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Relational Quantum Mechanics and Probability

M. Trassinelli

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Abstract We present a discussion on the three postulates of Relational Quantum Mechanics (RQM) and the definition of probability within this framework. The first two RQM postulates are based on the information that can be extracted from interaction of different systems, and the third postulate defines the properties of the probability function. Here we demonstrate that from a rigorous definition of the conditional probability for the possible outcomes of different measurements, the third postulate is unnecessary and the Born's rule naturally emerges from the first two postulates by applying the Gleason's theorem. We demonstrate in addition that the probability function is uniquely defined for classical and quantum phenomena. The presence or not of interference terms is demonstrated to be related to the precise formulation of the conditional probability where distributive property on its arguments cannot be taken for granted. In the particular case of Young's slits experiment, the two possible argument formulations correspond to the possibility or not to determine the particle passage through a particular path.

Keywords quantum mechanics interpretation · relational quantum mechanics · information · conditional probability · Young's slits · distributive property · orthomodular lattice · yes/no experiment

1 Introduction

After more than 100 years from the first attempt to formulate Quantum Mechanics (QM) by postulates [1], its interpretation is still in discussion in the scientific community. The origin of this open issue partially lies in the fact that our process of understanding is strongly based on the ability to make analogies with something familiar to us. In the case of quantum phenomena,

this approach fails because in our daily environment these kinds of events are not evident. In the Young's slit experiment for example, where a massive particle or photons pass through the slits and form an interference pattern on a screen, we compare this phenomenon to common systems such as a ping-pong ball or sea waves. The non-compatibility and contradictions between these two analogies create the impression to deal with paradoxes producing a sense of discomfort. This difficulty to deal with quantum phenomena is well manifested by the famous phrase of R. Feynman who, in his sixth lecture *The character of physical law* at Cornell University in 1964 [2], provocatively stated "I think I can safely say that nobody understands quantum mechanics". Without going into the discussion of the meaning of *understanding*, different interpretations of quantum phenomena and different choices of QM postulates can be discussed. Several approaches have been formulated since the early days of Quantum Mechanics and have been amply discussed in the literature (see e.g. Ref. [3] and references therein). New approaches continue to be proposed also in recent years [4, 5, 6, 7, 8, 9, 10] demonstrating a strong interest in this subject still nowadays.

To choose among the different QM interpretations and postulates formulations, a pragmatic approach is to consider two important criteria (i) the simplicity of the theory, following the Ockham's razor philosophical principle¹, and (ii) an interpretation that uses as ingredients for the postulates formulation, the concepts that are most familiar to our common experience. These two criteria are, to the author's opinion, fully satisfied by the Relational Quantum Mechanics (RQM) approach formulated by Rovelli in 1996 [11]. In RQM framework, no system is privileged (there is no observable and observer, both are considered simply different systems) and everything depends on the reciprocal relationship through the mutual interaction, hence the name "Relational" in RQM. The aspect of the quantisation of nature is dictated in the first postulate by imposing a maximum limit on the information that can be extracted from a system (e.g. the spin of an atom in a Stern-Gerlach type experiment, discrete atomic spectra, etc.). The probabilistic aspect of the theory is introduced by the second postulate that states that, even when all possible information is available, new one can be extracted from a system (e.g. in measurements of magnetic spin particle with a series of Stern-Gerlach type apparatus or light and polarisers). These first two postulates are based on the concept of information, a concept familiar to contemporary human beings and then easier to accept with respect to other abstract entities. The third and last postulate, less intuitive, defines the way to compute probabilities, imposing, accordingly to experimental observations (Young's slits for ex.), the calculation via squares of amplitudes and sums of amplitudes (Born's rule).

In this article, after a presentation of the first two postulates of RQM and their connection with the logical structure of series of measurements, we will

¹ From the English Franciscan friar William of Ockham statement "*Non sunt multiplicanda entia sine necessitate*", "Entities must not be multiplied beyond necessity" from William of Ockham (1287-1347), which can be interpreted in a more modern form as "Among competing hypotheses, the one with the fewest assumptions should be selected".

demonstrate that the third RQM postulate is not necessary to reconstruct QM formalism. We will demonstrate in particular how to build a probability function and how the Born's rule emerges from the first two RQM postulates and the Gleason's theorem. To clearly present different aspects of this approach, we will consider the example of Young's slits experiment and we will calculate probabilities in cases of distinguishability or indistinguishability of the particle path through the slits pair. We will in particular demonstrate that the probability function is uniquely defined for both cases. The presence or not of interference terms is related to the precise formulation of the conditional probability where distributive property on its arguments cannot be taken for granted.

The article is organised as follows: in Sec. 2 we present the first two RQM postulates and their relation to the measurement processes and Hilbert spaces. In Sec. 3, we discuss how we can build a probability function and we will study its properties. Section 4 analyses the Young's slits experiment in the framework of the formalism developed. In Sec. 5 we will discuss on the interpretation and consequences. We will end the article with the sections of conclusions (Sec. 6) and an appendix for an introduction to orthomodular lattices and their connections with "yes/no" experiments.

2 Relational Quantum Mechanics postulates, measurements algebra and Hilbert space

2.1 Interacting systems and measurement processes

The basic assumption of RQM is that the world can be decomposed into a collection of systems, each of which can be equivalently considered as an observing system or as an observed system. The information is exchanged via physical interaction between systems. The process of acquisition of information can be described as a "question" that a system (observing system) asks another system (observed system). Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information by a series of "yes/no" experiment, i.e. a click or not of a specific measurement apparatus. An answer to a question correspond then to one bit of acquired information.

On another hand, a set of "yes/no" measurements on a same system can be seen as a set of propositions linked to each other by an ordered relationship " \succeq " [12, 13, 14, 15]. If i, j represent two possible measurement results, the relationship $i \succeq j$ indicates that each time we have a certain yes outcome for i also we have a certain yes outcome for j . The ensemble of propositions and the order relation between them form a structure called *partially ordered set (poset)* with some particular properties (orthocomplementation) that will be essential to define from it a Hilbert space.

It is not the intent of this article to make an introduction to poset and orthocomplemented lattices. Here we will present only their basic properties in

the relation with RQM postulates. For the readers that are not familiar to these structures, their properties and their connection to “yes/no” measurements of classical and quantum phenomena, a short introduction is presented in the appendix at the end of the article. An exhaustive presentation on this topic can be found in Refs. [14, 15].

2.2 First postulate of RQM and orthocomplemented lattices

The first postulate of RQM is [11]

Postulate 1 (Limited information): there is a maximum amount of relevant information that can be extracted from a system.

If there is a maximal amount of information that can be extracted from the system, it exists an ensemble of N' questions Q^i that completely describes the system. These questions can be represented by different experimental results: the detection of a particle with spin up or down or the measurement of a particle momentum within a value interval. If the information is limited, this means that exist a list of limited yes/no questions Q^i with $i = 1, \dots, N \leq N'$ independent from each other that describes completely the system. With the operations of conjunction “ \wedge ” (“AND”), disjunction “ \vee ” (“OR”) and negation “ \neg ” (“NOT”), we can in addition build another list of mutually exclusive questions Q_α^i where $Q_\alpha^i \wedge Q_\alpha^j = \emptyset$ for $i \neq j$ [11], where the Greek letter indicate belonging to the same complete set of questions and “ \emptyset ” defines an empty set. For each pair of questions, the combinations $Q_\alpha^i \wedge Q_\alpha^j$ and $Q_\alpha^i \vee Q_\alpha^j$ are univocally defined. Because we have a complete set of questions, we have that $\bigvee_i Q_\alpha^i = \mathbb{1}$, the ensemble of all possible positive answers, defines certainty (represented here by “ $\mathbb{1}$ ”).

For each question Q_α^i (e.g. “is the spin up?”) its positive answer correspond to a statement i (“the spin is up”). A negative answer corresponds to its complementary statement i^\perp (“the spin is not up”), which correspond to a positive answer to the question $\neg Q_\alpha^i$ opposite to Q_α^i .² The different possible answers $\{i, i^\perp\}$ together with the operations “ \wedge ”, “ \vee ” constitute a logical structure called *lattice*, indicated by \mathcal{L} . This lattice is orthocomplemented because for each element i (a positive answer), it exist is complement i^\perp (a negative answer to the same question, with the properties $i \vee i^\perp = \mathbb{1}$ and $i \wedge i^\perp = \emptyset$, where $\mathbb{1}, \emptyset$ are also part of the lattice \mathcal{L} (a complete lattice).

Another important property of the lattice defined by a complete series of mutually exclusive questions is the distributivity, which corresponds to the property

$$i \wedge (j \vee k) = (i \wedge j) \vee (i \wedge k), \quad i \vee (j \wedge k) = (i \vee j) \wedge (i \vee k) \quad (1)$$

for any triplet $i, j, k \in \mathcal{L}$. As we will see in the following sections where we will consider more than one set of complete questions, distributivity, or more

² In other words we have $\neg Q_\alpha^i = Q_\alpha^{i^\perp}$

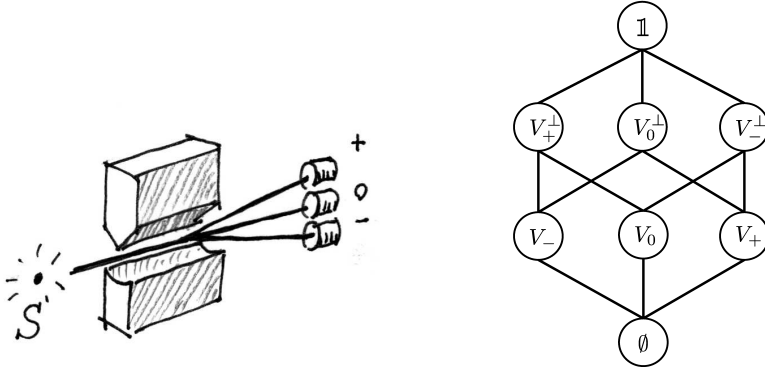


Fig. 1 Left: artistic scheme of a Stern-Gerlach experiment with vertically aligned magnets. Right: graph of the orthocomplemented lattice corresponding to the detection of the three emerging beams.

precisely its absence, will play an important role to calculate interference phenomena in the probability calculation. Note, a distributive orthocomplemented lattice is also called Boolean lattice.

A simple example of a complete set of questions can be built considering a spin-1 particle passing through a Stern-Gerlach apparatus with a defined axis orientation, vertical for example (see Fig. 1 left). Here the spin orientation of the particle is determined by the deflection of the particle itself passing through a non-homogeneous magnetic field. The complete set of question is constituted by a series of three particle detectors that correspond to the questions $\{Q_V\} = Q_V^+, Q_V^0$ and Q_V^- , one for each spin projection. The ensemble of the corresponding answers $\{V_+, V_0, V_-, V_+^\perp, V_0^\perp, V_-^\perp\}$ forms a orthocomplemented distributive (Boolean) lattice represented in Fig. 1 right (see Appendix for more details on lattices properties).

From the output of the three detectors we can extract only a limited amount of information. In opposite, in a classical case as the measurement of egg holes by different calibers (see Appendix), from the use of additional holes with different diameters, additional information can be always be extracted.³

2.3 Second postulate of RQM and orthomodular lattices

The second postulate of RQM is [11]

Postulate 2 (Unlimited questions): it is always possible to acquire new information about a system.

When a particle coming out from a Stern-Gerlach apparatus, vertically oriented for example, is injected into another Stern-Gerlach apparatus with a

³ If we do not consider the atomic structure of the eggs and calibers, the extractable information is in fact limitless.

different orientation, e.g. horizontal, the new output can be used to extract new information with a loss of old information because its total quantity is bounded. The new results $\{H_+, H_0, H_-, H_+^\perp, H_0^\perp, H_-^\perp\}$ correspond to a new set of questions $\{Q_H\} = Q_H^+, Q_H^0$ (which corresponding to a new context in the framework of the Context-System-Modality developed by Auffèves et Granger [9, 16]). As for the vertical case, the new possible outcomes alone have a mutual relation similar to the scheme in Fig. 1 (right). But when horizontal and vertical measurement apparatus are considered together, they form a lattice that is no more distributive, contrary to classical cases as a egg measured by two different sets of calibers (see Appendix).

The non-distributivity of the lattice is related to the fact that each set of questions is complete, but the sets are not compatible each other because each apparatus can extract an information I_{\max} and the maximal quantity of information that can be extracted is also $I = I_{\max}$. The absence of the distributive property has major consequences. It implies in fact that the order on which the questions/measurements are made matters and is related to the commutation of operators in standard QM.

As pointed in the early years of QM [12], proposition lattices defined by set of measurements of classical or quantum cases, have still in common an important property, *orthomodularity*, which is a weaker property than distributivity⁴ consisting in

$$i = j \vee (i \wedge j^\perp) \quad (2)$$

for each pair of elements $i, j \in \mathcal{L}$ with $i \succeq j$. In the next subsection we will see that this property is essential to build a Hilbert space.

A final comment to the second postulate is that it can be actually considered, as a lower bound on the information, in complement to the first postulate that determine the higher bound. Absolute certainty corresponds to a zero information gain. Postulate 2 of RQM assure that we can extract always information larger than zero.

2.4 Reconstruction of the Hilbert space

It can be demonstrated that if we have an orthomodular lattice, which is in addition atomic (that correspond essentially to the existence of pure states) and has covering properties (that correspond to the requirement that a pure state transforms in a pure state) an Hilbert space \mathcal{H} can be reconstructed [3, 14]. This is the case of the lattices we can built from yes/no measurements (see Appendix). The choice of the numbered field of the Hilbert space (reals, complex, quaternion, p-adic, etc.) is not yet defined. However, it can be additionally demonstrated that the complex case satisfies the basic requirements for describing quantum phenomena without redundancy [14].

⁴ Orthomodularity derives from the modularity property, $i \vee (j \wedge k) = (i \vee j) \wedge k$ if $k \succeq i$ a special case of distributivity, and the property of orthocompleteness [12, 14, 15]. All Boolean lattices, which are distributive, are automatically orthomodular.

Because a system can be described by a complete set of questions, each complete set $\{Q_\alpha^i\}$ can be associated to a basis $\{|i_\alpha\rangle\}$ of \mathcal{H} . Moreover, from any $|i_\alpha\rangle$ we can build a projector $P_\alpha^i = |i_\alpha\rangle\langle i_\alpha|$. Different sets of questions $\{|i_\alpha\rangle\}, \{|j_\beta\rangle\}$ correspond to different complete bases that are related to each other via unitary matrices U with $|j_\beta\rangle = U|i_\alpha\rangle$. The unitarity of U is guaranteed by the completeness and orthogonality of each basis element [17]. In this context, a pure state, $|i_\alpha\rangle$, correspond to a certain positive answer to a question of a complete set. A mixed state is the results of an uncertainty between answers of among questions.

3 Probability function and properties

3.1 Probability function construction

Taking into account two complete families of questions $\{Q_\alpha^i\}, \{Q_\beta^j\}$ and the second RQM postulate, we can consider the probability for obtaining a positive answer to the question Q_β^j after a positive answer to the question Q_α^i from subsequent interactions between the system we want to study and our system, on which the form of our question depends. This is a perfect example of conditional probability that can be indicated by $p[Q_\beta^j|Q_\alpha^i]$. To underline that we are dealing with measurements that corresponds to answers to questions, in the previous expression and in the following paragraphs we implicitly indicates by Q^i its corresponding positive answer.

Different approaches can be used to define the probability function p [14, 15, 18, 18, 19]. In the case of RMQ framework, the sets possible answers to the different sets of questions, can be considered as a set of propositions that can be true or false. In this general context, if we require consistency and that p is represented by a positive number, the probability can be defined by the following axioms [18, 20, 21]

$$0 \leq p[Q_\beta^j|Q_\alpha^i] \leq 1, \quad (3a)$$

$$p\left[\bigvee_j Q_\beta^j|Q_\alpha^i\right] = 1, \quad (3b)$$

$$p[Q_\beta^j \vee Q_\beta^k|Q_\alpha^i] = p[Q_\beta^j|Q_\alpha^i] + p[Q_\beta^k|Q_\alpha^i], \quad (3c)$$

for a defined set of questions Q_β^j knowing a positive question Q_α^i . The second axiom states that the probability of *certainty*, here represented by the conjunction of all questions $\bigvee_j Q_\beta^j \equiv \mathbb{1}$ of the the complete set $\{Q_\beta^j\}$, is equal to one, which is equivalent to the statement $p[Q_\beta^j|Q_\beta^j] = 1$. The last axiom is valid for mutually exclusive questions only, i.e. for questions with $Q_\beta^j \wedge Q_\beta^k = \emptyset$ corresponding to $p[Q_\beta^j \wedge Q_\beta^k|Q_\alpha^i] = 0$, which is exactly our case. A fourth axiom can be provided for conditional probabilities for specific circumstances, on which Bayes' theorem is based and on which we will discuss below. For a given answer of the question Q_α^i the probability function $p[\cdot|Q_\alpha^i]$ defined by

these three axioms is compatible with Kolmogorov's axiomatic definition [22], more commonly used for physical quantities. In particular, p respects the basic probability properties to be used in the Gleason's theorem [23], the implementation of which is discussed in the next section. Differently from Kolmogorov approach, we will see that the explicit dependency on the prior answers provides a clear use of p in the different contexts.

3.2 The Gleason's theorem

In the previous sections we saw that from the first and the second postulates of RQM we can build a Hilbert space. Considering the definition of the probability function with Eqs. (3), we can then apply the Gleason's theorem [23] reported here from an adaption of Refs. [3, 14]:

In a separable Hilbert space \mathcal{H} of dimension ≥ 3 , whether real, complex or quaternion, every probability measure (that respect Eqs. (3) n.a.) on a closed subspace \mathcal{H}_S can be written in the form

$$p[H_S] = \text{tr}[\rho P_S], \quad (4)$$

where P_S denotes the orthogonal projection on \mathcal{H}_S and ρ is a density operator.

In our notation, for a pure state, defined by a positive answer to a question Q_α^i for example, the probability for having a positive answer to the question Q_β^j is

$$p[Q_\beta^j | Q_\alpha^i] = \text{tr}[\rho_\alpha P_\beta^j] = |\langle j_\beta | i_\alpha \rangle|^2 = |U_{ij}|^2, \quad (5)$$

with

$$\rho_\alpha = |i_\alpha\rangle \langle i_\alpha|, \quad (6)$$

and where U is the unitary matrix allowing the transformation between the complete bases $\{|i_\alpha\rangle\}$ and $\{|j_\beta\rangle\}$ complete bases transformation. Thus, we see that Born's rule can be deduced solely from the first two postulates and from Gleason's theorem, without the need of any further assumption: a third postulate devoted to probabilities does not appear necessary. Furthermore, we have

$$p[Q_\beta^j | Q_\alpha^i] = p[Q_\alpha^i | Q_\beta^j], \quad (7)$$

a property that was only assumed as valid in the original formulation of RQM [11].

Considering the previous equation and Eqs. (3), we have in addition

$$\sum_i p[Q_\beta^j | Q_\alpha^i] = \sum_j p[Q_\beta^j | Q_\alpha^i] = p[\bigvee_j Q_\beta^j | Q_\alpha^i] = 1, \quad (8)$$

which is completely compatible with the unitary properties of U in Eq. (5) [11, 17].

3.3 Conditional probability and Bayes' theorem

Other non-trivial properties of the conditional probability function p can be discussed considering three sets of complete questions. If we have three subsequent specific questions $Q_\alpha^i, Q_\beta^j, Q_\gamma^k$ from three different sets of complete questions, we can consider the probability $p[Q_\gamma^k|Q_\beta^j \wedge Q_\alpha^i]$ for having a positive answer for Q_γ^k knowing a positive answer for Q_β^j *after* a positive answer for Q_α^i . To take into account the order of the questions we introduce the new non-symmetric relation operator “ \wedge ”, similar to the “AND” logical operator “ \wedge ” but where $Q^i \wedge Q^j \neq Q^j \wedge Q^i$. We underline the importance of the sequence of the questions because it is related to the non-commutativity of the operators P_α^i and P_β^j , i.e. $P_\alpha^i P_\beta^j \neq P_\beta^j P_\alpha^i$. As we will see, when the sequence of the questions is not important, the operator “ \wedge ” reduces to the conjugation operator “ \wedge ”.

From Gleason's theorem (Eq. (5)) and the properties of the density operator, and in particular Lüders' rule for conditional probabilities, we have for pure states

$$\begin{aligned} p[Q_\gamma^k|Q_\beta^j \wedge Q_\alpha^i] &= \frac{\text{tr}[P_\beta^j \rho_\alpha P_\beta^j P_\gamma^k]}{\text{tr}[\rho_\alpha P_\beta^j]} = \frac{\text{tr}[P_\beta^j P_\alpha^i P_\beta^j P_\gamma^k]}{\text{tr}[P_\alpha^i P_\beta^j]} = \\ &= \frac{\text{tr}[P_\beta^j P_\alpha^i P_\beta^j P_\gamma^k]}{p[Q_\alpha^i|Q_\beta^j]} = \frac{|\langle k_\gamma | P_\beta^j | i_\alpha \rangle|^2}{|\langle j_\beta | i_\alpha \rangle|^2}. \end{aligned} \quad (9)$$

Because of the trace properties $\text{tr}[MNL] = \text{tr}[NLM] = \text{tr}[LMN]$ (but $\neq \text{tr}[MLN]$), the nominator of Eq. (9) is completely symmetric with respect to the interchange of P_α^i and P_γ^k . The difference between $p[Q_\gamma^k|Q_\beta^j \wedge Q_\alpha^i]$ and $p[Q_\gamma^k \wedge Q_\beta^j|Q_\alpha^i]$ is only the denominator,

$$p[Q_\gamma^k \wedge Q_\beta^j|Q_\alpha^i] = \text{tr}[P_\beta^j P_\alpha^i P_\beta^j P_\gamma^k] = |\langle k_\gamma | P_\beta^j | i_\alpha \rangle|^2, \quad (10)$$

which is required for the probability normalisation. In other words, we have

$$p[Q_\gamma^k|Q_\beta^j \wedge Q_\alpha^i] = \frac{p[Q_\gamma^k \wedge Q_\beta^j|Q_\alpha^i]}{p[Q_\alpha^i|Q_\beta^j]}, \quad (11)$$

which is very similar to Bayes' theorem for conditional probability except for the presence of the operator “ \wedge ” instead of the conjunction “ \wedge ”.

If the projectors relative to questions commute⁵ additional simplifications can be applied [14]. In this case we have $P_\beta^j P_\gamma^k = P_\gamma^k P_\beta^j$ that, together with

⁵ This conditions is satisfied if the orthomodular lattice associated to the set of measurements is distributive.

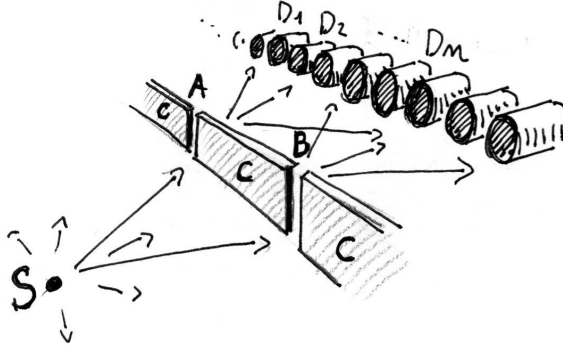


Fig. 2 Artistic scheme of the Young's slit experiment.

Eq. (9) gives

$$\begin{aligned}
 p[Q_\gamma^k | Q_\alpha^i \wedge Q_\beta^j] &= \frac{\text{tr}[P_\beta^j \rho_\alpha P_\beta^j P_\gamma^k]}{\text{tr}[\rho_\alpha P_\beta^j]} = \frac{\text{tr}[\rho_\alpha P_\beta^j P_\gamma^k]}{\text{tr}[\rho_\alpha P_\beta^j]} = \\
 &= \frac{p[Q_\gamma^k \wedge Q_\beta^j | Q_\alpha^i]}{p[Q_\beta^j | Q_\alpha^i]} \equiv p[Q_\gamma^k | Q_\alpha^i \wedge Q_\beta^j], \quad (12)
 \end{aligned}$$

which is nothing else the Bayes' theorem with “ \wedge ” equivalent to “ \cap ”. We note that the demonstration of the Bayes' theorem provided by Cox and Jaynes [21, 24] is based on the assumption that the conditional probability $p[Q_\gamma^k \wedge Q_\beta^j | Q_\alpha^i]$ can be written as function of $p[Q_\gamma^k | Q_\alpha^i \wedge Q_\beta^j]$ and $p[Q_\beta^j | Q_\alpha^i]$ [18, 21, 24]. This is clearly not true for the general case described by Eq. (9) where the conjunction ‘ \wedge ’ between non-compatible questions is not defined [20]. In the other way around, we note also that Bayes' theorem can be easily derived using the distributive property, as demonstrated by D'Agostini [25] (page 61), which underlines once more the important role of distributivity.

4 An example: the Young's slits

To see the implication of the previous formulas and their interpretation, we consider the classical example of the a ‘Young's slits’-like experiment, extensively considered in the literature (see e.g. [14, 15, 26]).

We consider here the ideal experiment represented in Fig. 2 where a particle is emitted by a source S and detected by a series of detectors D_0, \dots, D_N after passing through a pair of slits, A and B . We assume that the walls of the slits themselves, indicated by C , can detect the arrival of particles.

The initial state, i.e. the initial question $Q_\alpha^i = Q_\alpha^S$, corresponds to the emission or not of a particle from the source S . The detection in proximity of

the slits correspond to the questions Q_β^j with $j = \{A, B, C\}$ for the passage through the slits A, B or the hitting of the slits walls C . The final set of questions corresponds to the detection on the the different detectors on the screen with Q_γ^k with $k = \{D_0, \dots, D_N\}$. Each set of questions $\{Q_\beta^A, Q_\beta^B, Q_\beta^C\}$ and $\{Q_\gamma^{D_0}, \dots, Q_\gamma^{D_N}\}$ are complete and corresponds to complete bases of Hilbert space $\{|A_\beta\rangle, |B_\beta\rangle, |C_\beta\rangle\}$ and $\{|D_0\rangle_\gamma, \dots, |D_N\rangle_\gamma\}$.

We consider two different cases. In the first one we consider the possibility to distinguish the passage of a particle through A or B (for ex. following the trajectory of the particle if the particle is a ball or another macroscopic object) before being detected on the screen. In the second case, we consider that only the information from the slits wall C is available and we cannot access to the information of the passage through A or B . In both case we will study the detection from one particular detector only $D = D_n \in \{D_0, \dots, D_N\}$.

In the first case, because we can distinguish a path between A or B , we are considering a detection that correspond to the question structure $(Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B)$: we want to determine the probability to detect a particle from D passed through A or to detect a particle from D passed through B . We are only interested to the detection in D (from here the operator “ \vee ”) but we could in principle detect from which slit the particle passed (from here the operator “ \wedge ”). The probability we are interested is then $p[(Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B) | Q_\alpha^S]$ where $Q_\alpha^S = \text{TRUE}$ correspond to a particle emission from the source. Detection of a passage through A or B correspond to mutually exclusive questions $Q_\beta^A \wedge Q_\beta^B = \emptyset$. Any combination with these questions, $(Q_\gamma^D \wedge Q_\beta^A)$ and $(Q_\gamma^D \wedge Q_\beta^B)$ in our case, is also mutually exclusive as well.⁶ Applying Eq. (3c), we have then

$$\begin{aligned} p[(Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B) | Q_\alpha^S] &= p[Q_\gamma^D \wedge Q_\beta^A | Q_\alpha^S] + p[Q_\gamma^D \wedge Q_\beta^B | Q_\alpha^S] = \\ &= |\langle D_\gamma | P_\beta^A | S_\alpha \rangle|^2 + |\langle D_\gamma | P_\beta^B | S_\alpha \rangle|^2 = \\ &= |\langle D_\gamma | A_\beta \rangle \langle A_\beta | S_\alpha \rangle|^2 + |\langle D_\gamma | B_\beta \rangle \langle B_\beta | S_\alpha \rangle|^2. \end{aligned} \quad (13)$$

The final probability is the sum of the individual probabilities (composed each by the square product of different terms) to pass through the slit A and the slit B .

In the second case we consider that we cannot distinguish in principle the passage through one specific slit. The non-detection on the walls C after an emission from S corresponds however to the certainty of the passage of a particle through A or B . Symbolically it means that $\neg Q_\beta^C \equiv Q_\beta^{C^\perp} \equiv Q_\beta^A \vee Q_\beta^B$. In this case, using Eq. (10), the probability to detect a particle from D is

$$p[Q_\gamma^D \wedge (Q_\beta^A \vee Q_\beta^B) | Q_\alpha^S] = |\langle D_\gamma | P_\beta^{A \vee B} | S_\alpha \rangle|^2 \quad (14)$$

⁶ Because we can distinguish the trajectories, each pair $(Q_\gamma^D \wedge Q_\beta^A)$ and $(Q_\gamma^D \wedge Q_\beta^B)$ can be considered related to two distinct detectors to which correspond the questions Q^{DA} and Q^{DB} with $Q^{DA} \wedge Q^{DB} = \emptyset$.

with $P_\beta^{A \vee B}$ the projector operator corresponding to the positive answer to the question “*Is the particle passed through A or B?*”.

Considering that $\{|j_\beta\rangle\}$ basis is orthonormal (formed by mutual exclusive questions), we have that $\langle j_\beta | k_\beta \rangle = P_\beta^j P_\beta^k = 0$ for $j \neq k$. Then [27]

$$P_\beta^{A \vee B} = I - P_\beta^{A^\perp \wedge B^\perp} = I - (I - P_\beta^A)(I - P_\beta^B) = P_\beta^A + P_\beta^B. \quad (15)$$

We have then that the final probability is

$$\begin{aligned} p[Q_\gamma^D \wedge (Q_\beta^A \vee Q_\beta^B) | Q_\alpha^S] &= |\langle D_\gamma | (P_\beta^A + P_\beta^B) | S_\alpha \rangle|^2 = \\ &= |\langle D_\gamma | A_\beta \rangle \langle A_\beta | S_\alpha \rangle + \langle D_\gamma | B_\beta \rangle \langle B_\beta | S_\alpha \rangle|^2. \end{aligned} \quad (16)$$

Compared to the discernible case (Eq. (13)), we have the square of the sum of two terms with the possibility of interference terms. We have then the well known property of probability calculation of QM without introducing additional postulates or axioms.⁷

5 Discussions

As presented in the previous section, the interference effect on the final probability by one of the detectors D_0, D_1, \dots, D_N is directly related to the possibility to distinguish from where the particle passed, i.e. the possibility or not to access to the information relative to the passage through a particular slit. Because of the first RQM postulate, this information can be in fact intrinsically limited. The possibility to distinguish or not the particle path correspond to different conditional probability expressions, Eqs. (13) and (16) respectively. Their equivalence cannot be assumed a priori in the slits experiment example because the distributive property, habitually assumed as valid as in common (classical) cases, is not justified and can be violated, with in our specific case

$$Q_\gamma^D \wedge (Q_\beta^A \vee Q_\beta^B) \neq (Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B). \quad (17)$$

This property is in fact non-valid in question/experiment lattices involving a limited amount of information and it cannot be assumed by default, as presented in a specific example in the Appendix. If distributive property is not given for granted, we have to distinguish the different cases as in the example of Young’s slits experiment. Different cases correspond to different expressions of the probability function p , which is uniquely defined by Eqs. (3). What it changes is the proposition combination (question combination) for which we are calculating the probability.

Because of the contradiction with daily experience phenomena where the distributive property is valid, its abandon is, to the opinion of the author, the largest intellectual difficulty in this framework to reconstruct QM formalism

⁷ An alternative demonstration of the derivation of the superposition property can be found in Ref. [14] (Section 26.3).

from RQM postulates. The other advantage of RQM formulation is the possibility to transpose the series of questions/measurements to a set of propositions that can be studied then in the context of general logic. In the case of the slits experiment, once a particle has been emitted and detected, the sentence “*The particle passed through the slit A OR the particle passed through the slit B*” is always true. In opposite the sentence “*The particle passed through both slits A and B*” is always *false* because questions is always false. Q_β^A and Q_β^B are in fact compatible and mutually exclusive and $Q_\beta^A \wedge Q_\beta^B = \emptyset$ ($P_\beta^A P_\beta^B = 0$). Finally, the sentences “*The particle passed through the slit A*” and “*The particle passed through the slit B*” are completely indeterminate if we do not measure where the particle passed through. This is similar to an analog case discussed by Aristotle (reported in Ref. [28]) well before the formulation of quantum mechanics. The sentences “*Tomorrow there will be a sea-battle*” and its negation “*Tomorrow there will not be a sea-battle*” have today no definite truth-value (they are indeterminate), but their disjunction “*Either tomorrow there will be a sea-battle or tomorrow there will not be a sea-battle*” is true. As discussed in Refs. [3, 28], this structure can be related to the Łukasiewicz three-valued logic approach [29] where in addition to the *true* and *false* logical values for propositions, an *indeterminate* or *unknown* third value is considered.

6 Conclusions

In conclusion, after a critical analysis and discussion of the postulates of Relational Quantum Mechanics, we demonstrate that the third one is unnecessary. From the first two postulates and the logical structure of “yes/no” measurements, the Hilbert space can be in fact reconstructed, and a rigorous definition of the conditional probability for the possible outcomes of different measurements can be built. The Born’s rule and the more general rule with the density operator can be then derived from the Hilbert space structure applying Gleason’s theorem. The presence or not of interference terms, which characterises Born’s rule, is demonstrated to be related to the precise formulation of the conditional probability where distributive property cannot be taken for granted.

In the specific case of Young’s slits, this is translated to a difference between the probability *to detect a particle that passed through the pair of slits A and B* ($Q_\gamma^D \wedge (Q_\beta^A \vee Q_\beta^B)$) and the probability *to detect a particle that passed through slit A OR to detect a particle that passed through slit B* ($(Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B)$). Even if the two propositions are apparently equivalent in common language, there is a important difference linked to the possibility or not to determine the particle passage through a particular slit bringing to the disequality $Q_\gamma^D \wedge (Q_\beta^A \vee Q_\beta^B) \neq (Q_\gamma^D \wedge Q_\beta^A) \vee (Q_\gamma^D \wedge Q_\beta^B)$. The non-validity of distributive property in the logical structure of “yes/no” measurements derives from the boundaries of the information that can be extracted from a system and from the possibility to always obtain new information. This is related to the well know Quantum Mechanics structure with compatible or non-compatible measurements (operators) and their commutativity.

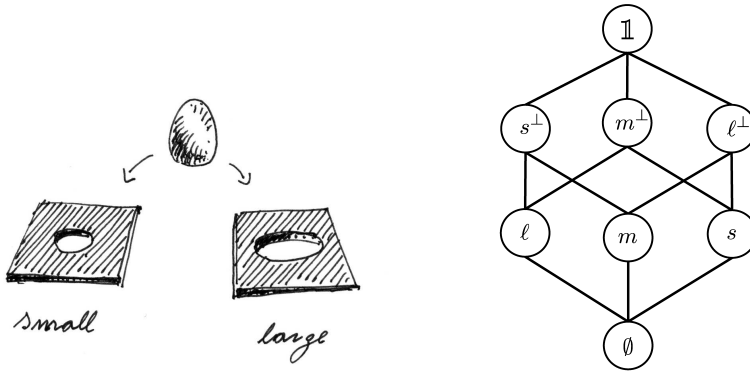


Fig. 3 Left: artistic scheme of an egg size measurement with two calibers. Right: graph of the orthocomplemented lattice corresponding to the possible outcomes on the egg size.

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Appendix A: Yes/No measurements and orthomodular lattices

A1 A simple case of distributive and orthocomplemented lattice

To present orthomodular lattices, their properties and their link to a set of yes/no experiments, we will consider two simple cases: a classic case relative to the measurement of the size of eggs using two calibers (an example introduced originally by Piron [27]), and a case of measurement of a spin-1 particle with a Stern-Gerlach apparatus.

In the case of the egg measurement, the questions $\{Q_\alpha\}$ associated to these measurements are composed by $Q^s = \text{“is the egg small?”}$, which positive answer correspond to an egg passing through the small hole with diameter d_s , and by $Q^\ell = \text{“is the egg large?”}$ corresponding to an egg not passing through the large hole with diameter $d_\ell > d_s$ (see Fig. 3 left). From the combination of these two questions we can deduce three sizes: “ s ” *small* (Q^s), “ l ” *large* (Q^ℓ) and “ m ” *medium* from the combination $Q^m \equiv \neg Q^s \wedge \neg Q^\ell$.

The different answers (positive or negative) can be interpreted as propositions that can be true or false and are related to each other. As example, if we have a positive answer to the question Q^s , this corresponds to the proposition “*the egg is small*” $\equiv s$. If the we have a negative answer (positive to its negation $\neg Q^s$), this corresponds to the proposition “*the egg is not small*” $\equiv s^\perp$, where “ \perp ” indicates complementation (orthogonality) of negative answers with respect to positive answers. In addition to the complementation “ \perp ” we can

introduces the operations of conjunction “ \wedge ” (“AND”), disjunction “ \vee ” (“OR”) between the propositions $s, m, \ell, s^\perp, m^\perp, \ell^\perp$.

The different answers/propositions and their relation constitute a logical structure of a *lattice*. As example, a negative answer to the question Q^ℓ , i.e. $\neg Q^\ell$, implies a positive answer to Q^m or Q^s . In this specific example, we have $\ell^\perp \implies m \vee s$ (“not large implies medium or small”). This means that for every egg measurement result giving as answers “*the egg is medium*” or “*the egg is small*”, we are sure that the proposition “*the egg is not large*” is true. This order relation (related to the logical implication \implies) can be indicated by “ \succeq ”. In the specific example considered, we have that $\ell^\perp \succeq m$, $\ell^\perp \succeq s$ and $\ell^\perp \succeq m \vee s$. This order relation is represented by a graph (Fig. 3 right) by descending lines and where the elements $i \wedge j$ and $i \vee j$ can be determined joining the two lower or upper lines, respectively, generating from the elements i and j .

Due to the question/answer basic structure, each element i of the partially ordered set so defined has a complement i^\perp with the properties $i \vee i^\perp = \mathbb{1}$ and $i \wedge i^\perp = \emptyset$. $\mathbb{1}, \emptyset$ are also part of the set with $\mathbb{1}$ corresponding to all type of eggs and \emptyset to no one. Because of these properties, our partially ordered set is actually a orthocomplemented lattice⁸ \mathcal{L} . Another important property of the algebra of the lattice defined by this series of measurement is the distributivity (Eq. (1)).

Another simple example is the case of a spin-1 particle passing through a Stern-Gerlach apparatus discussed in Sec. 2.2. The complete set of the answers to the questions $\{Q_{\bar{V}}\}$ has the same structure as the eggs measurement.

A2 A more complex case; orthomodularity and distributivity

In the classic case of the egg-size determination, a new set of questions $\{Q_\beta\}$ corresponds to a new set of calibers with different diameters. We can consider for example, slightly larger holes d_S, d_L with $d_s < d_S < d_\ell < d_L$ that correspond to Q^S and Q^L questions and that generate a new set of possible five measurable sizes.⁹ The corresponding lattice is represented in Fig. 4 (left). It is more complex but still distributive.¹⁰ As discussed by Piron [27, 30] and Beltrametti and Cassinelli [14], this reflects the fact that the two set of questions (two pairs of calibers) are compatible with each other and that the final measurement output does not depend on the order on which the two pairs of calibers are implemented.

⁸ The considered poset is a lattice because for any elements i, j , we can define the element combinations $i \wedge j$ and $i \vee j$ are part also of the considered ensemble of propositions (answers). It is a orthocomplemented lattice because for any element i , there is an element i^\perp with the properties $i \vee i^\perp = \mathbb{1}$ and $i \wedge i^\perp = \emptyset$ and $(i^\perp)^\perp = i$ and $i \succeq j \implies j^\perp \succeq i^\perp$.

⁹ As in the previous example, the measurement “*m*” medium come from the combination $Q^m \equiv \neg Q^s \wedge \neg Q^\ell$, with the set of questions Q^s, Q^S, Q^ℓ, Q^L different sizes can be defined as e.g. “*not very small*” corresponding to $\neg Q^s \wedge \neg Q^S$ the intermediate size between the two smaller diameters (the egg pass through the S hole but not the s one).

¹⁰ Similar example of such a lattice can be found in Ref. [15].

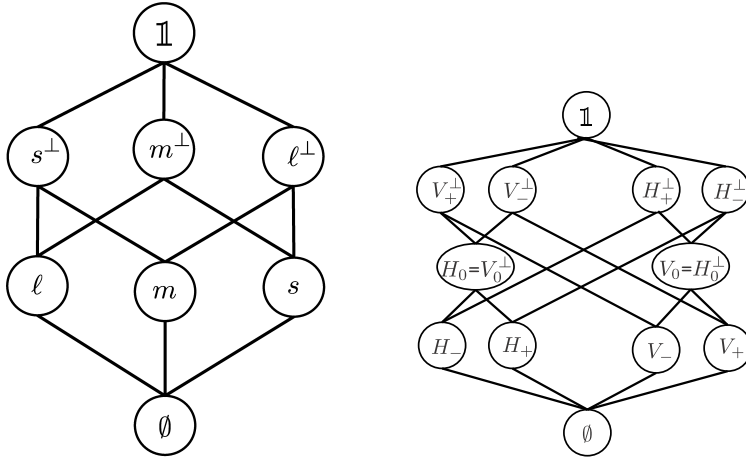


Fig. 4 Graphs of the orthocomplemented lattice corresponding to the measurement of egg size with four different calibers (left) and to the measurement with two Stern-Gerlach apparatus, one vertical and one horizontal.

In the case of the spin-1 particle, in addition to a vertically Stern-Gerlach apparatus, we can consider a horizontally oriented one. This new measurement apparatus is related to a new set of questions $\{Q_H\} = \{Q_H^+, Q_H^0, Q_H^-\}$ that corresponds to new possible measurement results $\{H_+, H_0, H_-, H_+^\perp, H_0^\perp, H_-^\perp\}$. When the two sets of measurements are considered together, they form a non-trivial but relatively simple lattice represented in Fig. 4 (right). The perpendicularity of the two apparatus axes determines the fact that a particle giving the measurement result H_0 in the horizontal apparatus (V_0) corresponds to a measurement $V_+ \vee V_-$ in the vertical apparatus ($H_+ \vee H_-$ in the horizontal one). The most important characteristic is that the lattice is no more distributive. For example, if we consider the possible outcomes H_+, V_-, V_-^\perp , we have that

$$H_+ \vee (V_- \wedge V_-^\perp) \neq (H_+ \vee V_-) \wedge (H_+ \vee V_-^\perp). \quad (18)$$

In fact¹¹ the left term is equal to H_+ and the right term is equal to V_-^\perp .

Additional examples of non-distributive orthomodular lattices can be found in the literature, as the lattice corresponding to the measurement of polarised light [27] or a quantised vector in the real space [15].

As discussed by [27], the distributive property is related to the logical structure of the orthomodular lattice \mathcal{L} corresponding to the ensemble of the measurements. For two elements (yes/no measurements) $i, j \in \mathcal{L}$, we have that

$$\text{"}i \text{ true"} \text{ or "}j \text{ true"} \implies \text{"}i \vee j \text{ true"} \text{"}, \quad (19)$$

but

$$\text{"}i \vee j \text{ true"} \text{"} \implies \text{"}i \text{ true"} \text{ or "}j \text{ true"} \text{"}, \quad (20)$$

¹¹ In the right term, $V_- \wedge V_-^\perp = \emptyset$ because of the complementarity and then $H_+ \vee \emptyset = H_+$. In the left term $H_+ \vee V_- = \mathbb{1}$ (there is no other common element) and $H_+ \vee V_-^\perp = V_-^\perp$ because of the lattice hierarchy.

is valid only if the lattice \mathcal{L} is also distributive, which is not the case for measurements reaching the limit of the extractable information, i.e, the quantisation aspects [12, 13, 27] (a simple presentation of this aspect in a Hilbert space can be found in [28, p. 37] and in [15, p. 190]).

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