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HOMOGENIZATION FOR A MULTI-SCALE MODEL OF MAGNETORHEOLOGICAL SUSPENSION

GRIGOR NIKA AND BOGDAN VERNESCU

ABSTRACT. Using the homogenization method we obtain a model describing the behavior of the suspension of solid magnetizable particles in a viscous non-conducting fluid in the presence of an externally applied magnetic field. We use the quasi-static Maxwell equations coupled with the Stokes equations to capture the magnetorheological (MR) effect. The model generalizes the one introduced by Neuringer and Rosensweig [14], for quasistatic phenomena. The macroscopic constitutive properties are given explicitly in terms of the solutions of the local problems. We determine the homogenized constitutive parameters for an aqueous MR fluid with magnetite particles using the finite element method. The Poiseuille flow, for the solution of our homogenized coupled system, approaches the Bingham flow profile for large values of the magnetic field. The stress-strain curves obtained for the Couette flow exhibit a yield stress close to the one determined experimentally.

INTRODUCTION

Magneto-rheological (MR) fluids are a suspension of non-colloidal, ferromagnetic particles in a non-magnetizable carrier fluid. The particles are often of micron size ranging anywhere from $0.05 - 10 \mu m$ with particle volume fraction from $10 - 40\%$. They were discovered by J. Rabinow in 1948 [16]. Around the same time W. Winslow discovered electrorheological (ER) fluids, a closely related counterpart.

MR fluids respond to an external magnetic field by a rapid, reversible change in their properties. They can transform from a liquid to a semi solid state in a matter of milliseconds. Upon the application of a magnetic field, the dipole interaction of adjacent particles aligns the particles in the direction of the magnetic field lines. Namely particles attract one another along the magnetic field lines and repel one another in the direction perpendicular to them. This leads to the formation of aggregate structures. Once these aggregate structures are formed, the MR fluid exhibits a yield stress that is dependent and controlled by the applied external magnetic field [11], [5].

The formation of these aggregates means that the behavior of the fluid is non-Newtonian. In many works, the Bingham constitutive law is used as an approximation to model the response of the MR and ER fluids, particularly in shear experiments [15], [4], [6]. Although the Bingham model has proven itself useful in characterizing the behavior of MR fluids, it is not always sufficient. Recent experimental data show that true MR fluids exhibit departures from the Bingham model [22], [6].

Another member of the magnetic suspensions family are ferrofluids. Ferrofluids are stable colloidal suspensions of nanoparticles in a non-magnetizable carrier fluid. The initiation into the hydrodynamics of ferrofluids began with Neuringer and Rosensweig in 1964 [14] and by

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a series of works by Rosensweig and co-workers summarized in [17]. The model introduced in [14] assumes that the magnetization is collinear with the magnetic field and has been very useful in describing quasi-stationary phenomena. This work was extended by Shliomis [20] by avoiding the collinearity assumption of the magnetization and the magnetic field and by considering the rotation of the nanoparticles with respect to the fluid they are suspended in.

The models mentioned above have all been derived phenomenologically. The first attempt to use homogenization mechanics to describe the behavior of MR \ ER fluids was carried out in [8], [9] and [15]. In the works [8], [9] the influence of the external magnetic field is introduced as a volumic density force acting on each particle and as a surface density force acting on the boundary of each particle. The authors in [15] extend the work in [9], for ER fluids, by presenting a more complete model that couples the conservation of mass and momentum equations with Maxwell's equations through the Maxwell stress tensor. As an application they consider a uniform shearing of the ER fluid submitted to a uniform electric field boundary conditions in a two dimensional slab and they recover that the stress tensor at the macroscopic scale has exactly the form of the Bingham constitutive equation.

The authors in [15], [18], [17] use models that decouple the conservation of mass and momentum equations from the Maxwell equations. Thus in principle one can solve the Maxwell equations and use the resulting magnetic or electric field as a force in the conservation of mass and momentum equations.

The present work focuses on a suspension of rigid magnetizable particles in a Newtonian viscous fluid with an applied external magnetic field. We assume the fluid to be electrically non-conducting. Thus, we use the quasi-static Maxwell equations coupled with the Stokes equations through Ohm's law to capture the magnetorheological effect. In doing so we extend the model of [15]. Thus the Maxwell and the balance of mass and momentum equations must be simultaneously solved.

In Section 1. we introduce the problem in the periodic homogenization framework. The particles are periodically distributed and the size of the period is of the same order as the characteristic length of the particles. We assume the fluid velocity is continuous across the particle interface and that the particles are in equilibrium in the presence of the magnetic field.

The two scale expansion is carried out in Section 2. where we obtain a decoupled set of problems at order $\mathcal{O}(\epsilon^{-1})$.

In Section 3. and in Section 4. we study the local problems that arise from the contribution of the bulk magnetic field as well as the bulk velocity and provide new constitutive laws for Maxwell's equations.

In Section 5. we provide the governing effective equations of the MR fluid which include, in addition to the viscous stresses, a "Maxwell type" stress. Furthermore, we provide formulas for the effective viscosity and effective magnetic permeabilities for the Maxwell type stress that generalize those in [9].

Section 6. is devoted to comparing the results of the proposed model against experimental data. We compute the constitutive coefficients for an aqueous MR fluid with magnetite particles using the finite element method, we obtain the velocity profiles of both Poiseuille and Couette flows for this MR fluid and plot the stress vs shear rate curve for different values of the applied magnetic field, that exhibit a yield stress comparable to the one obtained in experiments (e.g. [22])

Notation. Throughout the paper we are going to be using the following notation: I indicates the $n \times n$ identity matrix, $e(\mathbf{u})$ will indicate the strain rate tensor defined by, $e(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$, where often times we will use subscript to indicate the variable of differentiation. The inner product between matrices is denoted by $A:B = \text{tr}(A^\top B) = \sum_{ij} a_{ij} b_{ji}$ and throughout the paper we employ the Einstein summation notation for repeated indices.

1. PROBLEM STATEMENT

For the homogenization setting of the suspension problem we define $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, to be a bounded open set with sufficiently smooth boundary $\partial\Omega$, $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ be the unit cube in \mathbb{R}^n , and \mathbb{Z}^n is the set of all n -dimensional vectors with integer components. For every positive ϵ , let N^ϵ be the set of all points $\ell \in \mathbb{Z}^n$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with sufficiently smooth boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon(\ell + T)$. The set T_ℓ^ϵ represents one of the rigid particles suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface (see FIG. 1). We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \bar{\Omega}_{1\epsilon}.$$

In what follows T_ℓ^ϵ will represent the magnetizable rigid particles, $\Omega_{1\epsilon}$ is the domain occupied by the rigid particles and $\Omega_{2\epsilon}$ the domain occupied by the surrounding fluid of viscosity ν . By \mathbf{n} we indicate the unit normal on the particle surface pointing outwards and by $[[\cdot]]$ we indicate the jump discontinuity between the fluid and the rigid part.

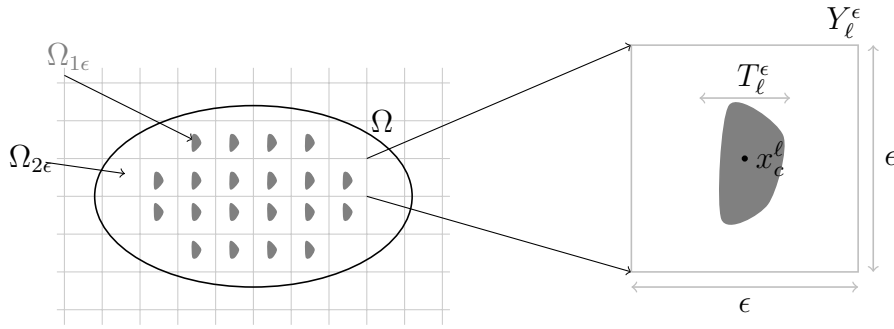


FIGURE 1. Schematic of the periodic suspension of rigid magnetizable particles in non-magnetizable fluid

The description of the problem is,

$$\rho \frac{\partial \mathbf{v}^\epsilon}{\partial t} + \rho (\mathbf{v}^\epsilon \cdot \nabla) \mathbf{v}^\epsilon - \text{div } \boldsymbol{\sigma}^\epsilon = \rho \mathbf{f}, \quad \text{where } \boldsymbol{\sigma}^\epsilon = 2\nu e(\mathbf{v}^\epsilon) - p^\epsilon I \quad \text{in } \Omega_{2\epsilon}, \quad (1.1a)$$

$$\text{div } \mathbf{v}^\epsilon = 0, \quad \text{div } \mathbf{B}^\epsilon = 0, \quad \text{curl } \mathbf{H}^\epsilon = \mathbf{0} \quad \text{in } \Omega_{2\epsilon}, \quad (1.1b)$$

$$e(\mathbf{v}^\epsilon) = 0, \quad \text{div } \mathbf{B}^\epsilon = 0, \quad \text{curl } \mathbf{H}^\epsilon = \eta \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \quad \text{in } \Omega_{1\epsilon}, \quad (1.1c)$$

where $\mathbf{B}^\epsilon = \mu^\epsilon \mathbf{H}^\epsilon$ with boundary conditions on the surface of each particle T_ℓ^ϵ ,

$$[[\mathbf{v}^\epsilon]] = \mathbf{0}, \quad [[\mathbf{B}^\epsilon \cdot \mathbf{n}]] = 0, \quad [[\mathbf{n} \times \mathbf{H}^\epsilon]] = \mathbf{0} \quad \text{on } S_\ell^\epsilon, \quad (1.2)$$

and outer boundary conditions

$$\mathbf{v}^\epsilon = \mathbf{0}, \quad \mathbf{H}^\epsilon = \mathbf{b} \quad \text{on } \partial\Omega, \quad (1.3)$$

where ρ is the density of the fluid, \mathbf{v}^ϵ represents the velocity field, p^ϵ the pressure, $e(\mathbf{v}^\epsilon)$ the strain rate, \mathbf{f} the body forces, \mathbf{n} the exterior normal to the particles, \mathbf{H}^ϵ the magnetic field, μ^ϵ is the magnetic permeability of the material, $\mu^\epsilon(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_{2\epsilon}$, η the electric conductivity of the rigid particles, and \mathbf{b} is an applied constant magnetic field on the exterior boundary of the domain Ω . When the MR fluid is submitted to a magnetic field, the rigid particles are subjected to a force that makes them behave like a dipole aligned in the direction of the magnetic field. This force can be written in the form,

$$\mathbf{F}^\epsilon = -\frac{1}{2} |\mathbf{H}^\epsilon|^2 \nabla \mu^\epsilon,$$

where $|\cdot|$ represents the standard Euclidean norm. The force can be written in terms of the Maxwell stress $\tau_{ij}^\epsilon = \mu^\epsilon H_i^\epsilon H_j^\epsilon - \frac{1}{2} \mu^\epsilon H_k^\epsilon H_k^\epsilon \delta_{ij}$ as $\mathbf{F}^\epsilon = \text{div } \tau^\epsilon + \mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon$. Since the magnetic permeability is considered constant in each phase, it follows that the force is zero in each phase. Therefore, we deduce that

$$\text{div } \tau^\epsilon = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_{2\epsilon} \\ -\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon & \text{if } \mathbf{x} \in \Omega_{1\epsilon}. \end{cases} \quad (1.4)$$

Lastly, we remark that unlike the viscous stress σ^ϵ , the Maxwell stress is present in the entire domain Ω . Hence, we can write the balance of forces and torques in each particle as,

$$\begin{aligned} \int_{T_\ell^\epsilon} \rho \frac{d\mathbf{u}^\epsilon}{dt} d\mathbf{x} &= \int_{S_\ell^\epsilon} (\sigma^\epsilon \mathbf{n} + [[\tau^\epsilon \mathbf{n}]]) ds + \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon d\mathbf{x} + \int_{T_\ell^\epsilon} \rho \mathbf{f} d\mathbf{x}, \\ \int_{T_\ell^\epsilon} \rho (\mathbf{x} - \mathbf{x}_c^\ell) \times \frac{d\mathbf{u}^\epsilon}{dt} d\mathbf{x} &= \int_{S_\ell^\epsilon} (\sigma^\epsilon \mathbf{n} + [[\tau^\epsilon \mathbf{n}]]) \times (\mathbf{x} - \mathbf{x}_c^\ell) ds \\ &+ \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon) \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x} + \int_{T_\ell^\epsilon} \rho \mathbf{f} \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x}, \end{aligned} \quad (1.5)$$

where \mathbf{x}_c^ℓ is the center of mass of the rigid particle T_ℓ^ϵ .

1.1. Dimensional Analysis. Before we proceed further we non-dimensionalize the problem. Denote by $t^* = t/L$, $x^* = x/L$, $\mathbf{v}^* = \mathbf{v}/V$, $p^* = p/\nu \frac{V}{L}$, $\mathbf{H}^* = \mathbf{H}/H$, $\mathbf{f}^* = \mathbf{f}/\frac{V^2}{L}$, and $\mu^{\epsilon*} = \mu^\epsilon/\mu_2$. Here L is a characteristic length, V is a characteristic velocity, p is a characteristic pressure, \mathbf{f} is a characteristic force and H is a characteristic unit of the magnetic field. Substituting the above expressions into (1.1) as well as in the balance of forces and torques, and using the fact that the flow is assumed to be at low Reynolds numbers, we obtain

$$\begin{aligned} Re \left(\frac{\partial \mathbf{v}^{\epsilon*}}{\partial t} + (\mathbf{v}^{\epsilon*} \cdot \nabla) \mathbf{v}^{\epsilon*} \right) - \text{div } {}^* \sigma^{\epsilon*} &= Re \mathbf{f}^*, \quad \text{where } \sigma^{\epsilon*} = 2e(\mathbf{v}^{\epsilon*}) - p^{\epsilon*} I \quad \text{in } \Omega_{2\epsilon}, \\ \text{div } {}^* \mathbf{v}^{\epsilon*} &= 0, \quad \text{div } {}^* \mathbf{B}^{\epsilon*} = 0, \quad \text{curl } {}^* \mathbf{H}^{\epsilon*} = \mathbf{0} \quad \text{in } \Omega_{2\epsilon}, \\ e^*(\mathbf{v}^{\epsilon*}) &= 0, \quad \text{div } {}^* \mathbf{B}^{\epsilon*} = 0, \quad \text{curl } \mathbf{H}^{\epsilon*} = R_m \mathbf{v}^{\epsilon*} \times \mathbf{B}^{\epsilon*} \quad \text{in } \Omega_{1\epsilon}, \end{aligned}$$

where $\mathbf{B}^{\epsilon*} = \mu^{\epsilon*} \mathbf{H}^{\epsilon*}$ and with boundary conditions on the surface of each particle T_ℓ^ϵ ,

$$\begin{aligned} \llbracket \mathbf{v}^{\epsilon*} \rrbracket &= \mathbf{0}, & \llbracket \mathbf{B}^{\epsilon*} \cdot \mathbf{n} \rrbracket &= 0, & \llbracket \mathbf{n} \times \mathbf{H}^{\epsilon*} \rrbracket &= \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{v}^{\epsilon*} &= \mathbf{0}, & \mathbf{H}^{\epsilon*} &= \mathbf{b}^* & & & \text{on } \partial\Omega. \end{aligned}$$

together with the balance of forces and torques,

$$\begin{aligned} Re \int_{T_\ell^\epsilon} \frac{d\mathbf{u}^{\epsilon*}}{dt^*} d\mathbf{x}^* &= \int_{S_\ell^\epsilon} \sigma^{\epsilon*} \mathbf{n} ds^* + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^{\epsilon*} \mathbf{n} \rrbracket ds^* + \alpha \int_{T_\ell^\epsilon} \mathbf{B}^{\epsilon*} \times \text{curl } \mathbf{H}^{\epsilon*} d\mathbf{x}^* + Re \int_{T_\ell^\epsilon} \mathbf{f}^* d\mathbf{x}^*, \\ Re \int_{T_\ell^\epsilon} (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) \times \frac{d\mathbf{u}^{\epsilon*}}{dt^*} d\mathbf{x}^* &= \int_{S_\ell^\epsilon} \sigma^{\epsilon*} \mathbf{n} \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) ds^* + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^{\epsilon*} \mathbf{n} \rrbracket \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) ds^* \\ &+ \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^{\epsilon*} \times \text{curl } \mathbf{H}^{\epsilon*}) \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) d\mathbf{x}^* + Re \int_{T_\ell^\epsilon} \mathbf{f}^* \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) d\mathbf{x}^*, \end{aligned}$$

where $Re = \frac{\rho V L}{\nu}$ is the Reynolds number, $\alpha = \frac{\mu_2 H^2 L}{\nu V}$ is the Alfven number, and $R_m = \eta \mu_1 L V$ is the magnetic Reynolds number.

In what follows we drop the star for simplicity. Moreover, for low Reynolds numbers the preceding equations become,

$$-\text{div } \sigma^\epsilon = \mathbf{0}, \quad \text{where } \sigma^\epsilon = 2e(\mathbf{v}^\epsilon) - p^\epsilon I \quad \text{in } \Omega_{2\epsilon}, \quad (1.6a)$$

$$\text{div } \mathbf{v}^\epsilon = 0, \quad \text{div } \mathbf{H}^\epsilon = 0, \quad \text{curl } \mathbf{H}^\epsilon = \mathbf{0} \quad \text{in } \Omega_{2\epsilon}, \quad (1.6b)$$

$$e(\mathbf{v}^\epsilon) = 0, \quad \text{div } \mathbf{H}^\epsilon = 0, \quad \text{curl } \mathbf{H}^\epsilon = R_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \quad \text{in } \Omega_{1\epsilon}, \quad (1.6c)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^\epsilon \rrbracket &= \mathbf{0}, & \llbracket \mathbf{B}^\epsilon \cdot \mathbf{n} \rrbracket &= 0, & \llbracket \mathbf{n} \times \mathbf{H}^\epsilon \rrbracket &= \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0}, & \mathbf{H}^\epsilon &= \mathbf{b} & & & \text{on } \partial\Omega, \end{aligned} \quad (1.7)$$

together with the balance of forces and torques,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket ds + \alpha \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon d\mathbf{x}, \\ 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} \times (\mathbf{x} - \mathbf{x}_c^\ell) ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket \times (\mathbf{x} - \mathbf{x}_c^\ell) ds \\ &+ \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon) \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x}. \end{aligned} \quad (1.8)$$

In the next section we will use a two scale expansion on the velocity, pressure and the magnetic field.

2. TWO SCALE EXPANSIONS

We assume the particles are periodically distributed in Ω and thus consider the two scale expansion on \mathbf{v}^ϵ , \mathbf{H}^ϵ and p^ϵ ,

$$\mathbf{v}^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} \epsilon^i \mathbf{v}^i(\mathbf{x}, \mathbf{y}), \quad \mathbf{H}^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} \epsilon^i \mathbf{H}^i(\mathbf{x}, \mathbf{y}), \quad p^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} \epsilon^i p^i(\mathbf{x}, \mathbf{y}) \quad \text{with } \mathbf{y} = \frac{\mathbf{x}}{\epsilon}.$$

where $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathbb{R}^n$. One can show that \mathbf{v}^0 is independent of \mathbf{y} and can thus obtain the following problem at order ϵ^{-1} ,

$$-\frac{\partial \sigma_{ij}^0}{\partial y_j} = 0 \quad \text{in } Y_f, \quad (2.1a)$$

$$\sigma_{ij}^0 = -p^0 \delta_{ij} + 2\nu (e_{ijx}(\mathbf{v}^0) + e_{ijy}(\mathbf{v}^1)) \quad (2.1b)$$

$$\frac{\partial v_j^0}{\partial x_j} + \frac{\partial v_j^1}{\partial y_j} = 0 \quad \text{in } Y_f, \quad (2.1c)$$

$$e_{ijx}(\mathbf{v}^0) + e_{ijy}(\mathbf{v}^1) = 0 \quad \text{in } T, \quad (2.1d)$$

$$\frac{\partial B_j^0}{\partial y_j} = 0, \quad \epsilon_{ijk} \frac{\partial H_k^0}{\partial y_j} = 0 \quad \text{where } B_i^0 = \mu H_i^0 \quad \text{in } Y, \quad (2.1e)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^1 \rrbracket = \mathbf{0}, \quad \llbracket \mathbf{B}^0 \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^0 \rrbracket = \mathbf{0} \quad \text{on } S, \\ \mathbf{v}^1, \quad \mathbf{H}^0 \text{ are } Y\text{-periodic.} \end{aligned} \quad (2.2)$$

Here Y_f and T denote the fluid, respectively the particle part of Y ; and S denotes the surface of T . At order of ϵ^2 and ϵ^3 we obtain from (1.8) the balance of forces and torques for the particle T respectively,

$$\begin{aligned} 0 &= \int_S \sigma^0 \mathbf{n} ds + \alpha \int_S \llbracket \tau^0 \mathbf{n} \rrbracket ds - \alpha \int_T \mathbf{B}^0 \times \text{curl}_y \mathbf{H}^0 dy, \\ 0 &= \int_S \mathbf{y} \times \sigma^0 \mathbf{n} ds + \alpha \int_S \mathbf{y} \times \llbracket \tau^0 \mathbf{n} \rrbracket ds - \alpha \int_T \mathbf{y} \times (\mathbf{B}^0 \times \text{curl}_y(\mathbf{H}^0)) dy, \end{aligned} \quad (2.3)$$

where by τ_{ij}^0 :

$$\tau_{ij}^0 = \mu H_i^0 H_j^0 - \frac{1}{2} \mu H_k^0 H_k^0 \delta_{ij}, \quad (2.4)$$

we denote the Maxwell stress. We remark that since from (2.1e) $\text{curl}_y(\mathbf{H}^0) = \mathbf{0}$ in Y , the balance of forces and torques simplify to the following,

$$0 = \int_S \sigma^0 \mathbf{n} + \alpha \int_S \llbracket \tau^0 \mathbf{n} \rrbracket ds \quad \text{and} \quad 0 = \int_S \mathbf{y} \times \sigma^0 \mathbf{n} ds + \alpha \int_S \mathbf{y} \times \llbracket \tau^0 \mathbf{n} \rrbracket ds. \quad (2.5)$$

Remark 1. *At first order, in the problem (2.1)-(2.5) the Stokes and Maxwell equations are decoupled. Hence, in principle one could solve the Maxwell equations (2.1e) and once a solution is obtained then solve the Stokes problem (2.1a)-(2.1c), albeit with an extra known force added to the balance of forces and torques (2.5).*

3. CONSTITUTIVE RELATIONS FOR MAXWELL'S EQUATIONS

3.1. Study of the local problem. Using the results from the two scale expansions, (2.1e), we can see that $\text{curl}_y(\mathbf{H}^0) = \mathbf{0}$ in Y and thus there exists a function $\psi = \psi(\mathbf{x}, \mathbf{y})$ with average $\tilde{\psi} = 0$ such that

$$H_i^0 = -\frac{\partial \psi(\mathbf{x}, \mathbf{y})}{\partial y_i} + \tilde{H}_i^0(\mathbf{x}), \quad (3.1)$$

where $\tilde{\cdot} = \frac{1}{|Y|} \int_Y \cdot d\mathbf{y}$. Using the fact $\operatorname{div}_y \mathbf{B}^0 = 0$ in Y , $B_i^0 = \mu H_i^0$ and the boundary conditions (1.7) we have,

$$\begin{aligned} -\frac{\partial}{\partial y_i} \left(\mu \left(-\frac{\partial \psi}{\partial y_i} + \tilde{H}_i^0 \right) \right) &= 0 \quad \text{in } Y, \\ \left[\mu \left(-\frac{\partial \psi}{\partial y_i} + \tilde{H}_i^0 \right) n_i \right] &= 0 \quad \text{on } S, \\ \psi \text{ is } Y\text{-periodic, } \tilde{\psi} &= 0. \end{aligned} \tag{3.2}$$

Introducing the space of periodic functions, with zero average

$$W_{per}(Y) = \{w \in H_{per}^1(Y) \mid \tilde{w} = 0\},$$

then the variational formulation of (3.2) is

$$\begin{aligned} \text{Find } \psi \in W_{per}(Y) \text{ such that} \\ \int_Y \mu \frac{\partial \psi}{\partial y_i} \frac{\partial v}{\partial y_i} d\mathbf{y} = \tilde{H}_i^0 \int_Y \mu \frac{\partial v}{\partial y_i} d\mathbf{y} \text{ for any } v \in W_{per}(Y). \end{aligned} \tag{3.3}$$

Since we have imposed that ψ has zero average over the unit cell Y , the solution to (3.3) can be determined uniquely by a simple application of the Lax-Milgram lemma.

Let ϕ^k be the unique solution of

$$\begin{aligned} \text{Find } \phi^k \in W_{per}(Y) \text{ such that} \\ \int_Y \mu \frac{\partial \phi^k}{\partial y_i} \frac{\partial v}{\partial y_i} d\mathbf{y} = \int_Y \mu \frac{\partial v}{\partial y_k} d\mathbf{y} \text{ for any } v \in W_{per}(Y). \end{aligned} \tag{3.4}$$

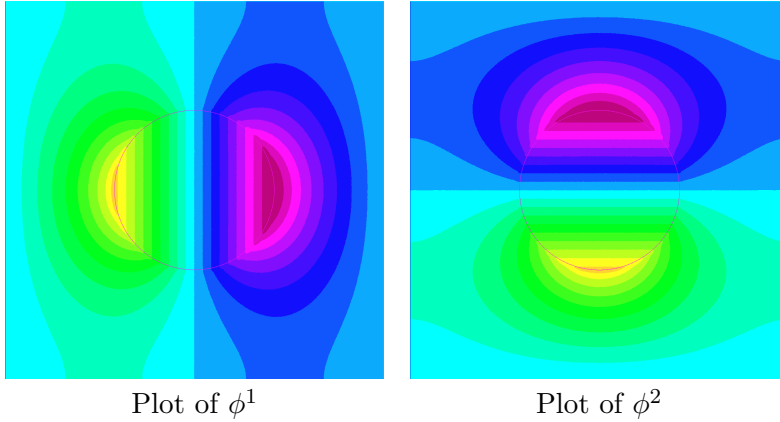


FIGURE 2. Plot of the solution ϕ^k in (3.4) for magnetite nanoparticles of volume fraction $\phi = 0.14$ with magnetic permeability $\mu = 8.41946 \times 10^{-6} N/A^2$ using FreeFem++.

By virtue of linearity of (3.3) we can write

$$\psi(\mathbf{x}, \mathbf{y}) = \phi^k(\mathbf{y}) \tilde{H}_k^0(\mathbf{x}) + C(\mathbf{x}).$$

In principle, once \tilde{H}_k^0 is known, we can determine ψ up to an additive function of \mathbf{x} . Hence, combining (3.1) and the above relationship between ψ and ϕ^k we obtain the following constitutive law between the magnetic induction and the magnetic field,

$$\tilde{B}_i^0 = \mu_{ik} \tilde{H}_k^0, \text{ where } \mu_{ik} = \int_Y \mu \left(-\frac{\partial \phi^k}{\partial y_i} + \delta_{ik} \right) d\mathbf{y}. \quad (3.5)$$

One can show (see [19]) that the homogenized magnetic permeability tensor is symmetric, $\mu_{ik} = \mu_{ki}$. Moreover, if we denote by $A_{i\ell}(\mathbf{y}) = \left(-\frac{\partial \phi^\ell(\mathbf{y})}{\partial y_i} + \delta_{i\ell} \right)$ one can see from (3.1) that $H_i^0 = A_{i\ell} \tilde{H}_\ell^0$ and thus the Maxwell stress (2.4) takes the following form,

$$\tau_{ij}^0 = \mu A_{i\ell} A_{jm} \tilde{H}_\ell^0 \tilde{H}_m^0 - \frac{1}{2} \mu A_{mk} A_{\ell k} \delta_{ij} \tilde{H}_m^0 \tilde{H}_\ell^0 = \mu A_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0.$$

Here $A_{ij}^{m\ell} = \frac{1}{2} (A_{i\ell} A_{jm} + A_{j\ell} A_{im} - A_{mk} A_{\ell k} \delta_{ij})$ and has the following symmetry, $A_{ij}^{m\ell} = A_{ji}^{m\ell} = A_{ij}^{\ell m}$. Recall that the $\text{div } \tau^\epsilon = 0$ in $\Omega_{2\epsilon}$ and $\text{div } \tau^\epsilon = -\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon$ in $\Omega_{1\epsilon}$. From the two scale expansion, at order ϵ^{-1} from equation (1.4) we obtain,

$$\text{div } {}_y\tau^0 = 0 \text{ in } Y. \quad (3.6)$$

4. FLUID VELOCITY AND PRESSURE

4.1. Study of the local problems. Problem (2.1)-(2.2), (2.5) is an elliptic problem in the variable $\mathbf{y} \in Y$ with forcing terms $\mathbf{v}^0(\mathbf{x})$ and $\tilde{\mathbf{H}}^0(\mathbf{x})$ at the macroscale. We can decouple the contributions of $\mathbf{v}^0(\mathbf{x})$ and $\tilde{\mathbf{H}}^0(\mathbf{x})$ and split \mathbf{v}^1 and p^0 in two parts: a part that is driven by the bulk velocity, and a part that comes from the bulk magnetic field.

$$v_k^1(\mathbf{x}, \mathbf{y}) = \chi_k^{m\ell}(\mathbf{y}) e_{m\ell}(\mathbf{v}^0) + \xi_k^{m\ell}(\mathbf{y}) \tilde{H}_m^0 \tilde{H}_\ell^0 + A_k(\mathbf{x}), \quad (4.1)$$

$$p^0(\mathbf{x}, \mathbf{y}) = p^{m\ell}(\mathbf{y}) e_{m\ell}(\mathbf{v}^0) + \pi^{m\ell}(\mathbf{y}) \tilde{H}_m^0 \tilde{H}_\ell^0 + \bar{p}^0(\mathbf{x}), \quad (4.2)$$

where $\int_{Y_f} p^{m\ell}(\mathbf{y}) d\mathbf{y} = 0$ and $\int_{Y_f} \pi^{m\ell}(\mathbf{y}) d\mathbf{y} = 0$.

Here, $\chi^{m\ell}$ satisfies

$$\begin{aligned} -\frac{\partial}{\partial y_j} \varepsilon_{ij}^{m\ell} &= 0 \quad \text{in } Y_f, \\ \varepsilon_{ij}^{m\ell} &= -p^{m\ell} \delta_{ij} + 2(C_{ijm\ell} + e_{ijy}(\chi^{m\ell})) \\ -\frac{\partial \chi_i^{m\ell}}{\partial y_i} &= 0 \quad \text{in } Y_f, \\ \llbracket \chi^{m\ell} \rrbracket &= 0 \quad \text{on } S, \\ C_{ijm\ell} + e_{ijy}(\chi^{m\ell}) &= 0 \quad \text{in } T, \\ \chi^{m\ell} &\text{ is } Y\text{-periodic, } \tilde{\chi}^{m\ell} = \mathbf{0} \text{ in } Y, \end{aligned} \quad (4.3)$$

together with the balance of forces and torques,

$$\int_S \varepsilon_{ij}^{m\ell} n_j ds = 0 \text{ and } \int_S \epsilon_{ijk} y_j \varepsilon_{kp}^{m\ell} n_p ds = 0, \quad (4.4)$$

where $C_{ijm\ell} = \frac{1}{2}(\delta_{im}\delta_{j\ell} + \delta_{i\ell}\delta_{jm}) - \frac{1}{n}\delta_{ij}\delta_{m\ell}$.

The variational formulation problem of (4.3)-(4.4) is

$$\begin{aligned} & \text{Find } \boldsymbol{\chi}^{m\ell} \in \mathcal{U} \text{ such that} \\ & \int_{Y_f} 2 e_{ijy}(\boldsymbol{\chi}^{m\ell}) e_{ijy}(\boldsymbol{\phi} - \boldsymbol{\chi}^{m\ell}) d\mathbf{y} = 0, \text{ for all } \boldsymbol{\phi} \in \mathcal{U}_{ad}, \end{aligned} \quad (4.5)$$

where \mathcal{U} is the closed, convex, non-empty subset of $H_{per}^1(Y)^n$ defined by

$$\mathcal{U} = \{ \mathbf{u} \in H_{per}^1(Y)^n \mid \operatorname{div} \mathbf{u} = 0 \text{ in } Y_f, e_{ijy}(\mathbf{u}) = -C_{ijm\ell} \text{ in } T, \llbracket \mathbf{u} \rrbracket = \mathbf{0} \text{ on } S, \tilde{\mathbf{u}} = \mathbf{0} \text{ in } Y \}$$

We remark that if we define $B_k^{ij} = \frac{1}{2}(y_i \delta_{jk} + y_j \delta_{ik}) - \frac{1}{n} y_k \delta_{ij}$, then $e_{ij}(\mathbf{B}^{m\ell}) = C_{ijm\ell}$. Existence and uniqueness of a solution follows from classical theory of variational inequalities [7].

In similar fashion we can derive the local problem for $\boldsymbol{\xi}^{m\ell}$,

$$\begin{aligned} & -\frac{\partial}{\partial y_j} \Sigma_{ij}^{m\ell} = 0 \quad \text{in } Y_f, \\ & \Sigma_{ij}^{m\ell} = -\pi^{m\ell} \delta_{ij} + 2 e_{ijy}(\boldsymbol{\xi}^{m\ell}) \\ & -\frac{\partial \xi_i^{m\ell}}{\partial y_i} = 0 \quad \text{in } Y_f, \\ & \llbracket \boldsymbol{\xi}^{m\ell} \rrbracket = 0 \quad \text{on } S, \\ & e_{ijy}(\boldsymbol{\xi}^{m\ell}) = 0 \quad \text{in } T, \\ & \boldsymbol{\xi}^{m\ell} \text{ is } Y\text{-periodic, } \tilde{\boldsymbol{\xi}}^{m\ell} = 0. \end{aligned} \quad (4.6)$$

Using (3.6) the balance of forces reduces to

$$\int_S \Sigma_{ij}^{m\ell} n_j ds = 0, \quad (4.7)$$

together with the balance of torques

$$\int_S \epsilon_{ijk} y_j (\Sigma_{kp}^{m\ell} + \alpha \llbracket \mu A_{kp}^{m\ell} \rrbracket) n_p ds = 0. \quad (4.8)$$

We can formulate (4.6)–(4.8) variationally as

$$\begin{aligned} & \text{Find } \boldsymbol{\xi}^{m\ell} \in V_{per}(Y) \text{ such that} \\ & \int_{Y_f} 2 e_{ijy}(\boldsymbol{\xi}^{m\ell}) e_{ijy}(\boldsymbol{\phi}) d\mathbf{y} + \int_Y A_{ij}^{m\ell} e_{ijy}(\boldsymbol{\phi}) d\mathbf{y} = 0, \text{ for all } \boldsymbol{\phi} \in V_{per}(Y), \end{aligned} \quad (4.9)$$

where

$$V_{per}(Y) = \{ \mathbf{v} \in H_{per}^1(Y)^n \mid \operatorname{div} \mathbf{v} = 0 \text{ in } Y_f, e_{ijy}(\mathbf{v}) = 0 \text{ in } T, \llbracket \mathbf{v} \rrbracket = \mathbf{0} \text{ on } S, \tilde{\mathbf{v}} = \mathbf{0} \text{ in } Y \},$$

is a closed subspace of $H_{per}^1(Y)^n$. Existence and uniqueness follows from an application of the Lax-Milgram lemma. Below we plot the streamlines of the solutions $\boldsymbol{\chi}^{m\ell}$ of (4.5) and $\boldsymbol{\xi}^{m\ell}$ of (4.9).

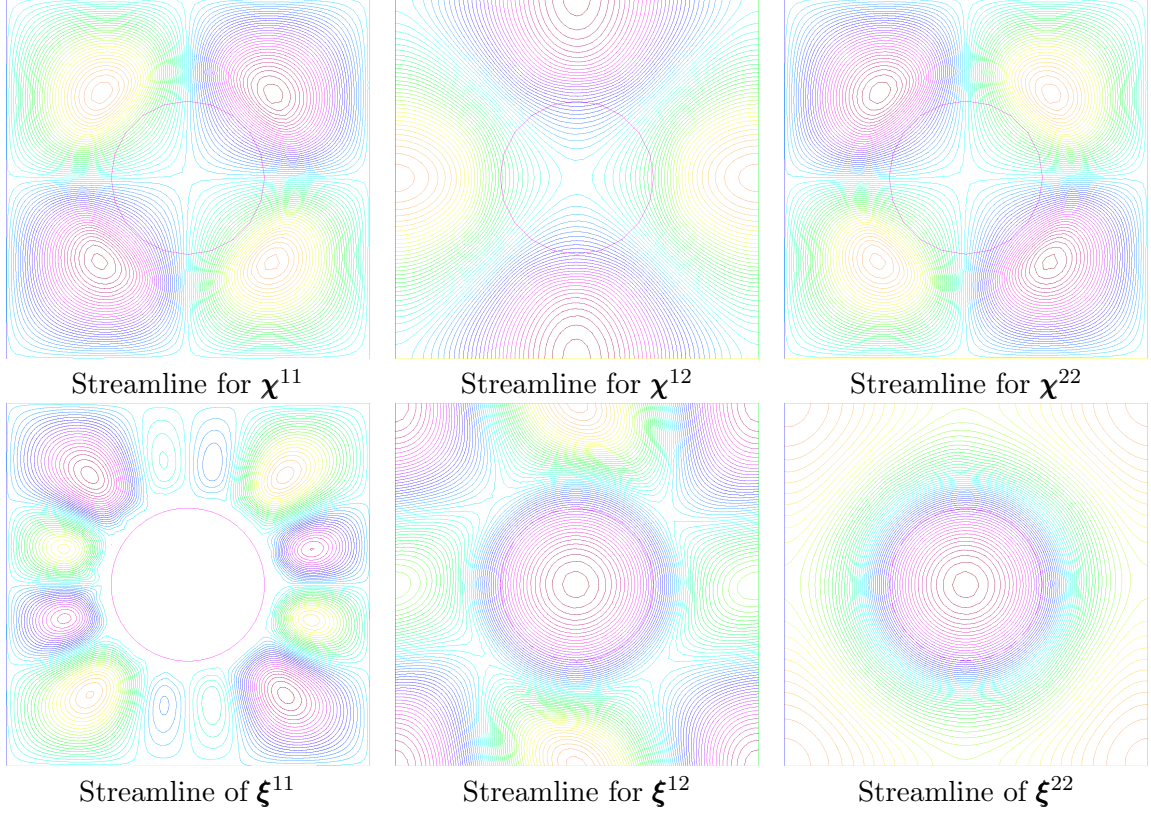


FIGURE 3. On the top are the streamlines of the solution $\chi^{m\ell}$ in (4.5) and on the bottom are the corresponding streamlines of the solution $\xi^{m\ell}$ in (4.9) for spherical magnetite nanoparticles of volume fraction $\phi = 0.14$ generated using FreeFem++.

Remark 2. From the second line of the plots above, we can observe that the only driving force that makes the solution $\xi^{m\ell}$ non zero in (4.9) are the rotations induced by the magnetic field through the fourth order tensor $A_{ij}^{m\ell}$.

5. HOMOGENIZED EQUATIONS OF THE MAGNETO-RHEOLOGICAL FLUID

At the ϵ^0 order we obtain the following problems,

$$-\operatorname{div}_x \sigma^0 - \operatorname{div}_y \sigma^1 = \mathbf{0} \quad \text{in } Y_f, \quad (5.1a)$$

$$\operatorname{div}_x \mathbf{v}^1 + \operatorname{div}_y \mathbf{v}^2 = 0 \quad \text{in } Y_f, \quad (5.1b)$$

$$\operatorname{div}_x \mathbf{B}^0 + \operatorname{div}_y \mathbf{B}^1 = 0 \quad \text{in } Y, \quad (5.1c)$$

$$\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1 = 0 \quad \text{in } Y_f, \quad (5.1d)$$

$$\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1 = R_m \mathbf{v}^0 \times \mathbf{B}^0 \quad \text{in } T, \quad (5.1e)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^2 \rrbracket = \mathbf{0}, \quad \llbracket \mathbf{B}^1 \cdot \mathbf{n} \rrbracket = 0 \quad \llbracket \mathbf{n} \times \mathbf{H}^1 \rrbracket = \mathbf{0} \quad \text{on } S, \\ \mathbf{v}^2, \quad \mathbf{H}^1 \text{ are } Y\text{-periodic.} \end{aligned} \quad (5.2)$$

In each period, we consider a Taylor expansion, around the center of mass of the rigid particle, both of the viscous stress and the Maxwell stress of the form (see [10]),

$$\begin{aligned}\sigma^\epsilon(\mathbf{x}) &= \sigma^0(\mathbf{x}_c^\ell, \mathbf{y}) + \frac{\partial \sigma^0(\mathbf{x}_c^\ell, \mathbf{y})}{\partial x_\alpha} (x_\alpha - x_{c,\alpha}^\ell) + \epsilon \sigma^1(\mathbf{x}_c^\ell, \mathbf{y}) + \epsilon \frac{\partial \sigma^1(\mathbf{x}_c^\ell, \mathbf{y})}{\partial x_\alpha} (x_\alpha - x_{c,\alpha}^\ell) + \dots \\ \tau^\epsilon(\mathbf{x}) &= \tau^0(\mathbf{x}_c^\ell, \mathbf{y}) + \frac{\partial \tau^0(\mathbf{x}_c^\ell, \mathbf{y})}{\partial x_\alpha} (x_\alpha - x_{c,\alpha}^\ell) + \epsilon \tau^1(\mathbf{x}_c^\ell, \mathbf{y}) + \epsilon \frac{\partial \tau^1(\mathbf{x}_c^\ell, \mathbf{y})}{\partial x_\alpha} (x_\alpha - x_{c,\alpha}^\ell) + \dots\end{aligned}$$

where the expansion of the Maxwell stress occurs both inside the rigid particle and the fluid.

Using this method we can expand the balance of forces, (1.8), and obtain at order ϵ^3 ,

$$\begin{aligned}0 &= \int_S \left(\frac{\partial \sigma_{ij}^0}{\partial x_k} y_k + \sigma_{ij}^1 \right) n_j ds + \alpha \int_S \left[\left(\frac{\partial \tau_{ij}^0}{\partial x_k} y_k + \tau_{ij}^1 \right) n_j \right] ds \\ &\quad - \alpha \int_T (\mathbf{B}^0 \times (\text{curl}_x \mathbf{H}^0 + \text{curl}_y \mathbf{H}^1))_i d\mathbf{y}.\end{aligned}\tag{5.3}$$

Integrate (5.1a) over Y_f and add to (5.3) obtain the following,

$$\begin{aligned}0 &= \int_{Y_f} \frac{\partial \sigma_{ij}^0}{\partial x_j} d\mathbf{y} + \int_S \frac{\partial \sigma_{ij}^0}{\partial x_k} y_k n_j ds + \alpha \int_S \left[\left(\frac{\partial \tau_{ij}^0}{\partial x_k} y_k + \tau_{ij}^1 \right) n_j \right] ds \\ &\quad - \alpha \int_T (\mathbf{B}^0 \times (\text{curl}_x \mathbf{H}^0 + \text{curl}_y \mathbf{H}^1))_i d\mathbf{y}.\end{aligned}\tag{5.4}$$

At order ϵ^0 we obtain, $\text{div}_x \tau^0 + \text{div}_y \tau^1 = 0$ in Y_f and $\text{div}_x \tau^0 + \text{div}_y \tau^1 = -\mathbf{B}^0 \times (\text{curl}_x \mathbf{H}^0 + \text{curl}_y \mathbf{H}^1)$ in T . Combining the aforementioned results and the divergence theorem we can rewrite (5.4) the following way,

$$0 = \int_{Y_f} \frac{\partial \sigma_{ij}^0}{\partial x_j} d\mathbf{y} + \int_S \frac{\partial \sigma_{ik}^0}{\partial x_j} y_j n_k ds + \alpha \int_S \left[\left[\frac{\partial \tau_{ik}^0}{\partial x_j} y_j n_k \right] \right] ds + \alpha \int_Y \frac{\partial \tau_{ij}^0}{\partial x_j} d\mathbf{y}.\tag{5.5}$$

Using the decomposition of \mathbf{v}^1 and \mathbf{p}^0 in (4.1) and (4.2) we can re-write σ_{ij}^0 and τ_{ij}^0 ,

$$\sigma_{ij}^0 = -\bar{p}^0 \delta_{ij} + \varepsilon_{ij}^{m\ell} e_{m\ell x}(\mathbf{v}^0) + \Sigma_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0, \quad \tau_{ij}^0 = \mu A_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0.$$

Moreover, equations (2.1b), (2.4), (4.3) and (4.6) allow us to retain the only symmetric part of (5.5).

Hence the homogenized fluid equations (5.5) become,

$$\begin{aligned}0 &= \frac{\partial}{\partial x_j} \left(-\bar{p}^0 \delta_{ij} + \left\{ \int_{Y_f} 2 e_{ijy}(\mathbf{B}^{m\ell} + \boldsymbol{\chi}^{m\ell}) d\mathbf{y} + \int_S \varepsilon_{pk}^{m\ell} B_p^{ij} n_k ds \right\} e_{m\ell x}(\mathbf{v}^0) \right. \\ &\quad \left. + \left\{ \int_{Y_f} 2 e_{ijy}(\boldsymbol{\xi}^{m\ell}) d\mathbf{y} + \int_S \Sigma_{pk}^{m\ell} B_p^{ij} n_k ds + \alpha \int_Y \mu A_{ij}^{m\ell} d\mathbf{y} + \alpha \int_S \left[\mu A_{pk}^{m\ell} \right] B_p^{ij} n_k \right\} \tilde{H}_m^0 \tilde{H}_\ell^0 \right).\end{aligned}\tag{5.6}$$

Furthermore, using (2.1c)–(2.1d) and the divergence theorem we can obtain the incompressibility condition, $\text{div}_x \mathbf{v}^0 = 0$.

Denote by

$$\nu_{ijm\ell} = \left\{ \int_{Y_f} 2 e_{ijy}(\mathbf{B}^{m\ell} + \boldsymbol{\chi}^{m\ell}) d\mathbf{y} + \int_S \varepsilon_{pk}^{m\ell} B_p^{ij} n_k ds \right\},$$

and

$$\beta_{ijml} = \left\{ \int_{Y_f} 2 e_{ijy}(\boldsymbol{\xi}^{ml}) d\mathbf{y} + \int_S \Sigma_{pk}^{ml} B_p^{ij} n_k ds + \alpha \int_Y \mu A_{ij}^{ml} d\mathbf{y} + \alpha \int_S \llbracket \mu A_{pk}^{ml} \rrbracket B_p^{ij} n_k \right\}.$$

then the homogenized equation (5.6) becomes

$$0 = \frac{\partial}{\partial x_j} \left(-\bar{p}^0 \delta_{ij} + \nu_{ijml} e_{mlx}(\mathbf{v}^0) + \beta_{ijml} \tilde{H}_m^0 \tilde{H}_l^0 \right).$$

Using local problem (4.3) we can re-write the ν_{ijml} the following way,

$$\nu_{ijml} = \int_{Y_f} 2 e_{pq}(\mathbf{B}^{ml} + \boldsymbol{\chi}^{ml}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y}. \quad (5.7)$$

In a similar fashion, using local problem (4.6) and the kinematic condition in (4.3) we can re-write β_{ijml} as follows

$$\beta_{ijml} = \int_{Y_f} 2 e_{pq}(\boldsymbol{\xi}^{ml}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \alpha \int_{Y_f} \mu A_{pq}^{ml} e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \alpha \int_Y \mu A_{ij}^{ml} d\mathbf{y}. \quad (5.8)$$

It is now clear that ν_{ijml} possesses the following symmetry, $\nu_{ijml} = \nu_{jiml} = \nu_{mlij}$. While for β_{ijml} , we have $\beta_{ijml} = \beta_{jiml} = \beta_{ijlm}$.

To obtain the homogenized Maxwell equations, average (5.1c), (5.1d), and (5.1e) over Y , Y_f , and T respectively and use equation (3.5) to obtain,

$$\frac{\partial (\mu_{ik} \tilde{H}_k^0)}{\partial x_j} = 0, \quad \epsilon_{ijk} \frac{\partial \tilde{H}_k^0}{\partial x_j} = R_m \epsilon_{ijk} v_j^0 \mu_{kp}^S \tilde{H}_p^0 \quad \text{in } \Omega,$$

where

$$\mu_{ik}^S = \int_T \mu \left(-\frac{\partial \phi^k}{\partial y_i} + \delta_{ik} \right) d\mathbf{y} \quad (5.9)$$

with boundary conditions,

$$\tilde{H}_i^0 = b_i, \quad v_i^0 = 0 \quad \text{on } \partial\Omega.$$

The effective coefficients are computed as the angular averaging of the tensors ν_{ijml} and β_{ijml} . This is done by introducing the projection on hydrostatic fields, P_b , and the projection on shear fields P_s (see [12]). The components of the projections in three dimensional space are given by:

$$(P_b)_{ijkl} = \frac{1}{n} \delta_{ij} \delta_{kl}, \quad (P_s)_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{n} \delta_{ij} \delta_{kl}$$

Let us make the following notations:

$$\begin{aligned} \nu_b &= \text{tr}(P_b \nu) = \frac{1}{n} \nu_{ppqq}, & \nu_s &= \text{tr}(P_s \nu) = \left(\nu_{ppqq} - \frac{1}{n} \nu_{ppqq} \right), \\ \beta_b &= \text{tr}(P_b \beta) = \frac{1}{n} \beta_{ppqq}, & \beta_s &= \text{tr}(P_s \beta) = \left(\beta_{ppqq} - \frac{1}{n} \beta_{ppqq} \right). \end{aligned}$$

Then we can re-write the homogenized coefficients ν_{ijml} and β_{ijml} as follows:

$$\nu_{ijml} = \frac{1}{n} (\nu_b - \nu_s) \delta_{ij} \delta_{ml} + \frac{1}{2} \nu_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

$$\beta_{ijml} = \frac{1}{n} (\beta_b - \nu_s) \delta_{ij} \delta_{ml} + \frac{1}{2} \beta_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

Gathering all the equations we have that the homogenized equations governing the MR fluid form the following coupled system between the Stokes equations and the quasistatic Maxwell equations,

$$\begin{aligned} \frac{\partial}{\partial x_j} (\sigma_{ij}^H + \tau_{ij}^H) &= 0, \quad \frac{\partial v_i^0}{\partial x_i} = 0 && \text{in } \Omega, \\ \sigma_{ij}^H + \tau_{ij}^H &= -\bar{p}^0 \delta_{ij} + \nu_s e_{ij}(\mathbf{v}^0) + \frac{1}{n} (\beta_b - \beta_s) \delta_{ij} \left| \tilde{\mathbf{H}}^0 \right|^2 + \beta_s \tilde{H}_i^0 \tilde{H}_j^0 && (5.10) \\ \frac{\partial (\mu_{jk} \tilde{H}_k^0)}{\partial x_j} &= 0, \quad \epsilon_{ijk} \frac{\partial \tilde{H}_k^0}{\partial x_j} = R_m \epsilon_{ijk} v_j^0 \mu_{kp}^S \tilde{H}_p^0 && \text{in } \Omega, \\ v_i^0 &= 0, \quad \tilde{H}_i^0 = b_i && \text{in } \Omega. \end{aligned}$$

Remark 3. We should remark here that the effective constitutive properties consist of the homogenized viscosity, ν_{ijml} , and three homogenized magnetic permeabilities, μ_{ij} , μ_{ij}^s , and β_{ijml} , which all depend on the geometry of the suspension, the volume fraction, and the magnetic permeability μ . In addition the new coefficient β_{ijml} depends also on the Alfvén number α .

6. VELOCITY PROFILE OF THE MAGNETO-RHEOLOGICAL FLUID

In this section we compute the cross sectional velocity profiles of Poiseuille and Couette flow for spherical suspensions of rigid particles. We denote by $\mathbf{v} = (v_1, v_2)$ the two dimensional velocity and by $\mathbf{H} = (H_1, H_2)$ the two dimensional magnetic field. We remark that in two dimensions the tensors $C_{ijmm} = 0$ and $\mathbf{B}^{mm} = \mathbf{0}$. Then due to the linearity of local problem (4.3) we have $\chi^{mm} = \mathbf{0}$. Thus, $\nu_{mmii} = 0$ which implies that $\nu_b = 0$. Using a similar argument, we further note that $\beta_{mmii} = 0$ which implies that $\beta_b = 0$. Hence, the two dimensional stresses of (5.10) reduce to

$$\sigma_{ij}^H + \tau_{ij}^H = -\bar{p}^0 \delta_{ij} + \nu_s e_{ij}(\mathbf{v}^0) - \frac{1}{2} \beta_s \delta_{ij} \left| \tilde{\mathbf{H}}^0 \right|^2 + \beta_s \tilde{H}_i^0 \tilde{H}_j^0$$

Thus, the two dimensional MR equations in (5.10) reduce to the following:

$$\frac{\nu_s}{2} \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} \right) - \frac{\partial \pi^0}{\partial x_1} + \frac{\partial}{\partial x_1} \left(\frac{1}{2} \beta_s (H_1^2 - H_2^2) \right) + \frac{\partial}{\partial x_2} (\beta_s H_1 H_2) = 0, \quad (6.1a)$$

$$\frac{\nu_s}{2} \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) - \frac{\partial \pi^0}{\partial x_2} + \frac{\partial}{\partial x_1} (\beta_s H_1 H_2) + \frac{\partial}{\partial x_2} \left(\frac{1}{2} \beta_s (H_2^2 - H_1^2) \right) = 0, \quad (6.1b)$$

$$\frac{\partial}{\partial x_1} (\mu H_1) + \frac{\partial}{\partial x_2} (\mu H_2) = 0, \quad (6.1c)$$

$$\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = \eta \mu^S (v_1 H_2 - v_2 H_1), \quad (6.1d)$$

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0. \quad (6.1e)$$

6.1. Poiseuille flow. We consider the problem of a steady flow due to a pressure gradient between two infinite, parallel, stationary plates that are non-conducting and non-magnetizable with one plate aligned along the x_1 -axis while the second plate is of distance one unit apart. We apply a stationary magnetic field \mathbf{H} on the bottom plate. Since we are dealing with infinite plates, the velocity \mathbf{v} depends only on x_2 . Using (6.1e) we immediately obtain that v_2 is constant and since the plates are stationary $v_2 = 0$. Since the flow is unidirectional, we expect that the the magnetic field will depend only on the height x_2 . Hence, using (6.1c) we obtain $H_2(x_2) = K$, while the component parallel to the flow depends on the fluid velocity. Therefore the equations in (6.1) reduce to the following,

$$\frac{\nu_s}{2} \frac{\partial^2 v_1}{\partial x_1^2} + \beta_s K \frac{\partial H_1}{\partial x_2} = \frac{\partial \pi^0}{\partial x_1}, \quad (6.2a)$$

$$-\frac{\partial \pi^0}{\partial x_2} - \frac{1}{2} \beta_s \frac{\partial H_1^2}{\partial x_2} = 0, \quad (6.2b)$$

$$-\frac{\partial H_1}{\partial x_2} = \eta \mu^s K v_1. \quad (6.2c)$$

Making use of (6.2b) we obtain that $\pi^0(x_1, x_2) + \frac{1}{2} \beta_s H_1(x_2)^2$ is a function of only x_1 and therefore by differentiating the expression with respect to x_1 we get that $\frac{\partial \pi^0}{\partial x_1}$ is a function only x_1 . Therefore, on (6.2a) the left hand side is a function of x_2 and the right hand side is a function of x_1 . Thus they have to be constant. Substituting (6.2c) in (6.2a) we obtain the following differential equations,

$$\frac{d^2 v_1}{dx_2^2} - \lambda^2 v_1 = C_p, \quad (6.3a)$$

$$\frac{\partial \pi^0}{\partial x_1} = C_p, \quad (6.3b)$$

with $\lambda = \sqrt{\frac{2\eta\mu^s\beta_s}{\nu_s} K}$.

The general solution of (6.3a) is

$$v_1(x_2) = c_1 e^{\lambda x_2} + c_2 e^{-\lambda x_2} + \frac{C_p}{\nu \lambda^2}.$$

Given that $v_1(0) = v_1(1) = 0$ we have,

$$v_1(x_2) = \frac{C_p}{\nu \lambda^2} \left(\frac{\sinh(\lambda x_2) - \sinh(\lambda(x_2 - 1))}{\sinh(\lambda)} - 1 \right). \quad (6.4)$$

Once the velocity $v_1(x_2)$ is known, we can use (6.2c) to compute $H_1(x_2)$ with boundary condition $H_1(0) = K_1$ and obtain,

$$H_1(x_2) = \eta \mu^s K \frac{C_p}{\nu \lambda^3 \sinh(\lambda)} (-\cosh(\lambda x_2) + \cosh(\lambda(x_2 - 1)) - \cosh(\lambda) + 1) + K_1.$$

Remark 4. As K tends to zero, λ also tends to zero and we have

$$\lim_{K \rightarrow 0} v_1(x_2) = \frac{C_p}{2\nu} x_2(x_2 - 1),$$

which is precisely the profile of Poiseuille flow with stationary plates at $x_2 = 0$ and $x_2 = 1$.

6.2. Couette flow. The setting and calculations for the unidirectional Couette flow are the same as Poiseuille flow. In a similar way, we can carry out computations for the plane Couette flow. For simplicity we assumed the bottom plate is the x_1 axis and the top plate is at $x_2 = 1$ and the pressure gradient is zero. A shear stress $\dot{\gamma}$ is applied to the top plate while the bottom plate remains fixed. Thus, we solve (6.3a) with initial conditions $v_1(0) = 0$ and $v_1'(1) = \dot{\gamma}$ and obtain

$$v_1(x_2) = \frac{\dot{\gamma} \nu \lambda \sinh(\lambda x_2) + C_p \cosh(\lambda (x_2 - 1))}{\nu \lambda^2 \cosh(\lambda)} - \frac{C_p}{\nu \lambda^2} \quad (6.5)$$

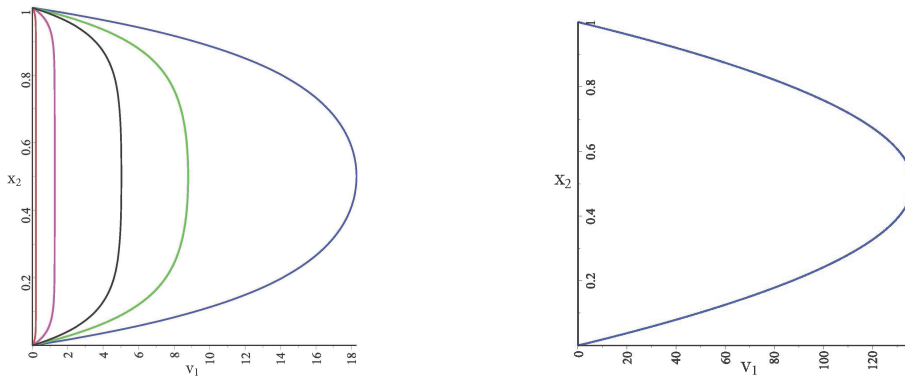
Remark 5. *Again, as before, we note that as K approaches zero, λ also approaches zero and*

$$\lim_{\lambda \rightarrow 0} \frac{C_p x_2^2 + 2 \dot{\gamma} x_2 \nu_s - 2 C_p x_2}{2 \nu_s}$$

To compute H_1 we use (6.2c) to obtain

$$H_1(x_2) = \frac{\dot{\gamma}}{\lambda^2 \cosh(\lambda)} (\cosh(\lambda x_2) - 1) + \frac{C_p}{\nu \lambda^3 \cosh(\lambda)} (\sinh(\lambda (x_2 - 1)) - \sinh(\lambda)) - \frac{C_p x_2}{\nu \lambda^2} + K_1.$$

6.3. Magnetite nanoparticles. In this section we consider a suspension of spherical magnetite nanoparticles in de-ionized water of viscosity 0.001 Pa with volume fraction $\phi = 0.07$. The electrical conductivity of the nanoparticles is assumed to be 20,000 S/m, while the magnetic permeability is $8.41946 \times 10^{-6} N/A^2$ for the nanoparticles and $1.25662 \times 10^{-6} N/A^2$ for the water. Carrying out explicit computations of the effective coefficients in (5.7), (5.8) and (5.9) we obtain $\nu_s = 0.006 Pa$, $\beta_s = 2.59 \times 10^{-6} N/A^2$, $\mu^s = 3.28 \times 10^{-7} N/A^2$. In the case of Poiseuille flow we can plot the profile (6.4) of the MR flow, for a constant pressure gradient and different values of the magnetic field and compare them with the Poiseuille flow profile in the absence of a magnetic field (FIG. 4). We can see that the damping force increases with B_2 ; the profile is close to flat in the middle region for high B_2 , but is not parabolic close to the walls as in the case of Bingham flows.



Velocity profile for MR Poiseuille flow Velocity profile for regular Poiseuille flow

FIGURE 4. The plots on the left represent the velocity profile for $B_2=0.05, 0.02, 0.01, 0.0075, 0.005$ T (left to right). The plot on the right is the velocity profile for $B_2=0$ T.

Likewise, in Couette flow regime we can plot and compare the velocity profile (6.5) of MR Couette flow against the Couette flow in the absence of a magnetic field, for zero pressure gradient. The plots in FIG. 5 show that as b_2 increases the flow region is smaller close to the upper plate. Thus an “apparent” yield stress is present. However, the velocity profile is not linear like in the case of Bingham fluids.

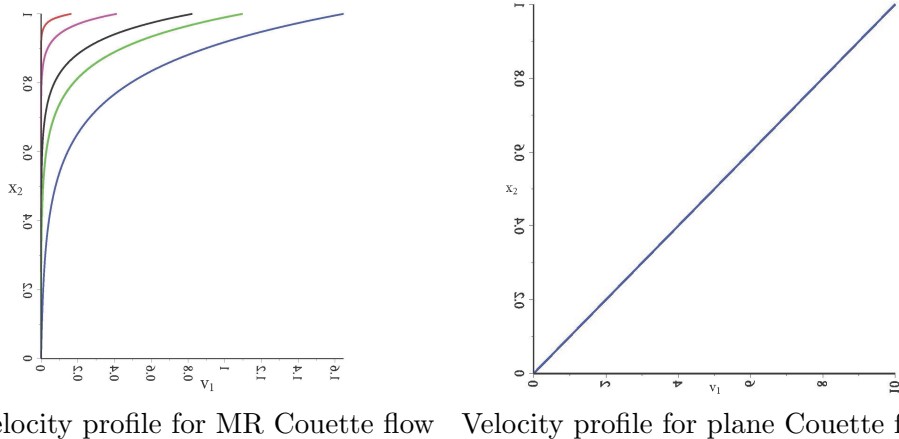


FIGURE 5. The plots on the left represent the velocity profile for $B_2=0.05, 0.02, 0.01, 0.0075, 0.005$ T (left to right). The plot on the right is the velocity profile for $B_2=0$ T.

Remark 6. *For shear experiments, the response of magneto-rheological fluids is often modeled using a Bingham constitutive law [4], [5], [15]. Although the Bingham constitutive law measures the response of the magneto-rheological fluid quite reasonably, actual magneto-rheological fluid behavior exhibits departures from the Bingham model [6], [22]. In Fig. 4 and FIG 5 we see that for low values of the magnetic field, the Bingham constitutive law is not adequate, however, it appears that for higher values of the magnetic field the flow gets closer to resembling a Bingham fluid.*

The plot in FIG. 6 depicts the stress vs shear rate curve relationship measured at $x_2 = 1$. When $K_1 = 0$ there is no yield stress present. However, for very small non-zero values of K_1 we obtain the results of [22] for the linear portion of the stress vs shear rate curve at high shear rates. Additionally, we are able to match their extrapolated Bingham yield stress values.

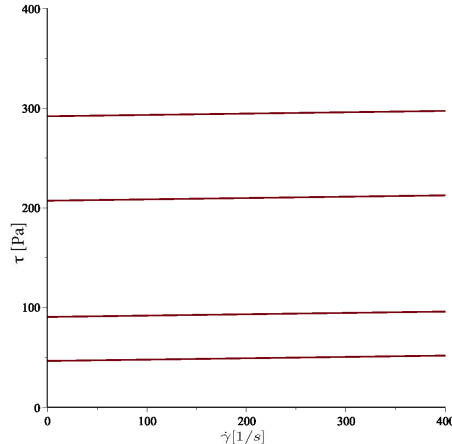


FIGURE 6. The stress versus the shear rate curve for four different magnetic fields, $B_2 = 0.288, 0.230, 0.173, 0.058$.

7. CONCLUSIONS

We consider a suspension of rigid magnetizable particles in a non-magnetizable, non-conducting aqueous viscous fluid. In (3.4), (4.5), (4.9) we derive the local problems that arise from the Maxwell equations, the bulk velocity and the bulk magnetic field and obtain new constitutive laws. The effective equations governing the behavior of the MR fluid are presented in (5.10). The proposed model generalizes the one in [14] by coupling the velocity field with the magnetic field intensity. Moreover, we obtain formulae for the effective coefficients that can be numerically computed and identify three different magnetic permeabilities governing the effective behavior. Unidirectional velocity profiles of Poiseuille and Couette flows are computed for magnetite nanoparticles of volume fraction $\phi = 0.14$ to validate against experimental data for the stress-strain relationship of MR flows.

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