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Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation

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Abstract

In this paper, we study the exponential stabilization of a shock steady state for the inviscid Burgers equation on a bounded interval. Our analysis relies on the construction of an explicit strict control Lyapunov function. We prove that by appropriately choosing the feedback boundary conditions, we can stabilize the state as well as the shock location to the desired steady state in H^2 -norm, with an arbitrary decay rate.

1 Introduction

The problem of asymptotic stabilization for hyperbolic systems using boundary feedback control has been studied for a long time. We refer to the pioneer work due to Rauch and Taylor [31] and Russell [32] for linear coupled hyperbolic systems. The first important result of asymptotic stability concerning quasilinear hyperbolic equations was obtained by Slemrod [34] and Greenberg and Li [15]. These two works dealt with local dissipative boundary conditions. The result was established by using the method of characteristics, which allows to estimate the related bounds along the characteristic curves in the framework of C^1 solutions. Another approach to analyze the dissipative boundary conditions is based on the use of Lyapunov functions. Especially, Coron, Bastin and Andrea-Novel [11] used this method to study the asymptotic behavior of the nonlinear hyperbolic equations in the framework of H^2 solutions. In particular, the Lyapunov function they constructed is an extension of the entropy and can be made strictly negative definite by properly choosing the boundary conditions. This method has been later on widely used for hyperbolic conservation laws in the framework of C^1 solutions [9, 16, 17] or H^2 solutions [2, 4, 5, 8, 10, 13, 18] (see [3] for an overview of this method).

But all of these results concerning the asymptotic stability of nonlinear hyperbolic equations focus on the convergence to regular solutions, i.e., on the stabilization of regular solutions to a desired regular steady state. It is well known, however, that for quasilinear hyperbolic partial differential equations, solutions may break down in finite time when their first derivatives break up even if the initial condition is smooth [24]. They give rise to the phenomena of shock waves with numerous important applications in physics and fluid mechanics. Compared to classical case, very few results exist on the stabilization of less regular solutions. Among which, we refer to [6, 30] in the scalar case and [7] for a general hyperbolic system of conservation laws. In [6, 7, 30], by acting suitably on both sides of the interval, one can steer asymptotically any initial data with sufficiently small total variations to any close constant steady states. Recently, it was shown in [12] the existence and exponential convergence of an entropy BV solution to the null-steady-state with small BV initial conditions. To our knowledge, no results exist concerning the boundary stabilization of steady states with jump discontinuities for nonlinear hyperbolic systems.

Hyperbolic systems have a wide application in fluid dynamics, and hydraulic jump is one of the best known examples of shock waves as it is frequently observed in open channel flow such as rivers and spillways. In the literature, Burgers equation often appears as a simplification of the dynamical model of flows. Burgers turbulence has been investigated both analytically and numerically by many

authors either as a preliminary approach to turbulence prior to an occurrence of the Navier-Stokes turbulence or for its own sake since the Burgers equation describes the formation and decay of weak shock waves in a compressible fluid [21, 27, 36]. From a mathematical point of view, it turns out that the study of Burgers equation leads to many of the ideas that arise in the field of nonlinear hyperbolic equations. It is therefore a natural first step to develop methods for the control of this equation. For the boundary stabilization problem of Burgers equation, we refer to works by Krstic et al. [23, 35] for the stabilization of regular steady states and [22] for the stabilization of null-steady-state.

In this paper, we study the exponential asymptotic stability of a shock steady state of the Burgers equation. To this end, we construct an explicit Lyapunov function with a strict negative definite time derivative by properly choosing the boundary conditions. The first problem is to deal with the well-posedness of the corresponding initial boundary value problem (IBVP) on a bounded domain. The existence of the weak solution to the initial value problem (IVP) of Burgers equation was first studied by Hopf by using vanishing viscosity [19]. The uniqueness of the entropy solution was then studied by Oleinik [29]. One can refer to [24] for a comprehensive study of the well-posedness of hyperbolic conservation laws in piecewise continuous entropy solution case and also to [14] in the class of entropy BV functions. Although there are many results for the well-posedness of the (IVP) for hyperbolic conservation laws, the problem of (IBVP) is less studied due to the difficulty of handling the boundary condition. In [1], the authors studied (IBVP) but in the quarter plane, i.e., $x > 0, t > 0$. By requiring that the boundary condition at $x = 0$ is satisfied in a weak sense, they can apply the method introduced by LeFloch [25] and obtain the explicit formula of the solution. However, our case is more complicated since we consider the Burgers equation defined on a bounded interval.

The organization of the paper is the following. In Section 2, we formulate the problem and state our main results. In Section 3, we prove the well-posedness of the Burgers equation in the framework of piecewise continuously differentiable entropy solutions, which is one of the main results in this paper. Based on this well-posedness result, we then prove in Section 4 by a Lyapunov approach that for appropriately chosen boundary conditions, we can achieve the exponential stability in H^2 -norm of a shock steady state with any given arbitrary decay rate and with an exact exponential stabilization of the desired shock location.

2 Problem statement and main result

We consider the following nonlinear inviscid Burgers equation on a bounded domain

$$y_t(t, x) + y(t, x)y_x(t, x) = 0 \quad (1)$$

with initial condition

$$y(0, x) = y_0(x), \quad x \in (0, L), \quad (2)$$

and boundary controls

$$y(t, 0^+) = u_0(t), \quad y(t, L^-) = u_L(t). \quad (3)$$

In this article, we will be exclusively concerned with the case where the controls $u_0(t) > 0$, $u_L(t) < 0$ have opposite signs and the state $y(t, \cdot)$ at each time t has a jump discontinuity as illustrated in Figure 1. The discontinuity is a shock wave that occurs at position $x_s(t) \in (0, L)$. According to the Rankine-Hugoniot condition, the shock wave moves with the speed

$$\dot{x}_s(t) = \frac{y(t, x_s(t)^+) + y(t, x_s(t)^-)}{2} \quad (4)$$

which satisfies the Lax entropy condition [24]

$$y(t, x_s(t)^+) < \dot{x}_s(t) < y(t, x_s(t)^-), \quad (5)$$

together with the initial condition

$$x_s(0) = x_{s0}. \quad (6)$$

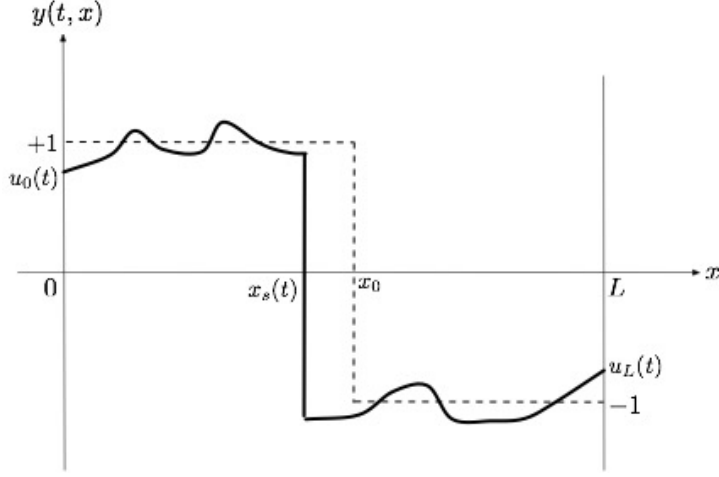


Figure 1: Entropy solution to the Burgers equation with a shock wave.

Under a constant control $u_0(t) = -u_L(t) = 1$ for all t , for any $x_0 \in (0, L)$, the system (1), (3), (4) has a steady state (y^*, x_s^*) defined as follows:

$$y^*(x) = \begin{cases} 1, & x \in [0, x_0), \\ -1, & x \in (x_0, L], \end{cases} \quad (7)$$

$$x_s^* = x_0.$$

These equilibria are clearly not isolated and, consequently, not asymptotically stable. It is therefore relevant to study the boundary feedback stabilization of the control system (1), (3), (4).

In this paper, our main contribution is precisely to show how we can exponentially stabilize any of the steady states defined by (7) with boundary feedback controls of the following form:

$$\begin{aligned} u_0(t) &= k_1 y(t, x_s(t)^-) + (1 - k_1) + b_1(x_0 - x_s(t)), \\ u_L(t) &= k_2 y(t, x_s(t)^+) - (1 - k_2) + b_2(x_0 - x_s(t)). \end{aligned} \quad (8)$$

Here, it is important to emphasize that, with these controls, we are able not only to guarantee the exponential convergence of the solution $y(t, x)$ to the steady state y^* but also to exponentially stabilize the location of the shock discontinuity at the exact desired position x_0 .

Before addressing the exponential stability issue, we first show that there exists a unique piecewise continuously differentiable entropy solution with $x_s(t)$ as its single shock for system (1)–(4), (6), (8) provided that y_0 and x_{s0} are in a small neighborhood of y^* and x_0 respectively.

For any given initial condition (2) and (6), we define the following first order compatibility conditions

$$\begin{aligned} y_0(0^+) &= k_1 y_0(x_{s0}^-) + (1 - k_1) + b_1(x_0 - x_{s0}), \\ y_0(L^-) &= k_2 y_0(x_{s0}^+) - (1 - k_2) + b_2(x_0 - x_{s0}), \end{aligned} \quad (9)$$

and

$$\begin{aligned} y_0(0^+)y_{0x}(0^+) &= k_1 y_0(x_{s0}^-)y_{0x}(x_{s0}^-) - k_1 y_{0x}(x_{s0}^-) \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2} + b_1 \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2}, \\ y_0(L^-)y_{0x}(L^-) &= k_2 y_0(x_{s0}^+)y_{0x}(x_{s0}^+) - k_2 y_{0x}(x_{s0}^+) \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2} + b_2 \frac{y_0(x_{s0}^-) + y_0(x_{s0}^+)}{2}. \end{aligned} \quad (10)$$

The first result of this paper deals with the well-posedness of system (1)–(4), (6), (8) and is stated in the following theorem.

Theorem 2.1. For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_{s0} \in (0, L)$ and $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$ satisfying the compatibility conditions (9)–(10) and

$$\begin{aligned} |y_0 - 1|_{H^2((0, x_{s0}); \mathbb{R})} + |y_0 + 1|_{H^2((x_{s0}, L); \mathbb{R})} &\leq \delta, \\ |x_{s0} - x_0| &\leq \delta, \end{aligned} \quad (11)$$

the system (1)–(4), (6), (8) has a unique piecewise continuously differentiable entropy solution $y \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R})) \cap H^2((x_s(t), L); \mathbb{R}))$ with $x_s \in C^1([0, T]; \mathbb{R})$ as its single shock. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0, T]$

$$\begin{aligned} |y(t, \cdot) - 1|_{H^2((0, x_s(t)); \mathbb{R})} + |y(t, \cdot) + 1|_{H^2((x_s(t), L); \mathbb{R})} + |x_s(t) - x_0| \\ \leq C(T) (|y_0 - 1|_{H^2((0, x_{s0}); \mathbb{R})} + |y_0 + 1|_{H^2((x_{s0}, L); \mathbb{R})} + |x_{s0} - x_0|). \end{aligned} \quad (12)$$

The proof of this result is given in Section 3.

Our next result deals with the exponential stability of the steady state (7) for the H^2 -norm according to the following definition.

Definition 2.1. The steady state $(y^*, x_0) \in (H^2((0, x_0); \mathbb{R}) \cap H^2((x_0, L); \mathbb{R})) \times (0, L)$ of the system (1), (3), (4), (8) is exponentially stable for the H^2 -norm with decay rate γ , if there exists $\delta > 0$ and $C > 0$ such that for any $T > 0$ and for any $y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R})$ and $x_{s0} \in (0, L)$ satisfying

$$\begin{aligned} |y_0 - y_1^*(0, \cdot)|_{H^2((0, x_{s0}); \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2((x_{s0}, L); \mathbb{R})} &\leq \delta, \\ |x_{s0} - x_0| &\leq \delta \end{aligned} \quad (13)$$

and the compatibility conditions (9)–(10), the system (1)–(4), (6), (8) has a unique solution $(y, x_s) \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}) \cap H^2((x_s(t), L); \mathbb{R})) \times C^1([0, T]; \mathbb{R})$ and

$$\begin{aligned} |y(t, \cdot) - y_1^*(t, \cdot)|_{H^2((0, x_s(t)); \mathbb{R})} + |y(t, \cdot) - y_2^*(t, \cdot)|_{H^2((x_s(t), L); \mathbb{R})} + |x_s(t) - x_0| \\ \leq C e^{-\gamma t} (|y_0 - y_1^*(0, \cdot)|_{H^2((0, x_{s0}); \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2((x_{s0}, L); \mathbb{R})} + |x_{s0} - x_0|), \quad \forall t \in [0, T]. \end{aligned} \quad (14)$$

In (13) and (14),

$$\begin{aligned} y_1^*(t, x) &= y^* \left(x \frac{x_0}{x_s(t)} \right), \\ y_2^*(t, x) &= y^* \left(\frac{(x - L)x_0}{x_s(t) - L} \right). \end{aligned} \quad (15)$$

Remark 1. At first glance it could seem peculiar to define y_1^* and y_2^* and to compare $y(t, \cdot)$ with these functions. However as the shock may be moving around x_0 , such change of variables are needed to compare the solution y with the steady state on the same space interval.

We can now state the main result of this paper

Theorem 2.2. Let $\gamma > 0$. If the following conditions hold:

$$b_1 \in \left(\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}} \right), \quad b_2 \in \left(\gamma e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}} \right), \quad (16a)$$

$$k_1^2 < e^{-\gamma x_0} \left(1 - \frac{b_1}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \right), \quad (16b)$$

$$k_2^2 < e^{-\gamma(L-x_0)} \left(1 - \frac{b_2}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \right), \quad (16c)$$

then the steady state (y^*, x_0) of the system (1), (3), (4), (8) is exponentially stable for the H^2 -norm with decay rate $\gamma/4$.

The proof of this theorem is given in Section 4.

Remark 2. One can actually check that for any $\gamma > 0$ there exist parameters b_1 , b_2 and k_1 , k_2 satisfying (16) as, for $b_1 = \gamma e^{-\gamma x_0}$ and $b_2 = \gamma e^{-\gamma(L-x_0)}$, one has

$$\begin{aligned} & 1 - \frac{b_1}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \\ &= 1 - e^{-\gamma x_0} (2 - e^{-\gamma x_0} - e^{-\gamma(L-x_0)}) \\ &= e^{-2\gamma x_0} (e^{\gamma x_0} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{17}$$

Similarly, we get

$$\begin{aligned} & 1 - \frac{b_2}{\gamma} \left(b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \\ &= e^{-2\gamma(L-x_0)} (e^{\gamma(L-x_0)} - 1)^2 + e^{-\gamma L} > 0. \end{aligned} \tag{18}$$

Therefore, by continuity, there exist b_1 and b_2 , satisfying the condition (16a) such that there exist k_1 and k_2 satisfying (16b) and (16c). This implies that γ can be made arbitrarily large. And, from (16a)–(16c), we can note that for large γ the conditions on the k_i tend to

$$k_1^2 < e^{-\gamma x_0}, \quad k_2^2 < e^{-\gamma(L-x_0)}.$$

3 An equivalent system with shock-free solutions

Our strategy to analyze the existence and the exponential stability of the shock wave solutions to the scalar Burgers equation (1) is to use an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions. In order to set up this equivalent system, we define the two following functions

$$y_1(t, x) = y\left(t, x \frac{x_s(t)}{x_0}\right), \quad y_2(t, x) = y\left(t, L + x \frac{x_s(t) - L}{x_0}\right), \tag{19}$$

and the new state variables as follows:

$$\mathbf{z}(t, x) = \begin{pmatrix} z_1(t, x) \\ z_2(t, x) \end{pmatrix} = \begin{pmatrix} y_1(t, x) - 1 \\ y_2(t, x) + 1 \end{pmatrix}, \quad x \in (0, x_0). \tag{20}$$

We observe that the functions y_1 and y_2 in (19) correspond to the solution $y(t, x)$ on the time varying intervals $(0, x_s(t))$ and $(x_s(t), L)$ respectively, albeit with a time varying scaling of the space coordinate x which is driven by $x_s(t)$ and allows to define the new state variables (z_1, z_2) on the fixed time invariant interval $(0, x_0)$. Besides, from (20), the former steady state (y^*, x_0) corresponds now to the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ in the new variables. With these new variables, the dynamics of (y, x_s) can now be expressed as follows:

$$\begin{aligned} z_{1t} + \left(1 + z_1 - x \frac{\dot{x}_s}{x_0}\right) z_{1x} \frac{x_0}{x_s} &= 0, \\ z_{2t} + \left(1 - z_2 + x \frac{\dot{x}_s}{x_0}\right) z_{2x} \frac{x_0}{L - x_s} &= 0, \\ \dot{x}_s(t) &= \frac{z_1(t, x_0) + z_2(t, x_0)}{2}, \end{aligned} \tag{21}$$

with the boundary conditions:

$$\begin{aligned} z_1(t, 0) &= k_1 z_1(t, x_0) + b_1(x_0 - x_s(t)), \\ z_2(t, 0) &= k_2 z_2(t, x_0) + b_2(x_0 - x_s(t)), \end{aligned} \tag{22}$$

and with initial condition (\mathbf{z}^0, x_{s0}) , where $\mathbf{z}^0 = (z_1^0, z_2^0)^T$ and

$$\begin{aligned} z_1^0(x) &= y_0 \left(x \frac{x_{s0}}{x_0} \right) - 1, \\ z_2^0(x) &= y_0 \left(L + x \frac{x_{s0} - L}{x_0} \right) + 1. \end{aligned} \tag{23}$$

Furthermore, in the new variables, the compatibility conditions (9)–(10) are expressed as follows:

$$\begin{aligned} z_1^0(0) &= k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}), \\ z_2^0(0) &= k_2 z_2^0(x_0) + b_2(x_0 - x_{s0}), \end{aligned} \tag{24}$$

and

$$\begin{aligned} (1 + z_1^0(0)) z_{1x}^0(0) \frac{x_0}{x_{s0}} &= k_1 \left(1 + z_1^0(x_0) - \frac{z_1^0(x_0) + z_2^0(x_0)}{2} \right) z_{1x}^0(x_0) \frac{x_0}{x_{s0}} + b_1 \frac{z_1^0(x_0) + z_2^0(x_0)}{2}, \\ (1 - z_2^0(0)) z_{2x}^0(0) \frac{x_0}{L - x_{s0}} &= k_2 \left(1 - z_2^0(x_0) + \frac{z_1^0(x_0) + z_2^0(x_0)}{2} \right) z_{2x}^0(x_0) \frac{x_0}{L - x_{s0}} + b_2 \frac{z_1^0(x_0) + z_2^0(x_0)}{2}. \end{aligned} \tag{25}$$

Concerning the existence of the solution to the system (21)–(23), we have the following lemma.

Lemma 3.1. *For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_{s0} \in (0, L)$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ satisfying the compatibility conditions (24)–(25) and*

$$|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} \leq \delta, \quad |x_{s0} - x_0| \leq \delta, \tag{26}$$

the system (21)–(23) has a unique classical solution $(\mathbf{z}, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0, T]$

$$|\mathbf{z}(t, \cdot)|_{H^2((0, x_0); \mathbb{R}^2)} + |x_s(t) - x_0| \leq C(T) \left(|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} + |x_{s0} - x_0| \right). \tag{27}$$

Proof. The proof of Lemma 3.1 is given in Appendix. □

From this lemma, it is then clear that the proof of Theorem 2.1 follows immediately.

Proof of Theorem 2.1. The change of variables (19), (20) induces an equivalence between the classical solutions (\mathbf{z}, x_s) of the system (21)–(23) and the entropy solutions with a single shock (y, x_s) of the system (1)–(4), (6), (8). Consequently, from (20) and provided $|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)}$ and $|x_{s0} - x_0|$ are sufficiently small, the existence and uniqueness of a solution with a single shock (y, x_s) to the system (1)–(4), (6), (8) satisfying the entropy condition (5) when (y_0, x_{s0}) is in a sufficiently small neighborhood of (y^*, x_0) , follows directly from the existence and uniqueness of the classical solution (\mathbf{z}, x_s) to the system (21)–(23) which is guaranteed by Lemma 3.1. □

Remark 3. *Under the assumption in Lemma 3.1, if we assume furthermore that $\mathbf{z}^0 \in H^k((0, x_0); \mathbb{R}^2)$ with $k \geq 2$ satisfying the k -th order compatibility conditions (see the definition in [3, p.143]), then $(\mathbf{z}, x_s) \in C^0([0, T]; H^k((0, x_0); \mathbb{R}^2)) \times C^k([0, T]; \mathbb{R})$ and (27) still holds. This is a straightforward extension of the proof in Appendix, thus we will not give the details of this proof here.*

4 Exponential stability for the H^2 -norm

This section is devoted to the proof of Theorem 2.2 concerning the exponential stability of the steady state of system (1), (3), (4), (8). Actually, on the basis of the change of variables introduced in the previous section, we know that we only have to prove the exponential stability of the steady state of the auxiliary system (21)–(23) according to the following theorem which is equivalent to Theorem 2.2.

Theorem 4.1. For any $\gamma > 0$, under the conditions (16), there exist $\delta > 0$ and $C > 0$ such that for any $T > 0$ and for any $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$ and $x_{s0} \in (0, L)$ satisfying

$$|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} \leq \delta, \quad |x_{s0} - x_0| \leq \delta, \quad (28)$$

and the compatibility conditions (24)–(25), the system (21)–(23) has a unique classical solution $(\mathbf{z}, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$ such that

$$|\mathbf{z}(t, \cdot)|_{H^2((0, x_0); \mathbb{R}^2)} + |x_s(t) - x_0| \leq C e^{-\gamma t/4} (|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} + |x_{s0} - x_0|), \quad \forall t \in [0, T]. \quad (29)$$

When this theorem holds, we say that the steady state $(\mathbf{z} = \mathbf{0}, x_s = x_0)$ of the system (21)–(23) is exponentially stable for the H^2 -norm with convergence rate $\gamma/4$.

In order to prove Theorem 4.1, we introduce the following candidate Lyapunov function which is defined for all $\mathbf{z} = (z_1, z_2)^T \in H^2((0, x_0); \mathbb{R}^2)$ and $x_s \in (0, L)$:

$$V(\mathbf{z}, x_s) = V_1(\mathbf{z}) + V_2(\mathbf{z}, x_s) + V_3(\mathbf{z}, x_s) + V_4(\mathbf{z}, x_s) + V_5(\mathbf{z}, x_s) + V_6(\mathbf{z}, x_s) \quad (30)$$

with

$$V_1(\mathbf{z}) = \int_0^{x_0} p_1 e^{-\frac{\mu x}{\eta_1}} z_1^2 + p_2 e^{-\frac{\mu x}{\eta_2}} z_2^2 dx, \quad (31)$$

$$V_2(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{-\frac{\mu x}{\eta_1}} z_{1t}^2 + p_2 e^{-\frac{\mu x}{\eta_2}} z_{2t}^2 dx, \quad (32)$$

$$V_3(\mathbf{z}, x_s) = \int_0^{x_0} p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 + p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 dx, \quad (33)$$

$$V_4(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1(x_s - x_0) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2(x_s - x_0) dx + \kappa(x_s - x_0)^2, \quad (34)$$

$$V_5(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1t} \dot{x}_s dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2t} \dot{x}_s dx + \kappa(\dot{x}_s)^2, \quad (35)$$

$$V_6(\mathbf{z}, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} \ddot{x}_s dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} \ddot{x}_s dx + \kappa(\ddot{x}_s)^2. \quad (36)$$

In (31)–(36), $\mu, p_1, p_2, \bar{p}_1, \bar{p}_2$ are positive constants, $\kappa > 1, \eta_1 = 1$ and $\eta_2 = x_0/(L - x_0)$. Actually, in this section, we will need to evaluate $V(\mathbf{z}, x_s)$ only along the system solutions for which the variables $\mathbf{z}_t = (z_{1t}, z_{2t}), \mathbf{z}_{tt} = (z_{1tt}, z_{2tt}), \dot{x}_s$ and \ddot{x}_s that appear in the definition of V can be well defined as functions of $(\mathbf{z}, x_s) \in H^2((0, x_0); \mathbb{R}^2) \times (0, L)$ from the system (21)–(23) and their space derivatives. For example, z_{1t} and z_{2t} are defined as functions of (\mathbf{z}, x_s) by

$$z_{1t} := - \left(1 + z_1 - x \frac{z_1(x_0) + z_2(x_0)}{2x_0} \right) z_{1x} \frac{x_0}{x_s}, \quad (37)$$

$$z_{2t} := - \left(1 - z_2 + x \frac{z_1(x_0) + z_2(x_0)}{2x_0} \right) z_{2x} \frac{x_0}{L - x_s}, \quad (38)$$

and z_{1tt} and z_{2tt} as functions of (\mathbf{z}, x_s) by

$$z_{1tt} := - \left(1 + z_1 - x \frac{z_1(x_0) + z_2(x_0)}{2x_0} \right) (z_{1t})_x \frac{x_0}{x_s} - \left(z_{1t} - x \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2x_0} \right) z_{1x} \frac{x_0}{x_s} - z_{1t} \frac{z_1(x_0) + z_2(x_0)}{2x_s}, \quad (39)$$

$$z_{2tt} := - \left(1 - z_2 + x \frac{z_1(x_0) + z_2(x_0)}{2x_0} \right) (z_{2t})_x \frac{x_0}{L - x_s} + \left(z_{2t} - x \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2x_0} \right) z_{2x} \frac{x_0}{L - x_s} + z_{2t} \frac{z_1(x_0) + z_2(x_0)}{2(L - x_s)}. \quad (40)$$

The functions z_{1t} and z_{2t} which appear in (39) and (40) are supposed to be defined by (37) and (38) respectively.

Remark 4. The first three terms V_1, V_2, V_3 of this candidate Lyapunov function are identical to the Lyapunov function which is classically used to analyze the exponential stability of the steady states of 2×2 hyperbolic systems in H^2 (see e.g. [3, Section 4.4]). The three extra terms V_4, V_5 and V_6 are added to analyze the convergence of the shock position x_s towards the desired location x_0 .

For the proof Theorem 4.1, we first state the following lemma.

Lemma 4.1. *If*

$$\max\left(\frac{\bar{p}_1^2}{p_1} \frac{\eta_1}{\mu} \left(1 - e^{-\frac{\mu x_0}{\eta_1}}\right), \frac{\bar{p}_2^2}{p_2} \frac{\eta_2}{\mu} \left(1 - e^{-\frac{\mu x_0}{\eta_2}}\right)\right) < 2, \quad (41)$$

there exists $\beta > 0$ such that

$$\beta (|\mathbf{z}|_{H^2}^2 + |x_s - x_0|^2) \leq V \leq \frac{1}{\beta} (|\mathbf{z}|_{H^2}^2 + |x_s - x_0|^2) \quad (42)$$

for any $(\mathbf{z}, x_s) \in H^2((0, x_0); \mathbb{R}^2) \times (0, L)$ satisfying

$$|\mathbf{z}|_{H^2}^2 + |x_s - x_0|^2 < \beta^2. \quad (43)$$

Proof of Lemma 4.1. Let us start with

$$V_4 = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1(x_s - x_0) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2(x_s - x_0) dx + \kappa(x_s - x_0)^2. \quad (44)$$

Using Young's inequality we get

$$\begin{aligned} & -\frac{1}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 dx \right)^2 - \frac{(x_s - x_0)^2}{2} - \frac{1}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 dx \right)^2 - \frac{(x_s - x_0)^2}{2} \\ & + \kappa(x_s - x_0)^2 \leq V_4 \leq \frac{1}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 dx \right)^2 \\ & + \frac{(x_s - x_0)^2}{2} + \frac{1}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 dx \right)^2 + \frac{(x_s - x_0)^2}{2} + \kappa(x_s - x_0)^2. \end{aligned} \quad (45)$$

Hence, using the Cauchy-Schwarz inequality and the expression of V_1 given in (31),

$$\begin{aligned} & p_1 \left(1 - \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_1^2 dx + p_2 \left(1 - \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_2^2 dx \\ & (x_s - x_0)^2 (\kappa - 1) \leq V_1 + V_4 \leq p_1 \left(1 + \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_1^2 dx \\ & + p_2 \left(1 + \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_2^2 dx + (x_s - x_0)^2 (\kappa + 1), \end{aligned} \quad (46)$$

and similarly

$$\begin{aligned} & p_1 \left(1 - \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_{1t}^2 dx + p_2 \left(1 - \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_{2t}^2 dx \\ & (\dot{x}_s)^2 (\kappa - 1) \leq V_2 + V_5 \leq p_1 \left(1 + \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_{1t}^2 dx \\ & + p_2 \left(1 + \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_{2t}^2 dx + (\dot{x}_s)^2 (\kappa + 1), \end{aligned} \quad (47)$$

and also

$$\begin{aligned} & p_1 \left(1 - \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 dx + p_2 \left(1 - \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 dx \\ & + (\ddot{x}_s)^2 (\kappa - 1) \leq V_3 + V_6 \leq p_1 \left(1 + \frac{1}{2} \frac{\eta_1}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_1}}) \frac{\bar{p}_1^2}{p_1}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 dx \\ & + p_2 \left(1 + \frac{1}{2} \frac{\eta_2}{\mu} (1 - e^{-\frac{\mu x_0}{\eta_2}}) \frac{\bar{p}_2^2}{p_2}\right) \int_0^{x_0} e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 dx + (\ddot{x}_s)^2 (\kappa + 1). \end{aligned} \quad (48)$$

Hence, as $\kappa > 1$ and (41) is satisfied, there exists $\sigma > 0$ such that

$$\sigma \left(|\mathbf{z}|_{H_t^2((0, x_0); \mathbb{R}^2)}^2 + |x_s - x_0|^2 \right) \leq V \leq \frac{1}{\sigma} \left(|\mathbf{z}|_{H_t^2((0, x_0); \mathbb{R}^2)}^2 + |x_s - x_0|^2 \right), \quad (49)$$

where, for a function $\mathbf{z} \in H^2((0, x_0); \mathbb{R}^2)$, $|\mathbf{z}|_{H_t^2((0, x_0); \mathbb{R}^2)}$ is defined by

$$|\mathbf{z}|_{H_t^2((0, x_0); \mathbb{R}^2)} = \left(|\mathbf{z}|_{L^2((0, x_0); \mathbb{R}^2)}^2 + |\mathbf{z}_t|_{L^2((0, x_0); \mathbb{R}^2)}^2 + |\mathbf{z}_{tt}|_{L^2((0, x_0); \mathbb{R}^2)}^2 \right)^{1/2}, \quad (50)$$

with \mathbf{z}_t and \mathbf{z}_{tt} defined as (37)–(40). Let us point out that from (37)–(40), there exists $C > 0$ such that

$$\frac{1}{C} |\mathbf{z}|_{H^2((0, x_0); \mathbb{R}^2)} \leq |\mathbf{z}|_{H_t^2((0, x_0); \mathbb{R}^2)} \leq C |\mathbf{z}|_{H^2((0, x_0); \mathbb{R}^2)}, \quad (51)$$

if $(|\mathbf{z}|_{H^2((0, x_0); \mathbb{R}^2)}^2 + |x_s - x_0|^2) < 1/C$. It follows from (49) and (51) that $\beta > 0$ can be taken sufficiently small such that inequality (42) holds provided (43) is satisfied. This concludes the proof of Lemma 4.1. \square

Let us now prove Theorem 4.1. For the sake of simplicity, for any $\mathbf{z} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$, we denote from now on $|\mathbf{z}(t, \cdot)|_{H^2((0, x_0); \mathbb{R}^2)}$ by $|\mathbf{z}|_{H^2}$.

Proof of Theorem 4.1. From Lemma 4.1, all it remains to do is to show that for any $\gamma > 0$, under conditions (16) there exist μ, p_1, p_2, \bar{p}_1 and \bar{p}_2 satisfying (41) and such that, for any $T > 0$ and along any solution $(\mathbf{z}, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$ to the system (21)–(23), the Lyapunov function $V(\mathbf{z}, x_s)$ is exponentially decreasing with decay rate $\gamma/2$. Since the system solutions $(\mathbf{z}, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$, it follows that $V(\mathbf{z}, x_s)$ is a continuous function of t along the system trajectories.

Let us assume that (\mathbf{z}, x_s) is a C^3 solution to the system (21)–(23) (this assumption will be relaxed at the end of the proof), with initial condition $|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} \leq \delta$ and $|x_{s0} - x_0| \leq \delta$ respectively with $\delta > 0$ to be chosen later on. Let us examine the different components of the Lyapunov function. We start by studying V_1, V_2 and V_3 which can be treated similarly as in [3, Section 4.4]. Differentiating V_1 along the solution (\mathbf{z}, x_s) and integrating by parts, we have

$$\begin{aligned} \frac{dV_1}{dt} &= -2 \int_0^{x_0} \left(p_1 e^{\frac{-\mu x}{\eta_1}} z_1 (1 + z_1 - x \frac{\dot{x}_s}{x_0}) \frac{x_0}{x_s} z_{1x} + p_2 e^{\frac{-\mu x}{\eta_2}} z_2 (1 - z_2 + x \frac{\dot{x}_s}{x_0}) \frac{x_0}{L - x_s} z_{2x} \right) dx \\ &= -\mu V_1 - \left[p_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_1^2 + p_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_2^2 \right]_0^{x_0} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \end{aligned} \quad (52)$$

Here and hereafter, $O(s)$ means that there exist $\varepsilon > 0$ and $C_1 > 0$, both independent of \mathbf{z}, x_s, T and $t \in [0, T]$, such that

$$(s \leq \varepsilon) \implies (|O(s)| \leq C_1 s).$$

From (21), we have

$$\begin{aligned} z_{1tt} + (1 + z_1 - x \frac{\dot{x}_s}{x_0}) z_{1tx} \frac{x_0}{x_s} + (z_{1t} - x \frac{\ddot{x}_s}{x_0}) z_{1x} \frac{x_0}{x_s} + z_{1t} \frac{\dot{x}_s}{x_s} &= 0, \\ z_{2tt} + (1 - z_2 + x \frac{\dot{x}_s}{x_0}) z_{2tx} \frac{x_0}{L - x_s} - (z_{2t} - x \frac{\ddot{x}_s}{x_0}) z_{2x} \frac{x_0}{L - x_s} - z_{2t} \frac{\dot{x}_s}{L - x_s} &= 0. \end{aligned} \quad (53)$$

Therefore, similarly to (52), we can obtain

$$\frac{dV_2}{dt} = -\mu V_2 - \left[p_1 e^{\frac{-\mu x}{\eta_1}} \frac{x_0}{x_s} z_{1t}^2 + p_2 e^{\frac{-\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2t}^2 \right]_0^{x_0} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \quad (54)$$

From (53) and using (21), we get

$$\begin{aligned} z_{1ttt} + (1 + z_1 - x \frac{\dot{x}_s}{x_0}) z_{1ttx} \frac{x_0}{x_s} + 2(z_{1t} - x \frac{\ddot{x}_s}{x_0}) z_{1tx} \frac{x_0}{x_s} + \frac{\dot{x}_s}{x_s} (z_{1tt} + z_{1t} \frac{\dot{x}_s}{x_s}) \\ + (z_{1tt} - x \frac{\ddot{x}_s}{x_0}) z_{1x} \frac{x_0}{x_s} + z_{1tt} \frac{\dot{x}_s}{x_s} + z_{1t} \frac{\ddot{x}_s x_s - (\dot{x}_s)^2}{x_s^2} &= 0, \\ z_{2ttt} + (1 - z_2 + x \frac{\dot{x}_s}{x_0}) z_{2ttx} \frac{x_0}{L - x_s} - 2(z_{2t} - x \frac{\ddot{x}_s}{x_0}) z_{2tx} \frac{x_0}{L - x_s} + \frac{\dot{x}_s}{L - x_s} (-z_{2tt} + z_{2t} \frac{\dot{x}_s}{L - x_s}) \\ - (z_{2tt} - x \frac{\ddot{x}_s}{x_0}) z_{2x} \frac{x_0}{L - x_s} - z_{2tt} \frac{\dot{x}_s}{L - x_s} - z_{2t} \frac{\ddot{x}_s (L - x_s) + (\dot{x}_s)^2}{(L - x_s)^2} &= 0. \end{aligned} \quad (55)$$

Then differentiating V_3 along the system solutions and using (55), we have

$$\begin{aligned}
\frac{dV_3}{dt} \leq & - \left[p_1 e^{-\frac{\mu x}{\eta_1}} \frac{x_0}{x_s} (z_{1tt}^2) (1 + z_1 - x \frac{\dot{x}_s}{x_0}) \right]_0^{x_0} - \left[p_2 e^{-\frac{\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2tt}^2 (1 - z_2 + x \frac{\dot{x}_s}{x_0}) \right]_0^{x_0} \\
& - \mu \min \left(\frac{x_0}{x_s}, \frac{L - x_0}{L - x_s} \right) V_3 - \mu \int_0^{x_0} \left(\frac{x_0}{x_s} p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 z_1 - \frac{L - x_0}{L - x_s} p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 z_2 \right) dx \\
& + \mu \int_0^{x_0} \left(\frac{x_0}{x_s} p_1 e^{-\frac{\mu x}{\eta_2}} x z_{1tt}^2 \frac{\dot{x}_s}{x_0} - \frac{L - x_0}{L - x_s} p_2 e^{-\frac{\mu x}{\eta_1}} x z_{2tt}^2 \frac{\dot{x}_s}{x_0} \right) dx \\
& - 3 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 \frac{\dot{x}_s}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 \frac{\dot{x}_s}{L - x_s} \right) dx \\
& - \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt}^2 z_{1x} \frac{x_0}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt}^2 z_{2x} \frac{x_0}{L - x_s} \right) dx \\
& - 4 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} (z_{1t} - x \frac{\ddot{x}_s}{x_0}) z_{1tx} \frac{x_0}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} (z_{2t} - x \frac{\ddot{x}_s}{x_0}) z_{2tx} \frac{x_0}{L - x_s} \right) dx \\
& - 2 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} (z_{1tt} + z_{1t} \frac{\dot{x}_s}{x_s}) \frac{\dot{x}_s}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} (z_{2tt} - z_{2t} \frac{\dot{x}_s}{L - x_s}) \frac{\dot{x}_s}{L - x_s} \right) dx \\
& - 2 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} z_{1t} \frac{\ddot{x}_s x_s - (\dot{x}_s)^2}{x_c^2} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} z_{2t} \frac{\ddot{x}_s (L - x_s) + (\dot{x}_s)^2}{(L - x_s)^2} \right) dx \\
& - 2 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} (z_{1tt} - x \frac{\ddot{x}_s}{x_0}) z_{1x} \frac{x_0}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} (z_{2tt} - x \frac{\ddot{x}_s}{x_0}) z_{2x} \frac{x_0}{L - x_s} \right) dx.
\end{aligned} \tag{56}$$

Observe that, while previously all the cubic terms in \mathbf{z} could be bounded by $|\mathbf{z}|_{H^2}^3$, here in the last line in (56) we have \ddot{x}_s which is proportional to $\mathbf{z}_{tt}(t, x_0)$ and cannot be roughly bounded by the $|\mathbf{z}|_{H^2}$ norm. To overcome this difficulty, we transform these terms using Young's inequality and we get

$$\begin{aligned}
& 2 \int_0^{x_0} \left(p_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} (x \frac{\ddot{x}_s}{x_0}) z_{1x} \frac{x_0}{x_s} - p_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} (x \frac{\ddot{x}_s}{x_0}) z_{2x} \frac{x_0}{L - x_s} \right) dx \\
& \leq C |\mathbf{z}(t, \cdot)|_{C^1([0, x_0]; \mathbb{R}^2)} (z_{1tt}(t, x_0) + z_{2tt}(t, x_0))^2 + O \left(|\mathbf{z}(t, \cdot)|_{C^1([0, x_0]; \mathbb{R}^2)} |\mathbf{z}|_{H^2}^2 \right),
\end{aligned} \tag{57}$$

where C denotes a constant, independent of \mathbf{z} , x_s , T and $t \in [0, T]$. Note that the first term on the right is now proportional to $\mathbf{z}_{tt}^2(t, x_0)$ with a proportionality coefficient $C |\mathbf{z}(t, \cdot)|_{C^1([0, x_0]; \mathbb{R}^2)}$ that, by Sobolev inequality, can be made sufficiently small provided that $|\mathbf{z}|_{H^2}$ is sufficiently small and thus can be dominated by the boundary terms. More precisely, from (56) and (57) we have

$$\begin{aligned}
\frac{dV_3}{dt} \leq & - \mu V_3 - \left[p_1 e^{-\frac{\mu x}{\eta_1}} \frac{x_0}{x_s} (z_{1tt}^2) \right]_0^{x_0} - \left[p_2 e^{-\frac{\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2tt}^2 \right]_0^{x_0} \\
& + O \left(|\mathbf{z}|_{H^2} (z_{1tt}^2(t, x_0) + z_{2tt}^2(t, x_0)) + O \left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3 \right) \right).
\end{aligned} \tag{58}$$

Let us now deal with the term V_4 that takes into account the position of the jump. In the following, we use notations $\mathbf{z}(0)$ and $\mathbf{z}(x_0)$ instead of $\mathbf{z}(t, 0)$ and $\mathbf{z}(t, x_0)$ for simplicity. We have

$$\begin{aligned}
\frac{dV_4}{dt} = & - \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} (1 + z_1 - x \frac{\dot{x}_s}{x_0}) z_{1x} (x_s - x_0) \frac{x_0}{x_s} dx + \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 \dot{x}_s dx \\
& - \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} (1 - z_2 + x \frac{\dot{x}_s}{x_0}) z_{2x} (x_s - x_0) \frac{x_0}{L - x_s} dx \\
& + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 \dot{x}_s dx + 2\kappa \dot{x}_s (x_s - x_0) \\
= & - (x_s - x_0) \left[\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} \frac{x_0}{x_s} z_1 + \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_2 \right]_0^{x_0} - \mu (V_4 - \kappa (x_s - x_0)^2) \\
& + \frac{z_1(x_0) + z_2(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 dx \right) + \frac{z_1(x_0) + z_2(x_0)}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 dx \right) \\
& + \kappa (z_1(x_0) + z_2(x_0)) (x_s - x_0) + O \left((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3 \right).
\end{aligned} \tag{59}$$

According to Young's inequality, for any positive ε_1 and ε_2 , we have

$$\begin{aligned} \frac{z_1(x_0) + z_2(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 dx \right) &\leq \frac{\varepsilon_1}{4} \left(\frac{z_1(x_0) + z_2(x_0)}{2} \right)^2 + \frac{1}{\varepsilon_1} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_1 dx \right)^2, \\ \frac{z_1(x_0) + z_2(x_0)}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 dx \right) &\leq \frac{\varepsilon_1}{4} \left(\frac{z_1(x_0) + z_2(x_0)}{2} \right)^2 + \frac{1}{\varepsilon_1} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_2 dx \right)^2. \end{aligned} \quad (60)$$

Then using the boundary condition (22) and Cauchy-Schwarz inequality, (59) becomes

$$\begin{aligned} \frac{dV_4}{dt} &\leq -\mu V_4 - \bar{p}_1(x_s - x_0) \frac{x_0}{x_s} \left((e^{-\frac{\mu x_0}{\eta_1}} - k_1) z_1(x_0) + b_1(x_s - x_0) \right) \\ &\quad - \bar{p}_2(x_s - x_0) \frac{x_0}{L - x_s} \left((e^{-\frac{\mu x_0}{\eta_2}} - k_2) z_2(x_0) + b_2(x_s - x_0) \right) \\ &\quad + (\varepsilon_1 + \varepsilon_2) \frac{z_1^2(x_0) + z_2^2(x_0)}{8} + \max \left\{ \frac{1}{\varepsilon_1} \frac{\bar{p}_1^2}{p_1} \frac{\eta_1(1 - e^{-\frac{\mu x_0}{\eta_1}})}{\mu}, \frac{1}{\varepsilon_2} \frac{\bar{p}_2^2}{p_2} \frac{\eta_2(1 - e^{-\frac{\mu x_0}{\eta_2}})}{\mu} \right\} V_1 \\ &\quad + \kappa(x_s - x_0)(z_1(x_0) + z_2(x_0)) + \mu\kappa(x_s - x_0)^2 + O(|\mathbf{z}|_{H^2} + |x_s - x_0|)^3. \end{aligned} \quad (61)$$

Let us now consider V_5 . From (35) and (53), one has similarly

$$\begin{aligned} \frac{dV_5}{dt} &= - \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tx} \dot{x}_s \frac{x_0}{x_s} dx + \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1t} \ddot{x}_s dx \\ &\quad - \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tx} \dot{x}_s \frac{x_0}{L - x_s} dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2t} \ddot{x}_s dx + 2\kappa \dot{x}_s \dot{x}_s + O(|\mathbf{z}|_{H^2} + |x_s - x_0|)^3 \\ &= - \dot{x}_s \left[\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} \frac{x_0}{x_s} z_{1t} + \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} \frac{x_0}{L - x_s} z_{2t} \right]_0^{x_0} - \mu(V_5 - \kappa(\dot{x}_s)^2) \\ &\quad + \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1t} dx \right) + \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2t} dx \right) \\ &\quad + \kappa(z_{1t}(x_0) + z_{2t}(x_0)) \dot{x}_s + O(|\mathbf{z}|_{H^2} + |x_s - x_0|)^3. \end{aligned}$$

By differentiating (22) with respect to time, we have

$$\begin{aligned} z_{1t}(0) &= k_1 z_{1t}(x_0) - b_1 \dot{x}_s, \\ z_{2t}(0) &= k_2 z_{2t}(x_0) - b_2 \dot{x}_s, \end{aligned} \quad (62)$$

and therefore using Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned} \frac{dV_5}{dt} &\leq -\mu V_5 - \bar{p}_1 \dot{x}_s \frac{x_0}{x_s} \left((e^{-\frac{\mu x_0}{\eta_1}} - k_1) z_{1t}(x_0) + b_1 \dot{x}_s \right) \\ &\quad - \bar{p}_2 \dot{x}_s \frac{x_0}{L - x_s} \left((e^{-\frac{\mu x_0}{\eta_2}} - k_2) z_{2t}(x_0) + b_2 \dot{x}_s \right) \\ &\quad + (\varepsilon_1 + \varepsilon_2) \frac{z_{1t}^2(x_0) + z_{2t}^2(x_0)}{8} + \max \left\{ \frac{1}{\varepsilon_1} \frac{\bar{p}_1^2}{p_1} \frac{\eta_1(1 - e^{-\frac{\mu x_0}{\eta_1}})}{\mu}, \frac{1}{\varepsilon_2} \frac{\bar{p}_2^2}{p_2} \frac{\eta_2(1 - e^{-\frac{\mu x_0}{\eta_2}})}{\mu} \right\} V_2 \\ &\quad + \kappa \dot{x}_s (z_{1t}(x_0) + z_{2t}(x_0)) + \mu\kappa(\dot{x}_s)^2 + O(|\mathbf{z}|_{H^2} + |x_s - x_0|)^3. \end{aligned} \quad (63)$$

Furthermore, by differentiating (62) with respect to time, we have

$$\begin{aligned} z_{1tt}(0) &= k_1 z_{1tt}(x_0) - b_1 \ddot{x}_s, \\ z_{2tt}(0) &= k_2 z_{2tt}(x_0) - b_2 \ddot{x}_s, \end{aligned} \quad (64)$$

and therefore using also (55), one has

$$\begin{aligned}
\frac{dV_6}{dt} &= - \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} \ddot{x}_s \frac{x_0}{x_s} dx + \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} \ddot{x}_s dx \\
&\quad - \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} \ddot{x}_s \frac{x_0}{L-x_s} dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} \ddot{x}_s dx + 2\kappa \ddot{x}_s \ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} \ddot{x}_s \left(x \frac{\ddot{x}_s}{x_0}\right) z_{1x} \frac{x_0}{x_s} dx \\
&\quad - \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} \ddot{x}_s \left(x \frac{\ddot{x}_s}{x_0}\right) z_{2x} \frac{x_0}{L-x_s} dx + O\left(\left(\|\mathbf{z}\|_{H^2} + |x_s - x_0|\right)^3\right) \\
&= - \ddot{x}_s \left[\bar{p}_1 e^{-\frac{\mu x}{\eta_1}} \frac{x_0}{x_s} z_{1tt} + \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} \frac{x_0}{L-x_s} z_{2tt} \right]_0^{x_0} - \mu(V_6 - \kappa(\ddot{x}_s)^2) \\
&\quad + \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} z_{1tt} dx \right) + \frac{z_{1tt}(x_0) + z_{2tt}(x_0)}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} z_{2tt} dx \right) \\
&\quad + \kappa(z_{1tt}(x_0) + z_{2tt}(x_0)) \ddot{x}_s + \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{\eta_1}} \ddot{x}_s \left(x \frac{\ddot{x}_s}{x_0}\right) z_{1x} \frac{x_0}{x_s} dx \\
&\quad - \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{\eta_2}} \ddot{x}_s \left(x \frac{\ddot{x}_s}{x_0}\right) z_{2x} \frac{x_0}{L-x_s} dx + O\left(\left(\|\mathbf{z}\|_{H^2} + |x_s - x_0|\right)^3\right).
\end{aligned}$$

Note that, as above for V_3 , here appears again \ddot{x}_s which is proportional to $\mathbf{z}_{tt}(t, x_0)$ and cannot be bounded by $\|\mathbf{z}\|_{H^2}$. We therefore use Cauchy-Schwarz and Young's inequalities as previously and the boundary condition (64), to get

$$\begin{aligned}
\frac{dV_6}{dt} &\leq -\mu V_6 - \bar{p}_1 \ddot{x}_s \frac{x_0}{x_s} \left((e^{-\frac{\mu x_0}{\eta_1}} - k_1) z_{1tt}(x_0) + b_1 \ddot{x}_s \right) \\
&\quad - \bar{p}_2 \ddot{x}_s \frac{x_0}{L-x_s} \left((e^{-\frac{\mu x_0}{\eta_2}} - k_2) z_{2tt}(x_0) + b_2 \ddot{x}_s \right) \\
&\quad + (\varepsilon_1 + \varepsilon_2) \frac{z_{1tt}^2(x_0) + z_{2tt}^2(x_0)}{8} + \max \left\{ \frac{1}{\varepsilon_1} \frac{\bar{p}_1^2 \eta_1 (1 - e^{-\frac{\mu x_0}{\eta_1}})}{p_1 \mu}, \frac{1}{\varepsilon_2} \frac{\bar{p}_2^2 \eta_2 (1 - e^{-\frac{\mu x_0}{\eta_2}})}{p_2 \mu} \right\} V_2 \quad (65) \\
&\quad + \kappa \ddot{x}_s (z_{1tt}(x_0) + z_{2tt}(x_0)) + \mu \kappa (\ddot{x}_s)^2 + O\left(\|\mathbf{z}\|_{H^2}\right) (z_{1tt}^2(x_0) + z_{2tt}^2(x_0)) \\
&\quad + O\left(\left(\|\mathbf{z}\|_{H^2} + |x_s - x_0|\right)^3\right).
\end{aligned}$$

Hence, from (52), (61) and the boundary conditions (22), we have

$$\begin{aligned}
\frac{dV_1}{dt} + \frac{dV_4}{dt} &\leq -\mu(V_1 + V_4) \\
&\quad + \max \left\{ \frac{1}{\varepsilon_1} \frac{\bar{p}_1^2 \eta_1 (1 - e^{-\frac{\mu x_0}{\eta_1}})}{p_1 \mu}, \frac{1}{\varepsilon_2} \frac{\bar{p}_2^2 \eta_2 (1 - e^{-\frac{\mu x_0}{\eta_2}})}{p_2 \mu} \right\} V_1 \\
&\quad + \left[\frac{x_0}{x_s} p_1 (k_1^2 - e^{-\frac{\mu x_0}{\eta_1}}) + \frac{\varepsilon_1 + \varepsilon_2}{8} \right] z_1^2(x_0) \\
&\quad + \left[\frac{x_0}{L-x_s} p_2 (k_2^2 - e^{-\frac{\mu x_0}{\eta_2}}) + \frac{\varepsilon_1 + \varepsilon_2}{8} \right] z_2^2(x_0) \quad (66) \\
&\quad + \left[-2 \frac{x_0}{x_s} p_1 b_1 k_1 - \frac{x_0}{x_s} \bar{p}_1 (e^{-\frac{\mu x_0}{\eta_1}} - k_1) + \kappa \right] z_1(x_0) (x_s - x_0) \\
&\quad + \left[-2 \frac{x_0}{L-x_s} p_2 b_2 k_2 - \frac{x_0}{L-x_s} \bar{p}_2 (e^{-\frac{\mu x_0}{\eta_2}} - k_2) + \kappa \right] z_2(x_0) (x_s - x_0) \\
&\quad + \left[\frac{x_0}{x_s} p_1 b_1^2 + \frac{x_0}{L-x_s} p_2 b_2^2 - \frac{x_0}{x_s} \bar{p}_1 b_1 - \frac{x_0}{L-x_s} \bar{p}_2 b_2 + \mu \kappa \right] (x_s - x_0)^2 \\
&\quad + O\left(\left(\|\mathbf{z}\|_{H^2} + |x_s - x_0|\right)^3\right).
\end{aligned}$$

Let us now select ε_1 and ε_2 as follows:

$$\varepsilon_1 = 2 \frac{\bar{p}_1^2 \eta_1 (1 - e^{-\frac{\mu x_0}{\eta_1}})}{p_1 \mu^2}, \quad \varepsilon_2 = 2 \frac{\bar{p}_2^2 \eta_2 (1 - e^{-\frac{\mu x_0}{\eta_2}})}{p_2 \mu^2}. \quad (67)$$

Then (66) can be rewritten in the following compact form:

$$\frac{dV_1}{dt} + \frac{dV_4}{dt} \leq -\frac{\mu}{2}V_1 - \mu V_4 - \mathbf{Z}^T \mathbf{A} \mathbf{Z} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \quad (68)$$

This expression involves the quadratic form $\mathbf{Z}^T \mathbf{A} \mathbf{Z}$ with the vector \mathbf{Z} defined as

$$\mathbf{Z} = (z_1(x_0) \ z_2(x_0) \ (x_s - x_0))^T \quad (69)$$

and the matrix \mathbf{A} defined as follows:

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (70)$$

with

$$a_{11} = \frac{x_0}{x_s} p_1 (e^{-\frac{\mu x_0}{\eta_1}} - k_1^2) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \quad (71)$$

$$a_{13} = a_{31} = \frac{x_0}{x_s} p_1 b_1 k_1 + \frac{x_0 \bar{p}_1}{x_s} (e^{-\frac{\mu x_0}{\eta_1}} - k_1) - \frac{\kappa}{2}, \quad (72)$$

$$a_{22} = \frac{x_0}{L - x_s} p_2 (e^{-\frac{\mu x_0}{\eta_2}} - k_2^2) - \frac{\varepsilon_1 + \varepsilon_2}{8}, \quad (73)$$

$$a_{23} = a_{32} = \frac{x_0}{L - x_s} p_2 b_2 k_2 + \frac{x_0 \bar{p}_2}{L - x_s} (e^{-\frac{\mu x_0}{\eta_2}} - k_2) - \frac{\kappa}{2}, \quad (74)$$

$$a_{33} = -\frac{x_0}{x_s} p_1 b_1^2 - \frac{x_0}{L - x_s} p_2 b_2^2 + \frac{x_0}{x_s} \bar{p}_1 b_1 + \frac{x_0}{L - x_s} \bar{p}_2 b_2 - \mu \kappa. \quad (75)$$

Similarly, from (54) and (63), we get

$$\frac{dV_2}{dt} + \frac{dV_5}{dt} \leq -\frac{\mu}{2}V_2 - \mu V_5 - \mathbf{Z}_t^T \mathbf{A} \mathbf{Z}_t + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3), \quad (76)$$

while from (58) and (65), we have

$$\frac{dV_3}{dt} + \frac{dV_6}{dt} \leq -\frac{\mu}{2}V_3 - \mu V_6 - \mathbf{Z}_{tt}^T \mathbf{A}_1 \mathbf{Z}_{tt} + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3), \quad (77)$$

where

$$\mathbf{A}_1 = \begin{pmatrix} a_{11} + O(|\mathbf{z}|_{H^2}) & 0 & a_{13} \\ 0 & a_{22} + O(|\mathbf{z}|_{H^2}) & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \quad (78)$$

By continuity, if \mathbf{A} is positive definite, then \mathbf{A}_1 is also positive definite provided that $|\mathbf{z}|_{H^2}$ is sufficiently small.

Let us now show that there exist $\mu > \gamma$ and positive $p_1, p_2, \bar{p}_1, \bar{p}_2$ such that \mathbf{A} given by (70) is positive definite provided $|x_s - x_0|$ is sufficiently small. Let us first select p_1 and p_2 as

$$p_1 = \frac{\bar{p}_1}{2b_1}, \quad p_2 = \frac{\bar{p}_2}{2b_2}. \quad (79)$$

Then the cross terms (72), (74) become

$$a_{13} = a_{31} = \frac{x_0 \bar{p}_1}{x_s} e^{-\frac{\mu x_0}{\eta_1}} - \frac{\kappa}{2}, \quad a_{23} = a_{32} = \frac{x_0 \bar{p}_2}{L - x_s} e^{-\frac{\mu x_0}{\eta_2}} - \frac{\kappa}{2}. \quad (80)$$

Let \bar{p}_1 and \bar{p}_2 be selected as

$$\bar{p}_1 = \kappa e^{\frac{\mu x_0}{\eta_1}}, \quad \bar{p}_2 = \kappa \frac{L - x_0}{x_0} e^{\frac{\mu x_0}{\eta_2}}. \quad (81)$$

Then we have

$$a_{13} = a_{31} = \kappa \frac{x_0 - x_s}{2x_s}, \quad a_{23} = a_{32} = \kappa \frac{x_s - x_0}{2(L - x_s)}, \quad (82)$$

such that \mathbf{A} can now be rewritten as

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} + \kappa(x_s - x_0) \begin{pmatrix} 0 & 0 & \frac{-1}{2x_s} \\ 0 & 0 & \frac{1}{2(L-x_s)} \\ \frac{-1}{2x_s} & \frac{1}{2(L-x_s)} & 0 \end{pmatrix}. \quad (83)$$

Moreover from (79) and (81), we get

$$a_{33} = \frac{x_0}{x_s} \frac{\bar{p}_1}{2} b_1 + \frac{x_0}{L-x_s} \frac{\bar{p}_2}{2} b_2 - \mu\kappa = \frac{x_0}{x_s} \frac{\kappa}{2} b_1 e^{\frac{\mu x_0}{\eta_1}} + \frac{L-x_0}{L-x_s} \frac{\kappa}{2} b_2 e^{\frac{\mu x_0}{\eta_2}} - \mu\kappa. \quad (84)$$

As conditions (16) are strict inequalities, by continuity it follows that we can select $\mu > \gamma$ such that these conditions (16) are still satisfied with μ instead of γ such that

$$\mu e^{-\frac{\mu x_0}{\eta_1}} < b_1 < \frac{\mu e^{-\frac{\mu x_0}{\eta_1}}}{1 - e^{-\frac{\mu x_0}{\eta_1}}}, \quad \mu e^{-\frac{\mu x_0}{\eta_2}} < b_2 < \frac{\mu e^{-\frac{\mu x_0}{\eta_2}}}{1 - e^{-\frac{\mu x_0}{\eta_2}}}, \quad (85)$$

$$a_{33} > 0 \quad \text{provided } |x_s - x_0| \text{ is sufficiently small.} \quad (86)$$

From (67), (71), (73), (79) and (81), we have

$$a_{11} = \frac{x_0}{x_s} \frac{\kappa}{2b_1} (1 - k_1^2 e^{\frac{\mu x_0}{\eta_1}}) - \frac{\kappa}{2\mu^2} \left[b_1 \eta_1 (e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2 \eta_2 (e^{\frac{\mu x_0}{\eta_2}} - 1) \right], \quad (87)$$

$$a_{22} = \frac{L-x_0}{L-x_s} \frac{\kappa}{2b_2} (1 - k_2^2 e^{\frac{\mu x_0}{\eta_2}}) - \frac{\kappa}{2\mu^2} \left[b_1 \eta_1 (e^{\frac{\mu x_0}{\eta_1}} - 1) + b_2 \eta_2 (e^{\frac{\mu x_0}{\eta_2}} - 1) \right]. \quad (88)$$

Then, under assumptions (16), it can be checked that

$$a_{11} > 0, \quad a_{22} > 0 \quad \text{provided } |x_s - x_0| \text{ is sufficiently small.} \quad (89)$$

This implies that the first matrix in (83) is positive definite. Moreover the second matrix in (83) is only a perturbation which is proportional to $|x_s - x_0|$. Hence, we have shown that, under Lemma 3.1, there exists δ_1 such that, if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1$ and $|x_{s0} - x_0| \leq \delta_1$, then, along the solution of the system (21)–(23), the matrices \mathbf{A} and \mathbf{A}_1 are positive definite. Then from (68), (76) and (77), we have the inequality

$$\frac{dV}{dt} \leq -\frac{\mu}{2} V + O((|\mathbf{z}|_{H^2} + |x_s - x_0|)^3). \quad (90)$$

Now let us remark that from the condition (85) we have

$$\max \left(2 \frac{b_1 \eta_1}{\mu} e^{\frac{\mu x_0}{\eta_1}} \left(1 - e^{-\frac{\mu x_0}{\eta_1}} \right), 2 \frac{L-x_0}{x_0} \frac{b_2 \eta_2}{\mu} e^{\frac{\mu x_0}{\eta_2}} \left(1 - e^{-\frac{\mu x_0}{\eta_2}} \right) \right) < 2. \quad (91)$$

Therefore, there exists $\kappa > 1$ such that

$$\max \left(2\kappa \frac{b_1 \eta_1}{\mu} e^{\frac{\mu x_0}{\eta_1}} \left(1 - e^{-\frac{\mu x_0}{\eta_1}} \right), 2\kappa \frac{L-x_0}{x_0} \frac{b_2 \eta_2}{\mu} e^{\frac{\mu x_0}{\eta_2}} \left(1 - e^{-\frac{\mu x_0}{\eta_2}} \right) \right) < 2, \quad (92)$$

which means from (79) and (81) that (41) is satisfied.

Hence from Lemma 4.1, since $\mu > \gamma$, there exists $\delta \leq \delta_1$ such that, if $|\mathbf{z}^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta$ and $|x_{s0} - x_0| \leq \delta$, then

$$\frac{dV}{dt} \leq -\frac{\gamma}{2} V \quad (93)$$

along the C^3 solutions of the system (21)–(23).

Now, we show that (93) also holds in the distribution sense when $\mathbf{z} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$. Let $(\mathbf{z}^{0n}, x_{s0}^n) \in H^4((0, x_0); \mathbb{R}^2) \times (0, L)$, $n \in \mathbb{N}$ be a sequence of functions that satisfy the fourth order compatibility conditions and

$$|\mathbf{z}^{0n}|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta, \quad |x_{s0}^n - x_0| \leq \delta, \quad (94)$$

such that \mathbf{z}^{0n} converges to \mathbf{z}^0 in $H^2((0, x_0); \mathbb{R}^2)$ and $x_{s_0}^n$ converges to x_{s_0} . From Remark 3, there exists a unique solution $(\mathbf{z}^n, x_s^n) \in C^0([0, T]; H^4((0, x_0); \mathbb{R}^2)) \times C^4([0, T]; \mathbb{R})$ to (21)–(23) corresponding to the initial condition $(\mathbf{z}^{0n}, x_{s_0}^n)$ and for any $t \in [0, T]$, we have

$$|\mathbf{z}^n(t, \cdot)|_{H^2((0, x_0); \mathbb{R}^2)} + |x_s^n(t) - x_0| \leq C(T) (|\mathbf{z}^{0n}|_{H^2((0, x_0); \mathbb{R}^2)} + |x_{s_0}^n - x_0|). \quad (95)$$

Hence, from (94), the sequence (\mathbf{z}^n, x_s^n) is bounded in $C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$. By [33, Corollary 4], we can extract a subsequence, which we still denote by (\mathbf{z}^n, x_s^n) that converges to (\mathbf{u}, y_s) in $(C^0([0, T]; C^1([0, x_0]; \mathbb{R}^2)) \cap C^1([0, T]; C^0([0, x_0]; \mathbb{R}^2))) \times C^1([0, T]; \mathbb{R})$, which is a classical solution of (21)–(23). If we define

$$J(\mathbf{u}) = \begin{cases} +\infty, & \text{if } \mathbf{u} \notin L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)), \\ |\mathbf{u}|_{L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2))}, & \text{if } \mathbf{u} \in L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)), \end{cases} \quad (96)$$

then J is lower semi-continuous and we have

$$J(\mathbf{u}) \leq \varliminf_{n \rightarrow +\infty} |\mathbf{z}^n|_{C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))}, \quad (97)$$

thus from (95) and the convergence of $(\mathbf{z}^{0n}, x_{s_0}^n)$ in $H^2((0, x_0); \mathbb{R}^2) \times \mathbb{R}$, we have $J(\mathbf{u}) \in \mathbb{R}$ and $\mathbf{u} \in L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2))$. Moreover, as (\mathbf{u}, y_s) is a solution to (21)–(23), we get the extra regularity $\mathbf{u} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$. Hence, from the uniqueness of the solution given by Lemma 3.1, $\mathbf{u} = \mathbf{z}$ and consequently $y_s = x_s$, which implies that (\mathbf{z}^n, x_s^n) converges to (\mathbf{z}, x_s) in $(C^0([0, T]; C^1([0, x_0]; \mathbb{R}^2)) \cap C^1([0, T]; C^0([0, x_0]; \mathbb{R}^2))) \times C^1([0, T]; \mathbb{R})$. Now, we define $V^n(t) := V(\mathbf{z}^n(t, \cdot), x_s^n(t))$ with V given by (30). Note that $V(t) = V(\mathbf{z}(t, \cdot), x_s(t))$ is continuous with time t and well-defined as, from Lemma 3.1, $\mathbf{z} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$. As (\mathbf{z}^n, x_s^n) is C^3 , we have from (93)

$$\frac{dV^n}{dt} \leq -\frac{\gamma}{2} V^n, \quad (98)$$

thus V^n is decreasing on $[0, T]$. Therefore

$$V^n(t) - V^n(0) \leq -\frac{\gamma t}{2} V^n(t), \quad \forall t \in [0, T], \quad (99)$$

which implies that

$$\left(1 + \frac{\gamma t}{2}\right) V^n(t) \leq V^n(0), \quad \forall t \in [0, T]. \quad (100)$$

Using the lower semi-continuity of J , by (42) and the convergence of $(\mathbf{z}^{0n}, x_{s_0}^n)$ in $H^2((0, x_0); \mathbb{R}^2) \times \mathbb{R}$, we have

$$\left(1 + \frac{\gamma t}{2}\right) V(t) \leq V(0), \quad \forall t \in [0, T]. \quad (101)$$

Note that instead of approximating (\mathbf{z}^0, x_{s_0}) , we could have approximated $(\mathbf{z}(s, \cdot), x_s(s))$ where $s \in [0, T]$ and have done all the previous procedure. Therefore we have in fact for any $s \in [0, T]$

$$\left(1 + \frac{\gamma(t-s)}{2}\right) V(t) \leq V(s), \quad \forall t \in [s, T], \quad (102)$$

thus for any $0 \leq s < t \leq T$

$$\frac{V(t) - V(s)}{t - s} \leq -\frac{\gamma}{2} V(t), \quad (103)$$

which implies that (93) holds in the distribution sense. This concludes the proof of Theorem 4.1. \square

5 Conclusion

The stabilization of shock-free regular solutions of quasilinear hyperbolic systems has been the subject of a large number of publications in the recent scientific literature. In contrast, there are no results concerning the Lyapunov stability of solutions with jump discontinuities, although they occur naturally in the form of shock waves or hydraulic jumps in many applications of fluid dynamics. For instance, the inviscid Burgers equation provides a simple scalar example of a hyperbolic system having natural solutions with jump discontinuities. The main contribution of this paper is precisely to address the issue of the boundary exponential feedback stabilization of an unstable shock steady state for the Burgers equation over a bounded interval. Our strategy to solve the problem relies on introducing a change of variables which allows to transform the scalar Burgers equation with shock wave solutions into an equivalent 2×2 quasilinear hyperbolic system having shock-free solutions over a bounded interval. Then, by a Lyapunov approach, we show that, for appropriately chosen boundary conditions, the exponential stability in H^2 -norm of the steady state can be achieved with an arbitrary decay rate and with an exact exponential stabilization of the desired shock location. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the selection of an appropriate Lyapunov function is challenging because the equivalent system is parameterized by the time-varying position of the jump discontinuity. In particular, the standard quadratic Lyapunov function used in the book [3] has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position.

A Appendix

Proof of Lemma 3.1. We adapt the fixed point method used in [3, Appendix B] (see also [20, 26]). We first deal with the case where

$$T \in (0, \min(x_0, L - x_0)). \quad (104)$$

For any $\nu > 0$ and $\mathbf{z}^0 \in H^2((0, x_0); \mathbb{R}^2)$, let $C_\nu(\mathbf{z}^0)$ be the set of

$$\mathbf{z} \in L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1,\infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \cap W^{2,\infty}((0, T); L^2((0, x_0); \mathbb{R}^2))$$

such that

$$\|\mathbf{z}\|_{L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2))} \leq \nu, \quad (105)$$

$$\|\mathbf{z}\|_{W^{1,\infty}((0, T); H^1((0, x_0); \mathbb{R}^2))} \leq \nu, \quad (106)$$

$$\|\mathbf{z}\|_{W^{2,\infty}((0, T); L^2((0, x_0); \mathbb{R}^2))} \leq \nu, \quad (107)$$

$$\mathbf{z}(\cdot, x_0) \in H^2((0, T); \mathbb{R}^2), \quad \|\mathbf{z}(\cdot, x_0)\|_{H^2((0, T); \mathbb{R}^2)} \leq \nu^2, \quad (108)$$

$$\mathbf{z}(0, \cdot) = \mathbf{z}^0, \quad (109)$$

$$\mathbf{z}_t(0, \cdot) = -A(\mathbf{z}^0, x, x_s(\mathbf{z}(\cdot, x_0))(0))\mathbf{z}_x^0, \quad (110)$$

where we write $x_s(\mathbf{z}(\cdot, x_0))(t)$ in order to emphasize its dependence on $\mathbf{z}(\cdot, x_0)$ in the following proof and

$$x_s(\mathbf{z}(\cdot, x_0))(t) =: x_{s0} + \int_0^t \frac{z_1(s, x_0) + z_2(s, x_0)}{2} ds. \quad (111)$$

In (110),

$$A(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \begin{pmatrix} a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) & 0 \\ 0 & a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) \end{pmatrix} \quad (112)$$

with

$$a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 + z_1(t, x) - x \frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{x_s(\mathbf{z}(\cdot, x_0))(t)}, \quad (113)$$

$$a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 - z_2(t, x) + x \frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{L - x_s(\mathbf{z}(\cdot, x_0))(t)}. \quad (114)$$

The set $C_\nu(\mathbf{z}^0)$ is not empty and is a closed subset of $L^\infty((0, T); L^2((0, L); \mathbb{R}^2))$ provided that $|\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} \leq \delta$ and $|x_{s0} - x_0| \leq \delta$, with δ sufficiently small (see for instance [3, Appendix B]).

Let us define a mapping:

$$\begin{aligned} \mathcal{F} : C_\nu(\mathbf{z}^0) &\longrightarrow L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1, \infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \\ &\qquad \qquad \qquad \cap W^{2, \infty}((0, T); L^2((0, x_0); \mathbb{R}^2)) \\ \mathbf{v} = (v_1, v_2)^T &\longmapsto \mathcal{F}(\mathbf{v}) = \mathbf{z} = (z_1, z_2)^T \end{aligned} \quad (115)$$

where \mathbf{z} is the solution of the linear hyperbolic equation

$$\mathbf{z}_t + A(\mathbf{v}, x, x_s(\mathbf{v}(\cdot, x_0))(t))\mathbf{z}_x = 0, \quad (116)$$

$$\mathbf{z}(0, x) = \mathbf{z}^0(x), \quad (117)$$

with boundary conditions

$$z_1(t, 0) = k_1 z_1(t, x_0) + b_1 \psi(t), \quad (118)$$

$$z_2(t, 0) = k_2 z_2(t, x_0) + b_2 \psi(t), \quad (119)$$

where

$$\psi(t) = x_0 - x_s(\mathbf{v}(\cdot, x_0))(t). \quad (120)$$

In the following, we will treat z_1 in details. For the sake of simplicity, we denote

$$f_1(t, x) := a_1(\mathbf{v}(t, x), x, x_s(\mathbf{v}(\cdot, x_0))(t)). \quad (121)$$

It is easy to check from (113) that if ν is sufficiently small, then $f_1(t, x)$ is strictly positive for any $(t, x) \in [0, T] \times [0, x_0]$. Let us now define the characteristic curve $\xi_1(s; t, x)$ passing through (t, x) as

$$\begin{aligned} \frac{d\xi_1(s; t, x)}{ds} &= f_1(s, \xi_1(s; t, x)), \\ \xi_1(t; t, x) &= x. \end{aligned} \quad (122)$$

One can see that for every $(t, x) \in [0, T] \times [0, x_0]$, $\xi_1(\cdot; t, x)$ is uniquely defined on some closed interval in $[0, T]$. From (104), only two cases can occur (see Figure 2): If $\xi_1(t; 0, 0) < x \leq x_0$, there exists $\beta_1 \in [0, x_0]$ depending on (t, x) such that

$$\beta_1 = \xi_1(0; t, x). \quad (123)$$

If $0 < x < \xi_1(t; 0, 0)$, there exists $\alpha_1 \in [0, t]$ depending on (t, x) such that

$$\xi_1(\alpha_1; t, x) = 0, \quad (124)$$

and in this case, there exists $\gamma_1 \in [0, x_0]$ depending on α_1 such that

$$\gamma_1 = \xi_1(0; \alpha_1, x_0). \quad (125)$$

Moreover, we have the following lemma which will be used in the estimations hereafter (the proof can be found at the end of this appendix).

Lemma A.1. *There exist $\nu_0 > 0$ and $C > 0$ such that, for any T satisfying (104), for any $\nu \in (0, \nu_0]$ and for a.e. $t \in (0, T)$, we have*

$$|f_1(t, \cdot)|_0 \leq C, \quad |f_{1x}(t, \cdot)|_0 \leq C\nu, \quad |f_{1t}(t, \cdot)|_0 \leq C\nu, \quad (126)$$

$$|\partial_x \xi_1(s; t, \cdot)|_0 \leq C, \quad |\partial_t \xi_1(s; t, \cdot)|_0 \leq C, \quad s \in [0, t], \quad (127)$$

$$|\partial_x \beta_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \beta_1(t, \cdot))^{-1}|_0 \leq C, \quad (128)$$

$$|\partial_t \beta_1(t, \cdot)|_0 \leq C, \quad |(\partial_t \beta_1(t, \cdot))^{-1}|_0 \leq C, \quad (129)$$

$$|\partial_x \alpha_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \alpha_1(t, \cdot))^{-1}|_0 \leq C, \quad (130)$$

$$|\partial_x \gamma_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \gamma_1(t, \cdot))^{-1}|_0 \leq C, \quad (131)$$

$$\int_0^T |\partial_{tt}\beta_1(t, x_0)|^2 dt \leq C\nu, \quad (132)$$

$$\int_0^{x_0} |\partial_{xx}\alpha_1(t, x)|^2 dx \leq C\nu, \quad (133)$$

$$\int_0^{x_0} |\partial_{xx}\beta_1(t, x)|^2 dx \leq C\nu, \quad (134)$$

$$\int_0^{x_0} |\partial_{xx}\gamma_1(t, x)|^2 dx \leq C\nu. \quad (135)$$

In these inequalities, and hereafter in this section, $|f|_0$ denotes the C^0 -norm of a function f with respect to its variable and C may depend on x_0 , x_{s0} , ν_0 , k_1 , k_2 , b_1 and b_2 , but is independent of ν , T , \mathbf{v} and \mathbf{z} .

Our goal is now to use a fixed point argument to show the existence and uniqueness of the solution to (21)–(23). Firstly, we show that for ν and δ sufficiently small, \mathcal{F} maps $C_\nu(\mathbf{z}^0)$ into itself, i.e.,

$$\mathcal{F}(C_\nu(\mathbf{z}^0)) \subset C_\nu(\mathbf{z}^0).$$

Then, in a second step, we prove that \mathcal{F} is a contraction mapping.

1) \mathcal{F} maps $C_\nu(\mathbf{z}^0)$ into itself.

For any $\mathbf{v} \in C_\nu(\mathbf{z}^0)$, let $\mathbf{z} = \mathcal{F}(\mathbf{v})$, we prove that $\mathbf{z} \in C_\nu(\mathbf{z}^0)$. By the definition of \mathcal{F} in (115), using the method of characteristics, we can solve (116) to (119) for z_1 and obtain that

$$z_1(t, x) = \begin{cases} k_1 z_1^0(\gamma_1) + b_1 \psi(\alpha_1), & 0 < x < \xi_1(t; 0, 0), \\ z_1^0(\beta_1), & \xi_1(t; 0, 0) < x < x_0. \end{cases} \quad (136)$$

Obviously \mathbf{z} verifies the properties (109)–(110). Next, we prove that \mathbf{z} verifies the property (108). Using the change of variables and from (129), we have

$$\int_0^T z_1(t, x_0)^2 dt = \int_0^T z_1^0(\beta_1(t, x_0))^2 dt \leq C \int_0^{x_0} (z_1^0(x))^2 dx. \quad (137)$$

In (137) and hereafter, C denotes various constants that may depend on x_0 , x_{s0} , ν_0 , k_1 , k_2 , b_1 and b_2 , but are independent of ν , T , \mathbf{v} and \mathbf{z} . Similarly, by (129), we obtain

$$\int_0^T z_{1t}(t, x_0)^2 dt = \int_0^T (z_{1x}^0(\beta_1(t, x_0)) \partial_t \beta_1(t, x_0))^2 dt \leq C \int_0^T (z_{1x}^0(x))^2 dx. \quad (138)$$

From (132) and using Sobolev inequality, one has

$$\begin{aligned} \int_0^T z_{1tt}(t, x_0)^2 dt &= \int_0^T (z_{1xx}^0(\beta_1(t, x_0)) (\partial_t \beta_1(t, x_0))^2 + z_{1x}^0(\beta_1(t, x_0)) \partial_{tt} \beta_1(t, x_0))^2 dt \\ &\leq C \int_0^{x_0} (z_{1xx}^0(x))^2 dx + 2|z_{1x}^0|_0^2 \int_0^T (\partial_{tt} \beta_1(t, x_0))^2 dt \\ &\leq C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2. \end{aligned} \quad (139)$$

Combining (137)–(139), we get

$$|z_1(\cdot, x_0)|_{H^2((0, T); \mathbb{R})} \leq C|z_1^0|_{H^2((0, x_0); \mathbb{R})}. \quad (140)$$

Applying similar estimate to z_2 gives

$$|z_2(\cdot, x_0)|_{H^2((0, T); \mathbb{R})} \leq C|z_2^0|_{H^2((0, x_0); \mathbb{R})}. \quad (141)$$

From (140) and (141), we can select δ sufficiently small such that

$$|\mathbf{z}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)} \leq \nu^2, \quad (142)$$

which shows both the regularity and the boundedness property (108). We can again use the method of characteristics to prove properties (105)–(107). For a.e. $t \in (0, T)$,

$$z_{1x}(t, x) = \begin{cases} k_1 z_{1x}^0(\gamma_1) \partial_x \gamma_1 + b_1 \dot{\psi}(\alpha_1) \partial_x \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1x}^0(\beta_1) \partial_x \beta_1, & \xi_1(t; 0, 0) < x < x_0. \end{cases} \quad (143)$$

$$z_{1xx}(t, x) = \begin{cases} k_1 z_{1xx}^0(\gamma_1) \partial_{xx} \gamma_1 + k_1 z_{1xx}^0(\gamma_1) (\partial_x \gamma_1)^2 \\ + b_1 \ddot{\psi}(\alpha_1) (\partial_x \alpha_1)^2 + b_1 \dot{\psi}(\alpha_1) \partial_{xx} \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1xx}^0(\beta_1) \partial_{xx} \beta_1 + z_{1xx}^0(\beta_1) (\partial_x \beta_1)^2, & \xi_1(t; 0, 0) < x < x_0. \end{cases} \quad (144)$$

Note that the last equation is true in distribution sense but shows that z_1 belongs to $L^\infty((0, T); H^2((0, x_0); \mathbb{R}))$. We first estimate $\|\mathbf{z}\|_{L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2))}$. From (111) and (120), using Sobolev inequality, we get

$$|\psi|_0 \leq |x_{s0} - x_0| + C|\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}, \quad (145)$$

$$|\dot{\psi}|_0 \leq C|\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}, \quad (146)$$

$$|\ddot{\psi}|_0 \leq C|\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}. \quad (147)$$

From (136), (143) and (144), we can compute directly using (128), (130)–(131) and (133)–(135) that

$$\begin{aligned} \int_0^{x_0} z_1^2 dx &\leq (|\partial_x \beta_1(t, \cdot)|_0^{-1} + 2k_1^2 |\partial_x \gamma_1(t, \cdot)|_0^{-1}) \int_0^{x_0} (z_1^0(x))^2 dx + 2b_1^2 x_0 |\psi|_0^2, \\ &\leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |x_{s0} - x_0|^2 + |\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2), \end{aligned} \quad (148)$$

$$\begin{aligned} \int_0^{x_0} z_{1x}^2 dx &\leq (|\partial_x \beta_1(t, \cdot)|_0 + 2k_1^2 |\partial_x \gamma_1(t, \cdot)|_0) \int_0^{x_0} (z_{1x}^0(x))^2 dx + 2x_0 b_1^2 |\dot{\psi}|_0^2 |\partial_x \alpha_1(t, \cdot)|_0^2 \\ &\leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2), \end{aligned} \quad (149)$$

$$\begin{aligned} \int_0^{x_0} z_{1xx}^2 dx &\leq \left(2 |\partial_x \beta_1(t, \cdot)|_0^3 + 4k_1^2 |\partial_x \gamma_1(t, \cdot)|_0^3\right) \int_0^{x_0} (z_{1xx}^0)^2 dx \\ &\quad + 2 |z_{1x}^0|_0^2 \int_0^{x_0} |\partial_{xx} \beta_1(t, x)|^2 dx + 4k_1^2 |z_{1x}^0|_0^2 \int_0^{x_0} |\partial_{xx} \gamma_1(t, x)|^2 dx \\ &\quad + 4b_1^2 |\partial_x \alpha_1(t, x)|_0^4 \int_0^{x_0} |\ddot{\psi}(\alpha_1(t, x))|^2 dx + 4b_1^2 |\dot{\psi}|_0^2 \int_0^{x_0} |\partial_{xx} \alpha_1(t, x)|^2 dx \\ &\leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2). \end{aligned} \quad (150)$$

Combining (148)–(150), we obtain

$$\|z_1(t, \cdot)\|_{H^2((0, x_0); \mathbb{R})} \leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})} + |x_{s0} - x_0| + |\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}), \quad (151)$$

Similarly, one can get

$$\|z_2(t, \cdot)\|_{H^2((0, x_0); \mathbb{R})} \leq C(|z_2^0|_{H^2((0, x_0); \mathbb{R})} + |x_{s0} - x_0| + |\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}). \quad (152)$$

Noticing from $\mathbf{v} \in C_\nu(\mathbf{z}^0)$ that

$$|\mathbf{v}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)} \leq \nu^2,$$

thus by selecting δ and $\nu \in (0, \nu_0]$ sufficiently small, in addition to the previous hypothesis on δ , we have indeed

$$\|\mathbf{z}(t, \cdot)\|_{H^2((0, x_0); \mathbb{R}^2)} \leq \nu, \quad \text{a.e. } t \in (0, T), \quad (153)$$

which proves (105). The same method as to prove (153) enables us to show that z_1 verifies also (106) and (107). One only has to realize that

$$\begin{aligned} z_{1t}(t, x) &= \begin{cases} k_1 z_{1t}^0(\gamma_1) \partial_t \gamma_1 + b_1 \dot{\psi}(\alpha_1) \partial_t \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1t}^0(\beta_1) \partial_t \beta_1, & \xi_1(t; 0, 0) < x < x_0. \end{cases} \\ z_{1tt}(t, x) &= \begin{cases} k_1 z_{1tt}^0(\gamma_1) \partial_{tt} \gamma_1 + k_1 z_{1tt}^0(\gamma_1) (\partial_t \gamma_1)^2 + b_1 \ddot{\psi}(\alpha_1) (\partial_t \alpha_1)^2 + b_1 \dot{\psi}(\alpha_1) \partial_{tt} \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1tt}^0(\beta_1) \partial_{tt} \beta_1 + z_{1tt}^0(\beta_1) (\partial_t \beta_1)^2, & \xi_1(t; 0, 0) < x < x_0. \end{cases} \end{aligned}$$

$$z_{1tx}(t, x) = \begin{cases} k_1 z_{1x}^0(\gamma_1) \partial_x(\partial_t \gamma_1) + k_1 z_{1xx}^0(\gamma_1) (\partial_x \gamma_1 \partial_t \gamma_1) & 0 < x < \xi_1(t; 0, 0), \\ + b_1 \dot{\psi}(\alpha_1) (\partial_t \alpha_1 \partial_x \alpha_1) + b_1 \dot{\psi}(\alpha_1) \partial_x(\partial_t \alpha_1), & \\ z_{1x}^0(\beta_1) \partial_x(\partial_t \beta_1) + z_{1xx}^0(\beta_1) (\partial_x \beta_1 \partial_t \beta_1), & \xi_1(t; 0, 0) < x < x_0, \end{cases}$$

and to estimate $\int_0^{\xi_1(t; 0, 0)} |\partial_{tt} \alpha_1|^2 dx$, $\int_{\xi_1(t; 0, 0)}^{x_0} |\partial_{tt} \beta_1|^2 dx$, $\int_0^{\xi_1(t; 0, 0)} |\partial_{tt} \gamma_1|^2 dx$, $\int_0^{\xi_1(t; 0, 0)} |\partial_x(\partial_t \alpha_1)|^2 dx$, $\int_{\xi_1(t; 0, 0)}^{x_0} |\partial_x(\partial_t \beta_1)|^2 dx$ and $\int_0^{\xi_1(t; 0, 0)} |\partial_x(\partial_t \gamma_1)|^2 dx$ similarly as in (133)–(135) using the fact that \mathbf{v} belongs to $L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1, \infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \cap W^{2, \infty}((0, T); L^2((0, x_0); \mathbb{R}^2))$ with bound ν in these norms.

We can clearly perform similar estimates for z_2 . Consequently there exist δ and $\nu_1 \in (0, \nu_0]$ sufficiently small depending only on C such that, for any $\nu \in (0, \nu_1]$, $\mathbf{z} = \mathcal{F}(\mathbf{v})$ verifies properties (105)–(110) and therefore $\mathcal{F}(C_\nu(\mathbf{z}^0)) \subset C_\nu(\mathbf{z}^0)$.

2) \mathcal{F} is a contraction mapping.

Next, we prove that \mathcal{F} is a contraction mapping satisfying the following inequality:

$$\begin{aligned} & |\mathcal{F}(\mathbf{v}) - \mathcal{F}(\bar{\mathbf{v}})|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}^2))} + M |\mathcal{F}(\mathbf{v})(\cdot, x_0) - \mathcal{F}(\bar{\mathbf{v}})(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)} \\ & \leq \frac{1}{2} |\mathbf{v} - \bar{\mathbf{v}}|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}^2))} + \frac{M}{2} |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}, \end{aligned} \quad (154)$$

where $M > 0$ is a constant. We start with z_1 , and with the estimate of $|z_1 - \bar{z}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}$. For any chosen \mathbf{v} and $\bar{\mathbf{v}}$ from $C_\nu(\mathbf{z}^0)$, without loss of generality, we may assume that $\xi_1(t; 0, 0) < \bar{\xi}_1(t; 0, 0)$, where $\bar{\xi}_1$ is the characteristic defined in (122) associated to $\bar{\mathbf{v}}$. From (136), we have

$$\begin{aligned} & \int_0^{x_0} |z_1(t, x) - \bar{z}_1(t, x)|^2 dx \\ & = \int_0^{\xi_1(t; 0, 0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1) + b_1 \dot{\psi}(\alpha_1) - b_1 \dot{\psi}(\bar{\alpha}_1)|^2 dx \\ & \quad + \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \dot{\psi}(\bar{\alpha}_1))|^2 dx + \int_{\bar{\xi}_1(t; 0, 0)}^{x_0} |z_1^0(\beta_1) - z_1^0(\bar{\beta}_1)|^2 dx. \end{aligned} \quad (155)$$

From the definition of ψ in (120) and (111), using Sobolev and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} & \int_0^{\xi_1(t; 0, 0)} |b_1 \dot{\psi}(\alpha_1) - b_1 \dot{\psi}(\bar{\alpha}_1)|^2 dx \\ & = \int_0^{\xi_1(t; 0, 0)} b_1^2 \left| \int_0^{\alpha_1} \frac{v_1(s, x_0) + v_2(s, x_0)}{2} ds - \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s, x_0) + \bar{v}_2(s, x_0)}{2} ds \right|^2 dx \\ & \leq C |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + C |\bar{\mathbf{v}}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2 \int_0^{\xi_1(t; 0, 0)} |\alpha_1 - \bar{\alpha}_1|^2 dx. \end{aligned} \quad (156)$$

By the definition of γ_1 in (125) and the corresponding definition of $\bar{\gamma}_1$ and using (127), we obtain

$$\int_0^{\xi_1(t; 0, 0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\bar{\gamma}_1)|^2 dx \leq C |z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_0^{\xi_1(t; 0, 0)} |\alpha_1 - \bar{\alpha}_1|^2 dx. \quad (157)$$

Combining (155)–(157), we get

$$\begin{aligned} & \int_0^{x_0} |z_1(t, x) - \bar{z}_1(t, x)|^2 dx \\ & \leq C (|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |\bar{\mathbf{v}}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2) \int_0^{\xi_1(t; 0, 0)} |\alpha_1 - \bar{\alpha}_1|^2 dx \\ & \quad + |z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_{\xi_1(t; 0, 0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 dx + \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \dot{\psi}(\bar{\alpha}_1))|^2 dx \\ & \quad + C |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2. \end{aligned} \quad (158)$$

We estimate each term in (158) separately. By the definition of β_1 in (123) and the corresponding definition of $\bar{\beta}_1$, we have

$$\int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 dx = \int_{\bar{\xi}_1(t;0,0)}^{x_0} |\xi_1(0; t, x) - \bar{\xi}_1(0; t, x)|^2 dx. \quad (159)$$

Now, let us estimate $|\xi_1(0; t, x) - \bar{\xi}_1(0; t, x)|$. From the definition of x_s in (111) and the definitions of ξ_1 and $\bar{\xi}_1$, see (122), we get for any $s \in [0, t]$ that

$$\begin{aligned} & |\xi_1(s; t, x) - \bar{\xi}_1(s; t, x)| \\ &= \left| \int_s^t f_1(\theta, \xi_1(\theta; t, x)) d\theta - \int_s^t \bar{f}_1(\theta, \bar{\xi}_1(\theta; t, x)) d\theta \right| \\ &\leq \int_s^t \left(\left| \left(1 + v_1(\theta, \xi_1) - \xi_1 \frac{v_1(\theta, x_0) + v_2(\theta, x_0)}{2x_0} \right) \frac{x_0}{x_s(\mathbf{v}(\cdot, x_0))(\theta)x_s(\bar{\mathbf{v}}(\cdot, x_0))(\theta)} \right| \right. \\ &\quad \cdot \left. \int_0^\theta \left| \frac{v_1(\alpha, x_0) - \bar{v}_1(\alpha, x_0) + v_2(\alpha, x_0) - \bar{v}_2(\alpha, x_0)}{2} \right| d\alpha \right) d\theta \\ &\quad + \int_s^t \left| \frac{x_0}{x_s(\bar{\mathbf{v}}(\cdot, x_0))(\theta)} \right| \cdot \left| v_1(\theta, \xi_1) - \bar{v}_1(\theta, \bar{\xi}_1) + \bar{\xi}_1 \frac{\bar{v}_1(\theta, x_0) + \bar{v}_2(\theta, x_0)}{2x_0} - \xi_1 \frac{v_1(\theta, x_0) + v_2(\theta, x_0)}{2x_0} \right| d\theta \\ &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} + C\nu \int_s^t |\xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x)| d\theta \\ &\quad + C \int_s^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \end{aligned} \quad (160)$$

From (160), we get for $\nu \in (0, \nu_0]$ sufficiently small and for $\bar{\xi}_1(t; 0, 0) < x \leq x_0$ that

$$\begin{aligned} |\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([0,t];\mathbb{R})} &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} \\ &\quad + C \int_0^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \end{aligned} \quad (161)$$

Thus, from (159) and (161) we have

$$\begin{aligned} \int_{\bar{\xi}_1(t;0,0)}^{x_0} |\beta_1 - \bar{\beta}_1|^2 dx &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}^2 \\ &\quad + C \int_{\bar{\xi}_1(t;0,0)}^{x_0} \left(\int_0^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta \right)^2 dx \\ &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}^2 \\ &\quad + C \int_0^t \int_{\bar{\xi}_1(t;0,0)}^{x_0} |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))|^2 dx d\theta \\ &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}^2 \\ &\quad + C|v_1 - \bar{v}_1|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}^2. \end{aligned} \quad (162)$$

The last inequality is obtained using the change of variable $y = \bar{\xi}_1(\theta; t, x)$, well-defined for $0 \leq \theta \leq t \leq T$ and $\bar{\xi}_1(t; 0, 0) < x \leq x_0$. Let us now estimate $|\alpha_1 - \bar{\alpha}_1|_{L^2((0,\xi_1(t;0,0));\mathbb{R})}$. Without loss of generality, we may assume that $\alpha_1 \leq \bar{\alpha}_1$. By definition of α_1 in (124) and the corresponding definition of $\bar{\alpha}_1$, we have

$$\int_{\alpha_1}^t f_1(s, \xi_1(s; t, x)) ds = x = \int_{\bar{\alpha}_1}^t \bar{f}_1(s, \bar{\xi}_1(s; t, x)) ds. \quad (163)$$

Hence, similarly to (160), we get

$$\begin{aligned} |\alpha_1 - \bar{\alpha}_1| &\leq \frac{1}{\inf_{(t,x) \in [0,T] \times [0,x_0]} |f_1(t, x)|} \int_{\bar{\alpha}_1}^t |f_1(s, \xi_1(s; t, x)) - \bar{f}_1(s, \bar{\xi}_1(s; t, x))| ds \\ &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} + C\nu \int_{\bar{\alpha}_1}^t |\xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x)| d\theta \end{aligned}$$

$$+ C \int_{\bar{\alpha}_1}^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \quad (164)$$

Similarly to the proof of (161), for $\nu \in (0, \nu_0]$ sufficiently small, we can obtain that (note that $\xi_1(s; t, x)$ and $\bar{\xi}_1(s; t, x)$ for any $s \in [\bar{\alpha}_1, t]$ are well defined as we assume that $\alpha_1 \leq \bar{\alpha}_1$)

$$\begin{aligned} |\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([\bar{\alpha}_1, t]; \mathbb{R})} &\leq C |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)} \\ &\quad + C \int_{\bar{\alpha}}^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \end{aligned} \quad (165)$$

Using this inequality in (164) and performing similarly as in (162), we can obtain

$$\int_0^{\xi_1(t; 0, 0)} |\alpha_1 - \bar{\alpha}_1|^2 dx \leq C |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + C |v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2. \quad (166)$$

Let us now focus on the estimation of the term $\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \bar{\psi}(\bar{\alpha}_1))|^2 dx$ in (158). Using the compatibility condition (24), we have

$$\begin{aligned} &\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \bar{\psi}(\bar{\alpha}_1))|^2 dx \\ &= \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - z_1^0(0) + z_1^0(0) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \bar{\psi}(\bar{\alpha}_1))|^2 dx \\ &= \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - z_1^0(0) + k_1 z_1^0(x_0) + b_1(x_0 - x_{s0}) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \bar{\psi}(\bar{\alpha}_1))|^2 dx \\ &\leq C |z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\beta_1|^2 dx + C |z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |x_0 - \bar{\gamma}_1|^2 dx \\ &\quad + C \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} \left| \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s, x_0) + \bar{v}_2(s, x_0)}{2} ds \right|^2 dx. \end{aligned} \quad (167)$$

We first estimate $\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\beta_1|^2 dx$. As $\xi_1(s; t, x)$ is increasing with respect to $s \in [0, t]$, we have

$$|\beta_1| < |\xi_1(\bar{\alpha}_1; t, x)| = |\xi_1(\bar{\alpha}_1; t, x) - \bar{\xi}_1(\bar{\alpha}_1; t, x)| \leq |\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([\bar{\alpha}_1, t]; \mathbb{R})}, \quad (168)$$

then by (165) and performing the same proof as in (162), we get

$$\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\beta_1|^2 dx \leq C |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + C |v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2. \quad (169)$$

Let us now look at the second term in (167), from (127) and the definition of $\bar{\gamma}_1$, we have

$$\begin{aligned} \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |x_0 - \bar{\gamma}_1|^2 dx &= \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\xi}_1(0; 0, x_0) - \bar{\xi}_1(0; \bar{\alpha}_1, x_0)|^2 dx \\ &\leq |\partial_t \bar{\xi}_1|_0^2 \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\alpha}_1|^2 dx \leq C \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\alpha}_1|^2 dx. \end{aligned} \quad (170)$$

It is easy to deal with the last term in (167), one has

$$\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} \left| \int_0^{\bar{\alpha}_1} \frac{\bar{v}_1(s, x_0) + \bar{v}_2(s, x_0)}{2} ds \right|^2 dx \leq C |\bar{\mathbf{v}}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2 \int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\alpha}_1|^2 dx. \quad (171)$$

Thus, we only have to estimate $\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\alpha}_1|^2 dx$. Noticing that for any fixed (t, x) , the characteristic $\bar{\xi}_1(s; t, x)$ is increasing with respect to $s \in [\bar{\alpha}_1, t]$ and that $\xi_1^{-1}(\cdot; t, x)(\beta_1) = 0$, we obtain

$$\bar{\alpha}_1 < \bar{\xi}_1^{-1}(\cdot; t, x)(\beta_1) - \xi_1^{-1}(\cdot; t, x)(\beta_1).$$

Moreover,

$$\begin{aligned}\beta_1 &= x + \int_t^{\xi_1^{-1}(\cdot; t, x)(\beta_1)} f_1(s; \xi_1(s; t, x)) d\theta, \\ \beta_1 &= x + \int_t^{\bar{\xi}_1^{-1}(\cdot; t, x)(\beta_1)} \bar{f}_1(s; \bar{\xi}_1(s; t, x)) d\theta.\end{aligned}$$

Then similarly as for (164), we can prove that

$$\begin{aligned}|\bar{\xi}_1^{-1}(\cdot; t, x)(\beta_1) - \xi_1^{-1}(\cdot; t, x)(\beta_1)| &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2(0, T)} \\ &\quad + C\nu \int_{\bar{\xi}_1^{-1}(\cdot; t, x)(\beta_1)}^t |\xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x)| d\theta \\ &\quad + C \int_{\bar{\xi}_1^{-1}(\cdot; t, x)(\beta_1)}^t |v_1(\theta, \bar{\xi}_1(\theta; t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta.\end{aligned}$$

Thus, similarly as in the proof for (166), we get

$$\begin{aligned}\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |\bar{\alpha}_1|^2 dx &\leq C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 \\ &\quad + C|v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2.\end{aligned}\tag{172}$$

Finally, using estimations (169) and (170)–(172), (167) becomes

$$\begin{aligned}&\int_{\xi_1(t; 0, 0)}^{\bar{\xi}_1(t; 0, 0)} |z_1^0(\beta_1) - (k_1 z_1^0(\bar{\gamma}_1) + b_1 \psi(\bar{\alpha}_1))|^2 dx \\ &\leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |\bar{\mathbf{v}}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2) \left(|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + |v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2 \right).\end{aligned}\tag{173}$$

Combining (158), (162), (166) and (173), we get

$$\begin{aligned}&|z_1 - \bar{z}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2 \\ &\leq C(|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 + |\bar{\mathbf{v}}(\cdot, x_0)|_{H^2((0, T); \mathbb{R}^2)}^2) \left(|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + |v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2 \right) \\ &\quad + C|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2.\end{aligned}\tag{174}$$

We are left with estimating $|\mathbf{z}(\cdot, x_0) - \bar{\mathbf{z}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}$ in order to obtain (154). Here we give the estimation for z_1 . Using (161), we get

$$\begin{aligned}&\int_0^T |z_1(t, x_0) - \bar{z}_1(t, x_0)|^2 dt \\ &= \int_0^T |z_1^0(\xi_1(0; t, x_0)) - z_1^0(\bar{\xi}_1(0; t, x_0))|^2 \\ &\leq |z_{1x}^0|_0^2 \int_0^T |\xi_1(0; t, x_0) - \bar{\xi}_1(0; t, x_0)|^2 dt \\ &\leq C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 \\ &\quad + C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_0^T \int_0^t |v_1(\theta, \bar{\xi}_1(\theta; t, x_0)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x_0))|^2 d\theta dt \\ &\leq C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 |\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 \\ &\quad + C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \int_0^T \int_\theta^T |v_1(\theta, \bar{\xi}_1(\theta; t, x_0)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x_0))|^2 dt d\theta \\ &\leq C|z_1^0|_{H^2((0, x_0); \mathbb{R})}^2 \left(|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0, T); \mathbb{R}^2)}^2 + |v_1 - \bar{v}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}))}^2 \right).\end{aligned}\tag{175}$$

The last inequality is obtained by changing the variable $y = \bar{\xi}_1(\theta; t, x_0)$. Similar estimates can be done for z_2 . Hence, from (174) and (175), there exists $M > 0$ such that for δ sufficiently small and $\nu \in (0, \nu_2]$, where $\nu_2 \in (0, \nu_1]$ is sufficiently small and depends only on C , we have

$$\begin{aligned} & |\mathbf{z} - \bar{\mathbf{z}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + M|\mathbf{z}(\cdot, x_0) - \bar{\mathbf{z}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)} \\ & \leq \frac{1}{2}|\mathbf{v} - \bar{\mathbf{v}}|_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}^2))} + \frac{M}{2}|\mathbf{v}(\cdot, x_0) - \bar{\mathbf{v}}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}. \end{aligned} \quad (176)$$

Hence \mathcal{F} is a contraction mapping and has a fixed point $\mathbf{z} \in C_\nu(\mathbf{z}^0)$, i.e., there exists a unique solution $\mathbf{z} \in C_\nu(\mathbf{z}^0)$ to the system (21)–(23). Noticing (111), we get that $x_s \in C^1([0, T]; \mathbb{R})$. To get the extra regularity $\mathbf{z} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$, we adapt the proof given by Majda [28, p.44–46]. There, the author used energy estimates method for an initial value problem. Using this method for our boundary value problem, we have to be careful with the boundary terms when integrating by parts. Substituting \mathbf{v} by \mathbf{z} in $\psi(t)$ and $f_1(t, x)$ in the expression of z_{1x} , z_{1xx} in (143) and (144), noticing (105)–(107) and computing similar estimates as in (149) and (150), we can obtain the “hidden” regularity $\mathbf{z}_x(\cdot, x_0) \in L^2((0, T); \mathbb{R}^2)$ and $\mathbf{z}_{xx}(\cdot, x_0) \in L^2((0, T); \mathbb{R}^2)$ together with estimates on $|\mathbf{z}_x(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}$ and $|\mathbf{z}_{xx}(\cdot, x_0)|_{L^2((0,T);\mathbb{R}^2)}$, which are sufficient to take care of the boundary terms when integrating by parts. This concludes the proof of the existence and uniqueness of a classical solution $x_s(t) \in C^1([0, T]; \mathbb{R})$ and $\mathbf{z} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$ in $C_\nu(\mathbf{z}^0)$ to the system (21)–(23) for T satisfying (104).

The estimate (27) for $|\mathbf{z}(t, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ part can be obtained from estimates (151)–(152) by firstly replacing \mathbf{v} with \mathbf{z} and then applying (140)–(141). Noticing the definition of x_s in (111) and applying (140)–(141) again, the estimate for the $|x_s(t) - x_0|$ part follows.

Next, we show the uniqueness of the solution in $C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$. Suppose that there is another solution $\tilde{\mathbf{z}} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$, we prove that $\tilde{\mathbf{z}} \in C_\nu(\mathbf{z}^0)$, for δ sufficiently small. To that end, assume that $\mathbf{z}(t, \cdot) = \tilde{\mathbf{z}}(t, \cdot)$ for any $t \in [0, \tau]$ with $\tau \in [0, T]$. By (27), for δ sufficiently small and as $\tilde{\mathbf{z}} \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2))$, one can choose $\tau' \in (\tau, T)$ small enough such that $\tilde{\mathbf{z}} \in C_\nu(\mathbf{z}(\tau))$ with T is replaced by $\tau' - \tau$ and by considering τ as the new initial time. Thus, $\mathbf{z}(t, \cdot) = \tilde{\mathbf{z}}(t, \cdot)$ for any $t \in [0, \tau']$. As $|\tilde{\mathbf{z}}(t, \cdot)|_{H^2((0,x_0);\mathbb{R}^2)}$ is uniformly continuous on $[0, T]$, and as, moreover C and ν do not depend on T , we can repeat this process and finally get $\mathbf{z}(t, \cdot) = \tilde{\mathbf{z}}(t, \cdot)$ on $[0, T]$.

For general $T > 0$, one just needs to take T_1 satisfying (104) and, noticing that C and ν do not depend on T_1 , one can apply the above procedure at most $\lceil T/T_1 \rceil + 1$ times. This concludes the proof of Lemma 3.1. \square

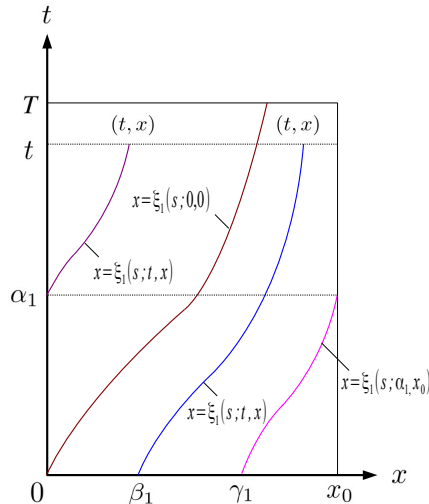


Figure 2: Demonstration of the characteristics.

Proof of Lemma A.1. From (122), we have

$$\begin{cases} \frac{\partial^2 \xi_1(s; t, x)}{\partial s \partial x} = f_{1x} \frac{\partial \xi_1(s; t, x)}{\partial x}, \\ \frac{\partial \xi_1(t; t, x)}{\partial x} = 1, \end{cases} \quad (177)$$

and

$$\begin{cases} \frac{\partial^2 \xi_1(s; t, x)}{\partial s \partial t} = f_{1x} \frac{\partial \xi_1(s; t, x)}{\partial t}, \\ \frac{\partial \xi_1(s; s, x)}{\partial s} + \frac{\partial \xi_1(s; s, x)}{\partial t} = 0. \end{cases} \quad (178)$$

Thus,

$$\partial_x \xi_1(s; t, x) = e^{-\int_s^t f_{1x}(\theta, \xi_1(\theta; t, x)) d\theta}, \quad (179)$$

$$\partial_t \xi_1(s; t, x) = -f_1(t, x) e^{-\int_s^t f_{1x}(\theta, \xi_1(\theta; t, x)) d\theta}. \quad (180)$$

From (179)–(180) and noticing $\beta_1 = \xi_1(0; t, x)$, we have

$$\frac{\partial \beta_1}{\partial t} = -f_1(t, x) e^{-\int_0^t f_{1x}(\theta, \xi_1(\theta; t, x)) d\theta}, \quad \frac{\partial \beta_1}{\partial x} = e^{-\int_0^t f_{1x}(\theta, \xi_1(\theta; t, x)) d\theta}. \quad (181)$$

From (179), noticing $\xi_1(\alpha_1; t, x) = 0$ and by chain rules, we have

$$\frac{\partial \alpha_1}{\partial x} = -\frac{1}{f_1(\alpha_1, 0)} e^{-\int_{\alpha_1}^t f_{1x}(s, \xi_1(s; t, x)) ds}, \quad (182)$$

and as $\gamma_1 = \xi_1(0; \alpha_1, x_0)$, we obtain from (180) that

$$\frac{\partial \gamma_1}{\partial x} = \frac{d\gamma_1}{d\alpha_1} \frac{\partial \alpha_1}{\partial x} = \frac{f_1(\alpha_1, x_0)}{f_1(\alpha_1, 0)} e^{-\int_0^{\alpha_1} f_{1x}(s, \xi_1(s; \alpha_1, x_0)) ds - \int_{\alpha_1}^t f_{1x}(s, \xi_1(s; t, x)) ds}. \quad (183)$$

Observe that for a.e. $s \in (0, T)$ and $x \in [0, x_0]$,

$$|v_1(s, x)| \leq \left| \int_{\theta}^x v_{1x}(s, l) dl \right| + |v_1(s, \theta)|, \quad \forall \theta \in [0, x_0] \quad (184)$$

and as v_1 is H^1 in x and its L^2 -norm is bounded by ν , there exists θ such that $|v_1(s, \theta)| \leq \nu/\sqrt{x_0}$, therefore

$$|v_1(s, x)| \leq C\nu, \quad (185)$$

and similarly as v_1 is H^2 in x with the same bound and v_{1t} is in $L^\infty((0, T); H^1((0, x_0); \mathbb{R}))$ with bound ν from (106)

$$\begin{aligned} x \in [0, x_0], \quad |v_{1x}(s, x)| &\leq C\nu, \text{ for a.e. } s \in (0, T), \\ x \in [0, x_0], \quad |v_{1t}(s, x)| &\leq C\nu, \text{ for a.e. } s \in (0, T). \end{aligned} \quad (186)$$

From the expression of f_1 defined in (121) and using (179)–(183) and (185)–(186), after some direct computations, estimates (126)–(131) can be obtained. We now demonstrate the estimate (132) in details, while (133)–(135) can be treated in a similar way, thus we omit them. From (181), we have

$$\partial_{tt} \beta_1 = \left(-f_{1t}(t, x) + f_1(t, x) \left(f_{1x}(t, x) + \int_0^t f_{1xx}(\theta, \xi_1(\theta; t, x)) \partial_t \xi_1(\theta; t, x) d\theta \right) \right) e^{-\int_0^t f_{1x}(\theta, \xi_1(\theta; t, x)) d\theta}.$$

Looking at (121), as \mathbf{v} is only in $L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1, \infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \cap W^{2, \infty}((0, T); L^2((0, x_0); \mathbb{R}^2))$, this equation is expressed a priori formally in the distribution sense. Thus, we have to be careful when we estimate (132). By (121) and using estimates (126), (127), we get by Cauchy-Schwarz inequality together with the change of variable $y = \xi_1(\theta; t, x_0)$ that

$$\int_0^T |\partial_{tt} \beta_1(t, x_0)|^2 dt \leq C\nu + C \int_0^T \left| \int_0^t f_{1xx}(\theta, \xi_1(\theta; t, x_0)) \partial_t \xi_1(\theta; t, x_0) d\theta \right|^2 dt$$

$$\begin{aligned}
&\leq C\nu + C \int_0^T \int_0^t v_{1xx}^2(\theta, \xi_1(\theta; t, x_0)) \partial_t^2 \xi_1(\theta; t, x_0) d\theta dt \\
&= C\nu + C \int_0^T \int_\theta^T v_{1xx}^2(\theta, \xi_1(\theta; t, x_0)) \partial_t^2 \xi_1(\theta; t, x_0) dt d\theta \\
&\leq C\nu + C \int_0^T \int_0^{x_0} v_{1xx}^2(\theta, y) dy d\theta \\
&\leq C\nu.
\end{aligned} \tag{187}$$

□

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