\ & \frac{-\log t - \frac{1-t^{-2\gamma_2+4d}}{-2\gamma_2+4d}}{\gamma_1+\gamma_2-2d-2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0, \gamma_1 - \gamma_2 + 2d = 0, \gamma_2 \neq 2d \\ & -\frac{t^{\gamma_1-\gamma_2+2d}}{\gamma_1-\gamma_2+2d} \log t + \frac{t^{\gamma_1-\gamma_2+2d}}{(\gamma_1-\gamma_2+2d)^2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \\ & + \left\{ \begin{aligned} & \frac{1-r^{-\gamma_2+\gamma_1+2d}}{-\gamma_2+\gamma_1+2d} \text{ if } \gamma_1 - \gamma_2 + 2d \neq 0 \\ & -\log r \text{ if } \gamma_1 - \gamma_2 + 2d = 0 \end{aligned} \right\} dt \end{aligned} \right. \quad (2.49) \\
& = r^{-\gamma_2} \left\{ \begin{aligned} & \frac{\frac{\frac{r^{2\gamma_2} - r^{\gamma_1+\gamma_2+2d}}{2\gamma_2 - \gamma_1 + \gamma_2 + 2d} - \frac{r^{2\gamma_2} - r^{2d}}{2\gamma_2 - 2d}}{\gamma_1 - \gamma_2 + 2d} - \frac{r^{2\gamma_2} - r^{2d}}{-\gamma_2 + 2d}}{\gamma_1 + \gamma_2 - 2d - 2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0, \gamma_1 - \gamma_2 + 2d \neq 0 \\ & \frac{\frac{r^{2\gamma_2}}{2\gamma_2} (-\log r) + \frac{r^{2\gamma_2}}{(2\gamma_2)^2} - \frac{r^{2\gamma_2} - r^{2d}}{-\gamma_2 + 2d}}{\gamma_1 + \gamma_2 - 2d - 2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0, \gamma_1 - \gamma_2 + 2d = 0 \\ & \frac{r^{\gamma_1+\gamma_2+2d}(-\log r)}{(\gamma_1+\gamma_2+2d)(\gamma_1-\gamma_2+2d)} + \frac{r^{\gamma_1+\gamma_2+2d}}{(\gamma_1+\gamma_2+2d)^2} + \frac{r^{\gamma_1+\gamma_2+2d}}{(\gamma_1+\gamma_2+2d)(\gamma_1-\gamma_2+2d)^2} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \\ & + \left\{ \begin{aligned} & \left(\frac{r^{\gamma_2}}{2\gamma_2} - \frac{r^{\gamma_2+\gamma_1+2d}}{\gamma_2+\gamma_1+2d} \right) \frac{1}{-\gamma_2+\gamma_1+2d} \text{ if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ & -\frac{r^{\gamma_2}}{2\gamma_2} \log r \text{ if } \gamma_1 + \gamma_2 - 2d - 2 = 0 \end{aligned} \right\} \end{aligned} \right. \\
& \leq Cr^{-\gamma_2+2}
\end{aligned}$$

where C depends only on \mathcal{K} .

Now we compute

$$r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-2\gamma_2+1} ds dt \leq \frac{r^{2-\gamma_2}}{2(2\gamma_2-2)} \leq Cr^{2-\gamma_2} \quad (2.50)$$

where C depends only on \mathcal{K} .

Then (2.49) and (2.50) give (2.21), and also

$$|\beta_1 - \beta_0|(r) \leq Cr^{-\gamma_2+2}.$$

We conclude for all $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$, there exists a solution (a_2, b_2) , satisfying the desired behavior at 0 and such that $(d, \gamma_1, \gamma_2) \mapsto (a_2(r), b_2(r))$ is continuous on \mathcal{D}_1 . The behavior of $(a_2'(r), b_2'(r))$ at 0 is left to the reader.

2.3.2 (a_2, b_2) , for γ_1 small.

Let us consider \mathcal{K} , a compact subset of \mathcal{D}_2 . For $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, we have in particular $\gamma_1 \leq c_0$, $-\gamma_2 + 2d + 1 \geq c_1$, $\gamma_2 \geq c_2$ and $-\gamma_1 - \gamma_2 + 2d + 2 \geq c_3$, where $c_0 < \frac{1}{4}$, $c_1 > 0$, $c_2 > 1$ and $c_3 > 0$ depend only on \mathcal{K} .

We have

Proposition 2.8 *There exists a solution (a_2, b_2) of (1.6) having the following property : for all compact set $\mathcal{K} \subset \mathcal{D}_2$, there exists R and C , depending only on \mathcal{K} , for which we have*

$$\text{for all } 0 < r \leq R \quad |b_2(r) - r^{-\gamma_2}| \leq Cr^{-\gamma_2+2} \quad (2.51)$$

and

$$\text{for all } 0 < r \leq R \quad |a_2(r)| \leq Cr^{-\gamma_2+2d+2} \quad (2.52)$$

Moreover, for all $0 < r < R$,

$$|a_2'(r)| \leq Cr^{-\gamma_2+2d+1} \quad \text{and} \quad |b_2'(r) + \gamma_2 r^{-\gamma_2-1}| \leq Cr^{-\gamma_2-1} \quad (2.53)$$

where C and R depend only on \mathcal{K} , and, for all $r > 0$

$$(d, \gamma_1, \gamma_2) \mapsto (a_2(r), a_2'(r), b_2(r), b_2'(r))$$

is continuous on \mathcal{D}_2

Proof Let us consider the following fixed point problem

$$\begin{cases} a &= r^{\gamma_1} \int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1} (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b &= r^{-\gamma_2} + r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_1^t s^{-\gamma_2+1} (f_d^2 a - (1 - 2f_d^2) b) ds dt. \end{cases} \quad (2.54)$$

We define $\zeta_1(r) = r^{-\gamma_2+2d}$ and $\zeta_2(r) = r^{-\gamma_2}$.

In order to prove (2.20), we verify that

$$r^{\gamma_1} \left(\int_0^r t^{-2\gamma_1-1} \int_0^t s^{\gamma_1+1} s^{2d-\gamma_2} ds dt \right) \leq Cr^{-\gamma_2+2d+2} \quad (2.55)$$

Then (2.55) gives (2.20) and gives also

$$|\alpha_1 - \alpha_0|(r) \leq Cr^{-\gamma_2+2d+2}.$$

Now, in order to prove (2.21), we compute

$$\begin{aligned} r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-\gamma_2+1} s^{2d-\gamma_2+2d} ds dt &= r^{-\gamma_2} \frac{\frac{r^{2\gamma_2}}{2\gamma_2} - \frac{r^{4d+2}}{4d+2}}{-2\gamma_2 + 4d + 2} \\ &\leq Cr^2 r^{-\gamma_2} \end{aligned} \quad (2.56)$$

and

$$r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-\gamma_2+1} s^{-2\gamma_2+1} ds dt = \frac{r^{-\gamma_2}}{-2\gamma_2 + 2} \left(\frac{r^{2\gamma_2}}{2\gamma_2} - \frac{r^2}{2} \right) \leq Cr^2 r^{-\gamma_2} \quad (2.57)$$

Then (2.56) and (2.57) give (2.21) and gives also

$$|\beta_1 - \beta_0|(r) \leq Cr^2r^{\gamma_2}.$$

So we have the existence and the continuity wrt $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$ of a solution (a_2, b_2) , and we have the desired behavior at $r = 0$. The behavior at $r = 0$ of (a'_2, b'_2) is left to the reader.

2.4 The solution (a_4, b_4) .

2.4.1 (a_4, b_4) when $\gamma_1 \neq 0$.

First we construct a solution (a_4, b_4) when $\gamma_1 > 0$. We have to prove

Proposition 2.9 *There exists a solution (a_4, b_4) of (1.6) having the following property : for all compact subset \mathcal{K} of \mathcal{D}_1 , there exist $R < 1$ and C depending only on \mathcal{K} , such that, for all $0 < r < R$*

$$|a_4(r) - r^{-\gamma_1}| \leq Cr^{-\gamma_1+2} \quad \text{and} \quad |b_4(r)| \leq C(r^2\tilde{\theta}(r) + r^{\gamma_2}),$$

where, for $0 < r < 1$, $\tilde{\theta}$ is defined by

$$\tilde{\theta}(r) = \begin{cases} \frac{-r^{\gamma_2-2} + r^{-\gamma_1+2d}}{\gamma_1 + \gamma_2 - 2d - 2} & \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0 \\ -r^{\gamma_2-2} \log r & \text{if } \gamma_1 + \gamma_2 - 2d - 2 = 0. \end{cases}$$

Moreover

$$|a'_4(r) + \gamma_1 r^{-\gamma_1-1}| \leq Cr^{-\gamma_1+1} \quad \text{and} \quad |b'_4(r)| \leq C(r\tilde{\theta} + r^{\gamma_2-1}).$$

And $(d, \gamma_1, \gamma_2) \mapsto (a_4(r), a'_4(r), b_4(r), b'_4(r))$ is continuous on \mathcal{D}_1 .

Proof Let us consider the following fixed point problem

$$\begin{cases} a = r^{-\gamma_1} + r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_1^t s^{-\gamma_1+1} (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b = r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_1^t s^{-\gamma_2+1} (f_d^2 a - (1 - 2f_d^2) b) ds dt. \end{cases} \quad (2.58)$$

Let us define $\zeta_1(r) = r^{-\gamma_1}$ and $\zeta_2(r) = \tilde{\theta}(r) + r^{\gamma_2-2}$.

Let \mathcal{K} be a compact subset of \mathcal{D}_1 . Then we have $\gamma_1 \geq c$, where $c > 0$ depends only on \mathcal{K} . In order to prove (2.20), we estimate

$$\begin{aligned} r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_t^1 s^{-2\gamma_1+1} ds dt &= r^{-\gamma_1} \begin{cases} \frac{r^{2\gamma_1} - r^2}{-2\gamma_1+2} & \text{if } \gamma_1 \neq 1 \\ -\frac{r^2}{2} \log r + \frac{r^2}{4} & \text{if } \gamma_1 = 1 \end{cases} \\ &= r^{-\gamma_1} \begin{cases} r^2 \frac{r^{2\gamma_1-2}-1}{(-2\gamma_1+2)2\gamma_1} + \frac{r^2}{4\gamma_1} & \text{if } \gamma_1 \neq 1 \\ -\frac{r^2}{2} \log r + \frac{r^2}{4} & \text{if } \gamma_1 = 1 \end{cases} \end{aligned}$$

$$\leq \begin{cases} Cr^{-\gamma_1} r^{2c_1} & \text{if } 0 < c_1 \leq \gamma_1 < 1 \\ Cr^{-\gamma_1} r^2 (-\log r) & \text{if } \gamma_1 = 1 \\ Cr^{-\gamma_1+2} & \text{if } \gamma_1 > 1 \end{cases} \quad (2.59)$$

Now, we return to (2.49). Exchanging γ_1 and γ_2 , we get

$$r^{-\gamma_1} \int_0^r t^{2\gamma_1-1} \int_t^1 s^{-\gamma_1+1+2d} (\tilde{\theta} + r^{\gamma_2-2}) ds dt \leq Cr^{-\gamma_1+2}. \quad (2.60)$$

Then, (2.60) and (2.59) give, in place of (2.20), and for some $c > 0$ depending only on \mathcal{K}

$$|\alpha_{k+1} - \alpha_k|(r) \leq C\zeta_1(r)r^c (\|\zeta_1^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])} + \|\zeta_2^{-1}(\beta_k - \beta_{k-1})\|_{L^\infty([0,r])})$$

and gives also

$$|\alpha_1 - \alpha_0|(r) \leq Cr^{-\gamma_1+c}.$$

Now to obtain (2.21), we return to (2.48). Exchanging γ_1 and γ_2 , we get

$$r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-\gamma_2+1+2d-\gamma_1} ds dt \leq Cr^2(\tilde{\theta} + r^{\gamma_2-2}) \quad (2.61)$$

where C depends only on \mathcal{K} .

Then, returning to (2.47), we get

$$r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_t^1 s^{-\gamma_2+1} (\tilde{\theta} + r^{\gamma_2-2}) ds dt \leq Cr^2(-\log r)(\tilde{\theta} + r^{\gamma_2-2}). \quad (2.62)$$

We have proved the existence of (a_4, b_4) , for all $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$, and the continuity of $(d, \gamma_1, \gamma_2) \mapsto (a_4(r), b_4(r))$. The behavior of $(a'_4(r), b'_4(r))$ at $r = 0$ is left to the reader.

2.4.2 (a_4, b_4) for small γ_1 .

Now we give $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$. We have

Proposition 2.10 *There exists a solution (a_4, b_4) of (1.6) having the following property : for all compact subset \mathcal{K} of \mathcal{D}_2 , there exist $R < 1$ and C and $c > 1$ depending only on \mathcal{K} , such that, for all $0 < r < R$*

$$|a_4(r) - \tau(r)| \leq Cr^c \tau(r) \quad \text{and} \quad |b_4(r)| \leq Cr^{2d+2} \tau(r),$$

where, for $0 < r < 1$, τ is defined by

$$\tau(r) = \begin{cases} \frac{r^{-\gamma_1-r\gamma_1}}{2\gamma_1} & \text{if } \gamma_1 > 0 \\ -\log r & \text{if } \gamma_1 = 0. \end{cases}$$

Moreover

$$|a'_4(r) - \tau'(r)| \leq Cr^{c-1} \tau(r) \quad \text{and} \quad |b'_4(r)| \leq Cr^{2d+1} \tau(r).$$

And $(d, \gamma_1, \gamma_2) \mapsto (a_4(r), a'_4(r), b_4(r), b'_4(r))$ is continuous on \mathcal{D}_2 .

Proof

In view of (2.26), let us consider the following fixed point problem

$$\begin{cases} a = \tau(r) + \tau(r) \int_0^r \frac{1}{t} \tau^{-2}(t) \int_0^t s \tau(s) (f_d^2 b - (1 - 2f_d^2) a) ds dt \\ b = r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_0^t s^{-\gamma_2+1} (f_d^2 a - (1 - 2f_d^2) b) ds dt \end{cases} \quad (2.63)$$

We define

$$\zeta_1(r) = \tau(r) \quad \text{and} \quad \zeta_2(r) = r^{2d} \tau(r).$$

Let us consider \mathcal{K} a compact subset of \mathcal{D}_2 . Then we have $\gamma_1 \leq c_1$ and $-\gamma_1 - \gamma_2 + 2d + 2 \geq c_2$, where $c_1 < \frac{1}{4}$ and $c_2 > 0$. But γ_1 can be equal to 0. Since $e^u \geq 1 + u$, we have for all $t \in]0, 1[$

$$t^{\gamma_1} (-\log t) \leq \tau(t) \leq t^{-\gamma_1} (-\log t) \quad (2.64)$$

In order to prove (2.20), we estimate, assuming that $r \leq \exp(-1)$

$$\begin{aligned} \tau(r) \int_0^r \frac{1}{t} \tau^{-2}(t) \int_0^t s \tau^2(s) ds dt &\leq \tau(r) \int_0^r \frac{t^{-2\gamma_1-1}}{\log^2 t} \int_0^t s^{-2\gamma_1+1} (\log s)^2 ds dt \\ &= \tau(r) \int_0^r \frac{t^{-2\gamma_1-1}}{\log^2 t} \left(\frac{t^{-2\gamma_1+2}}{-2\gamma_1+2} \log^2 t - 2 \frac{t^{-2\gamma_1+2}}{(-2\gamma_1+2)^2} \log t + 2 \frac{t^{-2\gamma_1+2}}{(-2\gamma_1+2)^3} \right) dt \\ &\leq Cr^{-4\gamma_1+2} \tau(r) \leq r^c \tau(r) \end{aligned} \quad (2.65)$$

where $C > 0$ and $c > 1$ are independent of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Following the same proof, we get, for $r \leq \exp(-1)$

$$\begin{aligned} \tau(r) \int_0^r \frac{1}{t} \tau^{-2}(t) \int_0^t s \tau(s) s^{2d} \tau(s) s^{2d} ds dt \\ \leq Cr^{-4\gamma_1+4d+2} \tau(r) \leq Cr^2 \tau(r). \end{aligned} \quad (2.66)$$

Then, (2.65) and (2.66) give (2.20), with r^c instead of r^2 , and give also

$$|\alpha_1 - \alpha_0| \leq Cr^c \tau(r).$$

Now, in order to prove (2.21), we compute

$$\begin{aligned} r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_0^t s^{-\gamma_2+1} s^{2d} \tau(s) ds dt \\ \leq r^{-\gamma_2} \int_0^r t^{2\gamma_2-1} \int_0^t s^{-\gamma_2+1} s^{2d-\gamma_1} (-\log s) ds dt \\ \leq Cr^{-\gamma_2} r^{\gamma_2-\gamma_1+2d+2} (-\log r) \leq Cr^{-\gamma_2+2d+2} \end{aligned} \quad (2.67)$$

where C depends only on \mathcal{K} . Then (2.56) gives (2.21) and gives also

$$|\beta_1 - \beta_0|(r) \leq Cr^{-\gamma_2+2d+2}.$$

We have proved the desired result for $(a_4(r), b_4(r))$. The proof concerning $(a'_4(r), b'_4(r))$ is left to the reader.

3 The possible behaviors at infinity and the dependence of the solutions wrt the parameters.

Our goal is to prove Theorem 1.4.

We use the system (1.7) and we construct a base of four solutions, (x_j, y_j) , $j = 1, \dots, 4$, characterized by their behavior at $+\infty$. Then the solutions (u_j, v_j) announced in Theorem 1.4 are obtained by $u_j = \frac{x_j + y_j}{2}$ and $v_j = \frac{x_j - y_j}{2}$.

Let us consider the first equation of (1.7). As is usual with regard to Bessel's equations, we let

$$\tilde{x}(r) = r^{\frac{1}{2}}x(r).$$

Then the system (1.7) becomes

$$\begin{cases} \tilde{x}'' - 2\tilde{x} + \frac{-\gamma^2 - 3d^2}{r^2}\tilde{x} + 3(1 - f_d^2 + \frac{d^2}{r^2})\tilde{x} + \frac{\xi^2}{r^{\frac{3}{2}}}y = 0 \\ y'' + \frac{y'}{r} - \frac{\gamma^2 - d^2}{r^2}y + \frac{\mu^2}{r^{\frac{5}{2}}}\tilde{x} + (1 - f_d^2 - \frac{d^2}{r^2})y = 0 \end{cases} \quad (3.68)$$

We can replace the first equation of this system by

$$(e^{2\sqrt{2}r}(\tilde{x}e^{-\sqrt{2}r})')' = e^{\sqrt{2}r}q(r)\tilde{x} - \frac{\xi^2}{r^2}y$$

or by

$$(e^{-2\sqrt{2}r}(\tilde{x}e^{\sqrt{2}r})')' = e^{-\sqrt{2}r}q(r)\tilde{x} - \frac{\xi^2}{r^2}y,$$

where

$$q(r) = \frac{-\gamma^2 - 3d^2}{r^2} + 3(1 - f_d^2 + \frac{d^2}{r^2}).$$

Let us suppose that $\gamma^2 - d^2 \geq 0$. Then we let

$$n = \sqrt{\gamma^2 - d^2}.$$

The second equation of the system (1.7) can be written as

$$(r^{2n+1}(r^{-n}y)')' = r^{n+1}(\frac{\mu^2}{r^2}x - (1 - f_d^2 - \frac{d^2}{r^2})y)$$

or

$$(r^{-2n+1}(r^n y)')' = r^{-n+1}(\frac{\mu^2}{r^2}x - (1 - f_d^2 - \frac{d^2}{r^2})y).$$

Finally, the system (1.7) can be written as

$$\begin{cases} (e^{\pm 2\sqrt{2}r}(r^{\frac{1}{2}}e^{\mp\sqrt{2}r}x)')' = r^{\frac{1}{2}}e^{\pm\sqrt{2}r}q(r)x - \frac{\xi^2}{r^2}y \\ (r^{\pm 2n+1}(r^{\mp n}y)')' = r^{\pm n+1}(\frac{\xi^2}{r^2}x - (1 - f_d^2 - \frac{d^2}{r^2})y) \end{cases} \quad (3.69)$$

We denote

$$J_+ = \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad J_- = \frac{e^{-\sqrt{2}r}}{\sqrt{r}} \quad \text{and} \quad n = \sqrt{\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2}.$$

We are going to construct four solutions of (3.69). The plan is almost the same for each solution. Let us explain it. First, for some given $R_0 > 0$, we define a fixed point problem of the form

$$(x, y) = \Phi(x, y),$$

for (x, y) defined on $[R_0, +\infty[$, and whose solutions are solutions of (3.69). The function Φ will depend on R_0 , except for one solution denoted by (x_2, y_2) (that vanishes exponentially at $+\infty$). Let us remark that the present construction does not allow us to construct the solutions (x_j, y_j) , $j \neq 2$ without taking into account a given compact subset

$$\mathcal{K} \subset \{(d, \gamma_1, \gamma_2); 0 \leq \gamma_1 < \gamma_2; \frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 > 0\}. \quad (3.70)$$

Indeed, R_0 depends on \mathcal{K} . Then we give a map ζ and we want to prove the existence of a fixed point (x, y) verifying, for some C depending only on \mathcal{K} , an estimate of the form

$$|x_j(r) - \zeta(r)| + |y_j(r)| \leq C\zeta(r)r^{-2} \quad \text{if } j = 1, 3,$$

$$\text{or } |x_j(r)| + |y_j(r) - \zeta(r)| \leq C\zeta(r)r^{-2} \quad \text{if } j = 2, 4.$$

Moreover we want $(d, \gamma_1, \gamma_2) \mapsto (a(r), a'(r), b(r), b'(r))$ to be continuous, and derivable wrt γ_1 and wrt γ_2 , for any given $r > R_0$.

We define by induction, for (x_1, y_1) and for (x_3, y_3)

$$(\alpha_0, \beta_0) = (\zeta, 0) \quad \text{and} \quad (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \quad (3.71)$$

For (x_2, y_2) and for (x_4, y_4) , we exchange the role of x and y , that gives

$$(\alpha_0, \beta_0) = (0, \zeta) \quad \text{and} \quad (\alpha_{k+1}, \beta_{k+1}) = \Phi(\alpha_k, \beta_k). \quad (3.72)$$

The proof of the continuity of $(d, \gamma_1, \gamma_2) \mapsto (\alpha_k, \alpha'_k, \beta_k, \beta'_k)(r)$, for all k follows from the Lebesgue Theorem and from an induction. We denote $\nu : r \mapsto r$.

Then we prove that there exists $C > 0$ depending only on \mathcal{K} and independent of R_0 , such that for all $r \geq R_0$ and all $k \geq 0$,

for $j = 1, 3$

$$\begin{aligned} |(\alpha_{k+1} - \alpha_k)\zeta^{-1}|(r) &\leq \frac{C}{r^2} (\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu^2\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.73)$$

and

$$\begin{aligned} r^2|(\beta_{k+1} - \beta_k)\zeta^{-1}|(r) &\leq \frac{C}{r^2} (\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\nu^2\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.74)$$

and for $j = 2, 4$

$$\begin{aligned} r^2|(\alpha_{k+1} - \alpha_k)\zeta^{-1}|(r) &\leq \frac{C}{r^2} (\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu^2\|_{L^\infty([R, +\infty[)} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\|_{L^\infty([R, +\infty[)}) \end{aligned} \quad (3.75)$$

and

$$\begin{aligned} |(\beta_{k+1} - \beta_k)\zeta^{-1}|(r) &\leq \frac{C}{r^2} (\|(\alpha_k - \alpha_{k-1})\zeta^{-1}\nu^2\|_{L^\infty([R, +\infty])} \\ &\quad + \|(\beta_k - \beta_{k-1})\zeta^{-1}\|_{L^\infty([R, +\infty])}) \end{aligned} \quad (3.76)$$

Then we define

$$x(r) = \alpha_0(r) + \sum_{k \geq 0} (\alpha_{k+1} - \alpha_k)(r) \quad \text{and} \quad y(r) = \beta_0(r) + \sum_{k \geq 0} (\beta_{k+1} - \beta_k)(r) \quad (3.77)$$

Since C is independent of R_0 , we choose $R_0 > 0$ such that $(CR_0^{-2}) < 1$, the sums $\zeta^{-1}x(r)$ and $\zeta^{-1}\nu^2 y(r)$ (or $\zeta^{-1}\nu^2 x(r)$ and $\zeta^{-1}y(r)$) converge, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Consequently, we get together the existence of a solution (x, y) having the desired behavior at $+\infty$ and the continuity of the map $(d, \gamma_1, \gamma_2) \mapsto (x(r), y(r))$.

Then we prove the continuity of $(d, \gamma_1, \gamma_2) \mapsto (x'(r), y'(r))$ in \mathcal{K} and the behavior of (x', y') at $+\infty$ by the uniform convergence of

$$\zeta^{-1} \sum_{k \geq 0} (\alpha'_{k+1} - \alpha'_k)(r) \quad \text{and} \quad \zeta^{-1}\nu^2 \sum_{k \geq 0} (\beta'_{k+1} - \beta'_k)(r).$$

We prove the derivability wrt γ_i , for $i = 1, 2$, of $(x(r), x'(r), y(r), y'(r))$ by the the uniform convergence of

$$\sum_{k \geq 0} \frac{\partial}{\partial \gamma_i} (\alpha_{k+1} - \alpha_k)(r), \quad \text{and} \quad \zeta^{-1}\nu^2 \sum_{k \geq 0} \frac{\partial}{\partial \gamma_i} (\beta_{k+1} - \beta_k)(r)$$

and

$$\zeta^{-1} \sum_{k \geq 0} \frac{\partial}{\partial \gamma_i} (\alpha'_{k+1} - \alpha'_k)(r), \quad \text{and} \quad \zeta^{-1}\nu^2 \sum_{k \geq 0} \frac{\partial}{\partial \gamma_i} (\beta'_{k+1} - \beta'_k)(r).$$

(For $j = 2, 4$ we change the place of ν^2).

We will use the following estimate, which is not difficult to prove, by an integration by part. Let $\alpha \in \mathbb{R}$ and $\beta > 0$ be given. Then

$$\int_t^{+\infty} s^\alpha e^{-\beta s} ds \leq \frac{2}{\beta} t^\alpha e^{-\beta t} \quad \text{for all } t \geq \frac{2\alpha}{\beta} \quad (3.78)$$

and

$$\int_R^t s^\alpha e^{\beta s} ds \leq \frac{2}{\beta} t^\alpha e^{\beta t} \quad \text{for all } t \geq R \geq \frac{-2\alpha}{\beta} \quad (3.79)$$

In what follows, \mathcal{K} is compact and is as in (3.70).

We will detail the proof of the construction only for the first solution (x_1, y_1) and we will only indicate the way to adapt it for the other three solutions.

3.1 The greatest behavior at $+\infty$: the solution (x_1, y_1) .

Proposition 3.11 *For every compact subset \mathcal{K} of $\{(d, \gamma_1, \gamma_2); 0 \leq \gamma_1 \leq \gamma_2; d \geq 0; \frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 > 0\}$, there exists a solution (x_1, y_1) of (3.69), such that there exist C and R_0 depending only on \mathcal{K} and such that*

$$|x_1(r) - \frac{e^{\sqrt{2}r}}{\sqrt{r}}| + |y_1(r)| \leq C \frac{e^{\sqrt{2}r}}{\sqrt{r}} r^{-2}$$

$$|x_1'(r) - (\frac{e^{\sqrt{2}r}}{\sqrt{r}})'| \leq Cr^{-3} \frac{e^{\sqrt{2}r}}{\sqrt{r}}, \quad |y_1'(r)| \leq Cr^{-4} \frac{e^{\sqrt{2}r}}{\sqrt{r}}$$

and for all $r > 0$

$$(d, \gamma_1, \gamma_2) \mapsto (x_1(r), x_1'(r), y_1(r), y_1'(r)) \text{ is continuous on } \mathcal{K}.$$

Moreover $(x_1, x_1', y_1, y_1')(r)$ is derivable wrt γ_1 and γ_2 and we have, for $i = 1, 2$

$$\text{for } r \geq R_0 \quad \left| \frac{\partial x_1}{\partial \gamma_i} \right|(r) + \left| \frac{\partial y_1}{\partial \gamma_i} \right|(r) \leq C \frac{e^{\sqrt{2}r}}{\sqrt{r}} r^{-2} \log r$$

and

$$\left| \frac{\partial x_1'}{\partial \gamma_i} \right|(r) + \left| \frac{\partial y_1'}{\partial \gamma_i} \right|(r) \leq C \frac{e^{\sqrt{2}r}}{\sqrt{r}} r^{-3} \log r \quad (3.80)$$

where C remains the same when $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Let us remark that at this stage, the solution (x_1, y_1) depends on \mathcal{K} .

Proof Let $R_0 > 0$ be given. Let us consider the following fixed point problem, with x and y defined in $[R_0, +\infty[$

$$\begin{cases} x &= J_+ + J_+ \int_{+\infty}^r (J_+)^{-2\frac{1}{t}} \int_{R_0}^t s J_+ (\frac{\xi^2}{s^2} y - 3(1 - f_d^2 - \frac{d^2}{s^2}) x) ds dt \\ y &= r^n \int_{R_0}^r t^{-2n-1} \int_{R_0}^t s^{n+1} (\frac{\xi^2}{s^2} x - (1 - f_d^2 - \frac{d^2}{s^2}) y) ds dt. \end{cases}$$

Let us denote it by

$$(x, y) = \Phi(x, y).$$

Let $\zeta = J_+$. We define (α_k, β_k) by (3.71). Let us denote

$$\mathcal{B} = \mathcal{C}(\mathcal{K}, \mathcal{C}([R_0, +\infty[))).$$

First, we prove that, for R_0 large enough,

$$\begin{aligned} &\text{when } ((\alpha_j - \alpha_{j-1})(J_+)^{-1}, (\beta_j - \beta_{j-1})(J_+)^{-1} \nu^2) \in \mathcal{B}^2 \quad \text{then} \\ &((\alpha_{j+1} - \alpha_j)(J_+)^{-1}, (\beta_{j+1} - \beta_j)(J_+)^{-1} \nu^2) \in \mathcal{B}^2 \end{aligned} \quad (3.81)$$

For this purpose, we write

$$\int_{R_0}^t s J_+ (\frac{\xi^2}{s^2} |\beta_k - \beta_{k-1}| + 3|1 - f_d^2 - \frac{d^2}{s^2}| |\alpha_k - \alpha_{k-1}|) ds$$

$$\leq \int_{R_0}^t s J_+ \frac{\xi^2}{s^2} J_+ s^{-2} ds \| (J_+)^{-1} \nu^2 (\beta_k - \beta_{k-1}) \|_{L^\infty([R_0, +\infty])} + \int_{R_0}^t s \frac{M}{s^4} (J_+)^2 \| (J_+)^{-1} (\alpha_k - \alpha_{k-1}) \|_{L^\infty([R_0, +\infty])}$$

where we have used $|1 - f_d^2 - \frac{d^2}{s^2}| \leq \frac{M}{s^4}$.

Then there exists C , depending only on \mathcal{K} such that $\xi^2 \leq C$. Using (3.79) and (3.78), we deduce that

$$(J_+)^{-2} \frac{1}{t} \int_{R_0}^t s J_+ \left(\frac{\xi^2}{s^2} (\beta_k - \beta_{k-1}) + 3 \left(1 - f_d^2 - \frac{d^2}{s^2} \right) (\alpha_k - \alpha_{k-1}) \right) ds$$

is integrable on $[r, +\infty[$, when $R_0 \geq \frac{6}{2\sqrt{2}}$, uniformly for $(d, \gamma_1, \gamma_2) \in \mathcal{K}$ and, by the Lebesgue Theorem, that $\alpha_{j+1} - \alpha_j$ is continuous wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Now, we write

$$\begin{aligned} & \int_{R_0}^t s^{n+1} \left| \frac{\xi^2}{s^2} (\alpha_k - \alpha_{k-1}) - \left(1 - f_d^2 - \frac{d^2}{s^2} \right) (\beta_k - \beta_{k-1}) \right| ds \\ & \leq \int_{R_0}^t C s^{n+1} J_+ \left(\frac{1}{s^2} \| (J_+)^{-1} (\alpha_k - \alpha_{k-1}) \|_{L^\infty([R_0, +\infty])} + \frac{M}{s^4} s^{-2} \| (J_+)^{-1} \nu^2 (\beta_k - \beta_{k-1}) \|_{L^\infty([R_0, +\infty])} \right) ds \end{aligned}$$

We use (3.79), with $\alpha = n - 1$ and for $\alpha = n - 3$. In any case, we have $|\alpha| \leq C$, for some $C > 0$ depending only on \mathcal{K} . Then, we chose $R_0 \geq \frac{-2(n-5-\frac{1}{2})}{\sqrt{2}}$ in order to conclude that

$$\begin{aligned} & t^{-2n-1} \int_{R_0}^t s^{n+1} \left| \frac{\xi^2}{s^2} (\alpha_k - \alpha_{k-1}) - \left(1 - f_d^2 - \frac{d^2}{s^2} \right) (\beta_k - \beta_{k-1}) \right| ds \\ & \leq C t^{-n-2} J_+ \left(\| (J_+)^{-1} (\alpha_k - \alpha_{k-1}) \|_{L^\infty([R_0, +\infty])} + \| (J_+)^{-1} \nu^2 (\beta_k - \beta_{k-1}) \|_{L^\infty([R_0, +\infty])} \right) \end{aligned}$$

where C depends only on \mathcal{K} . Moreover $3 \leq n + 2 \leq c$, where c depends only on \mathcal{K} . Then, this quantity is integrable in $[R_0, r[$, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. We deduce, by the Lebesgue Theorem, that $\beta_{j+1} - \beta_j$ is continuous wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

We have proved (3.81).

Now, in order to prove (3.73), we estimate, in view of (3.79) and for $R_0 \geq \frac{8}{2\sqrt{2}}$

$$\begin{aligned} & J_+ \int_r^{+\infty} (J_+)^{-2} \frac{1}{t} \int_{R_0}^t s (J_+)^2 s^{-4} ds dt \\ & \leq J_+ \int_r^{+\infty} (J_+)^{-2} \frac{1}{t} \frac{2}{2\sqrt{2}} t^{-4} e^{2\sqrt{2}t} dt \leq J_+ \int_r^{+\infty} \frac{2}{2\sqrt{2}} t^{-4} dt \leq C r^{-3} J_+. \end{aligned} \quad (3.82)$$

This gives (3.73), with $\zeta = J_+$, and this gives also

$$|\alpha_1 - \alpha_0|(r) \leq C r^{-3} J_+. \quad (3.83)$$

In order to prove (3.74), we estimate, for

$$R_0 \geq \frac{-2(n-\frac{7}{2})}{\sqrt{2}} \quad \text{and} \quad R_0 \geq \frac{2(n+\frac{7}{2})}{\sqrt{2}} \quad (3.84)$$

$$r^n \int_{R_0}^r t^{-2n-1} \int_{R_0}^t s^{n+1} \frac{J_+}{s^4} ds dt$$

$$= r^n \int_{R_0}^r t^{-2n-1} \left(\frac{2}{\sqrt{2}} t^{n-\frac{5}{2}} e^{\sqrt{2}t} \right) dt \leq r^n \int_{R_0}^r \frac{2}{\sqrt{2}} t^{-n-\frac{7}{2}} e^{\sqrt{2}t} dt \leq Cr^{-3} J_+. \quad (3.85)$$

This gives (3.74)
and gives also

$$|\beta_1 - \beta_0|(r) \leq Cr^{-3} J_+. \quad (3.86)$$

By (3.73) and (3.74) give, for all $k \geq 1$ and $r > R_0$

$$\begin{aligned} & (J_+)^{-1} |\alpha_{k+1} - \alpha_k|(r) + (J_+)^{-1} |\beta_{k+1} - \beta_k|(r) \\ & \leq Cr^{-3} (CR_0^{-3})^{k-1} (\|(J_+)^{-1}(\alpha_1 - \alpha_0)\|_{L^\infty([R_0, +\infty[)} + \|(J_+)^{-1}(\beta_1 - \beta_0)\|_{L^\infty([R_0, +\infty[)}). \end{aligned} \quad (3.87)$$

Defining x_1 and y_1 by (3.77) and choosing R_0 to have $CR_0^{-2} < 1$, the sums converge uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$ and then the continuity of $(x_1(r), y_1(r))$ wrt (d, γ_1, γ_2) is now complete. But while the first condition given in (3.84) remains to $R_0 \geq \frac{3}{\sqrt{2}}$ (since $n \geq 1$), the second one makes R_0 depending on \mathcal{K} , and consequently, the definition of (x_1, y_1) depends on \mathcal{K} . We remark also that C depends on \mathcal{K} , too.

Then, we write

$$\begin{aligned} |x_1(r) - J_+| & \leq \sum_{k \geq 0} |\alpha_{k+1} - \alpha_k|(r) \\ & \leq Cr^{-2} J_+ \sum_{k \geq 1} (CR_0^{-1})^{k-1} (\|(J_+)^{-1}(\alpha_1 - \alpha_0)\|_{L^\infty([R_0, +\infty[)} + \|(J_+)^{-1}(\beta_1 - \beta_0)\|_{L^\infty([R_0, +\infty[)} \\ & \quad + |\alpha_1 - \alpha_0|(r) \end{aligned}$$

and (3.83) gives the desired behavior at $+\infty$ for x_1 . A similar proof gives the desired behavior of y_1 at $+\infty$.

Now, let us turn to $(x'_1(r), y'_1(r))$. We write

$$\begin{aligned} (\alpha'_{k+1} - \alpha'_k)(r) & = (J_+)' J_+^{-1} (\alpha_{k+1} - \alpha_k)(r) \\ & \quad + \frac{J_+^{-1}}{r} \int_{R_0}^r s J_+ \left(\frac{\xi^2}{s^2} (\beta_k - \beta_{k-1}) - 3(1 - f_d^2 - \frac{d^2}{s^2}) (\alpha_k - \alpha_{k-1}) \right) ds. \end{aligned}$$

Consequently, using successively (3.79) and (3.87)

$$\begin{aligned} J_+^{-1} |\alpha'_{k+1} - \alpha'_k|(r) & \leq C J_+^{-1} |\alpha_{k+1} - \alpha_k|(r) \\ & \quad + Cr^{-2} (\|J_+^{-1} \nu^2 (\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty[)} + \|J_+^{-1} (\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty[)}) \\ & \leq Cr^{-3} (CR_0^{-3})^{k-1} (\|J_+^{-1} (\alpha_1 - \alpha_0)\|_{L^\infty([R_0, +\infty[)} + \|J_+^{-1} (\beta_1 - \beta_0)\|_{L^\infty([R_0, +\infty[)}). \end{aligned} \quad (3.88)$$

This gives the convergence of

$$\sum_{k \geq 1} J_+^{-1} (\alpha'_{k+1} - \alpha'_k)(r)$$

uniformly wrt $r \in [R_0, +\infty[$ and wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Then we directly estimate

$$|\alpha'_1(r) - \alpha'_0(r)| (J_+)^{-1} \leq (J_+)^{-1} |\alpha_1 - \alpha_0|(r) + Cr^{-4} J_+ (J_+)'^{-1} \leq Cr^{-3}$$

and we deduce the desired behavior at $+\infty$

$$J_+^{-1}|x'_1(r) - J'_+| \leq Cr^{-3}.$$

Now we write

$$(\beta'_{k+1} - \beta'_k)(r) = nr^{n-1}r^{-n}(\beta_{k+1} - \beta_k)(r) + r^{-n-1} \int_{R_0}^r s^{n+1} \left(\frac{\xi^2}{s^2} (\beta_k - \beta_{k-1}) - (1 - f_d^2 - \frac{d^2}{s^2}) (\alpha_k - \alpha_{k-1}) \right) ds$$

and consequently, using (3.79) and (3.87), we get for $k \geq 1$

$$\begin{aligned} r^3(J_+)^{-1}|\beta'_{k+1} - \beta'_k|(r) &\leq nr^2 J_+^{-1}|\beta_{k+1} - \beta_k|(r) \\ &+ Cr^{-2} (\|J_+^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])} + \|J_+^{-1}\nu^2(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])}) \\ &\leq Cr^{-2} (CR_0^{-3})^{k-1} (\|J_+^{-1}(\alpha_1 - \alpha_0)\|_{L^\infty([R_0, +\infty])} + \|J_+^{-1}\nu^2(\beta_1 - \beta_0)\|_{L^\infty([R_0, +\infty])}). \end{aligned} \quad (3.89)$$

This gives the convergence of

$$\sum_{k \geq 1} r^3(J_+)^{-1}|\beta'_{k+1} - \beta'_k|(r)$$

uniformly wrt r and to $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Now we estimate

$$|\beta'_1 - \beta'_0|(r) \leq nr^{-1}|\beta_1 - \beta_0|(r) + Cr^{-4}J_+ \leq Cr^{-4}J_+$$

and we deduce the desired estimate

$$|y'_1(r)| \leq Cr^{-4}J_+.$$

Let us turn to the derivability of $(x_1(r), y_1(r))$ wrt γ_1 and γ_2 .

Let us assume that $\alpha_k - \alpha_{k-1}$ and $\beta_k - \beta_{k-1}$ are derivable wrt γ_i , for $i = 1$ or $i = 2$ and that $(\log \nu)^{-1}(J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})$ and $(\log \nu)^{-1}(J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})$ belong to $L^\infty([R_0, +\infty])$ and are continuous wrt (d, γ_1, γ_2) .

On one hand, we write for $i = 1$ or $i = 2$, using $\frac{\partial \xi^2}{\partial \gamma_i} = (-1)^i \gamma_i$,

$$\begin{aligned} & \left| \frac{\partial}{\partial \gamma_i} \left(J_+ \left(\frac{\xi^2}{s^2} (\beta_k - \beta_{k-1}) - 3(1 - f_d^2 - \frac{d^2}{s^2}) (\alpha_k - \alpha_{k-1}) \right) \right) \right| \\ & \leq C(J_+)^2 s^{-4} (\|(J_+)^{-1}\nu^2(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} + \log s \|(\log \nu)^{-1}(J_+)^{-1}\nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])}) \\ & \quad + \log s \|(\log \nu)^{-1}(J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])}. \end{aligned}$$

Then we deduce that $\alpha_{k+1} - \alpha_k$ is derivable wrt γ_i , by the Lebesgue Theorem. Moreover, since C is independent of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, we have that $\frac{\partial}{\partial \gamma_i}(\alpha_{k+1} - \alpha_k)$ is continuous wrt (d, γ_1, γ_2) and we have, using (3.79)

$$(J_+)^{-1} \left| \frac{\partial}{\partial \gamma_i} (\alpha_{k+1} - \alpha_k) \right|(r) \leq \int_r^{+\infty} (J_+)^{-2} \frac{1}{t} \int_{R_0}^t s \left| \frac{\partial}{\partial \gamma_i} \left(J_+ \left(\frac{\xi^2}{s^2} (\beta_k - \beta_{k-1}) - 3(1 - f_d^2 - \frac{d^2}{s^2}) (\alpha_k - \alpha_{k-1}) \right) \right) \right| ds dt$$

$$\leq C \int_r^{+\infty} t^{-4} (\|(J_+^{-1}(\beta_k - \beta_{k-1}))\|_{L^\infty([R_0, +\infty])} + \log t (\|J_+^{-1}(\log \nu)^{-1} \nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} + \|J_+^{-1}(\log \nu)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])})) dt$$

that is

$$\begin{aligned} (J_+)^{-1} \left| \frac{\partial}{\partial \gamma_i}(\alpha_{k+1} - \alpha_k) \right|(r) &\leq C r^{-3} (\|(J_+^{-1} \nu^2(\beta_k - \beta_{k-1}))\|_{L^\infty([R_0, +\infty])} \\ &+ \log r (\|J_+^{-1}(\log \nu)^{-1} \nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} \\ &+ \|J_+^{-1}(\log \nu)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])})) \end{aligned} \quad (3.90)$$

where C is independent of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

On the other hand, we use $\frac{\partial n}{\partial \gamma_i} = \frac{1}{4} \frac{\gamma_i}{n}$ and we estimate, for $s \geq R_0 \geq e$

$$\begin{aligned} &\left| \frac{\partial}{\partial \gamma_i} (s^{n+1} \left(\frac{\xi^2}{s^2} (\alpha_k - \alpha_{k-1}) - (1 - f_d^2 - \frac{d^2}{s^2}) (\beta_k - \beta_{k-1}) \right)) \right| \\ &\leq C \frac{J_+}{s^2} s^{n+1} \log s (\|(J_+)^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])} + \|(\log \nu)^{-1} (J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])}) \\ &+ C \frac{J_+}{s^6} s^{n+1} \log s (\|(\log \nu)^{-1} (J_+)^{-1} \nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} + \|(J_+)^{-1} \nu^2(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])}). \end{aligned}$$

And then we use the Lebesgue Theorem to prove by induction that $\beta_{k+1} - \beta_k$ is derivable wrt γ_i and we use (3.79) to get

$$\begin{aligned} &t^{-2n-1} \left| \frac{\partial}{\partial \gamma_i} \int_{R_0}^t (s^{n+1} \left(\frac{\xi^2}{s^2} (\alpha_k - \alpha_{k-1}) - (1 - f_d^2 - \frac{d^2}{s^2}) (\beta_k - \beta_{k-1}) \right)) ds \right| \\ &\leq C J_+ t^{-n-2} \log t (\|(J_+)^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])} + \|(J_+)^{-1} \nu^2(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])}) \\ &+ \|(\log \nu)^{-1} (J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])} + \|(\log \nu)^{-1} (J_+)^{-1} \nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} \end{aligned}$$

with C independent of $(d, \gamma_1, \gamma_2) \in \mathcal{K}$.

Integrating this inequality on $[R_0, r]$, we get the same upper bound, with r in place of t . Then we can estimate

$$\int_{R_0}^r \frac{\partial}{\partial \gamma_i} (t^{-2n-1}) \left| \int_{R_0}^t (s^{n+1} \left(\frac{\xi^2}{s^2} (\alpha_k - \alpha_{k-1}) - (1 - f_d^2 - \frac{d^2}{s^2}) (\beta_k - \beta_{k-1}) \right)) ds dt \right|$$

by the same upper bound. Finally, we get, when $\log r > 1$

$$\begin{aligned} (J_+)^{-1} (\log r)^{-1} r^2 \left| \frac{\partial}{\partial \gamma_i}(\beta_{k+1} - \beta_k) \right|(r) &\leq C r^{-2} (\|(J_+)^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])} \\ &+ \|(J_+)^{-1} \nu^2(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])} + \|(\log \nu)^{-1} (J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_k - \alpha_{k-1})\|_{L^\infty([R_0, +\infty])}) \end{aligned}$$

$$+\|(\log \nu)^{-1}(J_+)^{-1}\nu^2 \frac{\partial}{\partial \gamma_i}(\beta_k - \beta_{k-1})\|_{L^\infty([R_0, +\infty])}. \quad (3.91)$$

But the sum of (3.87), (3.90) and (3.91) leads for all $k \geq 0$ to

$$\begin{aligned} & \| (J_+)^{-1}(\alpha_{k+1} - \alpha_k) \|_{L^\infty([R_0, +\infty])} + \| (J_+)^{-1}\nu^2(\beta_{k+1} - \beta_k) \|_{L^\infty([R_0, +\infty])} \\ & + \| (\log \nu)^{-1}(J_+)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_{k+1} - \alpha_k) \|_{L^\infty([R_0, +\infty])} + \| (\log \nu)^{-1}(J_+)^{-1}\nu^2 \frac{\partial}{\partial \gamma_i}(\beta_{k+1} - \beta_k) \|_{L^\infty([R_0, +\infty])} \\ & \leq C_0(CR_0^{-2})^k \end{aligned}$$

for some C and some C_0 depending only on \mathcal{K} .

So, using the sum of (3.90) and (3.91) again, we deduce, for all $k \geq 1$

$$\begin{aligned} & (\log r)^{-1}(J_+)^{-1} \left(\left| \frac{\partial}{\partial \gamma_i}(\alpha_{k+1} - \alpha_k) \right|(r) + r^2 \left| \frac{\partial}{\partial \gamma_i}(\beta_{k+1} - \beta_k) \right|(r) \right) \\ & \leq Cr^{-2}C_0(CR_0^{-2})^{k-1} \end{aligned}$$

Choosing R_0 large enough (since the constants are independent of R_0), we deduce that the sums

$$\begin{aligned} & \sum_{k \geq 0} \| (J_+)^{-1}(\log \nu)^{-1} \frac{\partial}{\partial \gamma_i}(\alpha_{k+1} - \alpha_k) \|_{L^\infty([R_0, +\infty])} \\ & \text{and } \sum_{k \geq 0} \| (\log \nu)^{-1}(J_+)^{-1}\nu^2 \frac{\partial}{\partial \gamma_i}(\beta_{k+1} - \beta_k) \|_{L^\infty([R_0, +\infty])} \end{aligned}$$

are convergent, uniformly wrt $(d, \gamma_1, \gamma_2) \in \mathcal{K}$. Recalling the definition of x_1 and y_1 by (3.77), we deduce that x and y are derivable wrt γ_1 and γ_2 . Moreover, since $\frac{\partial \alpha_0}{\partial \gamma_i} = \frac{\partial \beta_0}{\partial \gamma_i} = 0$, we get

$$\begin{aligned} & (\log r)^{-1}(J_+)^{-1} \left| \frac{\partial x_1}{\partial \gamma_i} \right|(r) \leq \sum_{k \geq 1} (\log r)^{-1}(J_+)^{-1}(\alpha_{k+1} - \alpha_k)(r) \\ & \leq Cr^{-2} \sum_{k \geq 1} C_0(CR_0^{-2})^k \end{aligned}$$

and this gives

$$(\log r)^{-1}(J_+)^{-1} \left| \frac{\partial x_1}{\partial \gamma_i} \right|(r) \leq Cr^{-2}.$$

The same proof works for $(\log r)^{-1}(J_+)^{-1}r^2 \left| \frac{\partial y_1}{\partial \gamma_i} \right|(r)$. This gives the first part of (3.80).

The proof of the behavior of $\frac{\partial x'_1}{\partial \gamma_i}$ and of $\frac{\partial y'_1}{\partial \gamma_i}$ at $+\infty$ is left to the reader.

3.2 The intermediate blowing up behavior at $+\infty$: the solution (x_3, y_3) .

Proposition 3.12 *For all \mathcal{K} as in (3.70) and for some $R_0 > 0$ depending on \mathcal{K} , there exists a solution (x_3, y_3) of (1.7), verifying the following property there exists C depending only on \mathcal{K} and such that for all $r > R_0$*

$$|x_3(r)| + |y_3(r) - r^n| \leq Cr^{n-2}, \quad |x'_3(r) + |y'_3(r) - nr^{n-1}| \leq Cr^{n-3}$$

and for all $r > 0$

$$(d, \gamma_1, \gamma_2) \mapsto (x_3(r), x'_3(r), y_3(r), y'_3(r)) \text{ is continuous on } \mathcal{K}.$$

Moreover, (x_3, y_3) is derivable wrt γ_1 and γ_2 and we have

$$\begin{aligned} \left| \frac{\partial x_3}{\partial \gamma_i} \right|(r) + \left| \frac{\partial y_3}{\partial \gamma_i} \right|(r) - r^n \log r &\leq C |\log r| r^{n-2} \\ \left| \frac{\partial x'_3}{\partial \gamma_i} \right|(r) + \left| \frac{\partial y'_3}{\partial \gamma_i} \right|(r) &\leq C |\log r| r^{n-1} \end{aligned} \quad (3.92)$$

for $r \geq R_0$ and for C depending only on \mathcal{K} .

Proof We follow the proof of Proposition 3.11, with the same notation. Let us indicate only what is different. We consider the following fixed point problem, with x and y defined in $[R_0, +\infty[$

$$\begin{cases} x &= J_+ \int_{+\infty}^r (J_+)^{-2\frac{1}{t}} \int_{R_0}^t s J_+ \left(\frac{\mu^2}{s^2} y - 3(1 - f_d^2 - \frac{d^2}{s^2}) x \right) ds dt \\ y &= r^n + r^n \int_{+\infty}^r t^{-2n-1} \int_{R_0}^t s^{n+1} \left(\frac{\mu^2}{s^2} x - (1 - f_d^2 - \frac{d^2}{s^2}) y \right) ds dt \end{cases}$$

We chose $\zeta(r) = r^n$ and $(\alpha_0, \beta_0) = (0, \zeta)$.

We let the reader prove, for all $r \geq R_0 \geq 1$

$$\begin{aligned} r^{-n+2} |\alpha_{j+1} - \alpha_j|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})\nu^{-n+2}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_j - \beta_{j-1})\nu^{-n}\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.93)$$

and

$$\begin{aligned} r^{-n} |\beta_{j+1} - \beta_j|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})\nu^{-n+2}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_j - \beta_{j-1})\nu^{-n}\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.94)$$

We also remark that the constant C is independent of R_0 and does depend on \mathcal{K} . In the course of the construction of (x_3, y_3) , we need the condition $CR_0^{-2} < 1$. We can conclude that the choice of R_0 , and consequently the solution (x_3, y_3) depend on \mathcal{K} . The end of the proof of Proposition 3.12 is left to the reader.

3.3 The least behavior at $+\infty$: the solution (x_2, y_2) .

Proposition 3.13 *There exists a solution (x_2, y_2) of (1.7) verifying, for all compact subset \mathcal{K} as in (3.70) there exist C and R_0 depending only on \mathcal{K} and such that for all $r > R_0$*

$$|x_2(r) - J_-| + |y_2(r)| \leq Cr^{-2}J_-, \quad |x_2'(r) - (J_-)'| + |y_2'(r)| \leq Cr^{-3}J_-$$

and, for all $r > 0$,

$(d, \gamma_1, \gamma_2) \mapsto (x_2(r), x_2'(r), y_2(r), y_2'(r))$ is continuous on $\{(d, \gamma_1, \gamma_2); 0 \leq \gamma_1 \leq \gamma_2; d \geq 0; \frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 > 0\}$.

Moreover, $(x_2(r), x_2'(r), y_2(r), y_2'(r))$ is derivable wrt γ_1 and γ_2 and, for $i = 1, 2$

$$\left| \frac{\partial x_2}{\partial \gamma_i} \right| + \left| \frac{\partial y_2}{\partial \gamma_i} \right| \leq CJ_-r^{-2}|\log r| \quad \text{and} \quad \left| \frac{\partial x_2'}{\partial \gamma_i} \right| + \left| \frac{\partial y_2'}{\partial \gamma_i} \right| \leq CJ_-r^{-3}|\log r| \quad (3.95)$$

for $r \geq R_0$ and for C depending only on \mathcal{K} .

Proof Let us consider the following fixed point problem

$$\begin{cases} x &= J_- + J_- \int_{+\infty}^r (J_-)^{-2} \frac{1}{t} \int_{+\infty}^t s J_- \left(\frac{\xi^2}{s^2} y - 3(1 - f_d^2 - \frac{d^2}{s^2}) x \right) ds dt \\ y &= r^{-n} \int_{+\infty}^r t^{2n-1} \int_{+\infty}^t s^{-n+1} \left(\frac{\xi^2}{s^2} x - 3(1 - f_d^2 - \frac{d^2}{s^2}) y \right) ds dt \end{cases}$$

We define, with the usual notation, $\zeta = J_-$ and $(\alpha_0, \beta_0) = (J_-, 0)$.

Let $(d, \gamma_1, \gamma_2) \in \mathcal{K}$, where \mathcal{K} is a compact set, as usual. We let the reader use (3.79) and (3.78) and verify that, if $R_0 > 0$ is large enough, depending only on \mathcal{K} , we have for all $r > R_0$

$$\begin{aligned} |(J_-)^{-1}(\alpha_{j+1} - \alpha_j)|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})(J_-)^{-1}\|_{L^\infty([R_0, +\infty])} \\ &\quad + \|(\beta_j - \beta_{j-1})(J_-)^{-1}\nu^2\|_{L^\infty([R_0, +\infty])}) \end{aligned} \quad (3.96)$$

and

$$\begin{aligned} r^2(J_-)^{-1}|(\beta_{j+1} - \beta_j)(J_+)^{-1}|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})(J_-)^{-1}\|_{L^\infty([R_0, +\infty])} \\ &\quad + \|(\beta_j - \beta_{j-1})(J_-)^{-1}\nu^2\|_{L^\infty([R_0, +\infty])}) \end{aligned} \quad (3.97)$$

where C depends only on \mathcal{K} and is independent of R_0 .

The rest of the proof is left to the reader, too.

We remark that, for this solution, the construction doesn't depend on \mathcal{K} .

3.4 The intermediate vanishing behavior at $+\infty$: the solution (x_4, y_4) .

Proposition 3.14 *For all compact subset \mathcal{K} as in (3.70), there exists a solution (x_4, y_4) of (1.7), verifying the following property there exists C and R_0 depending only on \mathcal{K} and such that for all $r \geq R_0$*

$$|x_4(r)| + |y_4(r) - r^{-n}| \leq Cr^{-n-2}, \quad |x_4'(r)| + |y_4'(r) + nr^{-n-1}| \leq Cr^{-n-3}$$

and, for all $r > 0$

$$(d, \gamma_1, \gamma_2) \mapsto (x_4(r), x'_4(r), y_4(r), y'_4(r)) \text{ is continuous on } \mathcal{K}.$$

Moreover, (x_4, x'_4, y_4, y'_4) is derivable wrt γ_1 and γ_2 and, for $i = 1, 2$

$$\left| \frac{\partial x_4}{\partial \gamma_i} \right| + \left| \frac{\partial y_4}{\partial \gamma_i} \right| \leq Cr^{-n-2} \log r \text{ and } \left| \frac{\partial x'_4}{\partial \gamma_i} \right| + \left| \frac{\partial y'_4}{\partial \gamma_i} \right| \leq Cr^{-n-3} \log r \quad (3.98)$$

for $r \geq R_0$ and for C depending only on \mathcal{K} .

Proof Let $R_0 > 0$ be given and let us consider the following fixed point problem

$$\begin{cases} x &= J_- \int_{R_0}^r (J_-)^{-2\frac{1}{t}} \int_{+\infty}^t s J_- \left(\frac{\xi^2}{s^2} y - 3(1 - f_d^2 - \frac{d^2}{s^2}) x \right) ds dt \\ y &= r^{-n} + r^{-n} \int_{+\infty}^r t^{2n-1} \int_{+\infty}^t s^{-n+1} \left(\frac{\xi^2}{s^2} x - 3(1 - f_d^2 - \frac{d^2}{s^2}) y \right) ds dt \end{cases}$$

We define $\zeta(r) = r^{-n}$ and $(\alpha_0, \beta_0) = (0, r^{-n})$.

Using (3.79) and (3.78), we can verify that for $R_0 > 0$ large enough depending on \mathcal{K} and for all $r \geq R_0$

$$\begin{aligned} r^{n+2} |\alpha_{j+1} - \alpha_j|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})\nu^{n+2}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_j - \beta_{j-1})\nu^n\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.99)$$

and

$$\begin{aligned} r^n |\beta_{j+1} - \beta_j|(r) &\leq \frac{C}{r^2} (\|(\alpha_j - \alpha_{j-1})\nu^{n+2}\|_{L^\infty([R_0, +\infty[)} \\ &\quad + \|(\beta_j - \beta_{j-1})\nu^n\|_{L^\infty([R_0, +\infty[)}) \end{aligned} \quad (3.100)$$

where C depends only on \mathcal{K} and is independent of R_0 . The proof of the proposition is left to the reader.

4 The smallest behavior at zero is relied with the greatest behavior at infinity

For all $(d, \gamma_1, \gamma_2) \in \mathcal{D}$, the solution that has the smallest behavior at 0, is well defined, to a multiplicative factor. In all what follows, we call ω_1 this solution, that is (a_1, b_1) in Theorem 1.3. In the same way, $\eta_2 = (u_2, v_2)$ is a solution that has the smallest behavior at $+\infty$, without ambiguity. Now we can enonce

Proposition 4.15 *When $d > 0$ and when $\gamma_2 \geq \gamma_1 \geq 0$, $(\gamma_2^2 + \gamma_1^2)/2 \geq d^2$, then the behavior of ω_1 at $+\infty$ is an exponentially increasing behavior.*

Proof Let us denote $\omega_1 = (a, b)$ and let us define $x = a + b$ and $y = a - b$. Using Theorem 1.3, we have $x(r) \sim_0 r^{\gamma_2}$ and $y(r) \sim_0 -r^{\gamma_2}$. Then we have $x(r) > 0$ and $y(r) < 0$ near $r = 0$.

Let us prove that for all r we have $x(r) > 0$ and $y(r) < 0$. Let us suppose that $x(r) > 0$ and $y(r) < 0$ in $[0, R[$. Combining the first equation of the system (1.7) with the equation (1.3), we get, for all $r \leq R$

$$[rx'f_d - rf_d'x]_0^r + \int_0^r \frac{-\gamma^2 + d^2}{s} x f_d ds + \mu^2 \int_0^r \frac{y}{s} f_d ds - 2 \int_0^r s f_d^3 x ds = 0.$$

We deduce that

$$rf_d^2 \left(\frac{x}{f_d} \right)'(r) \geq 2 \int_0^r s f_d^3 x ds. \quad (4.101)$$

then $\frac{x}{f_d}$ increases in $[0, R]$ and consequently $x > 0$ in $[0, R]$. We deduce that $x(R) > 0$. Moreover, combining the second equation of the system 1.7 and 1.3, we get

$$[ry'f_d - rf_d'y]_0^r + \int_0^r \frac{-\gamma^2 + d^2}{s} y f_d ds + \xi^2 \int_0^r \frac{x}{s} f_d ds = 0$$

and consequently

$$rf_d^2 \left(\frac{-y}{f_d} \right)'(r) \geq \int_0^r \frac{\gamma^2 - d^2}{s} y f_d ds. \quad (4.102)$$

Then $\frac{-y}{f_d}$ increases in $[0, R]$ and consequently $y < 0$ in $[0, R]$. We have proved that $x(r) > 0$ and $y(r) < 0$ for all $r > 0$. We have now that (4.101) and (4.102) are valid for all $r > 0$ and we know that $f_d \sim_{+\infty} 1$. Then the behavior of x and of $-y$ at $+\infty$ cannot be a polynomial increasing behavior. Now let us use Theorem 1.4 and let us identify the behavior of (x, y) at $+\infty$. Then x and y have an exponentially increasing behavior at $+\infty$.

We can now prove the following

Corollary 4.1 *When $d > 0$ and when $\gamma_2 \geq \gamma_1 \geq 0$, $(\gamma_2^2 + \gamma_1^2)/2 \geq d^2$, then the behavior of η_2 at 0 is the greater blowing up behavior.*

Proof Let (a, b) and (u, v) be two solutions of (1.6). Multiplying (1.6) and integrating by parts, we get easily, for all $r_1 > 0$ and $r_2 > 0$

$$[r(a'u - u'a + vb' - v'b)]_{r_1}^{r_2} = 0.$$

Then, if (a, b) and (u, v) correspond respectively to ω_1 and η_2 , we get a real number $C \neq 0$ such that

$$\lim_{+\infty} r(a'u - u'a + vb' - v'b)(r) = C$$

and consequently

$$\lim_0 r(a'u - u'a + vb' - v'b)(r) = C.$$

Considering that $(a, b) \sim_0 (o(r^{\gamma_2}), r^{\gamma_2})$ and in view of Theorem 1.3, that gives all the possible behaviors at 0, we conclude that the only fitting behavior at 0 for (u, v) is $(u, v) \sim_0 D(o(r^{\gamma_1}), r^{-\gamma_2})$, for some real number $D \neq 0$.

5 The eigenvalue problem

In this part, we give the proves of Proposition 1.1 and Proposition 1.2.

In what follows, we consider that $d > 0$, that $\gamma_2 > \gamma_1 \geq 0$ are given and we suppose that $\frac{\gamma_2^2 + \gamma_1^2}{2} > d^2$.

5.1 Proof of Proposition 1.1.

To begin with, using the notation of Proposition 1.1, we suppose that $\mu(\varepsilon) \rightarrow \mu$ and that $\tilde{\omega}_\varepsilon \rightarrow \omega_0$ on $[0, R]$, for each $R > 0$. If $\frac{\gamma_2^2 + \gamma_1^2}{2} - \mu d^2 > 0$, we define $n_0 = \sqrt{\frac{\gamma_2^2 + \gamma_1^2}{2} - \mu d^2}$.

Lemma 5.1 *If ω_0 blows up either exponentially, or like $(r^{n_0}, -r^{n_0})$ and if $\frac{\gamma_2^2 + \gamma_1^2}{2} - \mu d^2 > 0$, then we have $\frac{\mu(\varepsilon) - 1}{\varepsilon^2} > C$, for all ε small enough, where $C > 0$ is independent of ε .*

Proof Let $\omega_\varepsilon = (a_\varepsilon, b_\varepsilon) \in \mathcal{H}_{\gamma_1}$ be an eigenvector associated to $\mu(\varepsilon)$. Using (1.13), we write

$$\frac{\mu(\varepsilon)}{\varepsilon^2} \int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2) dr = \int_0^1 (r a_\varepsilon'^2 + r b_\varepsilon'^2 + \frac{\gamma_1^2}{r} a_\varepsilon^2 + \frac{\gamma_2^2}{r} b_\varepsilon^2 + \frac{r}{\varepsilon^2} f^2 (a_\varepsilon + b_\varepsilon)^2) dr.$$

We use the definition (1.17) of $m_0(\varepsilon)$ to get

$$\begin{aligned} & \frac{\mu(\varepsilon)}{\varepsilon^2} \int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2) dr \\ & \geq \frac{m_0(\varepsilon)}{\varepsilon^2} \int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2) dr + \int_0^1 \left(\frac{\gamma_1^2 - d^2}{r} a_\varepsilon^2 + \frac{\gamma_2^2 - d^2}{r} b_\varepsilon^2 + \frac{r}{\varepsilon^2} f^2 (a_\varepsilon + b_\varepsilon)^2 \right) dr. \end{aligned}$$

Now, we use the trick of TC Lin (see [6]). Letting $\tilde{b}_\varepsilon = \tau \tilde{a}_\varepsilon$, we consider the map

$$H : \tau \mapsto \frac{\gamma_1^2 - d^2}{r} + \frac{\gamma_2^2 - d^2}{r} \tau^2 + r f_d^2 (1 + \tau)^2 \quad (5.103)$$

and we minimize this map. The minimum is attained for τ_0 verifying

$$\tau_0 \left(\frac{\gamma_2^2 - d^2}{r} + r f_d^2 \right) + r f_d^2 = 0$$

and then

$$1 + \tau_0 = \frac{\frac{\gamma_2^2 - d^2}{r}}{\frac{\gamma_2^2 - d^2}{r} + r f_d^2}$$

and consequently

$$H(\tau_0) = \frac{\gamma_1^2 - d^2}{r} + \left(\frac{r f_d^2}{\frac{\gamma_2^2 - d^2}{r} + r f_d^2} \right)^2 \left(\frac{\gamma_2^2 - d^2}{r} \right) + r f_d^2 \left(\frac{\frac{\gamma_2^2 - d^2}{r}}{\frac{\gamma_2^2 - d^2}{r} + r f_d^2} \right)^2.$$

We have

$$H(\tau_0) \sim_{r \rightarrow +\infty} \frac{\gamma_1^2 + \gamma_2^2 - 2d^2}{r}.$$

Moreover

$$\text{for all } \tau > 0, H(\tau) \geq H(\tau_0)$$

Since we have suppose that $\frac{\gamma_1^2 + \gamma_2^2}{2} - d^2 > 0$, there exists some constants $C_1 > 0$ and $R_0 > 0$, independent of τ , such that for all $\tau > 0$

$$H(\tau) \geq \frac{C_1}{r} \text{ for all } r > R_0.$$

Then, for all $R > R_0$ and all $\varepsilon < \frac{1}{R}$, we write

$$\int_0^{\frac{1}{\varepsilon}} H(r) \tilde{a}_\varepsilon^2(r) dr \geq \int_0^{R_0} H(r) \tilde{a}_\varepsilon^2(r) dr + \int_{R_0}^R H(r) \tilde{a}_\varepsilon^2(r) dr.$$

Now a_0 blows up exponentially at $+\infty$, or as r^{n_0} . We can choose R_0 large enough and a constant $C_2 > 0$ to have also

$$a_0^2(r) \geq C_2 \left(\frac{e^{\sqrt{2}r}}{\sqrt{r}} \right)^2 \text{ or } C_2 r^{2n_0} \text{ for all } r > R_0.$$

Since $\tilde{a}_\varepsilon \rightarrow a_0$ as $\varepsilon \rightarrow 0$, uniformly in $[0, R_0]$, we can chose ε_0 such that for all $\varepsilon < \varepsilon_0$

$$\int_0^{R_0} H(r) \tilde{a}_\varepsilon^2(r) dr \geq \frac{1}{2} \int_0^{R_0} H(r) a_0^2(r) dr.$$

Moreover, for all $R > R_0$, $\tilde{a}_\varepsilon \rightarrow a_0$ as $\varepsilon \rightarrow 0$, uniformly in $[R_0, R]$. Then, there exists $\varepsilon(R)$ such that for all $\varepsilon < \varepsilon(R)$ we have

$$\int_{R_0}^R H(r) \tilde{a}_\varepsilon^2(r) dr \geq \frac{C_2}{2} \int_{R_0}^R \frac{1}{r} r^{2n_0} dr \quad \text{or} \quad \int_{R_0}^R H(r) \tilde{a}_\varepsilon^2(r) dr \geq \frac{C_2}{2} \int_{R_0}^R \frac{1}{r} \left(\frac{e^{\sqrt{2}r}}{\sqrt{r}} \right)^2 dr.$$

And finally, for $\varepsilon < \varepsilon(R)$, we have

$$\begin{aligned} \left(\frac{\mu(\varepsilon) - m_0(\varepsilon)}{\varepsilon^2} \right) \int_0^1 r(1-f^2)(a_\varepsilon^2 + b_\varepsilon^2) dr &\geq \frac{1}{2} \int_0^{R_0} H(r) a_0^2(r) dr \\ &+ \left\{ \begin{array}{l} \frac{C_1 C_2}{2} \int_{R_0}^R \frac{1}{r} r^{2n_0} dr \\ \text{or } \frac{C_1 C_2}{2} \int_{R_0}^R \frac{1}{r} \left(\frac{e^{\sqrt{2}r}}{\sqrt{r}} \right)^2 dr \end{array} \right. \end{aligned}$$

where C_1 and C_2 , given above, are independent of R and ε . But we can choose R such that the lhs is positive.

We deduce that

$$\mu(\varepsilon) - m_0(\varepsilon) > C_0,$$

for some $C_0 > 0$, independent of ε , and for ε small enough. Then we use Theorem 1.5 (i), that gives $\frac{m_0(\varepsilon) - 1}{\varepsilon^2} \geq C$. The lemma is proved.

Now let us enonce the following

Lemma 5.2 *If $\omega = (a, b)$ is a bounded solution of (1.6), then there exists an eigenvalue $\mu(\varepsilon)$ such that $(\mu(\varepsilon) - 1) \rightarrow 0$.*

Proof Let us suppose that $\omega = (a, b)$ is a solution of (1.6), a and b being real valued functions. Let $\frac{1}{2} < N < 1$ be given, let us define $\omega^{cut} = (a^{cut}, b^{cut})$ by

$$(a^{cut}, b^{cut})(r) = \begin{cases} (a, b)(r) & \text{for } 0 \leq r \leq \frac{N}{\varepsilon} \\ ((a, b)(r)(1 - h(r))) & \text{for } \frac{N}{\varepsilon} \leq r \leq \frac{1}{\varepsilon} \end{cases}$$

where

$$h(r) = \frac{(r - \frac{N}{\varepsilon})^3}{(\frac{1}{\varepsilon} - \frac{N}{\varepsilon})^3}.$$

We have

$$\omega^{cut} e^{id\theta} \in (H^2 \cap H_0^1)(B(0, \frac{1}{\varepsilon})).$$

We use, for ε small enough

$$|a(r)| \leq Cr^{-n} \text{ for } \frac{N}{\varepsilon} < r < \frac{1}{\varepsilon}, \quad |a^{cut}| \leq |a| \quad \text{and} \quad r(1 - f_d^2) = O(\frac{1}{r}) \quad \text{at } +\infty$$

and we verify that

$$\begin{aligned} & \langle \omega^{cut} - \omega, (1 - f_d^2)(\omega^{cut} - \omega) \rangle_{(L^2 \times L^2)(B(0, \frac{1}{\varepsilon}))} = \\ & \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(1 - f_d^2)((a - a^{cut})^2 + (b - b^{cut})^2) dr = O(\varepsilon^{2n}) \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (5.104)$$

Then, let us define

$$\bar{\omega}^{cut}(r) = \omega^{cut}(\frac{r}{\varepsilon}) \quad 0 < r < 1.$$

Let $(\zeta_i)_{i \in J}$ be a Hilbertian base of \mathcal{H}_{γ_1} , associated to the eigenvalues $\mu_i(\varepsilon)$, and such that

$$\langle \mathcal{C}\zeta_i, \zeta_i \rangle_{(L^2 \times L^2)(B(0,1))} = 1.$$

We have

$$\bar{\omega}^{cut} = \sum_{i \in J} \alpha_i \zeta_i$$

and

$$\langle \mathcal{C}\bar{\omega}^{cut}, \bar{\omega}^{cut} \rangle_{(L^2 \times L^2)(B(0,1))} = \langle (1 - f_d^2)\omega^{cut}, \omega^{cut} \rangle_{(L^2 \times L^2)(B(0, \frac{1}{\varepsilon}))} = \sum_{i \in J} \alpha_i^2.$$

Since

$$\langle (1 - f_d^2)\omega^{cut}, \omega^{cut} \rangle_{L^2(B(0, \frac{1}{\varepsilon}))} \longrightarrow \int_0^{+\infty} r(1 - f_d^2)(a^2 + b^2) dr \text{ as } \varepsilon \rightarrow +\infty,$$

there exists $I \subset J$, such that

$$I \neq \emptyset \text{ and for all } i \in I, \quad \alpha_i^2 \not\rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Now we write

$$-(\mathcal{T} + \mathcal{C})\bar{\omega}^{cut} = \sum_{i \in J} \alpha_i(\mu_i - 1)\mathcal{C}\zeta_i$$

that gives

$$\langle -(\mathcal{T} + \mathcal{C})\bar{\omega}^{cut}, \sum_{i \in J} \alpha_i (\mu_i - 1) \zeta_i \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} = \sum_{i \in J} \alpha_i^2 (\mu_i - 1)^2. \quad (5.105)$$

But $(\mathcal{T} + \mathcal{C})\bar{\omega}^{cut}$ is represented by a function of $L^2(B(0, 1)) \times L^2(B(0, 1))$, and $\mathcal{C} = \frac{1-f^2}{\varepsilon^2}$. So, using this identification, we can estimate the rhs of (5.105) as follows,

$$\begin{aligned} - \langle (\mathcal{T} + \mathcal{C})\bar{\omega}^{cut}, \sum_{i \in J} \alpha_i (\mu_i - 1) \zeta_i \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} &= \langle (\mathcal{T} + \mathcal{C})\bar{\omega}^{cut}, \frac{\varepsilon^2}{1-f^2} (\mathcal{T} + \mathcal{C})\bar{\omega}^{cut} \rangle_{(L^2 \times L^2)(B(0,1))} \\ &= \int_N^1 \frac{r\varepsilon^2}{1-f^2} \left((\bar{a}^{cut''} + \frac{\bar{a}^{cut'}}{r} - \frac{\gamma_1^2}{r^2} \bar{a}^{cut} - \frac{1}{\varepsilon^2} f^2 (\bar{a}^{cut} + \bar{b}^{cut}) + \frac{1}{\varepsilon^2} (1-f^2) \bar{a}^{cut})^2 \right. \\ &\quad \left. + (\bar{b}^{cut''} + \frac{\bar{b}^{cut'}}{r} - \frac{\gamma_2^2}{r^2} \bar{b}^{cut} - \frac{1}{\varepsilon^2} f^2 (\bar{a}^{cut} + \bar{b}^{cut}) + \frac{1}{\varepsilon^2} (1-f^2) \bar{b}^{cut})^2 \right) dr. \\ &= \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} \left((a^{cut''} + \frac{a^{cut'}}{r} - \frac{\gamma_1^2}{r^2} a^{cut} - f_d^2 (a^{cut} + b^{cut}) + (1-f_d^2) a^{cut})^2 \right. \\ &\quad \left. + (b^{cut''} + \frac{b^{cut'}}{r} - \frac{\gamma_2^2}{r^2} b^{cut} - f_d^2 (a^{cut} + b^{cut}) + (1-f_d^2) b^{cut})^2 \right) dr. \end{aligned}$$

Let us estimate each term, as $\varepsilon \rightarrow 0$.

We use

$$\frac{r}{1-f_d^2} = O(r^3) \quad \text{at } +\infty$$

to get

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f^2} \frac{(a^{cut})^2}{r^4} dr = O(\varepsilon^{2n}).$$

Taking advantage that $a + b = O(r^{-n-2})$ at $+\infty$, a similar estimate for $a^{cut} + b^{cut}$ gives

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} f_d^2 (a^{cut} + b^{cut})^2 dr = O(\varepsilon^{2n}).$$

Now

$$a^{cut'} = a'(1-h) + ah' \quad \text{and} \quad |a'| \leq Cr^{-n-1} \quad \text{and} \quad \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h'^2 dr = O(\varepsilon).$$

We deduce that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} \frac{(a^{cut'})^2}{r^2} dr = O(\varepsilon^{2n}).$$

Now, since

$$|a''| \leq Cr^{-n-2} \quad \text{and} \quad \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h''^2 dr = O(\varepsilon^3)$$

we get

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} (a^{cut''})^2 dr = O(\varepsilon^{2n}).$$

We have proved that

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} \left((a^{cut''} + \frac{a^{cut'}}{r} - \frac{\gamma_1^2}{r^2} a^{cut} - f_d^2 (a^{cut} + b^{cut}) - (1-f_d^2) a^{cut})^2 \right) = O(\varepsilon^{2n})$$

and with the same proof we have

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{r}{1-f_d^2} \left((b^{cut''} + \frac{b^{cut'}}{r} - \frac{\gamma_2^2}{r^2} b^{cut} - f_d^2 (a^{cut} + b^{cut}) - (1-f_d^2) b^{cut})^2 \right) dr = O(\varepsilon^{2n})$$

and finally

$$- \langle (\mathcal{T} + \mathcal{C}) \bar{\omega}^{cut}, \sum_{i \in J} \alpha_i (\mu_i - 1) \eta_i \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} = O(\varepsilon^{2n}) \quad (5.106)$$

But (5.106) and (5.105) give

$$\sum_{i \in J} \alpha_i^2 (\mu_i - 1)^2 = O(\varepsilon^{2n}).$$

So, for all $i \in J$ we have

$$|\alpha_i (\mu_i - 1)| = O(\varepsilon^n).$$

Since $n > 0$, we are led to

$$\mu_i(\varepsilon) - 1 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \text{ for all } i \in I.$$

We have proved the lemma.

Now, under the conditions of Lemma 5.2 and with the additional condition that $n \geq 1$ and that the least eigenvalue is greater than 1, we can improve the conclusion of Lemma 5.2 as follows.

Lemma 5.3 *If $n \geq 1$, if $m_{\gamma_1, \gamma_2}(\varepsilon) \geq 1$ and if there exists a bounded solution of (1.6), then there exists an eigenvalue $\mu(\varepsilon)$ such that*

$$\frac{\mu(\varepsilon) - 1}{\varepsilon^2} \rightarrow 0.$$

Proof Using the notation of the proof of Lemma 5.2, we write

$$\bar{\omega}^{cut} = \sum_{i \in J} \alpha_i \zeta_i$$

and consequently

$$- \langle (\mathcal{T} + \mathcal{C}) \bar{\omega}^{cut}, \bar{\omega}^{cut} \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} = \sum_{i \in J} \alpha_i^2 (\mu_i - 1).$$

To make the proof more easy, and since we don't need anymore the continuity of the second derivatives at $\frac{N}{\varepsilon}$, we use another definition for ω^{cut} , that is now

$$\omega^{cut}(r) = \begin{cases} \omega(r) & \text{if } 0 < r < \frac{N}{\varepsilon} \\ \omega(r)(1 - h(r)) & \text{if } \frac{N}{\varepsilon} < r < \frac{1}{\varepsilon} \end{cases}$$

where

$$h(r) = \left(\frac{r - \frac{N}{\varepsilon}}{\frac{1}{\varepsilon} - \frac{N}{\varepsilon}} \right)^2$$

and $N > \frac{1}{2}$. We will have to chose a suitable N , depending on ε . Then

$$\begin{aligned} & - \langle (\mathcal{T} + \mathcal{C})\bar{\omega}^{cut}, \bar{\omega}^{cut} \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} = - \langle (\mathcal{T} + \mathcal{C})(\bar{\omega}^{cut} - \bar{\omega}), (\bar{\omega}^{cut} - \bar{\omega}) \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} \\ & = \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} (r(a^{cut} - a)^2 + r(b^{cut} - b)^2 + \frac{\gamma_1^2}{r}(a^{cut} - a)^2 + \frac{\gamma_2^2}{r}(b^{cut} - b)^2 - r(1 - f_d^2)((a^{cut} - a)^2 + (b^{cut} - b)^2) \\ & \quad + r f_d^2(a^{cut} - a + b^{cut} - b)^2) dr. \end{aligned}$$

We have

$$\frac{|a^{cut} - a|^2}{r} \leq C\varepsilon^{2n+1}h^2(r)$$

where C is independent of N , when $\frac{1}{2} < N < 1$.

Since we have

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h^2(r) dr = \frac{1}{7} \left(\frac{1}{\varepsilon} - \frac{N}{\varepsilon} \right),$$

and

$$|a^{cut} + b^{cut} - a - b| \leq Cr^{-n-2}$$

we deduce that

$$\begin{aligned} & \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} \left(\frac{\gamma_1^2}{r}(a^{cut} - a)^2 + \frac{\gamma_2^2}{r}(b^{cut} - b)^2 + r(1 - f_d^2)((a^{cut} - a)^2 + (b^{cut} - b)^2) \right. \\ & \quad \left. + r(1 - f_d^2)(a^{cut} - a + b^{cut} - b)^2 \right) dr \leq C_1(1 - N)\varepsilon^{2n} \end{aligned}$$

where C_1 is independent of N .

Now

$$(a^{cut} - a)' = -a'h - ah'.$$

On one hand, we have

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} ra'^2 h^2 dr \leq C\varepsilon^{2n+1} \int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h^2(r) dr \leq C\varepsilon^{2n}(1 - N).$$

On the other hand, we use

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} h'^2(r) dr = \frac{4}{3} \frac{\varepsilon}{1 - N}$$

to get

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} a^2 h'^2 dr \leq C \varepsilon^{2n+1} \frac{1}{1-N}.$$

Finally

$$\begin{aligned} & | \langle (\mathcal{T} + \mathcal{C}) \bar{\omega}^{cut}, \bar{\omega}^{cut} \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} | \\ & \leq C_1 \varepsilon^{2n} (1-N) + C_2 \frac{\varepsilon^{2n+1}}{1-N}, \end{aligned}$$

where C_1 and C_2 are positive and independent of ε and of N , when $\frac{1}{2} < N < 1$. Then, we take

$$1 - N = \varepsilon^\alpha \quad \text{where } 0 < \alpha < 1,$$

to obtain

$$| \langle (\mathcal{T} + \mathcal{C}) \bar{\omega}^{cut}, \bar{\omega}^{cut} \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} | \leq C(\varepsilon^{2n+\alpha} + \varepsilon^{2n+1-\alpha}),$$

where $C > 0$ is independent of ε .

Then

$$\left| \sum_{i \in J} \alpha_i^2 (\mu_i(\varepsilon) - 1) \right| \leq C(\varepsilon^{2n+\alpha} + \varepsilon^{2n+1-\alpha}).$$

And, since we have supposed, for all i , that $\mu_i(\varepsilon) - 1 \geq 0$, we have

$$0 \leq \sum_{i \in J} \alpha_i^2 (\mu_i(\varepsilon) - 1) \leq C(\varepsilon^{2n+\alpha} + \varepsilon^{2n+1-\alpha}).$$

But we verify that we still have

$$\int_{\frac{N}{\varepsilon}}^{\frac{1}{\varepsilon}} r(1 - f_d^2) ((a^{cut})^2 + (b^{cut})^2) dr \rightarrow 0$$

to deduce, as in the proof of Lemma 5.2, that

$$\exists I \neq \emptyset, \quad \forall i \in I, \quad \alpha_i \not\rightarrow 0.$$

Then,

$$\forall i \in I, \quad 0 \leq \mu_i(\varepsilon) - 1 \leq C(\varepsilon^{2n+\alpha} + \varepsilon^{2n+1-\alpha}).$$

The lemma is proved.

The proof of Proposition 1.1 follows from Lemma 5.1 and Lemma 5.2.

5.2 Proof of Proposition 1.2.

The proof for $n = 2$ and $d = 2$ is originally in [8].

For $d \geq 1$ and $n \geq 1$, let $x = \frac{f'_d}{r^{n-1}}$ and $y = d \frac{f_d}{r^n}$. A calculus gives

$$\begin{cases} -(rx')' + \frac{\gamma^2}{r} x - \frac{\xi^2}{r} y - r(1 - 3f_d^2)x & = -2 \frac{n-1}{r^{n-1}} f_d (1 - f_d^2) \\ -(ry')' + \frac{\gamma^2}{r} y - \frac{\xi^2}{r} x - r(1 - f_d^2)y & = 0 \end{cases} \quad (5.107)$$

For $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$, we deduce that

$$\begin{cases} -(ra')' + \frac{\gamma_1^2}{r}a + f_d^2b - r(1 - 2f_d^2)a &= -\frac{n-1}{r^{n-1}}f_d(1 - f_d^2) \\ -(rb')' + \frac{\gamma_2^2}{r}b + f_d^2a - r(1 - 2f_d^2)b &= -\frac{n-1}{r^{n-1}}f_d(1 - f_d^2) \end{cases} \quad (5.108)$$

where, as usual, $\gamma_1 = |n - d|$, $\gamma_2 = n + d$, $\gamma^2 = \frac{\gamma_1^2 + \gamma_2^2}{2}$ and $\xi^2 = \frac{\gamma_2^2 - \gamma_1^2}{2}$.

We verify that

$$x \sim_0 y \sim_0 dr^{d-n} + O(r^{d-n+2}) \text{ and, at } +\infty, x = O(r^{-n}), \quad y = O(r^{-n}),$$

and consequently that

$$a \sim_0 2dr^{d-n} + O(r^{d-n+2}) \text{ and } b \sim_0 O(r^{d-n+2}).$$

let us suppose that $d \geq 1$ and that $1 < n < d + 1$. We can multiply the system (5.108) and integrate by parts. We obtain that

$$\begin{aligned} \int_0^{+\infty} (ra'^2 + rb'^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + rf_d^2(a+b)^2 - r(1 - f_d^2)(a^2 + b^2))dr \\ = \int_0^{+\infty} \frac{-(n-1)}{r^{n-1}}f_d(1 - f_d^2)(a+b)dr \end{aligned}$$

This gives

$$\frac{\int_0^{+\infty} (ra'^2 + rb'^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + rf_d^2(a+b)^2)dr}{\int_0^{+\infty} r(1 - f_d^2)(a^2 + b^2)dr} = 1 - C_n$$

with

$$C_n = \frac{\int_0^{+\infty} \frac{(n-1)}{r^{n-1}}f_d(1 - f_d^2)(a+b)dr}{\int_0^{+\infty} r(1 - f_d^2)(a^2 + b^2)dr} > 0.$$

Now we use an approximation argument, valid as soon as $n > 0$. For example for a given constant $0 < N < 1$ we define

$$(a_\varepsilon, b_\varepsilon)(r) = \begin{cases} (a, b)(\frac{r}{\varepsilon}) \text{ in } [0, N] \\ = (a(r)\frac{(1-r)^2}{(1-N)^2}, b(r)\frac{(1-r)^2}{(1-N)^2}) \text{ in } [N, 1]. \end{cases}$$

We have that $(a_\varepsilon, b_\varepsilon) \in \mathcal{H}_{|n-d|}$ and that

$$\begin{aligned} & \frac{\int_0^1 (ra'_\varepsilon{}^2 + rb'_\varepsilon{}^2 + \frac{\gamma_1^2}{r}a_\varepsilon^2 + \frac{\gamma_2^2}{r}b_\varepsilon^2 + r\frac{1}{\varepsilon^2}f^2(a_\varepsilon + b_\varepsilon)^2)dr}{\frac{1}{\varepsilon^2} \int_0^1 r(1 - f^2)(a_\varepsilon^2 + b_\varepsilon^2)dr} \\ &= \frac{\int_0^{\frac{N}{\varepsilon}} (ra'^2 + rb'^2 + \frac{\gamma_1^2}{r}a^2 + \frac{\gamma_2^2}{r}b^2 + rf_d^2(a+b)^2)dr + O(\varepsilon^{2n})}{\int_0^{\frac{N}{\varepsilon}} r(1 - f_d^2)(a^2 + b^2)dr + O(\varepsilon^{2n})} \\ & \rightarrow 1 - C_n, \text{ as } \varepsilon \text{ tends to } 0. \end{aligned} \quad (5.109)$$

We deduce that, if $1 < n < d + 1$

$$m_{d-n, d+n}(\varepsilon) < 1 - \frac{C_n}{2}$$

for ε small enough and the proof of the proposition is complete.

6 The proof of Theorem 1.4.

Let us consider $d > 1$. We can write the system (1.6) as

$$X' = MX \text{ with } X = (a, ra', b, rb')^t \quad (6.110)$$

and

$$M = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -r(1 - 2f_d^2) + \frac{\gamma_1^2}{r} & 0 & rf_d^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ rf_d^2 & 0 & -r(1 - 2f_d^2) + \frac{\gamma_2^2}{r} & 0 \end{pmatrix}$$

We are going to use a resolvent matrix for (6.110). First, we have

Lemma 6.4 *Let us suppose that there exists a bounded solution of (1.6). Then we can chose a base of solutions, X_1, X_2, X_3, X_4 , for (6.110), whose third vector is a bounded solution, and such that if we denote by $R(s)$ the resolvent matrix, whose columns are the vectors $X_i, i = 1, \dots, 4$ and if we denote the second and the fourth column of $R^{-1}(s)$ by \mathcal{C}_2 and \mathcal{C}_4 , we have*

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 < 0$

$$\mathcal{C}_2 = \begin{pmatrix} O(s^{\gamma_1}) \\ O(s^{\gamma_1+2\gamma_2}) \\ O(s^{-\gamma_1}) \\ O(s^{\gamma_1}) \end{pmatrix} \text{ and } \mathcal{C}_4 = \begin{pmatrix} O(s^{-\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{2\gamma_1+\gamma_2}) \end{pmatrix}$$

and

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 > 0$

$$\mathcal{C}_2 = \begin{pmatrix} O(s^{-\gamma_2+2d+2}) \\ O(s^{\gamma_2+2d+2}) \\ O(s^{-\gamma_1}) \\ O(s^{\gamma_1}) \end{pmatrix} \text{ and } \mathcal{C}_4 = \begin{pmatrix} O(s^{-\gamma_2}) \\ O(s^{\gamma_2}) \\ O(s^{-\gamma_1+2d+2}) \\ O(s^{\gamma_1+2d+2}) \end{pmatrix}$$

and

at 0 and when $(d, \gamma_1, \gamma_2) \in \mathcal{D}_2$

$$\mathcal{C}_2 = \begin{pmatrix} O(\tau(s)s^{-\gamma_2+\gamma_1+2d+2}) \\ O(\tau(s)s^{\gamma_1+\gamma_2+2d+2}) \\ O(\tau(s)) \\ O(s^{\gamma_1}) \end{pmatrix} \text{ and } \mathcal{C}_4 = \begin{pmatrix} O(s^{\gamma_1-\gamma_2}\tau(s)) \\ O(s^{\gamma_1+\gamma_2}\tau(s)) \\ O(s^{2d+2}\tau(s)) \\ O(s^{\gamma_1+2d+2}) \end{pmatrix}$$

and in any case, at $+\infty$

$$\mathcal{C}_2 \sim_{+\infty} \frac{1}{-16n\sqrt{2}} \begin{pmatrix} 4nJ_- \\ 4nJ_+ \\ -4\sqrt{2}s^n \\ -4\sqrt{2}s^{-n} \end{pmatrix} \text{ and } \mathcal{C}_4 \sim_{+\infty} \frac{1}{-16n\sqrt{2}} \begin{pmatrix} 4nJ_- \\ 4nJ_+ \\ 4\sqrt{2}s^n \\ 4\sqrt{2}s^{-n} \end{pmatrix}$$

where $-16n\sqrt{2}$ is the determinant of $R(s)$.

Proof We can choose $R(s)$ as follows

$$R(s) \sim_{+\infty} \begin{pmatrix} J_+ & J_- & s^{-n} & s^n \\ s(J_+)' & s(J_-)' & -ns^{-n} & ns^n \\ J_+ & J_- & -s^{-n} & -s^n \\ s(J_+)' & s(J_-)' & ns^{-n} & -ns^n \end{pmatrix}$$

where, as usual, the notation J_+ stands for $\frac{e^{\sqrt{2}s}}{\sqrt{s}}$ and the notation J_- stands for $\frac{e^{-\sqrt{2}s}}{\sqrt{s}}$. To give the behaviors at 0, we return to Theorem 1.3. We have, for some $c_i \neq 0$, $i = 1, \dots, 4$

$$\text{If } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1, \quad R(s) \sim_0 \begin{pmatrix} O(s^{\gamma_2+2d+2}) & O(s^{\tilde{\gamma}_1}) & c_3 s^{\gamma_1} & c_4 s^{-\gamma_1} \\ O(s^{\gamma_2+2d+2}) & O(s^{\tilde{\gamma}_1}) & c_3 \gamma_1 s^{\gamma_1} & -c_4 \gamma_1 s^{-\gamma_1} \\ c_1 s^{\gamma_2} & c_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(s^{\tilde{\gamma}_2}) \\ c_1 \gamma_2 s^{\gamma_2} & -c_2 \gamma_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(s^{\tilde{\gamma}_2}) \end{pmatrix}$$

where we use the notation

$$\tilde{\gamma}_1 = \min\{\gamma_1, -\gamma_2 + 2d + 2\} \text{ and } \tilde{\gamma}_1 = \min\{\gamma_2, -\gamma_1 + 2d + 2\} \quad \text{if } \gamma_1 + \gamma_2 - 2d - 2 \neq 0$$

(if $\gamma_1 + \gamma_2 - 2d - 2 = 0$, we have to replace $O(s^{\tilde{\gamma}_1})$ by $O(s^{\tilde{\gamma}_1} \log s)$ and $O(s^{\tilde{\gamma}_2})$ by $O(s^{\tilde{\gamma}_2} \log s)$)
and

$$\text{If } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2, \quad R(s) \sim_0 \begin{pmatrix} O(s^{\gamma_2+2d+2}) & O(s^{-\gamma_2+2d+2}) & c_3 s^{\gamma_1} & c_4 \tau(s) \\ O(s^{\gamma_2+2d+2}) & O(s^{-\gamma_2+2d+2}) & c_3 \gamma_1 s^{\gamma_1} & -c_4 s \tau'(s) \\ c_1 s^{\gamma_2} & c_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(\tau(s) s^{2d+2}) \\ c_1 \gamma_2 s^{\gamma_2} & -c_2 \gamma_2 s^{-\gamma_2} & O(s^{\gamma_1+2d+2}) & O(\tau(s) s^{2d+2}) \end{pmatrix}$$

where

$$\tau(s) = \begin{cases} \frac{s^{-\gamma_1} - s^{\gamma_1}}{2\gamma_1} & \text{if } \gamma_1 \neq 0 \\ -\log s & \text{if } \gamma_1 = 0 \end{cases}$$

The determinant W of $R(s)$ is independent of s , due to the fact that the matrix M of the differential system has a null trace. Moreover, $J_+ J_- = \frac{1}{s}$. Using the behavior at $+\infty$ of $R(s)$, given above, we deduce that W is the principal term, as $s \rightarrow +\infty$ of

$$\frac{1}{s} \begin{vmatrix} 1 & 1 & 1 & 1 \\ s\sqrt{2} & -s\sqrt{2} & -n & n \\ 1 & 1 & -1 & -1 \\ s\sqrt{2} & -s\sqrt{2} & n & -n \end{vmatrix}$$

that is

$$W = -16n\sqrt{2}.$$

A direct calculation of the suitable determinants gives the estimate of \mathcal{C}_2 and \mathcal{C}_4 .

Now let us enonce

Lemma 6.5 *Let $m_{\gamma_1, \gamma_2}(\varepsilon)$ be the first eigenvalue, and let m be such that $m_{\gamma_1, \gamma_2}(\varepsilon) \rightarrow m$ as $\varepsilon \rightarrow 0$. If there exists a bounded solution (a, b) of the system (1.6), then we have necessarily $m = 1$.*

Proof From the definition of $m_{\gamma_1, \gamma_2}(\varepsilon)$, we have that it decreases as ε decreases to 0, then we can define its limit $m \geq 0$. But we have supposed that there exists $\mu(\varepsilon) \rightarrow 1$, then we have $m \leq 1$. Moreover, we can define $\omega_\varepsilon \in \mathcal{H}_{\gamma_1}$ an eigenvector associated to $m_{\gamma_1, \gamma_2}(\varepsilon)$ such that there exists $\omega_0 = (a_0, b_0)$ such that $\tilde{\omega}_\varepsilon \rightarrow \omega_0$ on each compact subset of $[0, +\infty[$. The condition $m \leq 1$ gives $\frac{\gamma_1^2 + \gamma_2^2}{2} - md^2 \geq 0$. Since $a_0 \geq -b_0 \geq 0$, an examination of the proof of Theorem 1.4 gives that the possible behavior at $+\infty$ for (a_0, b_0) is

$$\text{either } (r^{-n_0}, -r^{-n_0}), \text{ or } (r^{n_0}, -r^{n_0})$$

where

$$n_0 = \sqrt{\frac{\gamma_1^2 + \gamma_2^2}{2} - md^2}. \quad (6.111)$$

In what follows, we suppose that $m < 1$, so we have $n_0 > n$, and we want to reach to a contradiction.

Since $m < 1$, we have by Lemma 5.1, that ω_0 has a bounded behavior at $+\infty$ and consequently

$$(a_0, b_0) \sim_0 (r^{-n_0}, -r^{-n_0})$$

and we recall that

$$a_0 + b_0 = O(r^{-n_0-2}).$$

At 0, in view of $a_0 \geq -b_0 \geq 0$, the only possible behavior is

$$(a_0, b_0) \sim_0 (cr^{\gamma_1}, O(r^{\gamma_1+2d+2})),$$

for some $c \neq 0$.

Let us denote $X_0 = (a_0, ra'_0, b_0, rb'_0)^t$, the vector corresponding to ω_0 . We have

$$X'_0 = MX_0 - (m-1)(1-f_d^2)(0, ra, 0, rb)^t.$$

let us define X_1, X_2, X_3 and X_4 as in Lemma 6.4. We are going to prove that there exist some constants C_i such that

$$X_0 = \sum_{i=1}^4 C_i X_i - (m-1) \sum_{i=1}^4 \hat{X}_i,$$

with

$$\hat{X}_i = X_0 O(r^2) \quad \text{at } 0 \quad (6.112)$$

and

$$\hat{X}_i = \begin{cases} X_0 O(r^{-2}) & \text{at } +\infty \text{ for } i = 1, 2 \\ X_i O(1) & \text{at } +\infty \text{ for } i = 3, 4 \end{cases} \quad (6.113)$$

We write

$$X_0 = \sum_{i=1}^4 A_i(r)X_i \quad (6.114)$$

We will name $A_i(r)X_i$ the i^{th} term of X_0 .

For all $i = 1, \dots, 4$, we have

$$A_i(r) = A_i - (m-1) \int_1^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix} ds]_i \quad (6.115)$$

where the notation $[]_i$ means the i^{th} line of the vector, and A_i is a constant.

Let us examine the behavior of each term $A_i(r)X_i$ at $+\infty$ and at 0, using Lemma 6.4.

For the first term, we use the first terms of \mathcal{C}_2 and \mathcal{C}_4 , given in Lemma 6.4, to obtain

$$[R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 \sim_{+\infty} O\left(\frac{1}{s}J_-(a_0 + b_0)\right)$$

and $\sim_0 \begin{cases} s(O(s^{\gamma_1}a_0 + O(s^{-\gamma_2}b_0))) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1, \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ s(O(s^{-\gamma_2+2d+2}a_0 + O(s^{-\gamma_2}b_0))) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1, \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ s(O(\tau(s)s^{\gamma_1-\gamma_2+2d+2}a_0 + O(\tau(s)s^{\gamma_1-\gamma_2}b_0))) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2. \end{cases}$
(6.116)

Let us define

$$B_1 = -(m-1) \int_1^{+\infty} [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 ds \quad \text{and} \quad \hat{X}_1 = X_1 \int_{+\infty}^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_1 ds$$

We can write

$$A_1(r)X_1 = (A_1 + B_1)X_1 - (m-1)\hat{X}_1$$

and, using Lemme (3.78), we see that

$$\hat{X}_1 = X_1 O(r^{-n_0-2}J_-) \quad \text{at } +\infty \quad \text{and} \quad \hat{X}_1 = X_1 O(1) \quad \text{at } 0. \quad (6.117)$$

For the second term, we obtain

$$[R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 \sim_{+\infty} O\left(\frac{1}{s}J_+(a_0 + b_0)\right)$$

and $\sim_0 \begin{cases} s(O(s^{\gamma_1+2\gamma_2}a_0) + O(s^{\gamma_2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ s(O(s^{\gamma_2+2d+2}a_0) + O(s^{\gamma_2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ s\tau(s)(O(s^{\gamma_1+\gamma_2+2d+2}a_0) + O(s^{\gamma_1+\gamma_2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases}$
(6.118)

Denoting

$$B_2 = -(m-1) \int_1^0 [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 ds \quad \text{and} \quad \hat{X}_2 = X_2 \int_0^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_2 ds$$

we get

$$A_2(r)X_2 = (A_2 + B_2)X_2 - (m-1)\hat{X}_2$$

with

$$\hat{X}_2 = X_2 O(r^{-n_0-3}J_+) \text{ at } +\infty$$

and, at 0

$$\hat{X}_2 = X_2 \begin{cases} O(r^{\gamma_1+2\gamma_2+2}a_0) + O(r^{\gamma_2+2}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ O(r^{\gamma_2+2d+4}a_0) + O(r^{\gamma_2+2}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ \tau(r)(O(r^{\gamma_1+\gamma_2+2d+4}a_0) + O(r^{\gamma_1+\gamma_2+2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases} \text{ at } 0. \quad (6.119)$$

For the third term, we obtain

$$[R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 \sim_{+\infty} \frac{-1}{16n\sqrt{2}} \frac{4\sqrt{2}d^2}{s} s^n (-a_0 + b_0)$$

and

$$\sim_0 \begin{cases} s(O(s^{-\gamma_1}a_0) + O(s^{\gamma_2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ s(O(s^{-\gamma_1}a_0) + O(s^{-\gamma_1+2d+2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ s(O(\tau(s)a_0) + O(\tau(s)s^{2d+2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases} \quad (6.120)$$

Letting

$$B_3 = -(m-1) \int_1^0 [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds \quad \text{and} \quad \hat{X}_3 = X_3 \int_0^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds$$

and keeping in mind $n - n_0 < 0$, we find

$$A_3(r)X_3 = (A_3 + B_3)X_3 - (m-1)\hat{X}_3$$

with

$$\hat{X}_3 = X_3 O(1) \text{ at } +\infty$$

and

$$\hat{X}_3 = X_3 \begin{cases} O(r^{-\gamma_1+2}a_0) + O(r^{\gamma_2+2}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ O(r^{-\gamma_1+2}a_0) + O(r^{-\gamma_1+2d+4}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ \tau(r)(O(r^2a_0) + O(r^{2d+4}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases} . \quad (6.121)$$

For the fourth term,

$$[R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 \sim_{+\infty} \frac{-1}{16n\sqrt{2}} \frac{4d^2\sqrt{2}}{s} s^{-n}(-a_0 + b_0)$$

and

$$\sim_0 \begin{cases} s(O(s^{\gamma_1}a_0) + O(s^{2\gamma_1+\gamma_2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ s(O(s^{\gamma_1}a_0) + O(s^{\gamma_1+2d+2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ s\tau(s)(O(s^{\gamma_1}a_0) + O(s^{\gamma_1+2d+2}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases} \quad (6.122)$$

Letting

$$B_4 = -(m-1) \int_1^0 [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds \text{ and } \hat{X}_4 = X_4 \int_0^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds$$

we find

$$A_4(r)X_4 = (A_4 + B_4)X_4 - (m-1)\hat{X}_4$$

with

$$\hat{X}_4 = X_4 O(1) \text{ at } +\infty$$

and

$$\hat{X}_4 = X_4 O \begin{cases} O(r^{\gamma_1+2}a_0) + O(r^{2\gamma_1+\gamma_2+2}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 < 0 \\ O(r^{\gamma_1+2}a_0) + O(r^{\gamma_1+2d+4}b_0) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_1 \text{ and } \gamma_1 + \gamma_2 - 2d - 2 > 0 \\ \tau(r)(O(r^{\gamma_1+2}a_0) + O(r^{\gamma_1+2d+4}b_0)) & \text{if } (d, \gamma_1, \gamma_2) \in \mathcal{D}_2 \end{cases} \quad (6.123)$$

Summing the four terms, we find

$$X_0 = \sum_{i=1}^4 (A_i + B_i)X_i - (m-1) \sum_{i=1}^4 \hat{X}_i.$$

We collect (6.117), (6.119), (6.121) and (6.123) and we use the expansions of X_1, X_2, X_3 and X_4 at 0 and at $+\infty$, given in the proof of Lemma 6.4 (the columns of $R(s)$). We get (6.112) and (6.113).

But X_0 is bounded at 0. We infer that $A_2 + B_2 = A_4 + B_4 = 0$.

But X_0 is bounded at $+\infty$, too. And \hat{X}_i is bounded at $+\infty$, for all $i \neq 4$. Since we have also $X_1 \gg \hat{X}_4$ at $+\infty$, we infer that $A_1 + B_1 = 0$ and that \hat{X}_4 must be bounded at $+\infty$. Returning to the definition of \hat{X}_4 , we must have

$$\int_0^{+\infty} [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds = 0$$

and consequently

$$\hat{X}_4 = X_4 \int_{+\infty}^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_4 ds$$

that gives

$$\hat{X}_4 = X_4 \int_{+\infty}^r [a_0 \mathcal{C}_2 + b_0 \mathcal{C}_4]_4 \sim_{+\infty} X_4 \int_{+\infty}^r \frac{-8\sqrt{2}}{-16n\sqrt{2}} s^{-n_0-n} \frac{d^2}{s} ds$$

and thus

$$\hat{X}_4 \sim_{+\infty} X_4 \frac{-1}{16n\sqrt{2}} \frac{8d^2\sqrt{2}}{n+n_0} r^{-n-n_0}. \quad (6.124)$$

Since we have now

$$X_0 = (A_3 + B_3)X_3 - (m-1) \sum_{i=1}^4 \hat{X}_i$$

and since $\hat{X}_1 + \hat{X}_2 \ll X_0$ at $+\infty$, we must have

$$X_0 \sim_{+\infty} (A_3 + B_3)X_3 - (m-1)\hat{X}_3 - (m-1)\hat{X}_4 \quad (6.125)$$

But, recalling (6.124) and recalling $n < n_0$, this implies that

$$(A_3 + B_3) - (m-1) \int_0^{+\infty} [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds = 0$$

and then

$$(A_3 + B_3)X_3 - (m-1)\hat{X}_3 = -(m-1)X_3 \int_{+\infty}^r [R^{-1}(s)s(1-f_d^2) \begin{pmatrix} 0 \\ a_0 \\ 0 \\ b_0 \end{pmatrix}]_3 ds$$

and consequently

$$(A_3 + B_3)X_3 - (m-1)\hat{X}_3 \sim_{+\infty} -(m-1)X_3 \frac{-1}{16\sqrt{2}} \frac{-8d^2\sqrt{2}}{n(n-n_0)} r^{n-n_0}. \quad (6.126)$$

Finally, we sum (6.124) and (6.126) to get, by (6.125)

$$a_0 \sim_{+\infty} (m-1) \frac{-1}{16} \frac{8d^2}{n} \left(\frac{-1}{n-n_0} + \frac{1}{n+n_0} \right) r^{-n_0}$$

and thus

$$(m-1) \frac{-1}{16} \frac{8d^2}{n} \left(\frac{-1}{n-n_0} + \frac{1}{n+n_0} \right) = 1.$$

But we have by (6.111)

$$n_0^2 - n^2 = (-m + 1)d^2.$$

We deduce that $n_0 = n$, that gives $m = 1$, that is in contradiction with $m < 1$. We have not written the proof of (6.112) and (6.113) for $(d, \gamma_1, \gamma_2) \in \mathcal{D}_1$ and $\gamma_1 + \gamma_2 - 2d - 2 = 0$, but this is true in this case, too.

Proof of Proposition 1.4 completed. With the notation of Lemma 6.5, we know that $m = 1$, that ω_0 verifies $a_0 \geq -b_0 \geq 0$ on $[0, +\infty[$ and that it is defined at 0. If ω_0 is not equal to ω , up to a multiplicative constant, we can find $C \neq 0$ such that $\omega_0 - C\omega$ has the behavior of ω_1 at 0 (that is the least vanishing behavior). But this implies an exponentially blowing up behavior at $+\infty$. Since ω is bounded, then ω_0 has an exponentially blowing up behavior at $+\infty$. But this is in contradiction with $a_0 \geq -b_0 \geq 0$. So ω_0 is a bounded solution of (1.6). The Proposition 1.4 is proved.

Proof of Proposition 1.5 (iv).

First let us prove the following

Lemma 6.6 *If $\max\{\frac{|\mu(\varepsilon)-1|}{\varepsilon^2}, \frac{|\mu(\varepsilon)-1|}{\varepsilon^n}\} \rightarrow 0$, then $\mu(\varepsilon)$ is an algebraically simple eigenvalue and no other eigenvalue can be such $\max\{\frac{|\mu(\varepsilon)-1|}{\varepsilon^2}, \frac{|\mu(\varepsilon)-1|}{\varepsilon^n}\} \rightarrow 0$.*

Proof Firstly, let us prove that if $\frac{\mu(\varepsilon)-1}{\varepsilon^n} \rightarrow 0$ and if $\tilde{\omega}_\varepsilon \rightarrow \omega$, where ω is a bounded solution of (1.6), then we can chose an eigenvector, still denoted by $\tilde{\omega}_\varepsilon$ and such that

$$\tilde{\omega}_\varepsilon = C_\varepsilon \omega_1 + \omega - (\mu(\varepsilon) - 1)\hat{\omega}_\varepsilon \quad (6.127)$$

for some constant C_ε and some function $\hat{\omega}_\varepsilon$, with the conditions

$$\hat{\omega}_\varepsilon \rightarrow \hat{\omega}_0 \quad \text{as } \varepsilon \rightarrow 0, \text{ for some limit function } \hat{\omega}_0 \text{ and uniformly on each } [0, R]$$

and

$$|C_\varepsilon| \leq C\varepsilon^n \sqrt{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}}. \quad (6.128)$$

Here $\omega_1 = (a_1, b_1)$, is, as usual, the solution defined in Theorem 1.3, that has a least behavior at 0 and blows up exponentially at $+\infty$.

In order to prove (6.127) and (6.128), we use $X_\varepsilon = (\tilde{a}_\varepsilon, r\tilde{a}'_\varepsilon, b_\varepsilon, r\tilde{b}'_\varepsilon)^t$ and a resolvent matrix, whose third vector is the bounded solution, and we write, with the notation of the proof of Lemma 6.5

$$X_\varepsilon = \sum_{i=1}^4 X_i (A_i - (\mu(\varepsilon) - 1) \int_1^r s(1 - f_d^2) [C_2 \tilde{a}_\varepsilon + C_4 \tilde{b}_\varepsilon]_i ds).$$

Then we use the analysis at 0 of each term, given in (6.116), (6.118), (6.120) and (6.122), in which we replace (a_0, b_0) by $(\tilde{a}_\varepsilon, \tilde{b}_\varepsilon)$. And we write

$$X_\varepsilon = \sum_{i=1}^4 (A_i + B_i) X_i - (\mu(\varepsilon) - 1) \sum_{i=1}^4 \hat{X}_i,$$

where

$$\text{for } i = 2, 3, 4, \quad \hat{X}_i = X_i \int_0^r s(1 - f_d^2)[\mathcal{C}_2 \tilde{a}_\varepsilon + \mathcal{C}_4 \tilde{b}_\varepsilon]_i ds = O(r^2 X_\varepsilon) \quad \text{as } r \rightarrow 0;$$

$$\hat{X}_1 = X_1 \int_{\frac{1}{\varepsilon}}^r s(1 - f_d^2)[\mathcal{C}_2 \tilde{a}_\varepsilon + \mathcal{C}_4 \tilde{b}_\varepsilon]_1 ds = O(X_1) \quad \text{as } r \rightarrow 0$$

and

$$\text{for } i = 2, 3, 4, \quad B_i = -(\mu(\varepsilon) - 1) \int_1^0 s(1 - f_d^2)[\mathcal{C}_2 \tilde{a}_\varepsilon + \mathcal{C}_4 \tilde{b}_\varepsilon]_i ds;$$

$$B_1 = -(\mu(\varepsilon) - 1) \int_1^{\frac{1}{\varepsilon}} s(1 - f_d^2)[\mathcal{C}_2 \tilde{a}_\varepsilon + \mathcal{C}_4 \tilde{b}_\varepsilon]_1 ds.$$

Now, in view of the behaviors at 0, we must have

$$A_2 + B_2 = A_4 + B_4 = 0.$$

The behavior at $+\infty$ given in Lemma 6.4, ie

$$[\mathcal{C}_2]_2 = O(J_-) \quad \text{and} \quad [\mathcal{C}_4]_2 = O(J_-)$$

gives a finite limit for $\hat{X}_1(r)$, as $\varepsilon \rightarrow 0$, when $r > 0$ is fixed. Indeed, the behavior of $(\tilde{a}_\varepsilon, \tilde{b}_\varepsilon)$ at $+\infty$ is at most (J_+, J_+) . So, we have

$$\text{for all } r > 0 \text{ and for all } i, (1 - \mu(\varepsilon))\hat{X}_i(r) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

We deduce that

$$A_3 + B_3 \rightarrow 1 \quad \text{and} \quad A_1 + B_1 \rightarrow 0.$$

By dividing the eigenvector in presence by $A_3 + B_3$, we are led to

$$A_3 + B_3 = 1.$$

Then let us give a large R and $\frac{1}{\varepsilon} > R$. Using Lemma 6.4 and (3.79) and (3.78), we obtain, for all $R < r < \frac{1}{\varepsilon}$, and some C , independent of r and ε (by $|X|$, we mean each component of X)

$$\begin{aligned} |\hat{X}_1| &\leq C|X_1| \frac{J_-(R)}{R} \max_{[R, \frac{1}{\varepsilon}]}(|\tilde{a}_\varepsilon| + |\tilde{b}_\varepsilon|); & |\hat{X}_2| &\leq C(1 + |X_2|) \frac{J_+(R)}{R} \max_{[R, \frac{1}{\varepsilon}]}(|\tilde{a}_\varepsilon| + |\tilde{b}_\varepsilon|); \\ |\hat{X}_3| &\leq C(1 + |X_3| r^n \max_{[R, \frac{1}{\varepsilon}]}(|\tilde{a}_\varepsilon| + |\tilde{b}_\varepsilon|)); & |\hat{X}_4| &\leq C(1 + |X_4| \max_{[R, \frac{1}{\varepsilon}]}(|\tilde{a}_\varepsilon| + |\tilde{b}_\varepsilon|)). \end{aligned} \quad (6.129)$$

Taking into account the behavior at $+\infty$ for each X_i , together with

$$X_\varepsilon = (A_1 + B_1)X_1 + X_3 - (\mu(\varepsilon) - 1) \sum_{i=1}^4 \hat{X}_i, \quad (6.130)$$

we deduce, for $r > R$

$$|\tilde{a}_\varepsilon + \tilde{b}_\varepsilon| \leq |A_1 + B_1||a_1 + b_1| + |a_3 + b_3| + C|\mu(\varepsilon) - 1| \left(1 + \frac{J_+(r)J_-(R)}{R} + \frac{J_+(R)J_-(r)}{R}\right)$$

$$+r^n) \max_{[R, \frac{1}{\varepsilon}]} (|\tilde{a}_\varepsilon| + |\tilde{b}_\varepsilon|).$$

This gives, since $\frac{\mu(\varepsilon)-1}{\varepsilon^n} \rightarrow 0$

$$|\tilde{a}_\varepsilon + \tilde{b}_\varepsilon| \leq C(|A_1 + B_1||a_1 + b_1| + |a_3 + b_3|). \quad (6.131)$$

This implies, for all $R < r < \frac{1}{\varepsilon}$

$$\sum_{i=2}^4 |\hat{a}_i|(r) \leq C(1 + \frac{1}{R^2} + \varepsilon^{-n})(|A_1 + B_1|J_+(r) + r^{-n}). \quad (6.132)$$

Now, we use

$$\tilde{a}_\varepsilon(\frac{1}{\varepsilon}) = 0$$

into (6.130) to get

$$0 = (A_1 + B_1)a_1(\frac{1}{\varepsilon}) + a_3(\frac{1}{\varepsilon}) - (\mu(\varepsilon) - 1) \sum_{i=2}^4 \hat{a}_i(\frac{1}{\varepsilon})$$

and using (6.132), we get, for R large enough and $R < r < \frac{1}{\varepsilon}$

$$|(A_1 + B_1)a_1(\frac{1}{\varepsilon}) + a_3(\frac{1}{\varepsilon})| \leq C \frac{|\mu(\varepsilon) - 1|}{\varepsilon^n} |(A_1 + B_1)a_1(\frac{1}{\varepsilon}) + a_3(\frac{1}{\varepsilon})|$$

and consequently

$$|A_1 + B_1| \leq C\varepsilon^n \sqrt{\varepsilon} e^{-\sqrt{2}r}.$$

We have proved (6.127) and (6.128).

Secondly, We use Theorem 1.3 to see that the vector space of the bounded solutions of (1.6) is at most one dimensional, spanned by some $\omega = (a, b)$. Then, in view of Lemma 5.1, if the property we have to prove is not true, there exist two eigenvalues $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ (that may be equal), associated to some eigenvectors ω_ε and η_ε and such that

$$\max\left\{\frac{|\mu_1(\varepsilon) - 1|}{\varepsilon^2}, \frac{|\mu_1(\varepsilon) - 1|}{\varepsilon^n}\right\} \rightarrow 0, \quad \max\left\{\frac{|\mu_2(\varepsilon) - 1|}{\varepsilon^2}, \frac{|\mu_2(\varepsilon) - 1|}{\varepsilon^n}\right\} \rightarrow 0,$$

$$\tilde{\omega}_\varepsilon \rightarrow \omega \quad \text{and} \quad \tilde{\eta}_\varepsilon \rightarrow \omega \quad \text{on each } [0, R]$$

and

$$\langle \mathcal{T}\omega_\varepsilon, \eta_\varepsilon \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} = 0. \quad (6.133)$$

and we have also

$$\langle \mathcal{C}\omega_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} = 0. \quad (6.134)$$

On one hand, we write

$$\begin{aligned} & \langle \mathcal{T}(\omega_\varepsilon - \eta_\varepsilon), \omega_\varepsilon - \eta_\varepsilon \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} \\ &= \mu_1(\varepsilon) \langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} + \mu_2(\varepsilon) \langle \mathcal{C}\eta_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} \end{aligned} \quad (6.135)$$

On the other hand, we let

$$\omega_\varepsilon - \eta_\varepsilon = (\alpha, \beta)$$

and, using the same trick as in the proof of Lemma 5.1, we get

$$\begin{aligned} & \langle \mathcal{T}(\omega_\varepsilon - \eta_\varepsilon), \omega_\varepsilon - \eta_\varepsilon \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} \\ & \geq \frac{m_0(\varepsilon)}{\varepsilon^2} \int_0^1 r(1 - f^2)(\alpha_\varepsilon^2 + \beta_\varepsilon^2) dr + \int_0^{\frac{1}{\varepsilon}} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr \end{aligned} \quad (6.136)$$

Where H is defined in (5.103). Defining $R_0 > 0$ such that $H(\tau_0) > 0$, for all $r \geq R_0$, we have

$$\int_0^{\frac{1}{\varepsilon}} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr \geq \int_0^{R_0} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr.$$

Moreover, by (6.134), we have

$$\frac{1}{\varepsilon^2} \int_0^1 r(1 - f^2)(\alpha_\varepsilon^2 + \beta_\varepsilon^2) dr = \langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} + \langle \mathcal{C}\eta_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))}$$

and consequently, (6.136) becomes

$$\begin{aligned} & \langle \mathcal{T}(\omega_\varepsilon - \eta_\varepsilon), \omega_\varepsilon - \eta_\varepsilon \rangle_{\mathcal{H}'_{\gamma_1}, \mathcal{H}_{\gamma_1}} \\ & \geq m_0(\varepsilon) (\langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} + \langle \mathcal{C}\eta_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))}) + \int_0^{R_0} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr. \end{aligned} \quad (6.137)$$

By collecting (6.135) and (6.137), we obtain

$$\begin{aligned} (m_0(\varepsilon) - \mu_1(\varepsilon)) \langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} + (m_0(\varepsilon) - \mu_2(\varepsilon)) \langle \mathcal{C}\eta_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} \\ \leq - \int_0^{R_0} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr. \end{aligned} \quad (6.138)$$

And we have also, for all $R > 0$ and $\varepsilon < \frac{1}{R}$

$$\langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} \geq \int_0^R r(1 - f_d^2)(a_\varepsilon^2 + b_\varepsilon^2) dr \rightarrow_{\varepsilon \rightarrow 0} \int_0^R r(1 - f_d^2)(a_0^2 + b_0^2) dr$$

So, there exists $C > 0$ such that, for ε small enough

$$\langle \mathcal{C}\omega_\varepsilon, \omega_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} > C \quad \text{and} \quad \langle \mathcal{C}\eta_\varepsilon, \eta_\varepsilon \rangle_{(L^2 \times L^2)(B(0,1))} > C.$$

Then we use (6.127) and (6.128) to get

$$\int_0^{R_0} H(\tau_0) \tilde{\alpha}_\varepsilon^2 dr = o(\varepsilon^4) \quad \text{as } \varepsilon \rightarrow 0.$$

Then (6.138) gives

$$\frac{m_0(\varepsilon) - \mu_1(\varepsilon)}{\varepsilon^2} \rightarrow 0 \quad \text{and} \quad \frac{m_0(\varepsilon) - \mu_2(\varepsilon)}{\varepsilon^2} \rightarrow 0.$$

Recalling that $\frac{m_0(\varepsilon)-1}{\varepsilon^2} \geq C$, for some $C > 0$ independent of ε , and that $\frac{\mu_i(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$, $i = 1, 2$, we infer that (6.138) leads to a contradiction.

The lemma is proved.

Now let us complete the proof of Proposition 1.5 (iv).

We have that $m_{d-1,d+1}(\varepsilon) > 1$ (see a sketch of the proof in the appendix) and consequently $\frac{m_{d-1,d+1}(\varepsilon)-1}{\varepsilon^2} > 0$. But, since F_d is a bounded solution of (1.6) and since $n \geq 1$, we know by Lemma 5.3 that there exists an eigenvalue $\mu(\varepsilon)$ verifying $\frac{\mu(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$. We deduce that $\frac{m_{d-1,d+1}(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$. By Lemma 6.6, we are led to $\mu(\varepsilon) = m_{d-1,d+1}(\varepsilon)$. Then we return to the end of the proof of Lemma 5.2, with F_d instead of ω . We have now that the set I defined there has one element. Denoting by i_0 this element, we have $\langle \mathcal{C}(F_d^{cut} - \alpha_{i_0}\zeta_{i_0}), F_d^{cut} - \alpha_{i_0}\zeta_{i_0} \rangle_{(L^2 \times L^2)(B(0,1))} \rightarrow 0$. This gives the proof of (iv), where $\omega_\varepsilon = \alpha_{i_0}\zeta_{i_0}$.

7 The proof of Theorem 1.2 completed

In this part, we consider $d \geq 1$ and $n \geq 1$ and $\gamma_1 = |n - d|$ and $\gamma_2 = n + d$.

First, we have

Proposition 7.16 *When $1 < n < d + 1$, there is no solution (a, b) of the system (1.6) such that $(ae^{i(n-d)\theta}, be^{i\theta}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.*

Proof This follows immediatly from Proposition 1.2 and from Proposition 1.4.

Now let $\eta_i = (x_i, y_i)$, $i = 1, 2, 3, 4$, be defined by Theorem 1.4. Theorem 1.3 allows us to use the solution $\omega_1 = (a_1, b_1)$, defined in Theorem 1.3, in place of (x_1, y_1) and to obtain a base $(\omega_1, \eta_2, \eta_3, \eta_4)$ of solutions of (1.6), whose behaviors at $+\infty$ are known.

Now, let $\omega_3 = (a_3, b_3)$ be defined in Theorem 1.3. Recall that ω_3 is continuous wrt $(d, \gamma_1, \gamma_2) \in \mathcal{D}$ and is derivable wrt γ_1 and γ_2 .

With these definitions, we can write

$$\omega_3 = C_1(n, d)\omega_1 + C_2(n, d)\eta_2 + C_3(n, d)\eta_3 + C_4(n, d)\eta_4.$$

Let us remark that ω_1 and $\omega_3 - C_1(n, d)\omega_1$ form a base of the bounded solutions at 0, and that $\omega_3 - C_1(n, d)\omega_1 = o(\omega_1)$ at $+\infty$. So the problem of the existence of some bounded solutions remains to the problem $C_3(n, d) = 0$.

We define $\gamma_1 = |n - d|$ and $\gamma_2 = n + d$. The real numbers $C_i(n, d)$ can be computed by the means of determinants involving only the values of the five solutions in presence, $(a, a', b, b')(r)$, for a given $r > 0$. So, as soon as (d, γ_1, γ_2) stays in a given compact subset of \mathcal{D} , C_i is continuous wrt (d, γ_1, γ_2) and consequently is continuous wrt (d, n) . Moreover, with the same condition, each C_i is derivable wrt γ_1 and wrt γ_2 and consequently is derivable wrt n . And $\frac{\partial C_i}{\partial n}$ is continuous wrt to (n, d) , for $i = 1, \dots, 4$.

Now, we are going to prove

Proposition 7.17 *There is no bounded solution of (1.6), when $d \geq 1$ and $n \geq d + 1$.*

In what follows, we suppose by contradiction that there exists (n_0, d_0) , $d_0 > 1$, $n_0 \geq d_0 + 1$, such that there exists a bounded solution of (1.6). By Theorem 1.1, we have $n_0 \leq 2d_0 - 1$.

From now on, we allow (n, d) to be such that $1 \leq d \leq d_0 + 1$ and $1 \leq n \leq 2d$. Clearly, $(d, |n - d|, n + d)$ stays in a compact subset of \mathcal{D} . This is sufficient for each solution η_i , $i = 1, 2, 3, 4$, to be defined without ambiguity, and consequently, for each C_i to be well defined too. And each C_i is smooth wrt (n, d) , as explained above.

Lemma 7.7 *With the notation above, if $C_3(n_0, d_0) = 0$, then there exists a continuous map $d \mapsto n(d)$, defined for d closed to d_0 and verifying $C_3(n(d), d) = 0$.*

Proof We can use the derivative of C_3 wrt n . If we have $\frac{\partial C_3}{\partial n}|_{n_0} = 0$, then $\frac{\partial}{\partial n}(\omega_3 - C_1(n, d)\omega_1)|_{n_0}$ is bounded at $+\infty$.

Let us denote $(a, b) = \omega_3 - C_1(n, d)\omega_1$. Then, we consider

$$\begin{aligned} & \int_0^{+\infty} r(a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2}a - f_d^2 b + (1 - 2f_d^2)a) \frac{\partial a}{\partial n} dr \\ & + \int_0^{+\infty} r(b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2}b - f_d^2 a + (1 - 2f_d^2)b) \frac{\partial b}{\partial n} dr \\ & - \int_0^{+\infty} r \frac{\partial}{\partial n} (a'' + \frac{a'}{r} - \frac{\gamma_1^2}{r^2}a - f_d^2 b + (1 - 2f_d^2)a) a dr \\ & - \int_0^{+\infty} r \frac{\partial}{\partial n} (b'' + \frac{b'}{r} - \frac{\gamma_2^2}{r^2}b - f_d^2 a + (1 - 2f_d^2)b) b dr. \end{aligned}$$

where the derivation is taken at n_0 , and $\gamma_1 = |d_0 - n_0|$ and $\gamma_2 = d_0 + n_0$.

Integrating by parts, and since $n_0 \geq d_0$, we get

$$\int_0^{+\infty} -2 \frac{n_0 - d_0}{r} a^2 - 2 \frac{n_0 + d_0}{r} b^2 dr = 0$$

and we conclude that $b = 0$, that is false.

So, we have proved that $\frac{\partial C_3}{\partial n}|_{n_0} \neq 0$. The Implicit Functions Theorem gives a continuous map $d \mapsto n(d)$ such that $C_3(n(d), d) = 0$, and defined in a neighborhood of d_0 , with values in a neighborhood of n_0 .

The proof of Proposition 7.17 completed.

With the definitions given above, let us define the set

$$\mathcal{E} = \{d \geq 1; \quad d \leq d_0 + 1; \quad \exists n \geq d + \frac{1}{2}, \quad C_3(n, d) = 0\}.$$

If $d \in \mathcal{E}$, then $n \leq 2d - 1$, by Theorem 1.1. Thus it is not difficult to see that \mathcal{E} is a closed subset of $[1, +\infty[$, thanks to the continuity of C_3 wrt (n, d) .

We have supposed that $d_0 \in \mathcal{E}$, then, letting $d_1 = \inf \mathcal{E}$, we deduce from Lemma 7.7 that we cannot have $d_1 > 1$ and thus $d_1 = 1$, and so $d_1 \notin \mathcal{E}$. This contradiction proves that $\mathcal{E} = \emptyset$.

The proof of Theorem 1.2 is complete.

8 Appendix

In this Appendix, we give a direct proof of Theorem 1.5, for the convenience of the reader. The original proof can be find in [8], [7].

I. Proof of $\frac{m_0(\varepsilon)-1}{\varepsilon^2} \geq C$. Using the Euler equation of the infimum problem (1.17), we have

$$a'' + \frac{a'}{r} - \frac{d^2}{r}a + \frac{1}{\varepsilon^2}(1 - f^2)a = -\frac{m_0(\varepsilon) - 1}{\varepsilon^2}(1 - f^2)a \quad (8.139)$$

where $r \in [0, 1]$, $f(r) = f_d(\frac{r}{\varepsilon})$ and $a(r) \geq 0$ and $a(1) = 0$. And we have, for the rescaled function \tilde{a}

$$\tilde{a}'' + \frac{\tilde{a}'}{r} - \frac{d^2}{r}\tilde{a} + (1 - f_d^2)\tilde{a} = -(m_0(\varepsilon) - 1)(1 - f_d^2)\tilde{a}. \quad (8.140)$$

Firstly, let us give a sketch of the proof of $m_0(\varepsilon) \rightarrow 1$.

Multiplying the equation (8.140), by f_d and integrating by parts on $[0, \frac{1}{\varepsilon}]$, we find $m_0(\varepsilon) > 1$, for all $\varepsilon > 0$. Then, using a truncation of f , with value 0 for $r \geq 1$, as a test function for the infimum $m_0(\varepsilon)$, and since we know the existence of the limit, we have that $\lim_{\varepsilon \rightarrow 0} m_0(\varepsilon) \leq 1$. This gives $m_0(\varepsilon) \rightarrow 1$.

Secondly, we use the same technics as in the proof of Theorem 1.3 to analyse the possible behaviors at 0. But in place of comparing the solution \tilde{a} with r^d , we compare it with f_d . More precisely, we know that f_d is one solution of the equation

$$a'' + \frac{a'}{r} - \frac{d^2}{r}a + (1 - f_d^2)a = 0. \quad (8.141)$$

Then, as usual in matter of ODE, we seek a solution of (8.140) of the form $f_d g$. We write

$$g'' f_d + 2g' f_d + \frac{g' f_d'}{r} = -(m_0(\varepsilon) - 1)(1 - f_d^2) f_d g,$$

that is

$$(g'(r f_d^2))' = -(m_0(\varepsilon) - 1)r(1 - f_d^2)f_d^2 g.$$

Letting $\tilde{a} = f_d g$, we are led to the following form of the equation (8.140)

$$(r f_d^2 (f_d^{-1} \tilde{a})')' = -(m_0(\varepsilon) - 1)r(1 - f_d^2) f_d \tilde{a}. \quad (8.142)$$

On the other hand, we define the fixed point problem

$$a = f_d - (m_0(\varepsilon) - 1) f_d \int_0^r \frac{f_d^{-2}}{t} \int_0^t s(1 - f_d^2) f_d a(s) ds. \quad (8.143)$$

We denote it by $\Phi(a) = a$.

In view of (8.142), each solution of this fixed point problem is a solution of (8.140).

As usual, we define by induction

$$\alpha_0 = f_d \quad \text{and} \quad \alpha_{k+1} = \Phi(\alpha_k)$$

and we write, for all $k \geq 1$

$$|\alpha_{k+1} - \alpha_k|(r) \leq f_d(r)(m_0(\varepsilon) - 1) \int_0^r \frac{f_d^{-2}(t)}{t} \int_0^t s f_d^2 ds dt \|f_d^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])}$$

and, using $f^{-2}(t) \leq f_d^{-2}(s)$ we get

$$|f_d^{-1}(\alpha_{k+1} - \alpha_k)|(r) \leq (m_0(\varepsilon) - 1) \frac{r^2}{4} \|f_d^{-1}(\alpha_k - \alpha_{k-1})\|_{L^\infty([0,r])}$$

and

$$|f_d^{-1}(\alpha_1 - \alpha_0)|(r) \leq (m_0(\varepsilon) - 1) \frac{r^2}{4}.$$

Consequently

$$\|f_d^{-1}(\alpha_{k+1} - \alpha_k)\|_{L^\infty([0,r])} \leq \left(\frac{m_0(\varepsilon) - 1}{4}\right)^{k+1} r^2.$$

Thus we can define

$$f_d + \sum_{k \geq 0} (\alpha_{k+1} - \alpha_k).$$

Since $m_0(\varepsilon) - 1 \rightarrow 0$, then for each $r > 0$, the sum is convergent for ε small enough, depending on r . This sum is a solution of (8.140). If we name it \tilde{a} , we have

$$|\tilde{a} - f_d|(r) \leq f_d(r) \frac{m_0(\varepsilon) - 1}{4} r^2 \frac{1}{1 - \frac{m_0(\varepsilon) - 1}{4} r^2}.$$

We remark that a similar proof gives the existence of a solution of (8.140) having the behavior r^{-d} at 0. Since the eigenvector \tilde{a} is defined at 0, it must be the solution defined above, to a multiplicative constant. We deduce two consequences.

Firstly, for all $R > 0$

$$|\tilde{a} - f_d|(r) \leq Cr^2 f_d(r)(m_0(\varepsilon) - 1), \quad \text{for all } \varepsilon < \varepsilon(R) \text{ and where } C \text{ depends only on } R,$$

and in particular, $\tilde{a} - f_d$ tends to 0, as $\varepsilon \rightarrow 0$, uniformly in $[0, R]$.

Secondly,

$$\text{if } \frac{m_0(\varepsilon) - 1}{\varepsilon^2} \rightarrow 0, \quad \text{then } \|\tilde{a} - f_d\|_{L^\infty([0, \frac{1}{\varepsilon}])} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

This second possibility cannot occur, since $\tilde{a}(\frac{1}{\varepsilon}) = 0$. We have proved that $\frac{m_0(\varepsilon) - 1}{\varepsilon^2} \geq C$.

The eigenvalue $m_{|d-1|,d+1}(\varepsilon)$.

Sketch of the proof of $m_{|d-1|,d+1}(\varepsilon) > 1$ and $m_{|d-1|,d+1}(\varepsilon) \rightarrow 1$.

Let (a, b) be an eigenvector associated to $m_{|d-1|,d+1}(\varepsilon)$. Let $F_d = (A, B)$ be defined in Theorem 1.1. Multiplying the system verified by (a, b) and the system verified by (A, B) and integrating by parts on $[0, \frac{1}{\varepsilon}]$ we get $m_{|d-1|,d+1}(\varepsilon) > 1$. This proof is in [8].

Then we can use the truncation of F_d and (5.109), where $n = 1$ and $C_n = 0$ or, alternatively, Proposition 1.4, to get that $m_{|d-1|,d+1}(\varepsilon) \rightarrow 1$.

Now, the proof of $\frac{m_{|d-1|,d+1}(\varepsilon)-1}{\varepsilon^2} \rightarrow 0$ is done in [7], by use of a suitable test function. The same proof works here, remarking that

$$\lambda_1(\varepsilon) = \frac{m_{|d-1|,d+1}(\varepsilon) - 1}{\varepsilon^2} \frac{\int_0^1 r(1-f^2)(a^2+b^2)dr}{\int_0^1 r(a^2+b^2)dr},$$

although the function f is not exactly the same one. This author use the fonction f defined by

$$f'' + \frac{f'}{r} - \frac{d^2}{r^2}f - f(1-f^2) = 0 \text{ in } [0, 1], \quad f(0) = 0; f(1) = 0.$$

that is also studied by Hervé-Hervé [4], and that makes no difference in the proof. An alternative proof is done in Part VI of the present paper.

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