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Finite time boundedness of neutral high-order Hopfield neural networks with time delay in the leakage term and mixed time delays

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Abstract

This article deals with the finite time boundedness (FTB) and FTB-stabilization problem for a general class of neutral high-order Hopfield neural networks (NHOHNNs) with time delay in the leakage term and mixed time delays. The mixed time delays consist of both discrete time-varying delays and infinite distributed delays. By using the topological degree theory, sufficient conditions are established to prove the existence of equilibrium points. Then, the Lyapunov-Krasovskii functional (LKF) method is used to prove sufficient conditions for the FTB. These conditions are in the form of linear matrix inequalities (LMIs) and can be numerically checked. Furthermore, a state feedback control is constructed to solve the FTB-stabilization problem. Finally, some numerical examples are presented to show the effectiveness of our main results.

Keywords: High-order neural networks, finite time boundedness, topological degree, stabilization, Lyapunov-Krasovskii functional, LMI, neutral systems, leakage delay .

1. Introduction

Neural Networks (NNs) have been widely studied due to their practical applications in lots of areas such as model identification, signal processing, image processing, pattern recognition, optimization problems, associative memories [1, 2, 3, 4]. The majority of these applications requires the stability of the designed NNs. It should be pointed out that the delay has a great effect on the system performances. Therefore, the stability analysis of delayed NNs has attracted the attention of many researchers and a lot of results have been obtained (see for instance [5, 6, 7, 8, 9, 10, 11]). In [10, 12], the authors discussed the case of constant delays and in [6, 9, 11, 13] the authors analyzed the stability of NNs with continuously distributed delays.

Recently, another kind of delay, variously known as leakage delay, is investigated in [14, 15, 16, 17]. It has been proved that this kind of delay tends to render the systems of NNs unstable. The effect of the leakage delay on stability is one of the important research topics in the field of the stability of NNs [18]. Many researchers analyzed the effect of the leakage delay on the Lyapunov stability of various kinds of NNs such that bidirectional associative memory NNs [19, 20] or impulsive NNs [21, 22]. However, it should be pointed out that manipulating this kind of delay is not easy. In addition, a kind of time delay systems, appointed neutral-type delay systems, is used by many authors due to their practical applications [23]. There are several results discussing the stability of neutral-type NNs, see for instance [24, 25, 26] and references therein.

In order to render the stability criteria less conservative with respect to the delays, several methods are developed in the literature [27, 28, 29, 30, 31, 32, 33, 34]. Most of these methods are based on a Lyapunov functional and associated LMIs because these inequalities can be numerically checked. However, it is well known that when a novel Lyapunov functional is designed for reducing the conservatism with respect to the delays a greater complexity in terms of inequalities and variables to be calculated can appear. Therefore, the problem of reducing simultaneously the number of decision variables and the conservatism with respect to the delays arises.

Most of the previous works on stability were mainly based on the classical Lyapunov stability which is associated with an infinite time interval. However, only a finite time interval is considered in practical applications [35]. In 1953, Kamenkov has introduced in [36] the concept of finite time boundedness (FTB). Dorato reported in [37] that FTB and Lyapunov stability are two independent concepts. Many studies have addressed the FTB problem of NNs, see for instance [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. For neutral-type NNs, the FTB was studied in [42, 47]. In [43, 49] and [41], the class of uncertain NNs with Markovian jumps and the complex-valued NNs are studied respectively. Also the impulsive NNs is investigated in [39] and the authors of [40, 38] deal with the problem of FTB for Memristive NNs and TS-fuzzy system with time-varying delay respectively. Moreover, all the previous results on FTB

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are for lower order NNs. This class of lower order NNs has several limitations (see for instance [50, 51]) which leads to consider the class of NNs with high-order connections. This class of high-order NNs has stronger approximation characteristics, important storage capacity, faster convergence rate and higher fault tolerance than lower-order NNs [52]. To the best of the authors' knowledge, the FTB and FTB-stabilization of neutral high-order Hopfield NNs (NHOHNNs) with a leakage delay and with both discrete and distributed delays, called mixed time delays, have not been fully investigated in the literature. Here is the main goal of our article. The delays considered in our article contain an infinite distributed delay which occurs in practice [23]. Indeed, there is no result for the FTB of NNs with infinite distributed delays. Despite important studies available yet to deal with the FTB concept [42, 43, 47, 48, 49], there is no result for the FTB of NNs with non-differentiable time-varying delay. Here is also a novelty of our article.

In our article, the FTB and FTB-stabilization are analyzed for a general class of NHOHNNs with time delay in the leakage term and mixed time delays. The main contributions are as follows:

(i) the LKF and LMIs techniques are used to study the effect of the leakage delay on the FTB results of the concerned NHOHNNs;

(ii) simplified LMI conditions are established to reduce the conservatism with respect to the delays and provide a less computational load simultaneously;

(iii) by using the FTB analysis, some conditions are given for the FTB-stabilization of the concerned NHOHNNs with a leakage delay. Moreover, the NHOHNNs considered in this article are subjected to non-differentiable time-varying delays.

The article is organized as follows. In Section 2, some preliminaries useful for the study of the class of NHOHNNs are presented. The existence of equilibrium points is addressed in Section 3. Then, the FTB and the FTB-stabilization are studied in Section 4. Numerical examples are presented in Section 5 to illuminate the validity of the proposed results. Finally, some concluding remarks are drawn in Section 6.

2. Preliminaries

In this article, we use the following notations:

- $\mathbb{R}$, $\mathbb{Z}^+$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ stand for the set of real numbers, the set of positive integers, the n-dimensional real space equipped with the euclidean norm $\|\,.\,\|$ and the set of $n\times n$ real matrices respectively;
- A^T, A^{-1}, $A>0$, $A<0$ and I means respectively the transpose of A, the inverse of A, the matrix A is positive definite, the matrix $-A$ is positive definite and the identity matrix with appropriate dimensions;
- $\lambda_{max}(A)$, $\lambda_{min}(A)$ and $*$ stand for the maximum eigenvalue of A, the minimum eigenvalue of A and the symmetric block in one symmetric matrix respectively;
- $X^+ = \left(|x_{ij}|\right)_{n\times n}$ where $X = (x_{ij})_{n\times n}$;
- for any interval $I\subset\mathbb{R}$ and $V\subset\ \mathbb{R}^k$ $(1\le k\le n)$, $C(I,\ V)$ and $C_b^1(I,\ V)$ stand respectively for the set $\{\phi : I\to V :\ \phi$ is continuous$\}$ and $\{\phi : I\to V :\ \phi$ is continuously differentiable and bounded$\}$.

We consider the following neutral high-order Hopfield neural networks (NHOHNNs) with time delay in the leakage term and mixed time delays

$$\begin{cases} \dot{x}_i(t) &= -c_i x_i(t-\sigma) + \sum\limits_{j=1}^{n} a_{ij} f_j\left(x_j\left(t-\tau(t)\right)\right) \\ &\quad + \sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n} T_{ijk} f_k\left(x_k(t-\tau(t))\right) f_j\left(x_j(t-\tau(t))\right) \\ &\quad + \sum\limits_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_j(t-s) f_j\left(x_j(s)\right) ds \\ &\quad + \sum\limits_{j=1}^{n} d_{ij}\dot{x}_j\left(t-h(t)\right) + J_i, \quad t>0,\, i=1,\ldots,n \\ x(s) &= \phi(s), \quad s\in(-\infty,\,0] \end{cases} \tag{1}$$

where $c_i>0$, $C = diag\left(c_1,\ldots,\ c_n\right)$, a_{ij}, b_{ij} and d_{ij} are the interconnection weight coefficients of the neurons, T_{ijk} the second-order synaptic weights of the NNs, J_i an external input vector, $x(t)\ = (x_1(t),\ \ldots,\ x_n(t))^T$ the neuron state vector of the NNs, $\dot{x}$ the time derivative of the neuron state,

$$f\left(x(.)\right) = \left(f_1\left(x_1(.)\right),\ \ldots\ ,\ f_n\left(x_n(.)\right)\right)^T$$

the neuron activation function such that $f_i(0)=0$ for all $1\le i\le n$, $\sigma\ge 0$ a constant which is the leakage delay,

$$K = diag\left(k_1(.),\ \ldots\ ,\ k_n(.)\right)$$

the delay kernel, $\tau(.)$ and $h(.)$ the time-varying transmission delays satisfying $0\le\tau(t)\le\tau$, $0\le h(t)\le\bar{h}$ and $\dot{h}(t)\le h^*<1$. The initial condition satisfies $\phi(.)\in C_b^1\left((-\infty,0],\mathbb{R}^n\right)$ where the norm is defined by

$$\|\phi\|_{\bar{h}} = \max\left\{\sup_{s\le 0}\|\phi(s)\|,\ \sup_{-\bar{h}\le s\le 0}\|\dot{\phi}(s)\|\right\}.$$

The term of the right-hand side of System (1) involving the time derivative of the state renders the system of neutral-type (see for instance [53]). Several studies have been done on System (1), especially in [8], where the global exponential stability is treated for $\sigma=0$ and with Markovian jump parameters. In [54], the authors studied the stability of the almost automorphic solutions for the same model but without taking into consideration infinite distributed delays. Note that such System (1) contains many of the well-known models as special cases.

Let us introduce the following three assumptions:

(H1) for all $x,\ y\in\mathbb{R}$, there exist constants $\omega>0$, M_j^- and M_j^+ such that

$$|f_j(x)|\le\omega_j \quad\text{and}\quad M_j^-\le\frac{|f_j(x)-f_j(y)|}{x-y}\le M_j^+,$$

(H2) for all $j = 1, \ldots, n$, the functions $k_j : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy

$$\int_0^{+\infty} k_j(u)\mathrm{d}u = \mathbf{k}_j \text{ and } \int_0^{+\infty} u\, k_j(u)\mathrm{d}u < \infty,$$

(H3) $C - A^+M - B^+K^+M$ is a M-matrix.

Remark 2.1. Under assumptions (H1) and (H2), the existence of the solutions of System (1) is guaranteed by using [53, Theorem 2.1]. It should also be pointed out that the constants M_j^+ and M_j^- can be negative or positive in the assumption (H1) which allows Lurie-type functions if we take $M_j^+, M_j^- > 0$ [55] or Lipschitz functions if we take $M_j^- = -M_j^+ < 0$. Therefore (H1) is weaker than the assumptions used for instance in [56, 57, 58]. Finally, if we only consider Lipschitz activation functions as in [12], the method used in [12] for proving the existence of an equilibrium point does not work for System (1) with the high-order terms.

Definition 2.1. *([12]) A nonsingular matrix $X = (x_{ij})_{n\times n}$ is a M-matrix if $X = rI - \beta$ with $\beta \geq 0$ and $r > \rho(\beta)$ where $\rho(\beta)$ stands for the spectral radius of the matrix β.*

Let

$$Z^{n\times n} = \left\{A = (a_{ij}) \in \mathbb{R}^{n\times n} : \ a_{ii} > 0, \ a_{ij} \leq 0, \text{ for all } i \neq j\right\}$$

then we have the following lemma:

Lemma 2.2. *([12]) The following claims are equivalent:*

(i) there exist $\delta_j > 0$ such that $\sum_{j=1}^{n} a_{ji}\delta_j > 0$ for all $i = 1, \ldots, n$;

(ii) $A \in Z^{n\times n}$ is a M-matrix;

(iii) there exist $\delta_j > 0$ such that $\sum_{j=1}^{n} a_{ij}\delta_j > 0$ for all $i = 1, \ldots, n$.

Let us give the definition of the topological degree.

Definition 2.3. *([12]) Assume that $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable function where $\Omega \subset \mathbb{R}^n$ is an open bounded set. If $J_F(u) = 0$ for any $u \in F^{-1}(p)$ and $p \notin F(\partial\Omega)$, where J_F denotes the Jacobian determinant relative to F, then the topological degree relative to Ω and p is given by*

$$\deg(F, \Omega, p) = \begin{cases} \sum\limits_{u\in F^{-1}(p)} sign\, J_F(u) & if \quad F^{-1}(p) \neq \emptyset \\ 0 & if \quad F^{-1}(p) = \emptyset \end{cases}$$

Let us introduce the notion of finite time boundedness.

Definition 2.4. *([59]) System (1) is said to be finite time bounded (FTB) with respect to (c_1, c_2, R, T) where $0 < c_1 \leq c_2$, $R > 0$ and $T > 0$ if for all $t \in [0, \ T]$ we have*

$$\max\left\{\sup_{\theta\leq 0}\{x^T(\theta)Rx(\theta)\}, \ \sup_{-\bar{h}\leq\theta\leq 0}\{\dot{x}^T(\theta)R\dot{x}(\theta)\}\right\} \leq c_1$$

implies that

$$x^T(t)Rx(t) < c_2.$$

Let us give two lemmas useful to prove our first result.

Lemma 2.5. *([15]) Given any real matrix $M = M^T > 0$ of appropriate dimension and a vector field $\omega : [a, \ b] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then we have*

$$\left[\int_a^b \omega(s)ds\right]^T M \int_a^b \omega(s)ds \leq (b-a)\int_a^b \omega^T(s)M\omega(s)ds.$$

Lemma 2.6. *([60]) Let $P \in \mathbb{R}^{n\times n}$ be a symmetric matrix, then we have*

$$\lambda_{\min}(P)x^Tx \leq x^TPx \leq \lambda_{\max}(P)x^Tx$$

for any $x \in \mathbb{R}^n$.

3. Existence of equilibrium points

In [15], the existence of equilibrium points for neutral lower order Hopfield NNs with time delay in the leakage term is discussed. Thanks to the high-order terms, the NNs given by (1) are more general. Compared with results in [15], we extend the existence of equilibrium points to the more general class of NHOHNNs.

We introduce the following notations:

$$M_j = \max\left\{|M_j^-|, \ |M_j^+|\right\}, \quad M = diag(M_1, \ \ldots \ , M_n)$$
$$A = [a_{ij}]_{n\times n}, \quad B = [b_{ij}]_{n\times n}, \quad D = [d_{ij}]_{n\times n}.$$

We present a sufficient condition which guarantee the existence of equilibrium points for the NHOHNNs given by (1).

Theorem 3.1. *Under assumptions $(H1) - (H2) - (H3)$, System (1) has at least one equilibrium point.*

PROOF. If $x^* = (x_1^*, \ \ldots, \ x_n^*)^T$ denotes an equilibrium point of System (1), then x^* satisfies for all $i = 1, \ldots, n$

$$\begin{aligned} &-c_ix_i^* + \sum_{j=1}^{n} a_{ij}f_j(x_j^*) + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}f_k(x_k^*)f_j(x_j^*) \\ &+ \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_j(t-s)f_j(x_j^*)\mathrm{d}s + \sum_{j=1}^{n} d_{ij}\dot{x}_j^* + J_i = 0 \end{aligned} \tag{2}$$

By using $(H2)$, Equation (2) is equivalent to

$$\begin{aligned} &-c_ix_i^* + \sum_{j=1}^{n} a_{ij}f_j(x_j^*) + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}f_k(x_k^*)f_j(x_j^*) \\ &+ \sum_{j=1}^{n} b_{ij}\mathbf{k}_jf_j(x_j^*) + J_i = 0. \end{aligned}$$

Let

$$l_i(x) = c_ix_i - \sum_{j=1}^{n}\left(a_{ij} + b_{ij}\mathbf{k}_j\right)f_j(x_j) - \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}f_k(x_k)f_j(x_j) - J_i. \tag{3}$$

Clearly, a solution of $l(x) = (l_1(x), \ \ldots, \ l_n(x))^T = 0$ is an equilibrium of System (1). Now, we define the homotopy mapping

$$F(x, \ \lambda) = (F_1(x), \ \ldots, \ F_n(x))^T$$

where $\lambda \in [0,\ 1]$ and $F_i(x) = \lambda l_i(x) + (1-\lambda)x_i$. It follows from assumptions $(H1)-(H2)$ that for all $1 \le i \le n$

$$\begin{aligned}
|F_i(x,\ \lambda)| &= \Big|\lambda\Big[c_i x_i - \sum_{j=1}^{n} a_{ij} f_j(x_j) - \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} f_k(x_k) f_j(x_j) \\
&\quad - \sum_{j=1}^{n} b_{ij}\mathbf{k}_j f_j(x_j) - J_i\Big] + (1-\lambda)x_i\Big| \\
&\ge |\lambda c_i x_i + (1-\lambda)x_i| - \lambda \sum_{j=1}^{n} |a_{ij}||f_j(x_j)| \\
&\quad - \lambda \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\,|f_k(x_k)|\,|f_j(x_j)| \\
&\quad - \lambda \sum_{j=1}^{n} |b_{ij}||\mathbf{k}_j||f_j(x_j)| - \lambda|J_i| \\
&\ge |\lambda c_i x_i + (1-\lambda)x_i| - \lambda \sum_{j=1}^{n} |a_{ij}|M_j|x_j| \\
&\quad - \lambda \sum_{j=1}^{n} |b_{ij}||\mathbf{k}_j|M_j|x_j| - \lambda\left[|J_i| + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\omega^2\right].
\end{aligned}$$

Since $C - A^+M - B^+K^+M$ is a M-matrix, Lemma 2.2 implies that there exists constants $\delta_i > 0$ with $i = 1,\ldots,n$ such that

$$\begin{aligned}
&\delta_i c_i - \sum_{j=1}^{n} \delta_j |a_{ij}|M_j - \sum_{j=1}^{n} \delta_j |b_{ij}||\mathbf{k}_j|M_j > 0 \\
&\text{or} \qquad\qquad (4)\\
&\delta_i c_i - \sum_{j=1}^{n} \delta_j |a_{ji}|M_i - \sum_{j=1}^{n} \delta_j |b_{ji}||\mathbf{k}_i|M_i > 0.
\end{aligned}$$

For $F(x) = (F_1(x_1),\ \ldots,\ F_n(x_n))^T$, we have

$$\begin{aligned}
\sum_{i=1}^{n} \delta_i |F_i(x,\ \lambda)| &\ge \sum_{i=1}^{n} \delta_i (1-\lambda)|x_i| + \lambda \sum_{i=1}^{n}\Big[\delta_i c_i |x_i| \\
&\quad - \delta_i \sum_{j=1}^{n} |a_{ij}|M_j|x_j| - \delta_i \sum_{j=1}^{n} |b_{ij}||\mathbf{k}_j|M_j|x_j|\Big] \\
&\quad - \lambda \sum_{i=1}^{n} \delta_i \left(|J_i| + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\omega^2\right) \\
&\ge \lambda \sum_{i=1}^{n}\Big[\delta_i c_i |x_i| - \delta_i \sum_{j=1}^{n} M_j |a_{ij}||x_j| \\
&\quad - \delta_i \sum_{j=1}^{n} M_j |b_{ij}||\mathbf{k}_j||x_j|\Big] \\
&\quad - \lambda \sum_{i=1}^{n} \delta_i \left(|J_i| + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\omega^2\right) \\
&= \lambda \sum_{i=1}^{n}\Big[\delta_i c_i - \sum_{j=1}^{n} \delta_j |a_{ji}|M_i \\
&\quad - \sum_{j=1}^{n} \delta_j |b_{ji}|M_i|\mathbf{k}_i|\Big]|x_i| \\
&\quad - \lambda \sum_{i=1}^{n} \delta_i \left(|J_i| + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\omega^2\right)
\end{aligned}$$

Define

$$\begin{aligned}
\delta_0 &= \min_{1\le i\le n}\left\{\delta_i c_i - \sum_{j=1}^{n} \delta_j |a_{ji}|M_i - \sum_{j=1}^{n} \delta_j |b_{ji}|M_i|\mathbf{k}_i|\right\} \\
\Delta_0 &= \max_{1\le i\le n}\left\{\delta_i \left(|J_i| + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|\omega^2\right)\right\}
\end{aligned}$$

Let

$$\Omega = \left\{x\ :\ |x_i| < \beta = \frac{n(\Delta_0+1)}{\delta_0}\right\} \tag{5}$$

then Ω is non-empty from (4). It follows from (5) that for any $x \in \partial\Omega$ there exists $1 \le i_0 \le n$ such that $|x_{i_0}| = \beta$. So we have

$$\begin{aligned}
\sum_{i=1}^{n} \delta_i \big|F_i(x,\ \lambda)\big| &\ge \lambda \sum_{i=1}^{n}\Big[\delta_{i_0} c_{i_0} - \sum_{j=1}^{n} \delta_j M_{i_0} |a_{ji_0}| \\
&\quad - \sum_{j=1}^{n} \delta_j M_{i_0} |b_{ji_0}||\mathbf{k}_{i_0}|\Big]|x_{i_0}| - \lambda \sum_{i=1}^{n} \Delta_0 \\
&\ge \lambda \delta_0 |x_{i_0}| - \lambda n \Delta_0 > 0.
\end{aligned}$$

For all $\lambda \in (0,\ 1]$, it means that $F(x,\ \lambda) \ne 0$ for any $x \in \partial\Omega$ and $\lambda \in (0,\ 1]$. If $\lambda = 0$, we have $F(x,\ \lambda) = x \ne 0$ for any $x \in \partial\Omega$. Therefore $F(x,\ \lambda) \ne 0$, for any $x \in \partial\Omega$, $\lambda \in [0,\ 1]$. We have $\deg(I_d, \Omega,\ 0) = 1$ where $I_d(x) = x$ and $\deg(I_d, \Omega,\ 0)$ is the topological degree. Thus from the homotopy invariance theorem given for instance in [61, page 13], we obtain

$$\deg(I_d, \Omega,\ 0) = \deg(l(.), \Omega,\ 0) = 1.$$

By using the topological degree theory [61], we can conclude that the equation $l(x) = 0$ has at least one solution in Ω. It implies that System (1) has at least one equilibrium point. □

4. Finite Time boundedness analysis

4.1. Finite time Boundedness

Compared with some existing results in [15, 24, 42], System (1) is more general to some extent. Moreover, the usual Lyapunov stability is studied in [24, 15] whereas the concept of FTB is discussed in this section. Assume that $x^* = (x_1^*,\ \ldots,\ x_n^*)^T$ is an equilibrium point of System (1). By a simple transformation

$$z(t) = x(t) - x^*$$

we can shift the equilibrium point x^* to the origin. By using (2), System (1) can be rewritten as (see [62]):

$$\begin{cases}
\dot{z}(t) &= -Cz(t-\sigma) + \left(A + \Gamma^T T^*\right) g\left(z(t-\tau(t))\right) \\
&\quad + B\int_{-\infty}^{t} K(t-s) g\left(z(s)\right) \mathrm{d}s + D\dot{z}\left(t-h(t)\right) \\
z(s) &= \phi(s) - x^*, \quad s \in (-\infty,\ 0]
\end{cases}$$

where

$$\begin{aligned}
g\left(z(.)\right) &= f\left(z(.) + x^*\right) - f(x^*), \quad T_i = [T_{ijk}]_{n\times n}, \\
T^* &= \left[T_1 + T_1^T,\ \ldots,\ T_n + T_n^T\right]^T, \\
\Gamma &= diag[\xi,\ \ldots,\ \xi],\ \xi = [\xi_1,\ \ldots,\ \xi_n]^T, \\
\xi_i &= \frac{T_{ijk}}{T_{ijk} + T_{ikj}} f_k\left(x_k(t-\tau(t)\right) + \frac{T_{ikj}}{T_{ijk} + T_{ikj}} f_k(x_k^*).
\end{aligned}$$

We will use this $z-$form of System (1) for the proof of the results of our article. We introduce the following notations:

$$\Gamma^+ = diag[\xi^+,\ ...,\ \xi^+],\ \xi^+ = [\omega_1,\ ...,\ \omega_n]^T,\ c^+ = \max_{1\le i\le n} c_i.$$

The methods used in [47, 48, 42, 63, 64] for ensuring the FTB of NNs require the differentiability and the boundedness of the derivative of the time-varying delays. For improving these results, we remove this restriction by establishing the following theorem where the time-varying delays are not necessary differentiable.

Theorem 4.1. *Under assumptions $(H1)-(H2)-(H3)$, System (1) is FTB with respect to (c_1, c_2, R, T) if there exist a positive scalar α, two $n\times n$ matrices $Q_1,\ Q_2$, three $n\times n$ symmetric positive definite matrices $P,\ Q_3,\ Q_4$, four $n\times n$ positive diagonal matrices $U_1,\ U_2,\ Q_5,\ Q_6$, and a $2n\times 2n$ matrix*

$$Q_7 = \begin{bmatrix} T_{11} & T_{12} \\ * & T_{22} \end{bmatrix} > 0$$

such that the following conditions hold:

$$\Xi = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & \Pi_{14} & 0 & \Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & 0 & 0 & \Pi_{28} & \Pi_{29} \\ * & * & \Pi_{33} & 0 & \Pi_{35} & \Pi_{36} & 0 & \Pi_{38} & \Pi_{39} \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & \Pi_{48} & 0 \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & \Pi_{68} & \Pi_{69} \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0 \quad (6)$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \omega_1^{-1} \end{bmatrix} > 0 \quad (7)$$

where

$$\begin{aligned}
&\Pi_{11} = -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1 - \alpha P,\\
&\Pi_{13} = PD,\quad \Pi_{14} = T_{12}^T,\quad \Pi_{16} = CPC + \alpha CP,\\
&\Pi_{17} = U_1\Sigma_2,\quad \Pi_{18} = P\tilde{A},\quad \Pi_{19} = PB,\\
&\Pi_{22} = \tau T_{22} + Q_6 - Q_1 - Q_1^T,\quad \Pi_{23} = Q_1 D + Q_2^T D,\\
&\Pi_{25} = -Q_1 C,\quad \Pi_{28} = Q_1\tilde{A},\quad \Pi_{29} = Q_1 B,\\
&\Pi_{33} = -Q_6(1-h^*) - D^T Q_2 D - D^T Q_2^T D,\quad \Pi_{35} = D^T Q_2 C,\\
&\Pi_{36} = -D^T PC,\quad \Pi_{38} = -D^T Q_2\tilde{A},\quad \Pi_{39} = -D^T Q_2 B,\\
&\Pi_{44} = \tau T_{11} - T_{12} - T_{12}^T - U_2\Sigma_1,\quad \Pi_{48} = U_2\Sigma_2,\quad \Pi_{55} = -Q_3,\\
&\Pi_{66} = -Q_4 - \alpha CPC,\quad \Pi_{68} = -CP\tilde{A},\quad \Pi_{69} = -CPB,\\
&\Pi_{77} = Q_5\kappa - U_1,\quad \Pi_{88} = -U_2,\quad \Pi_{99} = -Q_5,
\end{aligned}$$

and

$$\begin{aligned}
\omega_1 = &\Big[2\lambda_{\max}(P)(1+\sigma^2 c^{+2}) + \sigma\lambda_{\max}(Q_3)\\
&+ \bar{h}\lambda_{\max}(Q_6) + \sigma^3\lambda_{\max}(Q_4) + \tau^2\lambda_{\max}(T_{22})\\
&+ \sum_{j=1}^{n} q_j \mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u)\mathrm{d}u\Big]\\
&\times \left[\left(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)}\right)^{\frac{1}{2}} + (\lambda_{\min}(P))^{\frac{-1}{2}}\right]^2 cond(P)cond(\tilde{P})
\end{aligned}$$

with

$$\begin{aligned}
&\Sigma_1 = diag(M_1^- M_1^+,\ \ldots,\ M_n^- M_n^+),\\
&\Sigma_2 = diag\left(\frac{M_1^- + M_1^+}{2},\ \ldots,\ \frac{M_n^- + M_n^+}{2}\right),\\
&Q_5 = diag(q_1,\ \ldots,\ q_n),\quad \kappa = diag(\mathbf{k}_1^2,\ \ldots,\ \mathbf{k}_n^2),\\
&\tilde{A} = A + \Gamma^{+^T} T^*,\quad \tilde{P} = R^{\frac{1}{2}} P R^{\frac{1}{2}},\\
&cond(P) = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \text{ the condition number of } P.
\end{aligned}$$

The proof of Theorem 4.1 is inspired by the proof of Theorem 2 in [15].

PROOF. Let us consider the following LKF

$$V(t, z(t)) = \sum_{i=1}^{6} V_i(t, z(t)) \quad (8)$$

where

$$\begin{aligned}
V_1(t, z(t)) &= \left[z(t) - C\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right]^T P \left[z(t) - C\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right]\\
V_2(t,\ z(t)) &= \int_{t-\sigma}^{t} z^T(s) Q_3 z(s)\mathrm{d}s + \int_{t-h(t)}^{t} \dot{z}^T(s) Q_6 \dot{z}(s)\mathrm{d}s\\
V_3(t,\ z(t)) &= \sigma\int_{t-\sigma}^{t}\int_{s}^{t} z^T(u) Q_4 z(u)\mathrm{d}u\mathrm{d}s\\
V_4(t,\ z(t)) &= \int_0^t \int_{u-\tau(u)}^{u} \begin{bmatrix} z(u-\tau(u)) \\ \dot{z}(s) \end{bmatrix}^T Q_7 \begin{bmatrix} z(u-\tau(u)) \\ \dot{z}(s) \end{bmatrix} \mathrm{d}s\ \mathrm{d}u\\
V_5(t,\ z(t)) &= \int_{-\tau}^{0}\int_{t+u}^{t} \dot{z}^T(s) T_{22} \dot{z}(s)\mathrm{d}s\ \mathrm{d}u\\
V_6(t,\ z(t)) &= \sum_{j=1}^{n} q_j \mathbf{k}_j \int_0^\infty k_j(u) \int_{t-u}^{t} g_j^2(z_j(s))\mathrm{d}s\ \mathrm{d}u
\end{aligned}$$

By calculating $\dot{V}(t, z(t))$ (see AppendixA), we obtain

$$\dot{V}(t, z(t)) \le \zeta^T(t, z(t))\bar{\Xi}\zeta(t, z(t))$$

where

$$\begin{aligned}
\zeta(t, z(t)) = \Bigg[&z^T(t), \dot{z}^T(t), \dot{z}^T(t-h(t)), z^T(t-\tau(t)), z^T(t-\sigma),\\
&\left(\int_{t-\sigma}^{t} z(s)ds\right)^T, g^T(z(t)), g^T(z(t-\tau(t))),\\
&\left(\int_{-\infty}^{t} K(t-s)g(z(s))ds\right)^T\Bigg]^T
\end{aligned}$$

and

$$\bar{\Xi} = \begin{bmatrix} \bar{\Pi}_{11} & 0 & \Pi_{13} & \Pi_{14} & 0 & \bar{\Pi}_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & 0 & 0 & \Pi_{28} & \Pi_{29} \\ * & * & \Pi_{33} & 0 & \Pi_{35} & \Pi_{36} & 0 & \Pi_{38} & \Pi_{39} \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & \Pi_{48} & 0 \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \bar{\Pi}_{66} & 0 & \Pi_{68} & \Pi_{69} \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0 \quad (9)$$

with

$$\bar{\Pi}_{11} = -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1,\ \bar{\Pi}_{16} = CPC,\ \bar{\Pi}_{66} = -Q_4.$$

Therefore

$$\begin{aligned}\dot{V}(t,z(t)) &\le \zeta^T(t,z(t))\bar{\Xi}\zeta(t,z(t))\\ &\le \zeta^T(t,z(t))\Xi\zeta(t,z(t)) + \alpha V_1(t,z(t))\\ &\le \zeta^T(t,z(t))\Xi\zeta(t,z(t)) + \alpha V(t,z(t))\end{aligned}$$

with

$$\bar{\Xi} = \Xi + \begin{bmatrix} \alpha P & 0 & 0 & 0 & 0 & -\alpha CP & 0 & 0 & 0\\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & * & \alpha CPC & 0 & 0 & 0\\ * & * & * & * & * & * & 0 & 0 & 0\\ * & * & * & * & * & * & * & 0 & 0\\ * & * & * & * & * & * & * & * & 0 \end{bmatrix}$$

Since $\Xi < 0$, it leads to

$$\dot{V}(t,z(t)) \le \alpha V(t,z(t)). \tag{10}$$

Integrating (10) from 0 to $t \in [0,\ T]$, we obtain

$$V(t,z(t)) \le e^{\alpha t} V(z(0)) \tag{11}$$

where

$$\begin{aligned}V(z(0)) \le \Big[&2\lambda_{\max}(P)(1+\sigma^2 c^{+2}) + \sigma\lambda_{\max}(Q_3) + \bar{h}\lambda_{\max}(Q_6)\\ &+ \sigma^3\lambda_{\max}(Q_4) + \tau^2\lambda_{\max}(T_{22})\\ &+ \sum_{j=1}^{n} q_j \mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u)\mathrm{d}u\Big]\|\phi\|_{\bar{h}}^2.\end{aligned} \tag{12}$$

On one hand, by using the Cauchy-Schwartz inequality and Lemma 2.6 we have

$$\begin{aligned}\|z(t)\| &\le \left\|C\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right\| + \sqrt{\frac{V_1(t,z(t))}{\lambda_{\min}(P)}}\\ &\le \left\|C\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right\| + \sqrt{\frac{V(t,z(t))}{\lambda_{\min}(P)}}\end{aligned} \tag{13}$$

and since

$$\begin{aligned}\left\|\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right\|^2 &= \left(\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right)^T \left(\int_{t-\sigma}^{t} z(s)\mathrm{d}s\right)\\ &\le \sigma\int_{t-\sigma}^{t} z^T(s)z(s)\mathrm{d}s\\ &\le \frac{\sigma}{\lambda_{\min}(Q_3)}\int_{t-\sigma}^{t} z^T(s)Q_3 z(s)\mathrm{d}s\\ &\le \frac{\sigma}{\lambda_{\min}(Q_3)} V(t,z(t))\end{aligned}$$

we obtain

$$\|z(t)\| \le \left[\left(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)}\right)^{\frac{1}{2}} + \left(\lambda_{\min}(P)\right)^{\frac{-1}{2}}\right]\sqrt{V(t)}. \tag{14}$$

On the other hand, Lemma 2.6, implies that

$$\begin{aligned}\lambda_{\max}(\tilde{P})\|z(t)\|^2 &\ge z^T(t)\tilde{P}z(t) \ge \lambda_{\min}(P)z^T(t)Rz(t),\\ \lambda_{\min}(\tilde{P})\|z(0)\|^2 &\le z^T(0)\tilde{P}z(0) \le \lambda_{\max}(P)z^T(0)Rz(0).\end{aligned} \tag{15}$$

So (11)-(12)-(14) and (15) prove that

$$\begin{aligned}\|z(t)\| \le &\Big[\left(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)}\right)^{\frac{1}{2}} + \left(\lambda_{\min}(P)\right)^{\frac{-1}{2}}\Big] e^{\frac{\alpha}{2}T}\sqrt{V(z((0))}\\ &\le e^{\frac{\alpha}{2}T}\Big[2\lambda_{\max}(P)(1+\sigma^2 c^{+2}) + \sigma\lambda_{\max}(Q_3)\\ &+ \bar{h}\lambda_{\max}(Q_6) + \sigma^3\lambda_{\max}(Q_4) + \tau^2\lambda_{\max}(T_{22})\\ &+ \sum_{j=1}^{n} q_j\mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u)\mathrm{d}u\Big]^{\frac{1}{2}}\\ &\times \left[\left(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)}\right)^{\frac{1}{2}} + \left(\lambda_{\min}(P)\right)^{\frac{-1}{2}}\right]\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(\tilde{P})}c_1}\end{aligned} \tag{16}$$

and

$$\begin{aligned}z^T(t)Rz(t) \le e^{\alpha T}\Big[&2\lambda_{\max}(P)\left(1+\sigma^2 c^{+2}\right) + \sigma\lambda_{\max}(Q_3)\\ &+ \bar{h}\lambda_{\max}(Q_6) + \sigma^3\lambda_{\max}(Q_4) + \tau^2\lambda_{\max}(T_{22})\\ &+ \sum_{j=1}^{n} q_j\mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u)\mathrm{d}u\Big]\\ &\times \left[\left(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)}\right)^{\frac{1}{2}} + \left(\lambda_{\min}(P)\right)^{\frac{-1}{2}}\right]^2\\ &\times\ cond(P)cond(\tilde{P})c_1\end{aligned} \tag{17}$$

The inequality (17) and the Schur complement lemma, given for instance in [65], applied to condition (7) prove that

$$z(t)^T Rz(t) < c_2 \quad \text{for all } \ t \in [0,\ T].$$

So, the proof is completed. □

Remark 4.1. If conditions (6) and (7) of Theorem 1 are satisfied with $\alpha = 0$, then System (1) is asymptotically stable in the sense of Lyapunov. The LKF inequality (10) is used to deal with the problem of FTB of the NHOHNNs with infinite distributed delays and leakage delay. Compared with [15], some new sufficient conditions in terms of LMIs are established to make our results less conservative with respect to the delays. When the number of neurons n increases, the complexity increases strongly because $8.5n^2 + 4n + 1$ variables are involved in the LMIs that must be resolved. Such a complexity is caused by the use of the LKF inequality (10).

Remark 4.2. Condition (7) is not standard LMIs, however it

can be guaranteed by the following conditions

$$\lambda_1 I < P < \lambda_2 I,\ \lambda_3 I < Q_3 < \lambda_4 I,\ Q_4 < \lambda_5 I,\ Q_6 < \lambda_6 I,\ T_{22} < \lambda_7 I; \tag{18}$$

$$\begin{aligned} &-c_2 r_1^2 \lambda_1^2 e^{-\alpha T} + c_1 \Big[2\lambda_2 (1+\sigma^2 c^{+^2}) + \sigma\lambda_4 + \bar{h}\lambda_6 + \sigma^3\lambda_5 + \tau^2\lambda_7 \\ &+ \sum_{j=1}^{n} q_j \mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u) du \Big] \left[\frac{nc^+\sigma}{\sqrt{\lambda_3}} + \frac{1}{\sqrt{\lambda_1}} \right]^2 r_2^2 \lambda_2^2 < 0; \end{aligned} \tag{19}$$

where $r_1 = \lambda_{min}(R),\ r_2 = \lambda_{max}(R)$ and $\lambda_i, i = 1, \ldots 7$ are unknown positive variables. It should be pointed out that (19) is not a LMI w.r.t. $\lambda_i,\ i = 1,2,3$ because λ_i appears in a nonlinear fashion. Therefore, we first find the scalars λ_i from (6) and then we solve (19) which then becomes a LMI.

When $\tau(t) = 0$ and $K = 0$, System (1) turns into

$$\begin{cases} \dot{x}_i(t) &= -c_i x_i(t-\sigma) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &+ \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{n} T_{ijk} f_k(x_k(t)) f_j(x_j(t)) \\ &+ \sum\limits_{j=1}^{n} d_{ij} \dot{x}_j(t-h(t)) \\ x(s) &= \phi(s), \quad s \in (-\infty,\ 0] \end{cases} \tag{20}$$

Let us give the definition of finite time boundedness for System (20).

Definition 4.2. *([44]) System* (20) *is said to be Finite Time Bounded (FTB) with respect to* (c_1, c_2, R, T), $0 < c_1 \le c_2$, $T > 0$, $R > 0$ *if for all* $t \in [0,\ T]$

$$\max \left\{ \sup_{-\sigma<\theta\le 0} \{x^T(\theta) R x(\theta)\},\ \sup_{-\bar{h}\le\theta\le 0} \{\dot{x}^T(\theta) R \dot{x}(\theta)\} \right\} \le c_1$$

implies that

$$x^T(t) R x(t) < c_2.$$

According to Definition 4.2, we deduce the following corollary.

Corollary 4.3. *Under the assumptions and notations of Theorem 4.1, System* (20) *is FTB w.r.t.* (c_1, c_2, R, T) *if the following conditions hold:*

$$\tilde{\Xi} = \begin{bmatrix} \tilde{\Pi}_{11} & 0 & \Pi_{13} & 0 & \Pi_{16} & \tilde{\Pi}_{17} \\ * & \tilde{\Pi}_{22} & \Pi_{23} & \Pi_{25} & 0 & 0 \\ * & * & \Pi_{33} & \Pi_{35} & \Pi_{36} & 0 \\ * & * & * & \Pi_{55} & 0 & 0 \\ * & * & * & * & \Pi_{66} & 0 \\ * & * & * & * & * & -U_1 \end{bmatrix} < 0 \tag{21}$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \omega_2^{-1} \end{bmatrix} > 0 \tag{22}$$

where

$$\begin{aligned} &\tilde{\Pi}_{11} = -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1 - U_2\Sigma_1 - \alpha P, \\ &\tilde{\Pi}_{17} = (U_1 + U_2)\Sigma_2 + P\tilde{A}, \quad \tilde{\Pi}_{22} = Q_6 - Q_1 - Q_1^T. \end{aligned}$$

and

$$\begin{aligned} \omega_2 = e^{\alpha T} &\Big[2\lambda_{\max}(P)(1+\sigma^2 c^{+^2}) + \sigma\lambda_{\max}(Q_3) + \bar{h}\lambda_{\max}(Q_6) \\ &+ \sigma^3 \lambda_{\max}(Q_4) \Big] \left[\Big(\frac{nc^+\sigma}{\lambda_{\min}(Q_3)} \Big)^{\frac{1}{2}} + \big(\lambda_{\min}(P)\big)^{\frac{-1}{2}} \right]^2 \\ &\times\ cond(P) cond(\tilde{P}). \end{aligned}$$

PROOF. Consider the following LKF

$$V(t,\ z(t)) = \sum_{i=1}^{3} V_i(t, z(t)) \tag{23}$$

By using the LKF (23) and similar arguments to the ones of Theorem 4.1 we obtain easily the result. □

Remark 4.3. The lower order class of Hopfield NNs is investigated by many authors [15, 16, 17] and can be considered as a theoretical basis for solving optimization problems. As reported in [66], this class of NNs is expected to produce the poorest quality of solution with a great complexity as measured by the order of the network. Therefore, our work offers a theoretical basis for the design of the second-order class of NNs with mixed time delays more effective in the resolution of optimization problems thanks to the second order synaptic terms T_{ijk}.

Now, when the high-order terms $T_{ijk} = 0$, System (1) turns into

$$\begin{cases} \dot{x}_i(t) &= -c_i x_i(t-\sigma) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t-\tau(t))) \\ &+ \sum\limits_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_j(t-s) f_j(x(s)) ds + \sum\limits_{j=1}^{n} d_{ij} \dot{x}_j(t-h(t)) \\ x(s) &= \phi(s), \quad s \in (-\infty,\ 0] \end{cases} \tag{24}$$

Then, we have the following result.

Corollary 4.4. *Under the assumptions and notations of Theorem 4.1 , System* (24) *is FTB w.r.t.* (c_1, c_2, R, T) *if the following conditions hold:*

$$\hat{\Xi} = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & \Pi_{14} & 0 & \Pi_{16} & \Pi_{17} & \hat{\Pi}_{18} & \Pi_{19} \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & 0 & 0 & \hat{\Pi}_{28} & \Pi_{29} \\ * & * & \Pi_{33} & 0 & \Pi_{35} & \Pi_{36} & 0 & \hat{\Pi}_{38} & \Pi_{39} \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & \Pi_{48} & 0 \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & \hat{\Pi}_{68} & \Pi_{69} \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0 \tag{25}$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \omega_1^{-1} \end{bmatrix} > 0 \tag{26}$$

where $\hat{\Pi}_{i8} = \Pi_{i8} - P\Gamma^{+^T} T^*$ *for* $i = 1,\ 2,\ 3,\ 6$.

The proof of Corollary 4.4 is similar to the one that of Theorem 4.1, so it is omitted.

The computation can be simplified by reducing the parameters involved in the LKF. It is also possible to replace the LKF by a Lyapunov function with the Razumukhin techniques. However, this will have a great influence on the conservatism with respect to the delays, in particular on the upper bounds of the delays [67, 68].

In order to simplify the calculations without losing the conservatism with respect to the delays, we establish the following Corollary 4.5 which provides a simplified criterion for NHOHNNs without delay in the leakage term. This criterion is based on the Schur complement and has been presented for the Lyapunov stability of a special class of NHOHNNs in [69]. The impact of this criterion on the conservatism with respect to the delays will be illustrated in Example 5.3.

Corollary 4.5. *Under the assumptions and notations of Theorem 4.1, System* (1) *with* $\sigma = 0$ *is FTB w.r.t.* (c_1, c_2, T, R) *if the following conditions hold:*

$$\Lambda = \begin{bmatrix} -\Pi_{22} & -\Pi_{23} & 0 & 0 & -\Pi_{28} & -\Pi_{29} \\ * & -\Pi_{33} & 0 & 0 & -\Pi_{38} & -\Pi_{39} \\ * & * & -\Pi_{44} & 0 & -\Pi_{48} & 0 \\ * & * & * & -\Pi_{77} & 0 & 0 \\ * & * & * & * & -\Pi_{88} & 0 \\ * & * & * & * & * & -\Pi_{99} \end{bmatrix} > 0 \tag{27}$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \bar{\omega}_1 \end{bmatrix} > 0 \tag{28}$$

where

$$\bar{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}},$$

$$\bar{\omega}_1 = \frac{\lambda_{min}(P)}{cond(\bar{P})} \Big[2\lambda_{\max}(P) + \bar{h}\lambda_{\max}(Q_6) + \tau^2 \lambda_{\max}(T_{22}) + \sum_{j=1}^{n} q_j\, \mathbf{k}_j \max_j M_j^2 \int_0^\infty u\, k_j(u) du \Big]^{-1}.$$

PROOF. Let us consider the following LKF

$$V_0(t, z(t)) = \sum_{i=1}^{2} \bar{V}_i(t, z(t)) + \sum_{i=4}^{6} V_i(t, z(t)) \tag{29}$$

where

$$\bar{V}_1(t, z(t)) = z(t)^T P z(t),$$

$$\bar{V}_2(t,\, z(t)) = \int_{t-h(t)}^{t} \dot{z}^T(s) Q_6 \dot{z}(s) \mathrm{d}s,$$

$$V_4(t,\, z(t)) = \int_0^t \int_{u-\tau(u)}^{u} \begin{bmatrix} z(u-\tau(u)) \\ \dot{z}(s) \end{bmatrix}^T Q_7 \begin{bmatrix} z(u-\tau(u)) \\ \dot{z}(s) \end{bmatrix} \mathrm{d}s\mathrm{d}u;$$

$$V_5(t,\, z(t)) = \int_{-\tau}^{0} \int_{t+u}^{t} \dot{z}^T(s) T_{22} \dot{z}(s) \mathrm{d}s\mathrm{d}u,$$

$$V_6(t,\, z(t)) = \sum_{j=1}^{n} q_j \mathbf{k}_j \int_0^\infty k_j(u) \int_{t-u}^{t} g_j^2(z_j(s)) ds\, du.$$

By using the LKF (29) and similar arguments to the ones of Theorem 4.1, we obtain

$$\dot{V}_0(t, z(t)) \le -\zeta_0^T(t, z(t)) \bar{\Lambda} \zeta_0(t, z(t)) + \alpha V_0(t, z(t))$$

where

$$\zeta_0(t, z(t)) = \Big[z^T(t),\, \dot{z}^T(t),\, \dot{z}^T(t-h(t)),\, z^T(t-\tau(t)),\, g^T(z(t)),\, g^T(z(t-\tau(t))),\, \Big(\int_{-\infty}^{t} K(t-s) g(z(s)) ds \Big)^T \Big]^T$$

and

$$\bar{\Lambda} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & -\Pi_{14} & -\Pi_{17} & -\Pi_{18} & -\Pi_{19} \\ * & -\Pi_{22} & -\Pi_{23} & 0 & 0 & -\Pi_{28} & -\Pi_{29} \\ * & * & -\Pi_{33} & 0 & 0 & -\Pi_{38} & -\Pi_{39} \\ * & * & * & -\Pi_{44} & 0 & -\Pi_{48} & 0 \\ * & * & * & * & -\Pi_{77} & 0 & 0 \\ * & * & * & * & * & -\Pi_{88} & 0 \\ * & * & * & * & * & * & -\Pi_{99} \end{bmatrix}$$

with

$$\Lambda_{11} = PC + CP + U_1\Sigma_1 + \alpha P, \quad \Lambda_{12} = Q_1 C$$
$$\Lambda_{13} = -PD - CQ_2^T D$$

Clearly, if

$$\bar{\Lambda} > 0 \tag{30}$$

then we have

$$\dot{V}_0(t, z(t)) \le \alpha V_0(t, z(t)). \tag{31}$$

By using Schur complements, (30) can be expressed as

$$cR - \Gamma > 0 \tag{32}$$

where

$$R = \begin{bmatrix} \check{\Lambda}_{22} & \check{\Lambda}_{23} & 0 & 0 & \check{\Lambda}_{28} & \check{\Lambda}_{29} \\ * & \check{\Lambda}_{33} & 0 & 0 & \check{\Lambda}_{38} & \check{\Lambda}_{39} \\ * & * & \check{\Lambda}_{44} & 0 & \check{\Lambda}_{48} & 0 \\ * & * & * & \check{\Lambda}_{77} & 0 & 0 \\ * & * & * & * & \check{\Lambda}_{88} & 0 \\ * & * & * & * & * & \check{\Lambda}_{99} \end{bmatrix}$$

with

$$c > 0, \quad \check{Q}_i, > 0, i = 1, 2, 5, 7, \quad \check{U}_1 > 0, \quad \check{U}_2 > 0,$$
$$Q_1 = c\check{Q}_1, \quad Q_2 = c\check{Q}_2, \quad Q_5 = c\check{Q}_5, \quad Q_6 = c\check{Q}_6$$
$$Q_7 = c\check{Q}_7, \quad U_1 = c\check{U}_1, \quad U_2 = c\check{U}_2, \quad T_{11} = c\check{T}_{11},$$
$$T_{12} = c\check{T}_{12}, \quad T_{22} = c\check{T}_{22}, \quad \check{\Lambda}_{22} = -\tau\check{T}_{22} - \check{Q}_6 + \check{Q}_1 + \check{Q}_1^T,$$
$$\check{\Lambda}_{23} = -\check{Q}_1 D - \check{Q}_2^T D, \quad \check{\Lambda}_{28} = -\check{Q}_1\tilde{A}, \quad \check{\Lambda}_{29} = -\check{Q}_1 B,$$
$$\check{\Lambda}_{33} = \check{Q}_6(1-h^*) + D^T\check{Q}_2 D + D^T\check{Q}_2^T D, \quad \check{\Lambda}_{38} = -D^T\check{Q}_2\tilde{A},$$
$$\check{\Lambda}_{39} = D^T\check{Q}_2 B, \quad \check{\Lambda}_{44} = -\tau\check{T}_{11} + \check{T}_{12} + \check{T}_{12}^T + \check{U}_2\Sigma_1,$$
$$\check{\Lambda}_{48} = -\check{U}_2\Sigma_2, \quad \check{\Lambda}_{77} = -\check{Q}_5\kappa + \check{U}_1, \quad \check{\Lambda}_{88} = \check{U}_2, \quad \check{\Lambda}_{99} = \check{Q}_5.$$

and

$$\Gamma = \begin{bmatrix} \Lambda_{12}^T \\ -\Pi_{13}^T \\ -\Pi_{14}^T \\ -\Pi_{17}^T \\ -\Pi_{18}^T \\ -\Pi_{19}^T \end{bmatrix} (\Lambda_{11})^{-1} \begin{bmatrix} \Lambda_{12} & -\Pi_{13} & -\Pi_{14} & -\Pi_{17} & -\Pi_{18} & -\Pi_{19} \end{bmatrix}$$

Since $\Lambda_{11} > 0$ then it leads to $\Gamma > 0$ and consequently $R > \frac{1}{c}\Lambda > 0$ from condition (27). Therefore (31) is satisfied by taking $c > \frac{\lambda_{\max}(T)}{\lambda_{\max}(R)}$. Thus, condition (27) implies that $\dot{V}_0(t,z(t)) \leq \alpha V_0(t,z(t))$ and consequently $V(t,z(t)) \leq e^{\alpha T} V(z(0))$.

Furthermore, we have

$$\begin{aligned} \|x(0)\|^2 &\leq \frac{1}{\lambda_{\min}(P)} x^T(0)Px(0) \\ &\leq \frac{1}{\lambda_{\min}(P)} x^T(0)R^{\frac{1}{2}}R^{-\frac{1}{2}}PR^{-\frac{1}{2}}R^{\frac{1}{2}}x(0) \\ &\leq \frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(P)} x^T(0)Rx(0) \end{aligned}$$

Therefore

$$V(0) \leq \gamma c_1 \tag{33}$$

where

$$\begin{aligned} \gamma = \frac{\lambda_{max}(\bar{P})}{\lambda_{min}(P)} \Big[& 2\lambda_{\max}(P) + \bar{h}\lambda_{\max}(Q_6) + \tau^2 \lambda_{\max}(T_{22}) \\ & + \sum_{j=1}^{n} q_j \mathbf{k}_j \max_j M_j^2 \int_0^{\infty} u k_j(u) \mathrm{d}u \Big]. \end{aligned} \tag{34}$$

and

$$x^T(t)Rx(t) \leq \frac{1}{\lambda_{\min}(\bar{P})} x^T(t)Px(t) \leq \frac{1}{\lambda_{\min}(\bar{P})} V(t,x(t)).$$

So conditions (28) and (33) imply that $x^T(t)Rx(t) < c_2$ which achieves the proof. □

Remark 4.4. In [70], the problem of FTB for Markovian jumping NNs with time-varying delays is studied by using the reciprocally convex combination technique where a double integral term appears in the LKF. Very recently, the authors of [71] have used the same approach where a triple integral term is added in the LKF. However, these novel LKFs also lead to a greater complexity in terms of inequalities and variables to be calculated and this complexity can cause numerical problems with a large number of neurons ([33]). In our article we use a well know LKF (8), but based on the Jensen's integral inequality the upper bound of the LKF is estimated more tightly and then the conservatism with respect to the delays is reduced (see Example 5.1).

4.2. Finite time Boundedness stabilization

In this section, sufficient conditions are given for solving the FTB-stabilization problem of a general class of NHOHNNs with time delay in the leakage term and mixed time delays of the form

$$\begin{cases} \dot{x}_i(t) & = -c_i x_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau(t))) \\ & + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} f_k(x_k(t-\tau(t))) f_j(x_j(t-\tau(t))) \\ & + \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_j(t-s) f_j(x_j(s))\, ds \\ & + \sum_{j=1}^{n} d_{ij} \dot{x}_j(t-h(t)) + u, \quad t>0, i=1,\ldots,n \\ x(s) & = \phi(s), \quad s \in (-\infty, 0] \end{cases} \tag{35}$$

where u is the control variable. The following state feedback control is considered

$$u(x(t)) = K_2 x(t) \tag{36}$$

where K_2 is to be determined. The corresponding closed-loop $z-$system is given by

$$\begin{cases} \dot{z}(t) & = -Cz(t-\sigma) + Ag(z(t-\tau(t)) + \Gamma^T T^* g(z(t-\tau(t))) \\ & + B\int_{-\infty}^{t} K(t-s) g(z(s))\, ds + D\dot{z}(t-h(t)) K_2 z(t) \\ z(s) & = \phi(s) - x^*, \quad s \in (-\infty, 0] \end{cases} \tag{37}$$

We now introduce the definition of finite time bounded stabilization.

Definition 4.6. *([48]) The NNs described by System* (35) *is said to be FTB-stabilizable w.r.t.* (c_1, c_2, R, T) *if there exists a controller* $u(z(t))$ *of the form* (36) *such that the corresponding closed-loop* $z-$*System* (37) *is FTB with respect to* (c_1, c_2, R, T).

We can now state the main result of this section.

Theorem 4.7. *Under the assumptions and notations of Theorem 4.1, System* (35) *is FTB-stabilizable w.r.t.* (c_1, c_2, R, T) *if the following conditions hold:*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & \Pi_{15} & \Psi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & \Pi_{22} & \Pi_{23} & 0 & \Pi_{25} & 0 & 0 & \Pi_{28} & \Pi_{29} \\ * & * & \Pi_{33} & 0 & \Pi_{35} & \Pi_{36} & 0 & \Pi_{38} & \Pi_{39} \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & \Pi_{48} & \Pi_{49} \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & \Pi_{68} & \Pi_{69} \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0 \tag{38}$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \omega_1^{-1} \end{bmatrix} > 0 \tag{39}$$

where

$$\begin{aligned} &\tilde{C} = C - K_2, \\ &\Psi_{11} = -PC - CP + PK_2 + K_2^T P + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1 - \alpha P, \\ &\Psi_{12} = -Q_1 K_2, \quad \Psi_{13} = PD + D^T Q_2 K_2, \quad \Psi_{16} = CP\tilde{C} + \alpha CP, \end{aligned}$$

PROOF. According to Definition 4.6, if we apply Theorem 4.1 to the closed-loop System (37) then we can easily obtain the result. The details of the proof is left to the reader. □

When $\sigma = 0$, Theorem 4.7 cannot be directly used to design a feedback control of the form of (36). To overcome this obstacle, the following proposition is established.

Proposition 4.8. *Under assumptions $(H_1)-(H_3)$, System (35) without leakage delay is FTB- stabilizable w.r.t. (c_1,c_2,T,R) if there exist a positive scalar α, three $n\times n$ matrices $\tilde{Q}_1$, $\tilde{Q}_2$, Y, five $n\times n$ positive diagonal matrices X, $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{Q}_5$, Q_6 and a $2n\times 2n$ matrix*

$$\tilde{Q}_7 = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ * & T_{22} \end{bmatrix} > 0$$

such that the following conditions hold:

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & \Xi_{37} \\ * & * & * & \Xi_{44} & 0 & \Xi_{46} & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0 \quad (40)$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \bar{\omega}_2^{-1} \end{bmatrix} > 0 \quad (41)$$

where

$$\begin{aligned}
&\Xi_{11} = -CX - XC + Y + Y^T - \tilde{U}_1\Sigma_1 - \alpha X,\\
&\Xi_{12} = XC + Y,\ \Xi_{13} = D + D^T CX + D^T Y,\\
&\Xi_{14} = \tilde{T}_{12}^T,\ \Xi_{15} = \tilde{U}_1\Sigma_2,\ \Xi_{16} = \tilde{A}X,\ \Xi_{17} = BX,\\
&\Xi_{22} = \tau T_{22} + Q_6 - 2I,\ \Xi_{23} = XD + D^T X,\ ,\\
&\Xi_{26} = X\tilde{A},\ \Xi_{27} = XB,\ \Xi_{33} = -Q_6(1-h^*) - 2D^T D,\\
&\Xi_{36} = -XD^T\tilde{A},\ \Xi_{37} = -XD^T B,\\
&\Xi_{44} = \tau\tilde{T}_{11} - \tilde{T}_{12} - \tilde{T}_{12}^T - \tilde{U}_2\Sigma_1,\ \Xi_{46} = \tilde{U}_2\Sigma_2,\\
&\Xi_{55} = \tilde{Q}_5\kappa - \tilde{U}_1,\ \Xi_{66} = -\tilde{U}_2,\ \Xi_{77} = -\tilde{Q}_5.
\end{aligned}$$

and

$$\begin{aligned}
\bar{\omega}_2 = \frac{1}{\lambda_{min}(\bar{X})}\Big[&2\lambda_{\max}(X^{-1}) + \bar{h}\lambda_{\max}(Q_6)\\
&+ \tau^2\lambda_{\max}(T_{22}) + \sum_{j=1}^{n}\rho_j^{-1}\tilde{q}_j\rho_j^{-1}\mathbf{k}_j \max_j M_j^2 \int_0^\infty u\, k_j(u)du\Big]
\end{aligned}$$

with

$$\tilde{Q}_5 = diag(\tilde{q}_1,\dots,\tilde{q}_n),\ X = diag(\rho_1,\dots,\rho_n),\ \bar{X} = R^{\frac{-1}{2}}X^{-1}R^{\frac{-1}{2}}.$$

PROOF. Let $Q_1 = Q_2 = I$. Now, we make some transformations for the above inequalities (40)-(41). Pre and post-multiplying (40) by $diag(X^{-1},I,I,X^{-1},\dots,X^{-1})$ and letting

$$\begin{aligned}
&X = P^{-1},\quad Y = KX,\quad \tilde{Q}_5 = P^{-1}Q_5P^{-1};\\
&\tilde{U}_1 = P^{-1}U_1P^{-1},\ \tilde{U}_2 = P^{-1}U_2P^{-1},\ \tilde{T}_{12} = P^{-1}T_{12}P^{-1}. \quad (42)
\end{aligned}$$

we obtain $\Lambda > 0$. That is, (40) implies Condition (27). Then, by replacing (42) in Condition (41) we obtain easily (28). Therefore, the conditions of Corollary 4.5 are obtained which achieves the proof. □

Remark 4.5. Based on the inequality $S^{-1} \geq 2\xi I - \xi^2 S$ (S being a non-singular matrix), the following LMI conditions

$$\begin{bmatrix} -\lambda_1 I & I \\ * & -X_i \end{bmatrix} < 0,\quad X_i < \lambda_2 I,\quad \tilde{Q}_5 < 2\xi_1 I - \xi_1^2\lambda_3 I$$

$$\begin{bmatrix} -(2\xi_3 I - \xi_3^2\lambda_2)\pi_1 c_2 e^{-\alpha T} + c_1\left[2\lambda_1 + \bar{h}\lambda_6 + \tau^2\lambda_7\right] & \sqrt{c_1 r_3}\lambda_1 \\ * & -\lambda_3. \end{bmatrix} < 0$$

where $\pi_1 = \lambda_{min}(R^{-1})$, $r_3 = \sum_{j=1}^{n}\mathbf{k}_j \max_j M_j^2 \int_0^\infty u k_j(u)du$, ξ_1, ξ_3 are adjustable parameters and λ_i, $i = 1,2,3,6,7$ are unknown positive scalars, ensure Condition (41) [48].

When $\sigma \neq 0$, the following proposition can be applied directly to design a feedback control which is able to FTB-stabilize the NHOHNNs.

Proposition 4.9. *Under the assumptions $(H1)-(H2)-(H3)$, if there exist a positive scalar α, three positive symmetric definite matrices P, Q_3, Q_4, five $n\times n$ diagonal matrices X, $U_1 > 0$, $U_2 > 0$, $Q_5 > 0$, $Q_6 > 0$, and a $2n\times 2n$ matrix*

$$Q_7 = \begin{bmatrix} T_{11} & T_{12} \\ * & T_{22} \end{bmatrix} > 0$$

such that the following conditions hold:

$$\Theta = \begin{bmatrix} \Theta_{11} & X & \Pi_{13} & 0 & \Pi_{15} & \Theta_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & \Theta_{22} & \Theta_{23} & 0 & \Theta_{25} & 0 & 0 & \Theta_{28} & \Theta_{29} \\ * & * & \Theta_{33} & 0 & 0 & \Pi_{36} & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & \Pi_{48} & \Pi_{49} \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & \Pi_{68} & \Pi_{69} \\ * & * & * & * & * & * & \Pi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Pi_{99} \end{bmatrix} < 0 \quad (43)$$

and

$$\begin{bmatrix} c_2\, e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \omega_1^{-1} \end{bmatrix} > 0 \quad (44)$$

where

$$\begin{aligned}
&\Theta_{11} = -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1 - \alpha P + 2X,\ \Theta_{16} = CPC + \alpha CP - CX,\\
&\Theta_{22} = \tau T_{22} + Q_6 - 2P,\ \Theta_{23} = PD,\ \Theta_{25} = -PC,\ \Theta_{28} = P\tilde{A},\ \Theta_{29} = PB,\\
&\Theta_{33} = -Q_6(1-h^*)
\end{aligned}$$

and other parameters are the same as in Theorem 4.1, then the feedback control $u_2(x(t)) = P^{-1}Xx(t)$ FTB-stabilize System (35) w.r.t. (c_1,c_2,R,T).

PROOF. Let $Q_1 = P$ and $Q_2 = 0$. The proof of Proposition 4.9 is similar to the one of Theorem 4.7 so it is omitted here. □

Remark 4.6. In the results presented in[47, 48, 42, 63, 64], the feedback control given for ensuring the FTB-stabilization of the NNs cannot be designed with a leakage delay. When $\sigma \neq 0$, the conditions established in these works are not LMI conditions which renders the control algorithm more complicated.

To overcome these difficulties, the matrix gain should be of the following form $K = P^{-1}X$. In this case, it is possible to find LMI conditions ensuring the FTB-stabilization of NNs of the above mentioned works, even if there is a leakage delay.

5. Numerical examples

In this section, three numerical examples are presented to show the effectiveness of the results.

5.1. Example 1

Consider System (1) with $n = 2$ and

$$f_1(s) = f_2(s) = \tanh(s), \quad \tau(t) = 0.2 - 0.1\cos t,$$
$$h(t) = \sigma = 0.1, \quad k_1(s) = k_2(s) = e^{-s}, \quad J = (1,\ 2)^T,$$

and parameters C, A, T_1, T_2, B and D given as follows:

$$C = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \ A = \begin{bmatrix} 0.1 & 0.18 \\ -0.8 & -0.93 \end{bmatrix}, \ T_1 = \begin{bmatrix} 0.2 & 0.07 \\ 0.03 & 0.01 \end{bmatrix},$$
$$T_2 = \begin{bmatrix} 0.1 & 0.05 \\ 0.04 & 0.02 \end{bmatrix}, \ B = \begin{bmatrix} 0.7 & -0.2 \\ -0.2 & 0.5 \end{bmatrix}, \ D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

It leads to $\tau = 0.3$, $h^* = 0$, $M_j^- = 0$, $M_j^+ = 1$, $k_j = 1$ for $j = 1,\ 2$. Note that

$$C - A^+M - B^+K^+M = \begin{bmatrix} 3.2 & -0.38 \\ -1 & 2.57 \end{bmatrix}$$

is a M-matrix. By using Theorem 3.1, we know that System (1) has an equilibrium point.

For $c_1 = 0.35$, $T = 5$ and $R = I$, solving (6)-(7) with the Matlab LMI toolbox [72] with $\alpha = 0.02$ leads to the following solutions

$$P = \begin{bmatrix} 128.0037 & 19.7975 \\ 19.7975 & 62.5951 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 10.08 & 3.3704 \\ 3.3704 & 6.3560 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} 16.7695 & 1.7013 \\ 1.7013 & 7.4374 \end{bmatrix}, \ Q_3 = \begin{bmatrix} 220.9824 & 57.9710 \\ 57.9710 & 131.3136 \end{bmatrix}$$
$$Q_4 = \begin{bmatrix} 7.3255 & 1.2523 \\ 1.2523 & 5.1806 \end{bmatrix}, \ Q_5 = \begin{bmatrix} 110.8764 & 0 \\ 0 & 65.8265 \end{bmatrix}$$
$$Q_6 = \begin{bmatrix} 2.0464 & 0 \\ 0 & 0.8882 \end{bmatrix}, \ U_1 = \begin{bmatrix} 232.5544 & 0 \\ 0 & 156.5080 \end{bmatrix}$$
$$U_2 = \begin{bmatrix} 65.8755 & 0 \\ 0 & 58.9563 \end{bmatrix}, \ T_{11} = \begin{bmatrix} 112.1715 & 54.9582 \\ 54.9582 & 85.9251 \end{bmatrix}$$
$$T_{12} = \begin{bmatrix} 45.4573 & 20.3443 \\ 20.3443 & 31.9253 \end{bmatrix}, \ T_{22} = \begin{bmatrix} 19.7736 & 8.8448 \\ 8.8448 & 13.8120 \end{bmatrix}$$

The minimum value of c_2 satisfies $\min c_2 > 1.8157$ for $\alpha = 0.02$. Theorem 4.1 leads to the FTB of the considered system w.r.t. $(0.35, 3, I, 5)$. Time history of $x^T(t)x(t)$ is illustrated on Fig 1.

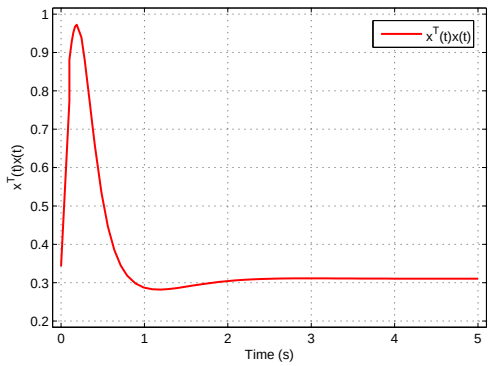

Figure 1: Time history of $x^T(t)x(t)$ with initial condition $(0.305, 0.5)^T$

Moreover, we see on Figure 2 that the considered system is not FTB w.r.t. $(0.35, 0.9, I, 100)$.

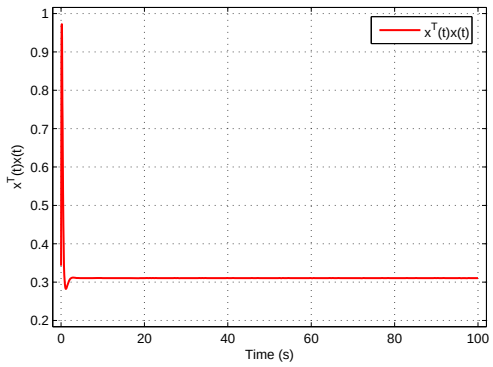

Figure 2: Time history of $x^T(t)x(t)$ with initial condition $(0.305,\ 0.5)^T$ for $T = 100$

Now, we fix $\alpha = 0.02$ and plot on Figure 3 the different values of the maximum of c_1 with respect to the parameters T and c_2 for having the FTB.

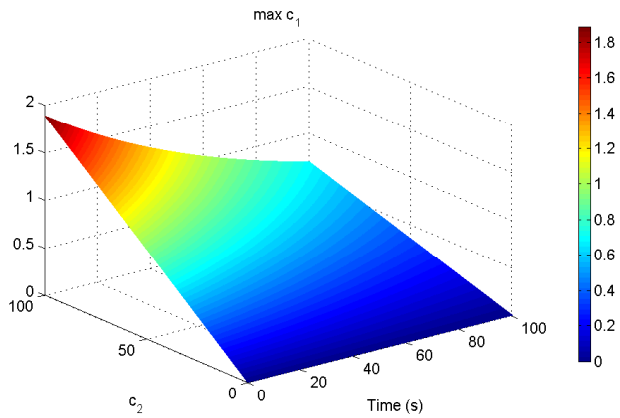

Figure 3: $\max c_1$ with respect to $(T,\ c_2)$

Finally for $\alpha = 0.01$, Figure 4 shows the minimum of c_2 with respect to $(T,\ c_1)$ for having the FTB.

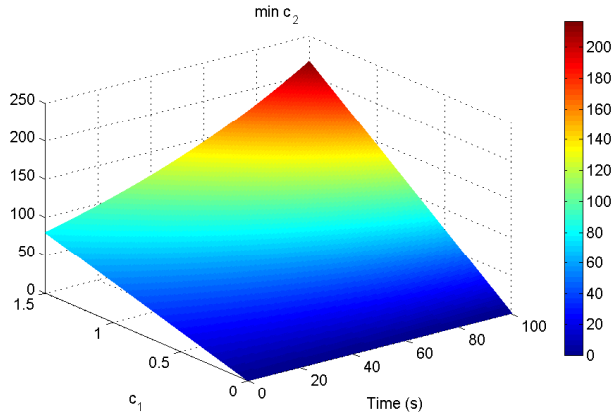

Figure 4: $\min c_2$ with respect to $(T,\ c_1)$

$h^* = 0$ and $\sigma = 0$	
Methods	$\tau_{\max}$
[73]	0.87
[15]	0.3527
[75]	0.4
Corollary 4.5	1.123

Table 1: The maximum allowable bounds $\tau_{\max}$ for $h^* = 0$ and $\sigma = 0$

When $\sigma = 0$, the upper bound of the delay τ for which LMIs (6) of Theorem 4.1 remain feasible is $\tau_{\max} = 1.088$. However, we obtain $\tau_{\max} = 1.123$ if we use the simplified criterion given in Corollary 4.5.

Remark 5.1. The stability criterion given in [15, 73, 74, 75] fails for small delays ($\tau_{\max} \leq 0.87$) and becomes infeasible when a leakage delay σ extends beyond 0.15. The method proposed in our work overcomes these difficulties by proving sufficient conditions that are able to ensure the FTB and even the asymptotic stability for a delay larger than the one given in [15, 73, 75]. Table 1 and Table 2 compare the maximum allowable bounds $\tau_{\max}$ of $\tau(t)$ derived from Theorem 4.1, Corollary 4.5 and [15, 73, 75, 74].

Remark 5.2. The method used in our article improves and extends the results given in [15, 74] by reducing the conservatism with respect to the delays and by simplifying the calculus simultaneously. This improvement is illustrated in Tables 3–4 and 5 by presenting a comparison of the computational load. Noted that FWM and DV stand respectively for the number of free-weight matrices and the number of decision variables used.

$h^* = 0$		
Methods	$\tau_{\max}$ for $\sigma = 0.15$	$\tau_{\max}$ for $\sigma = 0.2$
[74]	Infeasible	Infeasible
[Theorem 4.1]	0.245	0.095

Table 2: The maximum allowable bounds of $\tau_{\max}$ for $h^* = 0$

$\sigma = 0$			
Methods	FWM	DV	$\tau_{\max}$
[15]	2	$4n^2 + 4n$	0.3527
[Corollary 4.5]	2	$3n^2 + 4n + 1$	1.123

Table 3: A comparison of computational load when $\sigma = 0$

$\sigma = 0.15$			
Methods	FWM	DV	$\tau_{\max}$
[74]	2	$28n^2 + 16n$	Infeasible
[Theorem 4.1]	2	$8.5n^2 + 4n + 1$	0.245

Table 4: A comparison of computational load when $\sigma = 0.15$

5.2. *Example 2*

Consider System (1) with $n = 2$,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

and other parameters similar to Subsection 5.1. By setting $c_1 = 6.5$, $T = 5$ and $R = I$, we have that System (1) is FTB w.r.t. $(6.5, 1500, I, 5)$ but is not FTB w.r.t. $(6.5, 1500, I, 10)$. Moreover, we have $|x_i| \to +\infty$, $i = 1, 2$ when $t \to +\infty$ as shown if Figure 5 which proves that this system is not asymptotically stable.

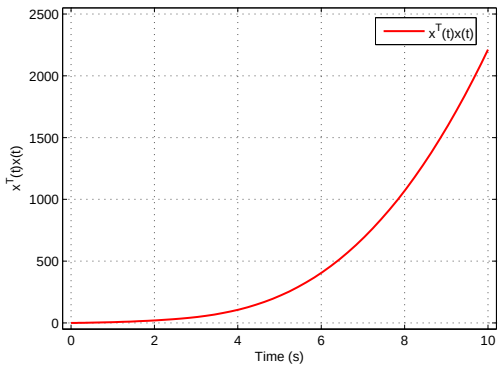

Figure 5: The history of $x^T(t)x(t)$ for System (1) in Exemple 2 with initial condition $(0.1,\ 0.1)^T$

$\sigma = 0.2$			
Methods	FWM	DV	$\tau_{\max}$
[74]	2	$28n^2 + 16n$	Infeasible
[Theorem 4.1]	2	$8.5n^2 + 4n + 1$	0.095

Table 5: The maximum allowable bounds of $\tau(t)$ when $\sigma = 0.2$

So, we have shown that FTB and Lyapunov asymptotic stability are two independent concepts.

Now we consider the controlled System (35) with the above numerical values. In order to FTB-stabilize System (35), we consider the feedback control according to the method proposed in Theorem 4.7. The feedback control is given by $u(x(t)) = K_1 x(t)$, with $K_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.042 \end{bmatrix}$. From Theorem 4.7, we deduce that System (35) is FTB stabilizable w.r.t. $(6.5,\ 1500,\ I,\ 10)$. Figure 6 shows the history of $x^T(t)x(t)$.

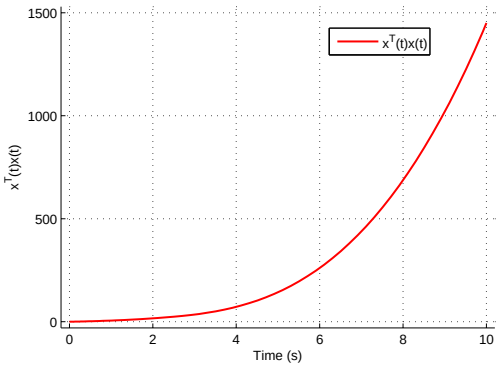

Figure 6: The history of $x^T(t)x(t)$ for System (35) with initial condition $(0.1,\ 0.1)^T$ under controller (36) with $K = K_1$

The state feedback control $u_2(x(t)) = K_2 x(t)$ with

$$K_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.9 \end{bmatrix}$$

FTB-stabilize the system but it also asymptotically stabilize the system if we consider an infinite time evolution. The history of $x^T(t)x(t)$ is then illustrated in Figure 7.

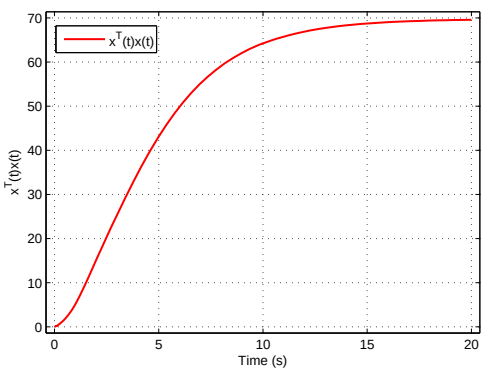

Figure 7: The history of $x^T(t)x(t)$ for System (37) with initial condition $(0.1,\ 0.1)^T$ under controller (36) with $K = K_2$

5.3. Example 3

Consider System (1) with $n = 2$,

$$C = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

and other parameters similar to Subsection 5.1. From condition (44), we have that $\min c_2 = 11.6515$. We choose $c_1 = 0.35$, $T = 10$ and $c_2 = 12$ and we see on Figure 8 that the open-loop System (1) is not FTB w.r.t. (c_1, c_2, I, T).

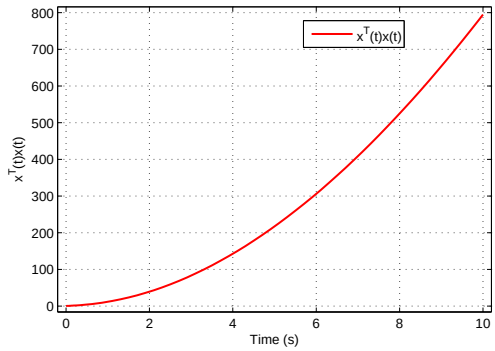

Figure 8: The history of $x^T(t)x(t)$ for System (37) without controller (36)

In order to FTB stabilize System (35) with the above numerical values, we build the following control $U(x(t)) = P^{-1}Xx(t)$ according to the strategy given in Subsection 4.2. Proposition 4.9 implies that the closed-loop System (37) is FTB-stabilizable w.r.t. (c_1, c_2, T, I) for $\alpha = 0.02$, with the following solutions

$$P = \begin{bmatrix} 6.5871 & 0 \\ 0 & 4.3602 \end{bmatrix},\ Q_3 = \begin{bmatrix} 0.3573 & -0.1380 \\ -0.1380 & 0.2511 \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} 4.8974 & -0.5024 \\ -0.5024 & 4.5153 \end{bmatrix},\ Q_5 = \begin{bmatrix} 6.1356 & 0 \\ 0 & 3.5997 \end{bmatrix}$$

$$Q_6 = \begin{bmatrix} 0.5134 & 0 \\ 0 & 0.3434 \end{bmatrix},\ U_1 = \begin{bmatrix} 12.0818 & 0 \\ 0 & 7.2642 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 11.2816 & 0 \\ 0 & 8.7200 \end{bmatrix},\ T_{11} = \begin{bmatrix} 4.6319 & 0.5660 \\ 0.5660 & 3.5918 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} 4.0306 & 1.0331 \\ 1.0331 & 3.1048 \end{bmatrix},\ T_{22} = \begin{bmatrix} 3.7924 & 1.3756 \\ 1.3756 & 2.8913 \end{bmatrix}$$

$$X = \begin{bmatrix} -14.7082 & 0 \\ 0 & -5.5260 \end{bmatrix}$$

and the controller $u(x(t)) = P^{-1}Xx(t)$ with

$$P^{-1}X = \begin{bmatrix} -2.2329 & 0 \\ 0 & -1.2674 \end{bmatrix}.$$

The history of $x^T(t)Rx(t)$ with the previous controller $u(x(t)) = P^{-1}Xx(t)$ is illustrated on Figure 9.

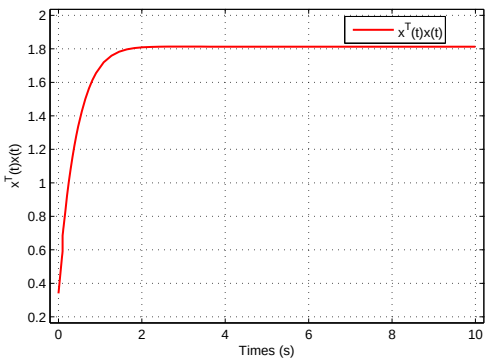

Figure 9: The history of $x^T(t)x(t)$ for System (37)under controller $P^{-1}Xx(t)$.

Now, if the time-varying delay $\tau(t)$ defined by the non-differentiable function is as follows:

$$\tau(t) = \begin{cases} 0.3 - 0.1\sin t & if \quad t \in \Im = \cup_{k\geq 0}\,[2k\pi, (2k+1)\pi]\,; \\ 0 & if \quad t \in \mathbb{R}^+ \backslash \Im. \end{cases} \tag{45}$$

System (35) stays FTB-stabilizable w.r.t. (c_1, c_2, T, I) which is illustrated in Figure 10.

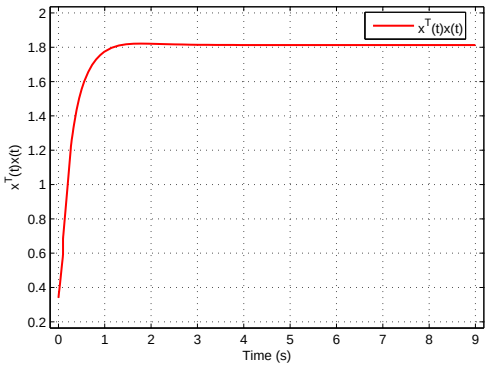

Figure 10: The history of $x^T(t)x(t)$ for System (37) under time-varying delay (45).

Remark 5.3. It should be pointed out that the results in [47, 48, 42, 63, 64] fail for System (1) with the time-varying delay (45) even without leakage delay because (45) is not differentiable. This proves the advantage of the proposed approach in our article.

6. Conclusion

This article deals with the problem of finite time boundedness and finite time boundedness stabilization of a general class of neutral high-order Hopfield neural networks with time delay in the leakage term and mixed time delays. By using the topological degree theory, sufficient conditions are given to prove the existence of equilibrium points. The Lyapunov-Krasovskii functional method and the LMIs technique are used to establish some sufficient conditions which ensure the finite time boundedness and finite time boundedness stabilization of the class of systems considered in our article. Finally, numerical examples are presented to show the effectiveness and the interest of our proposed results.

AppendixA. The calculus of $\dot{V}(t)$

System (1) has an equivalent form given by

$$\begin{cases} \frac{d}{dt}\left[z(t) - C\int_{t-\sigma}^{t} z(u)du\right] = -Cz(t) + (A+\Gamma^T T^*)g(z(t-\tau(t))) \\ +B\int_{-\infty}^{t} K(t-s)g(z(s))ds + D\dot{z}(t-h(t)) \\ z(s) = \phi(s) - x^* \end{cases}$$

Moreover, we have

$$\begin{aligned} \dot{V}_1(t,z(t)) &= 2\Big[z(t) - C\int_{t-\sigma}^{t} z(u)du\Big]^T P\Big[-Cz(t) + (A+\Gamma^T T^*)g(z(t-(\tau(t))) \\ &\quad + D\dot{z}(t-h(t)) + B\int_{-\infty}^{t} K(t-s)g(z(s))ds\Big] \\ &= -2z^T(t)PCz(t) + 2z^T(t)P(A+\Gamma^T T^*)g(z(t-\tau(t))) \\ &\quad + 2z^T(t)PB\int_{-\infty}^{t} K(t-s)g(z(s))ds + 2z^T(t)PD\dot{z}(t-h(t)) \\ &\quad + 2z^T(t)CPC\int_{t-\sigma}^{t} z(u)du - 2\Big[\int_{t-\sigma}^{t} z(u)du\Big]^T CPD\dot{z}(t-h(t)) \\ &\quad - 2\Big[\int_{t-\sigma}^{t} z(u)du\Big]^T CP(A+\Gamma^T T^*)g(z(t-(\tau(t))) \\ &\quad - 2\Big[\int_{t-\sigma}^{t} z(u)du\Big]^T CPB\int_{-\infty}^{t} K(t-s)g(z(s))ds \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \dot{V}_2(t,z(t)) &= z^T(t)Q_3z(t) - z^T(t-\sigma)Q_3z(t-\sigma) + \dot{z}^T(t)Q_6\dot{z}(t) \\ &\quad - \dot{z}^T(t-h(t))Q_6\dot{z}(t-h(t))(1-\dot{h}(t)) \\ &\leq z^T(t)Q_3z(t) - z^T(t-\sigma)Q_3z(t-\sigma) + \dot{z}^T(t)Q_6\dot{z}(t) \\ &\quad - \dot{z}^T(t-h(t))Q_6\dot{z}(t-h(t))(1-h^*) \end{aligned} \tag{A.2}$$

It follows from Lemma 2.5 that

$$\begin{aligned} \dot{V}_3(t,z(t)) &= \sigma^2 z^T(t)Q_4z(t) - \sigma\int_{t-\sigma}^{t} z^T(u)Q_4z(u)du \\ &\leq \sigma^2 z^T(t)Q_4z(t) - \Big[\int_{t-\sigma}^{t} z(u)du\Big]^T Q_4\Big[\int_{t-\sigma}^{t} z(u)du\Big] \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \dot{V}_4(t,z(t)) &= \int_{t-\tau(t)}^{t} \begin{bmatrix} g(z(t-\tau(t))) \\ \dot{z}(s) \end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} \\ * & T_{22} \end{bmatrix} \begin{bmatrix} g(z(t-\tau(t))) \\ \dot{z}(s) \end{bmatrix} ds \\ &= \tau(t)z^T(t-\tau(t))T_{11}z(t-\tau(t)) + 2z^T(t)T_{12}^T z(t-\tau(t)) \\ &\quad - 2z^T(t-\tau(t))T_{12}^T z(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{z}^T(s)T_{22}\dot{z}(s)ds \\ &\leq z^T(t-\tau(t))[\tau T_{11} - 2T_{12}^T]z(t-\tau(t))] \\ &\quad + 2z^T(t)T_{12}^T z(t-\tau(t)) + \int_{t-\tau}^{t} \dot{z}^T(s)T_{22}\dot{z}(s)ds \end{aligned} \tag{A.4}$$

and

$$\dot{V}_5(t,z(t)) = \tau\dot{z}^T(t)T_{22}\dot{z}(t) - \int_{-\tau}^{0} \dot{z}^T(t+u)T_{22}\dot{z}(t+u)du$$
$$= \tau\dot{z}^T(t)T_{22}\dot{z}(t) - \int_{t-\tau}^{t} \dot{z}^T(s)T_{22}\dot{z}(s)ds \tag{A.5}$$

and

$$\begin{aligned}\dot{V}_6(t,z(t)) &= \sum_{j=1}^{n} q_j k_j \int_0^{\infty} k_j(u)\big(g_j^2(z_j(t)) - g_j^2(z_j(t-u))\big)du \\ &\le g^T(z(t))Q_5\kappa g(z(t)) - \sum_{j=1}^{n} q_j \int_0^{\infty} \kappa_j(u)du \\ &\times \int_0^{\infty} \kappa_j(u) g_j^2(z_j(t-u))du \\ &\le g^T(z(t))Q_5\kappa g(z(t)) - \sum_{j=1}^{n} q_j \left(\int_0^{\infty} k_j(u) g_j(z_j(t-u))du\right)^2 \\ &\le g^T(z(t))Q_5\kappa g(z(t)) \\ &- \Big(\int_{-\infty}^{t} K(t-s)g(z(s))ds\Big)^T Q_5 \Big(\int_{-\infty}^{t} K(t-s)g(z(s))ds\Big)\end{aligned} \tag{A.6}$$

In addition, we note that

$$\begin{aligned}0 &= 2\dot{z}^T(t)Q_1\Big(-\dot{z}(t)+\dot{z}(t)\Big) \\ &= -2\dot{z}^T(t)Q_1\dot{z}(t) - 2\dot{z}^T(t)Q_1Cz(t-\sigma) + 2\dot{z}^T(t)Q_1(A+\Gamma^T T^*)g(z(t-\tau(t))) \\ &+ 2\dot{z}^T(t)Q_1D\dot{z}(t-h(t)) + 2\dot{z}^T(t)Q_1B\int_{-\infty}^{t} K(t-s)g(z(s))ds\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}0 &= 2\dot{z}^T(t-h(t))D^TQ_2\Big(-D\dot{z}(t-h(t)) + D\dot{z}(t-h(t))\Big) \\ &= 2\dot{z}^T(t-h(t))D^TQ_2\Big(-D\dot{z}(t-h(t)) + \dot{z}(t) + Cz(t-\sigma) \\ &- (A+\Gamma^T T^*)g(z(t-\tau(t)) - B\int_{-\infty}^{t} K(t-s)g(z(s))ds\Big) \\ &= -2\dot{z}^T(t-h(t))D^TQ_2D\dot{z}(t-h(t)) + 2\dot{z}^T(t-h(t))D^TQ_2D\dot{z}(t) \\ &+ 2\dot{z}^T(t-h(t))D^TQ_2Cz(t-\sigma) \\ &- 2\dot{z}^T(t-h(t))D^TQ_2(A+\Gamma^T T^*)g(z(t-\tau(t))) \\ &- 2\dot{z}^T(t-h(t))D^TQ_2B\int_{-\infty}^{t} K(t-s)g(z(s))ds\end{aligned} \tag{A.8}$$

By using $(H1)$, the following inequality holds

$$\begin{aligned}0 \quad &\le \begin{bmatrix} z(t) \\ g(z(t)) \end{bmatrix}^T \begin{bmatrix} -U_1\Sigma_1 & U_1\Sigma_2 \\ * & -U_1 \end{bmatrix} \begin{bmatrix} z(t) \\ g(z(t)) \end{bmatrix} \\ + \quad & \begin{bmatrix} z(t-\tau(t)) \\ g(z(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} -U_2\Sigma_1 & U_2\Sigma_2 \\ * & -U_2 \end{bmatrix} \begin{bmatrix} z(t-\tau(t)) \\ g(z(t-\tau(t))) \end{bmatrix}\end{aligned} \tag{A.9}$$

for any $n \times n$ diagonal matrices $U_1 > 0$, $U_2 > 0$. By using (A.1)-(A.9), we finally get

$$\dot{V}(t,z(t)) \le \zeta^T(t,z(t))\bar{\Xi}\zeta(t,z(t)).$$

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