Small-time global stabilization of the viscous Burgers equation with three scalar controls

Jean-Michel Coron, Shengquan Xiang

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Abstract

We construct explicit time-varying feedback laws leading to the global (null) stabilization in small time of the viscous Burgers equation with three scalar controls. Our feedback laws use first the quadratic transport term to achieve the small-time global approximate stabilization and then the linear viscous term to get the small-time local stabilization.

Keywords. Burgers equations, time-varying feedback laws, small-time global stabilization, controllability, backstepping.
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1 Introduction

A very classical problem for controllable system is the asymptotic stabilization issue. Let us first recall some results concerning systems in finite dimension. It was first pointed out in [63] that a system which is globally controllable may not be globally asymptotically stabilizable by means of continuous stationary feedback laws. In [10] a necessary condition for asymptotic stabilizability by means of continuous stationary feedback laws is established. See also [17]. There are controllable systems which do not satisfy this necessary condition. In order to overcome this problem two main strategies have been introduced, namely the use of discontinuous stationary feedback laws and the use of continuous (with respect to the state) time-varying feedback laws. For the first strategy, let us mention in particular [63] and [16]. Concerning the second strategy, which was introduced in [62] and [61], it is proved in [18] that many powerful sufficient conditions for small-time local controllability imply the existence of feedback laws which stabilize locally the system in small time.

Concerning control systems modelled by means of partial differential equations much less is known. The classical approach for local results is to first consider the linearized control system around the equilibrium of interest. If this linear system can be asymptotically stabilized by a linear feedback law one may expect that the same feedback law is going to stabilize asymptotically the initial nonlinear control system. This approach has been successfully applied to many control systems. Let us, for example, mention [3, 4, 5, 6, 7, 59, 60], which are dealing with the stabilization of the Navier-Stokes equations of incompressible fluids, equations which

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are close to the one we study here, i.e. the viscous Burgers equation. However this strategy does not work in two important cases, namely the case where the linearized system is not asymptotically stabilizable and the case where one is looking for a global result. In both cases one expects that the construction of (globally or locally) asymptotically stabilizing feedback laws heavily depends on the methods allowing to use the nonlinearity in order to prove the associated controllability property (global or local controllability). For the local controllability one of this method is the "power series expansion" method. See in particular [11, 12, 27] where an expansion to the order 2 and 3 is used in order to prove the local controllability of Korteweg-de Vries equations. This method can be indeed adapted to construct stabilizing feedback laws: see [33] for control systems in finite dimension and [34] for a Korteweg-de Vries control system and [15] for a Navier-Stokes equation.

Concerning the second case (global stabilization), even less is known. It is natural to expect that the construction of globally asymptotically stabilizable feedback laws depends strongly on the arguments allowing to prove this controllability. One of these arguments is the use of the return method together with scaling arguments (and, in some cases, a local controllability result) as introduced in [19, 20]. These arguments have been used to get global controllability results for

- The Euler equations of incompressible fluids in [20, 41],
- The Navier-Stokes equations of incompressible fluids in [14, 19, 28, 30, 40],
- Burgers equations in [13, 57],
- The Vlasov-Poisson system in [42, 46].

In some of these cases the "phantom tracking" method gives a possibility to get global stabilization. This method was introduced in [21] for the asymptotic stabilization of the Euler systems, then it has been used in various models [9, 43]. One can find a tutorial introduction to this method in [24]. However, it is not clear how to get finite-time stabilization with this method.

Concerning the stabilization in small time or even in finite time of partial differential equations very little is known. Let us mention

- The use of Krstic's backstepping method [51] to get stabilization in finite time of linear hyperbolic systems; see in particular [2, 29, 35, 36, 49],
- The small-time stabilization of $1-D$ parabolic equations [31],
- The small-time local stabilization of Korteweg-de Vries equations [65].

In this paper, we give the first small-time global stabilization result in a case where the global null controllability is achieved by using the return method together with scaling arguments and a local controllability result. We investigate the Burgers equation

$$y_t - y_{xx} + yy_x = a(t),\ y(t,0) = u_1(t),\ y(t,1) = u_2(t), \tag{1.1}$$

where, at time t, the state is $y(t,\cdot)$ and the controls are $a(t) \in \mathbb{R}$, $u_1(t) \in \mathbb{R}$, and $u_2(t) \in \mathbb{R}$. The Burgers equation has been very much studied for its important similarities with the Navier-Stokes equation as the appearance of boundary layers and the balance between the linear viscous term and the quadratic transport term.

Let us briefly recall some controllability results on the Burgers control system (1.1). When $a = 0$ and $u_1 = 0$, the small-time local null controllability is proved in [39] . When $a = 0$, it is proved in [47] that the small-time global null controllability does not hold. Before and after this, many related results were given in [1, 23, 37, 38, 44, 48, 58]. In [13] the return method and scaling arguments are used as in [19, 28] to prove that (1.1) is globally null controllable. The global null controllability in small time also holds if $u_2 = 0$ as proved in [57], even if in this case boundary layers appear when applying the return method. Moreover, it is proved in [56] that the small-time local controllability fails when $u_1 = u_2 = 0$.

This article is dealing with the small-time global stabilization of (1.1). To overcome some regularity issues we add an integration on the control variable a : now $a_t = \alpha(t)$ and $a(t)$ is part of the state. In other words, we consider a dynamical extension of (1.1) -see for example [22, p. 292]- with an extension with a variable of dimension only 1. Dynamical extensions are usually considered to handle output regulations. It can also be used to handle obstructions to full state stabilization for nonlinear systems even in finite dimension: see [32, Proposition 1]. In this paper, we therefore consider the following viscous Burgers controlled system:

$$\begin{cases} y_t - y_{xx} + yy_x = \alpha(t) & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = u_1(t) & \text{for } t \in (s,+\infty), \\ y(t,1) = u_2(t) & \text{for } t \in (s,+\infty), \\ a_t = \alpha(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{1.2}$$

where, at time t, the state is $(y(t,\cdot), a(t)) \in L^2(0,1) \times \mathbb{R}$, and the control is $(\alpha(t), u_1(t), u_2(t)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. (We could have considered $a_t = \beta(t)$ where $\beta(t)$ is a new control; however it turns out that one can just take $\beta(t) = \alpha(t)$.)

Before stating our results on stabilization, let us introduce the notion of feedback law, closed-loop system, proper feedback law, and flow associated to a proper feedback law. A feedback law is an application F

$$\left\{ \begin{array}{lcl} F : D(F) \subset \mathbb{R} \times L^2(0,1) \times \mathbb{R} & \to & \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ (t; y, a) & \mapsto & F(t; y, a) = (A(t; y, a), U_1(t; y, a), U_2(t; y, a)). \end{array} \right. \tag{1.3}$$

The closed-loop system associated to such a feedback law F is the evolution equation

$$\begin{cases} y_t - y_{xx} + yy_x = A(t; y, a) & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = U_1(t; y, a) & \text{for } t \in (s,+\infty), \\ y(t,1) = U_2(t; y, a) & \text{for } t \in (s,+\infty), \\ a_t = A(t; y, a) & \text{for } t \in (s,+\infty). \end{cases} \tag{1.4}$$

The feedback law F is called *proper* if the Cauchy problem associated to the closed-loop system (1.4) is well posed for every $s \in \mathbb{R}$ and for every initial data $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ at time s; see Definition 16 for the precise definition of a solution to this Cauchy problem and see Definition 17 for the precise definition of proper. For a proper feedback law, one can define the flow $\Phi : \Delta \times (L^2(0,1) \times \mathbb{R}) \to (L^2(0,1) \times \mathbb{R})$, with $\Delta := \{(t,s);\, t > s\}$ associated to this feedback law: $\Phi(t, s; y_0, a_0)$ is the value at $t > s$ of the solution (y, a) to the closed-loop system (1.4) which is equal to (y_0, a_0) at time s.

Let

$$V := L^2(0,1) \times \mathbb{R} \text{ with } \|(y,a)\|_V := \|y\|_{L^2} + |a|. \tag{1.5}$$

Our main result is the following small-time global stabilization result.

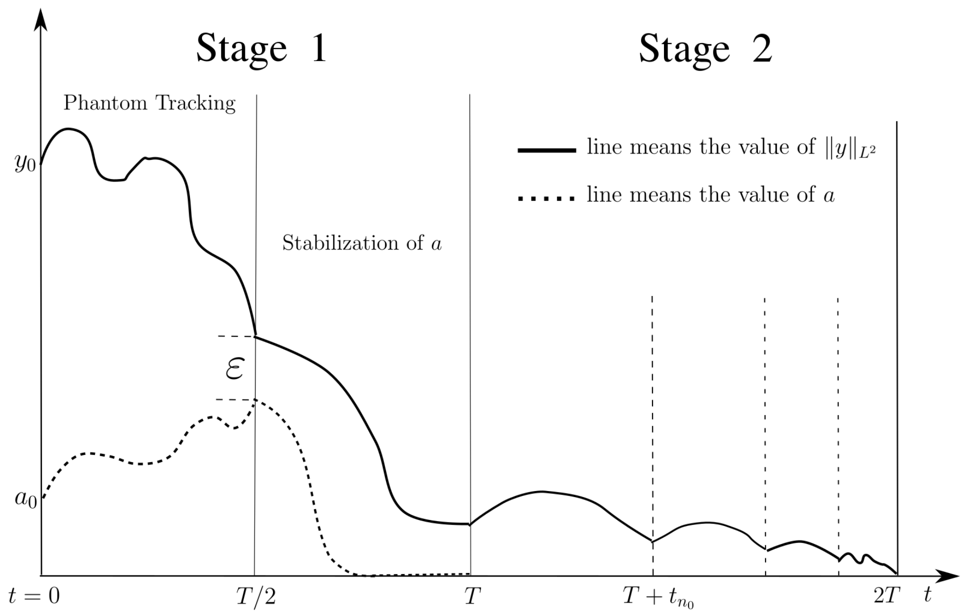

Figure 1: Small-time global stabilization of (y, a).

Theorem 1. *Let $T > 0$. There exists a proper $2T$-periodic time-varying feedback law for system (1.2) such that*

(i) $\Phi(4T + t, t; y_0, a_0) = 0, \ \ \forall t \in \mathbb{R}, \ \ \forall y_0 \in L^2(0, 1), \ \ \forall a \in \mathbb{R}.$

(ii) (Uniform stability property.) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|(y_0, a_0)\|_V \leqslant \eta) \Rightarrow \left(\|\Phi(t, t'; y_0, a_0)\|_V \leqslant \delta, \ \ \forall t' \in \mathbb{R}, \ \ \forall t \in (t', +\infty)\right). \tag{1.6}$$

Our strategy to prove Theorem 1 is to decompose the small-time global stabilization into two stages:

- Stage 1: Global "approximate stabilization", i.e., the feedback law steers the control system in a small neighborhood of the origin,
- Stage 2: Small-time local stabilization.

In the remaining part of this introduction, we heuristically describe these two stages (see Figure 1).

1.1 Global approximate stabilization

In this part we use the transport term yy_x and the "phantom tracking" strategy to get global approximate stabilization in small time, i.e. to get, for a given $\varepsilon > 0$, $\|y(t)\|_{L^2} + |a(t)| \leqslant \varepsilon$ for t larger than a given time. For this issue, let us perform the following change of variable

$$z := y - a. \tag{1.7}$$

Then (1.2) becomes

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = u_1(t) - a(t) & \text{for } t \in (s,+\infty), \\ z(t,1) = u_2(t) - a(t) & \text{for } t \in (s,+\infty), \\ a_t = \alpha(t) & \text{for } t \in (s,+\infty). \end{cases} \tag{1.8}$$

In this stage, we will always set

$$U_1(t;y,a) = U_2(t;y,a) = a. \tag{1.9}$$

Then the energy (i.e. the square of the L^2-norm) is dissipating:

$$\frac{d}{dt}\|z\|_{L^2}^2 \leqslant 0. \tag{1.10}$$

As we know, the "transport term" $a(t)z_x$ can lead to a small value for $\|z(T)\|_{L^2}$. For example letting $a(t) = C\|z\|_{L^2}$, one can expect that $\|z(T)\|_{L^2} \leqslant \varepsilon$ for $T > 0$ given, whatever the initial data is. However $|a(t)|$ can become larger. Thanks to the control of $a(t)$ (see (1.8)) and the dissipation of z (see (1.10)), a, as we will see, can be stabilized later on. In order to stabilize z only, we will try to find suitable feedback laws for system (1.8).

Using this strategy, we will get the following theorem, the proof of which is given in Section 3.

Theorem 2. *Let $T > 0, \varepsilon > 0$. There exists*

$$A : \mathbb{R} \times L^2(0,1) \times \mathbb{R} \to \bar{\mathbb{R}}, \ (t;y,a) \mapsto A(t;y,a), \tag{1.11}$$

such that the associated feedback law F_1 (see (1.3) and (1.9)) is proper for system (1.2) and such that the following properties hold, where Φ_1 denotes the flow associated to F_1,

($\mathcal{Q}_1$) The feedback law A is T-periodic with respect to time:

$$A(t;y,a) = A(T+t;y,a), \quad \forall (t,y,a) \in \mathbb{R} \times L^2(0,1) \times \mathbb{R}, \tag{1.12}$$

($\mathcal{Q}_2$) There exists a stationary feedback law $A_0 : L^2(0,1) \times \mathbb{R} \to \bar{\mathbb{R}}$, $(y,a) \mapsto A_0(y,a)$, such that

$$A(t;y,a) = A_0(y,a), \ \ \forall t \in [0,T/2), \ \ \forall (y,a) \in D(A_0), \tag{1.13}$$

($\mathcal{Q}_3$) There exists a stationary feedback law $A_1 : \mathbb{R} \to \mathbb{R}$, $a \mapsto A_1(a)$, such that

$$A(t;y,a) = A_1(a), \ \ \forall (t;y,a) \in [T/2,T) \times L^2(0,1) \times \mathbb{R}, \tag{1.14}$$

($\mathcal{Q}_4$) (Local uniform stability property.) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|(y_0,a_0)\|_V \leqslant \eta) \Rightarrow \left(\|\Phi_1(t,t';y_0,a_0)\|_V \leqslant \delta, \ \ \forall 0 \leqslant t' \leqslant t \leqslant T\right), \tag{1.15}$$

($\mathcal{Q}_5$) For every y_0 in $L^2(0,1)$ and for every $a_0 \in \mathbb{R}$,

$$\Phi_1(T,0;y_0,a_0) = (y(T),0) \ \textit{with} \ \|y(T)\|_{L^2(0,1)} \leqslant \varepsilon. \tag{1.16}$$

Theorem 2 is not a stabilization result, since we only get that $y(T)$ is "close to 0". For this reason we name this stage "global approximate stabilization".

1.2 Small-time local stabilization

Thanks to the first stage we now only need to get the small-time local stabilization. Since we already have $\Phi_1(T,0;y_0,a_0)=(y(T),0)$ with $\|y(T)\|_{L^2}\leqslant\varepsilon$, we can set $\alpha\equiv 0$. Inspired by the piecewise backstepping approach introduced in [31], we also set $u_1=0$. Hence the system becomes

$$\begin{cases} y_t-y_{xx}+yy_x=0 & \text{for } (t,x)\in(s,+\infty)\times(0,1),\\ y(t,0)=0 & \text{for } t\in(s,+\infty),\\ y(t,1)=u_2(t) & \text{for } t\in(s,+\infty).\end{cases} \tag{1.17}$$

We do not care about a since it does not change. In [31] the authors get small-time semi-global stabilization for the heat equation. Since we only need small-time local stabilization, the non-linear term yy_x could naturally be regarded as a small perturbation. However, by classical Lions–Magenes method, in order to have a $C^0([0,T];L^2(0,1))$ solution (to the system (1.17)), a $H^{1/4}(0,T)$ regularity of the control term is needed. For the control problem with the open-loop systems, the regularity condition on the control term is not a big obstacle. But when we consider the closed-loop system, it is hard to expect our feedback law will lead to a control in $H^{1/4}(0,T)$, especially when the feedback laws are given by some unbounded operators. Actually this problem also appears for the KdV system [65], where based on the special structure of KdV (leading to the Kato hidden regularity of $y_x(t,0)$), the "adding an integrator" method (i.e. the control is no longer u_2 but $\dot{u}_2$ in the framework of (1.17)) solved this problem. Nevertheless, this idea does not work for our case, since there is no such hidden regularity.

However, instead of the hidden regularity, we have now the maximum principle. With this principle we get that a control in $C^0([0,T])$ leads to a solution in $C^0([0,T];L^2(0,1))$. Hence we get a solution in $C^0([0,T];L^2(0,1))$ for the closed-loop system. We look for $U_2:\mathbb{R}\times L^2(0,1)\to\mathbb{R}$ satisfying the following properties

($\mathcal{P}_1$) The feedback law U_2 is T-periodic with respect to time:

$$U_2(t;y)=U_2(T+t;y), \tag{1.18}$$

($\mathcal{P}_2$) There exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that

$$t_0=0, \tag{1.19}$$

$$\lim_{n\to+\infty} t_n=T, \tag{1.20}$$

$$U_2 \text{ is of class } C^1 \text{ in } [t_n,t_{n+1})\times L^2(0,1), \tag{1.21}$$

($\mathcal{P}_3$) The feedback law U vanishes on $\mathbb{R}\times\{0\}$ and there exists a continuous function $M:[0,T)\to[0,+\infty)$ such that, for every $(t,y_1,y_2)\in[0,T)\times L^2(0,1)\times L^2(0,1)$,

$$|U(t;y_1)-U(t;y_2)|\leqslant M(t)(\|y_1-y_2\|_{L^2}), \tag{1.22}$$

($\mathcal{P}_4$) For every $(t,y)\in\mathbb{R}\times L^2(0,1)$, we have

$$|U(t;y)|\leqslant\min\{1,\sqrt{\|y\|_{L^2}}\}, \tag{1.23}$$

($\mathcal{P}_5$) If $\|y\|_{L^2(0,1)}\geqslant 1$, then, for every $t\in\mathbb{R}$, $U(t;y)=0$, and leading to the small-time local stability for the y variable if the feedback law $F = F_2$ is defined by

$$F_2(t; y, a) = (0, 0, U_2(t, y)). \tag{1.24}$$

More precisely, one has the following theorem.

Theorem 3. *Let $T > 0$. There exists $\varepsilon > 0$ and $U_2 : \mathbb{R} \times L^2(0,1) \to \mathbb{R}$ satisfying properties $(\mathcal{P}_1)$–$(\mathcal{P}_5)$, such that the feedback law F_2 defined by (1.24) is proper and, if the flow for the closed-loop system is denoted by Φ_2,*

(i) For every $y_0 \in L^2(0,1)$ and for every $a_0 \in \mathbb{R}$,

$$\Phi_2(T, 0; y_0, a_0) = (0, a_0) \text{ if } \|y_0\|_{L^2} \leqslant \varepsilon, \tag{1.25}$$

(ii) (Local uniform stability property.) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|(y_0, a_0)\|_V \leqslant \eta) \Rightarrow \left(\|\Phi_2(t, t'; y_0, a_0)\|_V \leqslant \delta, \ \ \forall 0 \leqslant t' \leqslant t \leqslant T\right). \tag{1.26}$$

This paper is organized as follows. Section 2 is dealing with the well-posedness of various Cauchy problems and the definition of proper feedback laws. Section 3 and Section 4 are on the global approximate stabilization and the small-time local stabilization. Then we define our time-varying feedback laws in Section 5. These feedbacks law lead to Theorem 1, which will be proved in Section 6. In the appendices, we prove some well-posedness results (for both open-loop systems and closed-loop systems), namely Proposition 7, Proposition 10, Theorem 24, Lemma 12, Lemma 27, and Lemma 29.

2 Well-posedness of the open-loop system (1.2) and proper feedback laws

In this section we briefly review results on the well-posedness of the open-loop system (1.2). Then we establish our new estimates which will be used for the well-posedness of the closed-loop systems. Finally we define *proper* feedback laws, i.e. feedback laws such that the closed-loop systems are well-posed in the context of our notion of solutions to (1.2).

Let us start with the linear Cauchy problem:

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = f(t,x) & \text{for } \ (t,x) \in (t_1, t_2) \times (0,1), \\ y(t,0) = \beta(t) & \text{for } \ t \in (t_1, t_2), \\ y(t,1) = \gamma(t) & \text{for } \ t \in (t_1, t_2), \\ y(t_1, \cdot) = y_0. \end{cases} \tag{2.1}$$

We use the following definition of solutions to the Cauchy problem (2.1) (solution in a transposition sense; see [22, 31, 54]).

Definition 4. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be such that $t_1 < t_2$. Let $y_0 \in H^{-1}(0,1), \beta$ and $\gamma \in L^2(t_1, t_2)$, and $f \in L^1(t_1, t_2; H^{-1}(0,1))$. A solution to the Cauchy problem (2.1) is a function*

y in $C^0([t_1,t_2];H^{-1}(0,1))$ such that

$$-\langle y_0, u(t_1,\cdot)\rangle_{H^{-1},H_0^1} + \langle y(s,\cdot), u(s,\cdot)\rangle_{H^{-1},H_0^1} + \int_{t_1}^{s} \gamma(t) u_x(t,1)dt \\ - \int_{t_1}^{s} \beta(t) u_x(t,0)dt - \int_{t_1}^{s} \langle f(t,x), u(t,x)\rangle_{H^{-1},H_0^1} dt = 0, \quad (2.2)$$

for every $s \in [t_1,t_2]$, for every $u \in L^2(t_1,t_2;H^2(0,1)) \cap H^1(t_1,t_2;H_0^1(0,1))$ such that

$$u_t(t,x) + u_{xx}(t,x) = 0 \text{ in } L^2((t_1,t_2)\times(0,1)). \quad (2.3)$$

This definition ensures the uniqueness (there exists at most one solution), but is not sufficient to get the existence of solutions. Concerning this existence of solutions, and therefore the well-posedness of the Cauchy problem (2.1), one has the following proposition.

Proposition 5. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2 < t_1 + 1$.*

(1) If $f = 0$, $\beta = \gamma = 0$, then, for every y_0 in $H^{-1}(0,1))$, the Cauchy problem (2.1) has a unique solution $y \in C^0([t_1,t_2];H^{-1}(0,1))$. Moreover, when $y_0 \in L^2(0,1)$, this solution is in

$$C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;H_0^1(0,1)), \quad (2.4)$$

and satisfies

$$\|y\|_{C^0L^2} \leqslant \|y_0\|_{L^2} \text{ and } \|y\|_{L^2H_0^1} \leqslant \|y_0\|_{L^2}. \quad (2.5)$$

(2) If $y_0 = 0, \beta = \gamma = 0$, and $f \in L^1(t_1,t_2;L^2(0,1)) \cup L^2(t_1,t_2;H^{-1}(0,1))$, the Cauchy problem (2.1) has a unique solution y. Moreover

$$\|y\|_{C^0L^2} \leqslant \|f\|_{L^1L^2} \text{ and } \|y\|_{L^2H_0^1} \leqslant \|f\|_{L^1L^2} \quad (2.6)$$

and there exists $C_1 \geqslant 1$ (which is independent of $0 < t_2 - t_1 < 1$) such that

$$\|y\|_{C^0L^2 \cap L^2H_0^1} \leqslant C_1 \|f\|_{L^2H^{-1}}. \quad (2.7)$$

(3) If $y_0 = 0$, $f = 0$, β, and $\gamma \in L^2(t_1,t_2)$, the Cauchy problem (2.1) has a unique solution y. If in addition β and $\gamma \in H^{3/4}(t_1,t_2)$, this solution is also in $C^0H^1 \cap L^2H^2$.

In this proposition and in the following, in order to simplify the notations, when there is no possible misunderstanding on the the time interval, C^0L^2 denotes the space $C^0([t_1,t_2];L^2(0,1))$, L^2L^∞ denotes the space $L^2(t_1,t_2;L^2(0,1))$ etc.

Properties (1) and (3) follow from classical arguments; see, for example, [22, Sections 2.3.1 and 2.7.1], [47, 57]. Property (2) follows from direct calculations and one can find similar results in [45]. Since we want to investigate the well-posedness of closed-loop systems, (3) is difficult to use. For that reason, we investigate the well-posedness with lower regularities on β and γ. For the heat equation, we have the maximum principle:

Lemma 6 (Maximum principle: linear case)**.** *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in H^{-1}$, $\beta \in L^2(t_1,t_2)$, $\gamma \in L^2(t_1,t_2)$, and $f \in L^2(t_1,t_2;H^{-1})$. Let $y \in C^0([0,T];H^{-1})$ be the solution of the Cauchy problem*

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = f & \text{for } (t,x) \in (t_1,t_2)\times(0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (t_1,t_2), \\ y(t,1) = \gamma(t) & \text{for } t \in (t_1,t_2), \\ y(t_1,\cdot) = y_0. \end{cases} \quad (2.8)$$

If

$$y_0 \geqslant 0, f \geqslant 0, \beta \geqslant 0, \text{ and } \gamma \geqslant 0, \tag{2.9}$$

then

$$y(t, \cdot) \geqslant 0, \quad \forall t \in [t_1, t_2]. \tag{2.10}$$

Thanks to the maximum principle, we get a new version of the well-posedness of system (2.1), the proof of which is given in Appendix A.

Proposition 7. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. If $f = 0, y_0 = 0$, β and $\gamma \in L^\infty(t_1, t_2)$, the unique solution y of the Cauchy problem (2.1) is in $L^\infty(t_1, t_2; L^2(0, 1)) \cap L^2(t_1, t_2; L^\infty(0, 1))$ and this solution is also in $C^0([t_1, t_2]; L^2(0, 1))$ provided that β and γ are in $C^0([t_1, t_2])$. Moreover, for every $T_0 > 0$, and for every $\eta > 0$, there exists a constant $C_{T_0,\eta} > 0$ such that, for every $t_1 \in \mathbb{R}$ and for every $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \leq t_1 + T_0$, for every β and for every $\gamma \in L^\infty(t_1, t_2)$, and for every $t \in (t_1, t_2]$,*

$$\|y\|_{L^\infty(t_1,t;L^2)\cap L^2(t_1,t;L^\infty)} \leqslant \left(\eta + C_{T_0,\eta}(t - t_1)^{1/2}\right) \left(\|\beta\|_{L^\infty(t_1,t)} + \|\gamma\|_{L^\infty(t_1,t)}\right). \tag{2.11}$$

Let us now turn to the nonlinear Cauchy problem

$$\begin{cases} y_t(t, x) - y_{xx}(t, x) + yy_x = f & \text{for } (t, x) \in (t_1, t_2) \times (0, 1), \\ y(t, 0) = \beta(t) & \text{for } t \in (t_1, t_2), \\ y(t, 1) = \gamma(t) & \text{for } t \in (t_1, t_2), \\ y(t_1, \cdot) = y_0. \end{cases} \tag{2.12}$$

The idea is to regard, in (2.12), $-yy_x = -(y^2)_x/2$ as a force term. Hence we adopt the following definition.

Definition 8. *Let $y_0 \in H^{-1}(0, 1)$, β and $\gamma \in L^2(t_1, t_2)$, and $f \in L^1(t_1, t_2; H^{-1}(0, 1))$. A solution to the Cauchy problem (2.12) is a function*

$$y \in L^\infty(t_1, t_2; L^2(0, 1)) \cap L^2(t_1, t_2; L^\infty(0, 1)) \tag{2.13}$$

which, in the sense of Definition 4, is a solution of (2.1) with

$$f := -(y^2)_x/2 + f \in L^1(t_1, t_2; H^{-1}(0, 1)). \tag{2.14}$$

Remark 9. *Let us point out that it would be better to write in (2.12) $(y^2)_x/2$ instead of yy_x. However, for the sake of better readability, we keep yy_x instead of $(y^2)_x/2$ here and in the following.).*

For this nonlinear system, thanks to Proposition 7, the classical well-posedness results, stability results, and the maximum principle on the Cauchy problem (2.12) can be modified into the following ones, which are more suitable for the stabilization problem and which are also proved in Appendix A.

Proposition 10. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in L^2(0, 1)$, β and $\gamma \in L^\infty(t_1, t_2)$. If β and γ are piecewise continuous the Cauchy problem (2.12) with $f = 0$ has one and only one solution. This solution is in $C^0([t_1, t_2]; L^2(0, 1))$.*

Moreover, for every $R > 0$, $r > 0$, and $\varepsilon > 0$, there exists $T^{\varepsilon}_{R,r} > 0$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \le t_1 + T^{\varepsilon}_{R,r}$ and for every $y_0 \in L^2(0,1)$, β and $\gamma \in L^\infty(t_1,t_2)$ (not necessary to be piecewise continuous) such that

$$\|y_0\|_{L^2} \leqslant R \text{ and } \|\beta\|_{L^\infty} + \|\gamma\|_{L^\infty} \leqslant r, \tag{2.15}$$

the Cauchy problem (2.12) with $f = 0$ has one and only one solution and this solution satisfies

$$\|y\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant 2R, \tag{2.16}$$

$$\|y\|_{L^2(t_1,t_2;L^\infty(0,1))} \leqslant \varepsilon R. \tag{2.17}$$

Remark 11. *The conditions on β and γ are for the existence of solutions: one can get the uniqueness of the solution with less regularity on β and γ.*

Lemma 12 (Maximum principle: nonlinear case)**.** *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0^\pm \in L^2(0,1)$, $\beta^\pm \in L^\infty(t_1,t_2)$ be piecewise continuous, and $\gamma^\pm \in L^\infty(t_1,t_2)$ be piecewise continuous. Let $y^\pm \in C^0([t_1,t_2];H^{-1}(0,1)) \cap C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;L^\infty(0,1))$ be solutions to the Cauchy problem*

$$\begin{cases} y^\pm_t(t,x) - y^\pm_{xx}(t,x) + y^\pm y^\pm_x = 0 & \text{for } (t,x) \in (t_1,t_2) \times (0,1), \\ y^\pm(t,0) = \beta^\pm(t) & \text{for } t \in (t_1,t_2), \\ y^\pm(t,1) = \gamma^\pm(t) & \text{for } t \in (t_1,t_2), \\ y^\pm(t_1,\cdot) = y^\pm_0. \end{cases} \tag{2.18}$$

If

$$y_0^- \leqslant y_0^+,\ \beta^- \leqslant \beta^+, \text{ and } \gamma^- \leqslant \gamma^+, \tag{2.19}$$

then

$$y^-(t,\cdot) \leqslant y^+(t,\cdot), \quad \forall t \in [t_1,t_2]. \tag{2.20}$$

Lemma 13. *For every $R > 0$, $r > 0$, and $\tau > 0$, there exists $C(R,r,\tau) > 0$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \le t_1 + \tau$, and for every $y_0^\pm \in L^2(0,1)$, $\beta^\pm \in L^\infty(t_1,t_2)$ piecewise continuous, and $\gamma^\pm \in L^\infty(t_1,t_2)$ piecewise continuous such that*

$$\|y_0^\pm\|_{L^2} \leqslant R \text{ and } \|\beta^\pm\|_{L^\infty} + \|\gamma^\pm\|_{L^\infty} \leqslant r, \tag{2.21}$$

the solution to the Cauchy problem (2.12) with $f = 0$ satisfies

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant C(R,r,\tau) \left(\|y_0^+ - y_0^-\|_{L^2(0,1)} + \|\beta^+ - \beta^-\|_{L^\infty} + \|\gamma^+ - \gamma^-\|_{L^\infty} \right). \tag{2.22}$$

Let us now come back to system (1.2). We start with the definition of a solution to the Cauchy problem associated to (1.2).

Definition 14. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in L^2(0,1)$, $a_0 \in \mathbb{R}$, $\alpha \in L^1(t_1,t_2)$, u_1 and $u_2 \in L^\infty(t_1,t_2)$. A solution (y,a) to (1.2) with initial data (y_0,a_0) at time t_1 is a (y,a) satisfying*

$$y \in C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;L^\infty(0,1)), \tag{2.23}$$

$$a \in C^0([t_1,t_2]),\ a_t = \alpha \text{ in the distribution sense, and } a(t_1) = a_0, \tag{2.24}$$

$$\text{(2.1) holds in the sense of Definition 4, with } f := (-yy_x) + a(t),\ \beta := u_1,\ \gamma := u_2. \tag{2.25}$$

Remark 15. *Let us point out that, with Definition 14, Proposition 10 does not imply the existence of a solution to the Cauchy problem (1.2) since, in Definition 14, u_1 and u_2 are assumed to be only in $L^\infty(t_1,t_2)$ and not necessarily in $C^0([t_1,t_2])$. However this proposition implies this existence if u_1 and u_2 are only piecewise continuous. We choose L^∞ condition for u_1 and u_2 precisely to cover this case, which will be useful in the framework of the well-posedness of the closed-loop systems that we are going to consider.*

Definition 14 allows to define the notion of solution to the Cauchy problem associated to the closed-loop system (1.4) as follows.

Definition 16. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let*

$$\begin{array}{rcc} F : [s_1,s_2] \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)). \end{array}$$

Let $t_1 \in [s_1,s_2]$, $t_2 \in (t_1,s_2]$, $a_0 \in \mathbb{R}$, and $y_0 \in L^2(0,1)$. A solution on $[t_1,t_2]$ to the Cauchy problem associated to the the closed-loop system (1.4) with initial data (y_0,a_0) at time t_1 is a couple $(y,a) : [t_1,t_2] \to L^\infty(0,1) \times \mathbb{R}$ such that

$$t \in (t_1,t_2) \mapsto a(t) := A(t; y(t,\cdot), a(t)) \in L^1(t_1,t_2), \tag{2.26}$$

$$t \in (t_1,t_2) \mapsto u_1(t) := U_1(t; y(t,\cdot), a(t)) \in L^\infty(t_1,t_2), \tag{2.27}$$

$$t \in (t_1,t_2) \mapsto u_2(t) := U_2(t; y(t,\cdot), a(t)) \in L^\infty(t_1,t_2), \tag{2.28}$$

$$(y,a) \text{ is a solution (see Definition 14) of (1.2) with initial data } (y_0,a_0) \text{ at time } t_1. \tag{2.29}$$

We can now define feedback laws such that the closed-loop system has a unique solution in the sense of Definition 14. These feedback laws are called *proper* and are defined as follows.

Definition 17. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1,s_2]$ is an application*

$$\begin{array}{rcc} F : [s_1,s_2] \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)) \end{array}$$

such that, for every $t_1 \in [s_1,s_2]$, for every $t_2 \in (t_1,s_2]$, for every $a_0 \in \mathbb{R}$, and for every $y_0 \in L^2(0,1)$, there exists a unique solution on $[t_1,t_2]$ to the Cauchy problem associated to the the closed-loop system (1.4) with initial data (y_0,a_0) at time t_1 (see Definition 16).

A proper feedback law is an application F

$$\begin{array}{rcc} F : (-\infty,\infty) \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)) \end{array}$$

such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ such that $s_1 < s_2$, the feedback law restricted to $[s_1,s_2] \times L^2(0,1) \times \mathbb{R}$ is a proper feedback law on $[s_1,s_2]$.

3 Global approximate stabilization

Let $T > 0$ be given. As explained in Section 1.1, throughout this section we work with (z,a) instead of (y,a), where z is defined by (1.7). The equation satisfied by (z,a) is

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \\ a_t = \alpha(t) & \text{for } t \in (0,T). \end{cases} \tag{3.1}$$

The idea is to use the "transport term" $a(t)z_x$. Following the idea of backstepping (see e.g.[22, Section 12.5]), we first regard the term $a(t)$ as a control term: we consider the system

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \end{cases} \tag{3.2}$$

where, at time $t \in [0,T]$, the state is $z(t,\cdot) \in L^2(0,1)$ and the control is $a(t) \in \mathbb{R}$. Inequality (1.10) shows that the L^2-norm of the state decays whatever is the control. However it does not provide any information on the decay rate of this L^2-norm. In order to get information on this decay rate, we consider the weighted energy (see e.g. [8, Chapter 2], [26])

$$V_1(z) := \int_0^1 z^2 e^{-x} dx. \tag{3.3}$$

With a slight abuse of notations, let $V_1(t) := V_1(z(t))$. Then, at least if

$$z \in C^1([0,T]; H^{-1}(0,1)) \cap C^0([0,T]; H_0^1(0,1)) \text{ and } a \in C^0([0,T]), \tag{3.4}$$

$V_1 \in C^1([0,T])$ and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}V_1 &= \langle z_t, ze^{-x}\rangle_{H^{-1},H_0^1} \\ &= \langle z_{xx} - (z^2/2)_x - a(t)z_x, ze^{-x}\rangle_{H^{-1},H_0^1} \\ &= -\int_0^1 z_x^2 e^{-x} dx + \left(\frac{1}{2} - \frac{a}{2}\right) V_1(z) - \frac{1}{3}\int_0^1 z^3 e^{-x} dx. \end{aligned} \tag{3.5}$$

In fact, as one easily sees, (3.5) holds in the distribution sense in $L^1(0,T)$ if

$$z \in H^1(0,T; H^{-1}(0,1)) \cap L^2(0,T; H_0^1(0,1)) \text{ and } a \in L^\infty(0,T). \tag{3.6}$$

From now on we assume that (3.6) holds. Since

$$\|z\|_{L^\infty} \leqslant 2\left(\int_0^1 z_x^2 e^{-x} dx\right)^{1/2}, \tag{3.7}$$

(3.5) leads to

$$\frac{d}{dt}V_1 \leqslant -\int_0^1 z_x^2 e^{-x} dx + V_1^2 + (1-a)V_1. \tag{3.8}$$

We choose $a := (k+1)V_1$, then

$$\frac{d}{dt}V_1 \leqslant V_1 - kV_1^2. \tag{3.9}$$

The positive "equilibrium" point (of V_1) of the right hand side of (3.9) is $1/k$. Hence if k large enough we have

$$V_1(T) \leqslant 2/\sqrt{k}, \tag{3.10}$$

whatever is the initial data as shown by the following lemma.

Lemma 18. *Let $T > 0$. There exists $k_T \in \mathbb{N}$ such that, for every $k \geqslant k_T$ and for every $V_1 \in C^0([0,T];[0,+\infty))$ satisfying (3.9) in the distribution sense in $(0,T)$,*

$$V_1(T) \leqslant 2/\sqrt{k}. \tag{3.11}$$

Proof of Lemma 18. It is easy to observe that, if for some time $t_0 \in [0,T]$, $V_1(t_0) \leqslant 2/\sqrt{k}$, then $V_1(t) \leqslant 2/\sqrt{k}$, for every $t \in [t_0,T]$. So, arguing by contradiction, we may assume that

$$V_1(t) > 2/\sqrt{k}, \ \ \forall t \in [0,T]. \tag{3.12}$$

Then

$$\dot{V_1}(t) \leqslant -\frac{k}{2}V_1^2(t), \tag{3.13}$$

which implies that

$$-\frac{1}{V_1(T)} + \frac{1}{V_1(0)} \leq -\frac{kT}{2}. \tag{3.14}$$

From (3.12) and (3.14), we get

$$\frac{k}{2}T \leqslant \frac{1}{V_1(t)} \leqslant \frac{\sqrt{k}}{2}, \tag{3.15}$$

which implies that

$$\sqrt{k}T \leqslant 1. \tag{3.16}$$

□

3.1 Construction of feedback laws

3.1.1 Phantom tracking stage

Let us come back to the system (3.1). For any T given, we consider the following Lyapunov function generated from the phantom tracking idea:

$$V_2(z,a) := V_1(z) + (a - \lambda V_1(z))^2, \tag{3.17}$$

with λ to be chosen later. The idea is to penalize $a \neq \lambda V_1(z)$; see [24]. Again, with a slight abuse of notations, we define $V_2(t) := V_2(z(t),a(t))$. Then, at least if z is in $C^1([0,T];H^{-1}(0,1)) \cap C^0([0,T];H_0^1(0,1))$ and $a \in C^1([0,T])$, V_2 is of class C^1 and

$$\begin{aligned}\frac{d}{dt}V_2 &= \frac{d}{dt}V_1 + 2(a-\lambda V_1(z))(\alpha - \lambda\frac{d}{dt}V_1)\\ &\leqslant -\int_0^1 z_x^2 e^{-x}dx + V_1^2 + V_1 - \lambda V_1^2 + 2\left(a - \lambda V_1(z)\right)\left(\alpha - \lambda\frac{d}{dt}V_1 - \frac{V_1}{2}\right).\end{aligned}$$

We choose

$$\begin{aligned}\alpha :=&\lambda\frac{d}{dt}V_1 + \frac{V_1}{2} - \frac{1}{2}\lambda(a-\lambda V_1(z))^3\\ =&\lambda\left(-2\int_0^1 z_x^2 e^{-x}dx + (1-a)\,V_1(z) - \frac{2}{3}\int_0^1 z^3 e^{-x}dx\right) + \frac{V_1}{2} - \frac{1}{2}\lambda(a-\lambda V_1(z))^3.\end{aligned} \tag{3.18}$$

Then, at least if z is in $C^1([0,T];H^{-1}(0,1)) \cap C^0([0,T];H^1_0(0,1))$ and $\alpha \in C^0([0,T])$

$$\begin{aligned}\frac{d}{dt}V_2 &\leqslant V_1^2 + V_1 - \lambda V_1^2 - \lambda(a-\lambda V_1(z))^4 \\ &\leqslant V_1 - \frac{\lambda-1}{2}V_2^2 \\ &\leqslant V_2 - \frac{\lambda-1}{2}V_2^2. \end{aligned} \tag{3.19}$$

In fact, as one easily sees, (3.19) holds in the distribution sense in $L^1(0,T)$ if

$$z \in H^1(0,T;H^{-1}(0,1)) \cap L^2(0,T;H^1_0(0,1)) \text{ and } \alpha \in L^\infty(0,T). \tag{3.20}$$

Let $\varepsilon > 0$. Using Lemma 18 and (3.19), one gets the existence of $\lambda_0 > 1$, independent of (z,a) satisfying (3.20), such that, for every $\lambda \in [\lambda_0, +\infty)$,

$$|V_2(T/2)| \leqslant \frac{3}{\sqrt{\lambda-1}} \leqslant \varepsilon. \tag{3.21}$$

Meanwhile, there exists a constant C_ε such that $|a(T/2)| \leqslant C_\varepsilon$.

We denote this stationary feedback law by A_0, i.e., $A_0 : L^\infty(0,1)\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ is defined by

$$A_0(y,a) := \alpha \text{ with } \alpha \text{ given in (3.18), where } z \text{ is defined by (1.7)}, \tag{3.22}$$

with the natural convention that $A_0(y,a) = -\infty$ if $y \notin H^1(0,1)$. This convention is justified by the fact that, by (3.7), there exists $C > 0$

$$\left|\int_0^1 z^3 e^{-x}dx\right| \leqslant \|z\|_{L^\infty}\|z\|^2_{L^2} \leqslant \frac{3}{2}\int_0^1 z_x^2 e^{-x}dx + C\|z\|^4_{L^2}, \quad \forall z \in H^1_0(0,1). \tag{3.23}$$

Remark 19. *Let us point out that A_0 is an unbounded operator on the state space, which is $L^2(0,1)\times\mathbb{R}$. The set where it takes finite value is $H^1(0,1)\times\mathbb{R} \subsetneq L^2(0,1)\times\mathbb{R}$. However, as we will see in Subsection 3.2, the feedback law*

$$F(t;y,a) := (A_0(y,a), a, a), \quad \forall (t,y,a) \in D(F) := \mathbb{R}\times H^1(0,1)\times\mathbb{R} \tag{3.24}$$

is proper.

From (3.21) we see that $V_1(T/2)$ is small thanks to λ. However, at the same time, because of λ, a will approach to $\lambda V_1(T/2)$. This is bounded by some constant, and unfortunately we can not expect more precise uniform bounds than the above one. To solve this problem, in the next phase we construct a (stationary) feedback law which makes a decay to 0, but keeps V_1 small.

Remark 20. *Similar a priori estimates could be obtained for L^p-norm cases. Even more, one could further get L^∞ type estimates by using the technique introduced in [8, Chapter 4] and [25].*

3.1.2 Small-time global stabilization of the variable a

In this section we construct the stationary feedback law A_1 (see ($\mathcal{Q}_3$)). Since L^2-norm of z decays whatever is the control α, we only need to find a feedback law which stabilizes "a". In this section, we give a feedback law which stabilizes "a" in small time. For that it suffices to define A_1 by

$$A_1(a) := -\mu(a^2 + \sqrt{|a|}) \cdot \mathrm{sgn(a)}. \tag{3.25}$$

Indeed, with this A_1, one can easily verify that there exists $\mu_T > 0$ such that, whatever is $a(T/2)$, $a(T) = 0$ if $\mu \geq \mu_T$ and $\dot{a} = A_1(a)$.

Remark 21. *The feedback law A_1 is continuous but not Lipschitz. However, for every $t_1 \in \mathbb{R}$, for every $t_2 \in [t_1, +\infty)$, and for every $a_0 \in \mathbb{R}$, the ordinary differential equation $a_t = A_1(a)$ has a unique solution on $[t_1, t_2]$ such that $a(t_1) = a_0$.*

Remark 22. *For our Burgers equation, thanks to the energy dissipation (1.10), we do not need to care of z during the interval of time $[T/2, T]$. For some other partial differential equations, such decay phenomenon may not hold. However, the same strategy would also work. Indeed, we can stabilize a in very small time so that the change of z keeps small.*

Remark 23. *Another idea to stabilize a in finite time is to design a time-varying feedback law of the form $A_1(a) = -\mu_n a$ for $t \in [t_n, t_{n+1})$. However, if for every solution of $\dot{a} = A_1(a)$ on $[T/2, T]$, one has $a(T) = 0$ whatever is $a(T/2)$, this feedback law has to be unbounded on $[T/2, T) \times (-\delta, \delta)$ for every $\delta > 0$, which is not the case of A_1 defined by (3.25).*

In this section, our feedback law $F = F_1$ is defined by ($\mathcal{Q}_1$), (1.9), (1.14), (3.22), and (3.25). Let us point out that it satisfies ($\mathcal{Q}_2$)–($\mathcal{Q}_3$).

3.2 Well-posedness and properties of the flow

This part is devoted to the properness of the feedback law F_1. From the definition of $F_1(t; y, a)$ for $t \in [T/2, T]$ (see (1.9), (1.14), and (3.25)), it follows from Proposition 10 that F_1 is proper on the interval of time $[T/2, T]$.

By the T-periodicity of F_1 it just remains to prove the properness of F_1 on the interval of time $[0, T/2]$. This properness is a consequence of the following lemma, the proof of which is given in Appendix B

Lemma 24. *For every $T \in (0, +\infty)$, every $z_0 \in L^2(0, 1)$, and every $a_0 \in \mathbb{R}$, there exists one and only one (z, a) satisfying*

$$z \in L^2(0, T; H_0^1(0, 1)), \tag{3.26}$$

such that $(y, a) := (z + a, a)$ is a solution to (1.2) (see Definition 14) with initial data $(z_0 + a_0, a_0)$ at time 0 with

$$\alpha(t) := A_0(y(t), a(t)), \text{ for almost every } t \in (0, T), \tag{3.27}$$

$$\beta(t) = a(t), \gamma(t) = a(t), \quad \forall t \in [0, T]. \tag{3.28}$$

At a first sight it seems that (3.26) is too strong compared to what is imposed by (2.26) for the properness of F_1. Indeed, (2.26) just impose that $z \in L^1(0, T; H^1(0, 1))$. However, it follows from (1) and (2) of Proposition 5 that, if (y, a) is as in Lemma 24 with $y \in L^1(0, T; H^1(0, 1))$, then $z := y - a \in L^2(0, T; H^1(0, 1))$.

3.3 Proof of Theorem 2

It only remains to give the proof of Properties ($\mathcal{Q}_4$) and ($\mathcal{Q}_5$) of Theorem 2.

We first look at ($\mathcal{Q}_5$). Let $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$. Note that (3.20) holds, and therefore (3.21) also holds:

$$|V_2(z(T/2), a(T/2))| \leqslant \frac{3}{\sqrt{\lambda - 1}} \leqslant \varepsilon. \tag{3.29}$$

From (3.29) one gets the existence of $\bar{\lambda} > 0$ independent of $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ such that

$$\|z(T/2)\|_{L^2(0,1)} + (a(T/2) - \lambda V_1(z(T/2)))^2 \leqslant \varepsilon \tag{3.30}$$

when $\lambda \geq \bar{\lambda}$.

Then the next stage (i.e. the evolution during the interval of time [T/2,T]; see Section 3.1.2) gives

$$\|z(T)\|_{L^2(0,1)} \leqslant \varepsilon \text{ and } a(T) = 0, \tag{3.31}$$

which concludes the proof of ($\mathcal{Q}_5$).

It only remains to prove Property ($\mathcal{Q}_4$). If $T/2 \leqslant t \leqslant T$, this property clearly holds, since both $\|z\|_{L^2}$ and $|a|$ decay as time is increasing. If $0 \leqslant t \leqslant T/2$, we only need to care about the case where $0 \leqslant t' \leqslant t \leqslant T/2$, thanks to the first case. Since (3.20) holds, (3.19) holds in $L^1(t', t)$. This shows that

$$\dot{V}_2 \leqslant V_2 \text{ on } [t', t] \tag{3.32}$$

if λ is larger than 1. Then, using (3.17),

$$\begin{aligned} \left(V_1(z(t) + (a(t) - \lambda V_1(z(t)))^2\right) &= V_2(t) \\ &\leqslant e^{T/2} V_2(t') = e^{T/2} \left(V_1(z(t') + (a(t') - \lambda V_1(z(t')))^2\right), \end{aligned} \tag{3.33}$$

which concludes the proof of Property ($\mathcal{Q}_4$).

4 Small-time local stabilization

The aim of this section is to get the small-time local stabilization (for the y variable). The small-time local (and even semi-global) stabilization of the heat equation is given in [31]. Here we follow [31] and regard yy_x term as a small force term (as in [65] for a KdV equation). Throughout this section we define $\alpha := 0$ and $u_1 := 0$ in (1.2), hence it is sufficient to consider

$$\begin{cases} y_t - y_{xx} + yy_x = 0 & \text{for } (t, x) \in (s, +\infty) \times (0, 1), \\ y(t, 0) = 0 & \text{for } t \in (s, +\infty), \\ y(t, 1) = u_2(t) & \text{for } t \in (s, +\infty). \end{cases} \tag{4.1}$$

We construct a time-varying feedback law satisfying ($\mathcal{P}_1$)–($\mathcal{P}_5$) leading to the small-time local stabilization of system (4.1).

4.1 Local rapid stabilization

At first, let us briefly recall (see [55], [51, Chapter 4] or [31]) how the backstepping approach is used to get rapid stabilization for the following heat equation:

$$\begin{cases} y_t - y_{xx} = 0 & \text{for } (t, x) \in (s, +\infty) \times (0, 1), \\ y(t, 0) = 0 & \text{for } t \in (s, +\infty), \\ y(t, 1) = u_2(t) & \text{for } t \in (s, +\infty). \end{cases} \tag{4.2}$$

Let $\lambda > 1$ given. Since Volterra type transformations will be considered, let us define

$$D := \{(x,v) \in [0,1]^2; v \leqslant x\}. \tag{4.3}$$

We define the feedback law by

$$u_2(t) := K_\lambda y = \int_0^1 k_\lambda(1,v)y(t,v)dv, \tag{4.4}$$

where the kernel function k_λ is the unique solution of

$$\begin{cases} k_{xx} - k_{vv} - \lambda k = 0 & \text{in } D, \\ k(x,0) = 0 & \text{in } [0,1], \\ k(x,x) = -\frac{\lambda x}{2} & \text{in } [0,1]. \end{cases} \tag{4.5}$$

Let us perform the following transformation $\Pi_\lambda : L^2(0,1) \to L^2(0,1)$, $y \mapsto z$,

$$z(x) = \Pi_\lambda(y(x)) := y(x) - \int_0^x k_\lambda(x,v)y(v)dv. \tag{4.6}$$

The kernel function k_λ is of class C^2 in D and satisfies the following estimate (see [31, Lemma 1]).

Lemma 25. *There exists a constant C_1 which is independent of $\lambda > 1$, such that*

$$\|k_\lambda\|_{C^2(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}. \tag{4.7}$$

Remark 26. *In fact [31, Lemma 1] is dealing with the H^1-norm (for more general equations). However, the proof can easily be adapted to get Lemma 25. Moreover in the case of (4.2), the kernel can be expressed in terms of the Bessel function:*

$$K(x,v) = -v\frac{I_1\left(\sqrt{\lambda(x^2-v^2)}\right)}{\sqrt{\lambda(x^2-v^2)}}, \tag{4.8}$$

where I_1 is the first order modified Bessel function of the first kind; see [50, (4.33)]. This explicit formula allows also to prove Lemma 25. Inequality (4.7) is related to the estimate given in [52, Proposition 1] by Lebeau and Robbiano. See also [53]. With no difficulty, the C^2-estimate can be generalized to C^n-estimates, $n \geqslant 3$, and one can prove the analyticity of the solution, which also follows from (4.8). For similar estimates for a Korteweg-de Vries equation, see [65, Lemma 2] and [64, Lemma 3].

In particular the transformation $\Pi_\lambda : L^2(0,1) \to L^2(0,1)$ is a bounded linear operator. This operator is also invertible. The inverse transformation, $\Pi_\lambda^{-1} : L^2(0,1) \to L^2(0,1)$, is given by

$$y(x) = \Pi_\lambda^{-1}(z(x)) := z(x) + \int_0^x l_\lambda(x,v)z(v)dv, \tag{4.9}$$

with the kernel l_λ satisfies

$$\begin{cases} l_{xx} - l_{vv} + \lambda l = 0 & \text{in } D, \\ l(x,0) = 0 & \text{in } [0,1], \\ l(x,x) = -\frac{\lambda x}{2} & \text{in } [0,1]. \end{cases} \tag{4.10}$$

The same estimate as (4.7) holds for l,

$$\|l_\lambda\|_{C^2(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}. \tag{4.11}$$

In fact one can even get better estimates than (4.11) (see [31, Corollary 2]). Let us denote $z := \Pi_\lambda y$ by z to simplify the notations. From (4.6) and (4.9), we know that

$$\|y\|_{L^2} \leqslant e^{3/2C_1\sqrt{\lambda}}\|z\|_{L^2} \text{ and } \|z\|_{L^2} \leqslant e^{3/2C_1\sqrt{\lambda}}\|y\|_{L^2}, \tag{4.12}$$

$$\|y\|_{H^1} \leqslant \|z\|_{H_0^1} + C\|z\|_{L^2} \text{ and } \|z\|_{H_0^1} \leqslant \|y\|_{H^1} + C\|y\|_{L^2}. \tag{4.13}$$

Then, following (4.2), (4.4), and (4.6), the solution y of (4.2) with (4.4), is transformed (via Π_λ) into a solution of

$$\begin{cases} z_t - z_{xx} + \lambda z = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,1) = 0 & \text{for } t \in (s,+\infty), \end{cases} \tag{4.14}$$

from which we get exponential decay of the energy of z with an exponential decay rate at least equal to 2λ.

Let us now consider the local rapid stabilization of the Burgers equation (4.1). The idea is to construct a stationary continuous locally supported feedback law which is given by (4.4) near the equilibrium point.

Suppose that y is a solution of (4.1) with feedback law (4.4), *i.e.* y is a solution of the Cauchy problem

$$\begin{cases} y_t - y_{xx} + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (s,+\infty), \\ y(t,1) = K_\lambda y = \int_0^1 k_\lambda(1,v) y(t,v)dv & \text{for } t \in (s,+\infty), \\ y(0,\cdot) = y_0, \end{cases} \tag{4.15}$$

with $y_0 \in L^2(0,1)$. Then $z := \Pi_\lambda(y)$ satisfies

$$\begin{cases} z_t - z_{xx} + \lambda z = I & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,1) = 0 & \text{for } t \in (s,+\infty), \\ z(0,\cdot) = z_0, \end{cases} \tag{4.16}$$

with

$$z_0 := \Pi_\lambda(y), \tag{4.17}$$

$$I := -\Pi_\lambda^{-1}(z)\left(\Pi_\lambda^{-1}(z)\right)_x + \int_0^x k_\lambda(x,v)(yy_x)(v)dv. \tag{4.18}$$

For the Cauchy problem (4.15) and (4.16), we have the following lemma, whose proof is given in Appendix D.

Lemma 27. *Let $\lambda > 1$, $R > 0$, and $s \in \mathbb{R}$. There exists $0 < T_R^{tr} < 1$ such that, for every $y_0 \in L^2(0,1)$ such that*

$$\|y_0\|_{L^2} \leqslant R, \tag{4.19}$$

the Cauchy problem (4.15) has one and only one solution. This solution is also in $C^0([s, s + T_R^{tr}]; L^2(0,1)) \cap L^2(s, s + T_R^{tr}; H^1(0,1))$. Moreover, this solution satisfies

$$\|y\|_{C^0L^2 \cap L^2\dot{H}^1} \leqslant 3e^{3C_1\sqrt{\lambda}}R. \tag{4.20}$$

By Lemma 27, for any $z_0 \in L^2(0,1)$, there exists $\tilde{T} > 0$ such that, the Cauchy problem (4.16) has a unique solution on $t[s, s+\tilde{T}]$ and this solution is also in $C^0([s, s+\tilde{T}]; L^2(0,1)) \cap L^2(s, s+\tilde{T}; H_0^1(0,1))$.
Since

$$\|w^2\|_{L^2L^2}^2 = \|w^4\|_{L^1L^1} \leqslant \|w\|_{L^\infty L^2}^2 \|w\|_{L^2L^\infty}^2 \leqslant C\|w\|_{C^0L^2}^2 \|w\|_{L^2H^1}^2, \tag{4.21}$$

we know form direct calculations that $I \in L^2(s, s+\tilde{T}; H^{-1}(0,1))$ and that

$$I = -\frac{1}{2}\Big(\big(\Pi_\lambda^{-1}(z)\big)^2\Big)_x - \frac{1}{2}\int_0^x k_{\lambda,v}(x,v)y^2(v)dv - \frac{\lambda x}{4}y^2(x). \tag{4.22}$$

Note that, since $z \in C^0([s, s+\tilde{T}]; L^2(0,1)) \cap L^2(s, s+\tilde{T}; H_0^1(0,1))$ and $I \in L^2(s, s+\tilde{T}; H^{-1}(0,1))$, we have

$$\Big\langle z, \Big(\big(\Pi_\lambda^{-1}(z)\big)^2\Big)_x \Big\rangle_{H_0^1, H^{-1}} = \Big\langle z_x, \big(\Pi_\lambda^{-1}(z)\big)^2 \Big\rangle_{L^2,L^2}, \tag{4.23}$$

and

$$\frac{d}{dt}\|z\|_{L^2}^2 = -2\int_0^1 z_x^2(x)dx - 2\lambda\|z\|_{L^2}^2 + 2\langle z, I\rangle_{L^2,L^2} \text{ in } L^1(s, s+\tilde{T}). \tag{4.24}$$

Thanks to (4.7), (4.22) and (4.23), there exists $C_0 > 1$, $C_2 > 2C_1$ and $C_3 > C_2$, independent of $\lambda > 1$ and z, such that

$$\begin{aligned} 2|\langle z, I\rangle_{L^2,L^2}| &\leqslant \|z_x\|_{L^2}\|\big(\Pi_\lambda^{-1}(z)\big)^2\|_{L^2} + C_0e^{2C_1\sqrt{\lambda}}\|y\|_{L^2}^2\|z\|_{L^2} + C_0e^{2C_1\sqrt{\lambda}}\|y\|_{L^2}^2\|z\|_{L^\infty} \\ &\leqslant e^{C_2\sqrt{\lambda}}\|z_x\|_{L^2}\left(\|z\|_{L^2}^2 + \|z\|_{L^4}^2\right) \\ &\leqslant \|z_x\|_{L^2}^2 + e^{C_3\sqrt{\lambda}}\left(\|z\|_{L^2}^4 + \|z\|_{L^2}^6\right). \end{aligned} \tag{4.25}$$

Here, we used the estimate

$$\|z_x\|_{L^2}\|z\|_{L^4}^2 \leqslant \|z_x\|_{L^2}\|z\|_{L^2}\|z\|_{L^\infty} \leqslant \|z_x\|_{L^2}^{3/2}\|z\|_{L^2}^{3/2} \leqslant \|z_x\|_{L^2}^2 + C\|z\|_{L^2}^6. \tag{4.26}$$

Therefore

$$\frac{d}{dt}\|z\|_{L^2}^2 \leqslant -2\lambda\|z\|_{L^2}^2 + e^{C_3\sqrt{\lambda}}\left(\|z\|_{L^2}^4 + \|z\|_{L^2}^6\right) \text{ in } L^1(s, s+\tilde{T}). \tag{4.27}$$

If the initial energy $\|z_0\|_{L^2}$ is smaller than $e^{-C_3\sqrt{\lambda}}$ (this is not a sharp bound), we then have an exponential decay of the energy

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \quad \forall t \in [s, s+\tilde{T}]. \tag{4.28}$$

Since the energy of z decays, we can continue to use Lemma 27 in order to get that the solution z of (4.16) is in $C^0([s, s+2\tilde{T}]; L^2(0,1)) \cap L^2(s, s+2\tilde{T}; H_0^1(0,1))$, and that

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \quad \forall t \in [s, s+2\tilde{T}]. \tag{4.29}$$

We continue such procedure as time goes to infinity to get that the unique solution z satisfies

$$z \in C^0([s,+\infty);L^2(0,1)) \cap L^2_{loc}([s,+\infty);H^1_0(0,1)), \tag{4.30}$$

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \ \ \forall t \in [s,+\infty). \tag{4.31}$$

This solves the local rapid stabilization problem. More precisely, we have the following theorem.

Theorem 28. *Let $\lambda > 1$ and $s \in \mathbb{R}$. For every $y_0 \in L^2(0,1)$ such that*

$$\|y_0\|_{L^2} \leqslant e^{-2C_3\sqrt{\lambda}}, \tag{4.32}$$

the Cauchy problem (4.15) has one and only one solution.
This solution is also in $C^0([s,+\infty);L^2(0,1)) \cap L^2_{loc}([s,+\infty);H^1(0,1))$. Moreover, this solution satisfies

$$\|y(t-s)\|_{L^2} \leqslant e^{3C_1\sqrt{\lambda}}e^{-\frac{\lambda(t-s)}{2}}\|y_0\|_{L^2}. \tag{4.33}$$

However, one also needs the feedback law to be proper. As it will be seen later on, it suffices to multiply the former feedback law by a suitable cut-off function (see, in particular, Lemma 29).

4.2 Construction of feedback laws: piecewise backstepping

Inspired by [31], we construct a piecewise continuous feedback law on $[0,T)$ such that properties $(\mathcal{P}_2)$–$(\mathcal{P}_5)$ hold.

Let us choose

$$n_0 := 1 + [\frac{2}{\sqrt{T}}], \tag{4.34}$$

$$t_n := 0, \lambda_n := 0 \text{ for } n \in \{0,1,\dots,n_0-1\}, \tag{4.35}$$

$$t_n := T - 1/n^2, \lambda_n := 2n^8 \text{ for } n \in \{n_0, n_0+1, \dots\}. \tag{4.36}$$

It is tempting to define $U_2 : (-\infty,+\infty) \times L^2(0,1)$ by

$$U_2(t;y) = K_{\lambda_n}(t,y),\ n \in \{n_0-1, n_0, \dots\},\ t \in [t_n, t_{n+1}),\ y \in L^2(0,1), \tag{4.37}$$

$$U_2(t+T;y) = U_2(t;y),\ t \in \mathbb{R},\ y \in L^2(0,1). \tag{4.38}$$

However with this definition U_2 is not locally bounded in a neighborhood of $[0,T) \times \{0\}$, which is a drawback for robustness issue with respect to measurement errors. In order to handle this problem, we introduce a Lipschitz cutoff function $\varphi_\lambda : \mathbb{R}^+ \to \mathbb{R}^+$:

$$\varphi_\lambda(x) := \left\{ \begin{array}{c} 1, \quad \text{if } x \in [0, e^{-C_3\sqrt{\lambda}}/5], \\ 2 - 5e^{C_3\sqrt{\lambda}}x, \quad \text{if } x \in (e^{-C_3\sqrt{\lambda}}/5, 2e^{-C_3\sqrt{\lambda}}/5), \\ 0, \quad \text{if } x \in [2e^{-C_3\sqrt{\lambda}}/5, +\infty), \end{array} \right. \tag{4.39}$$

and replace (4.37) by

$$U_2(t,y) = \mathcal{K}_{\lambda_n}(t,y),\ n \in \{n_0-1, n_0, \dots\},\ t \in [t_n, t_{n+1}),\ y \in L^2(0,1), \tag{4.40}$$

where, for $\lambda \in (1,+\infty)$, $\mathcal{K}_\lambda : L^2(0,1) \to L^2(0,1)$ is defined by

$$\mathcal{K}_\lambda(y) := \varphi_\lambda(\|y\|_{L^2})K_\lambda y,\ y \in L^2(0,1). \tag{4.41}$$

From (4.7), (4.4), (4.39), (4.40), and (4.41), one can easily verify that

$$|U_2(t,y)| \leqslant \min\{1, \sqrt{\|y\|_{L^2}}\},\ t \in [0,T],\ y \in L^2(0,1). \tag{4.42}$$

In particular $(\mathcal{P}_4)$ holds.

4.3 Proof of Theorem 3

Let us start this proof by stating a lemma, whose proof is given in Appendix C, giving a properness result on stationary feedback laws.

Lemma 29. *Let $M > 0$ and $G : L^2(0,1) \to \mathbb{R}$ be a (stationary) feedback law satisfying*

$$|G(y) - G(z)| \leqslant M\|y - z\|_{L^2(0,1)}, \ \ \forall y \in L^2(0,1), \ \ \forall z \in L^2(0,1) \text{ and } G(0) = 0, \tag{4.43}$$

$$|G(y)| \leqslant M, \ \ \forall y \in L^2(0,1). \tag{4.44}$$

Then, for every $y^0 \in L^2(0,1)$ and for every $T > 0$, the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) + yy_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = G(y(t,\cdot)) & \text{for } t \in (0,T), \\ y(0,\cdot) = y_0. \end{cases} \tag{4.45}$$

has a unique solution (in the sense of Definition 16 with $A = 0$ and $a_0 = a = 0$).

Similar to Lemma 13, we also have the following stability result, whose proof is omitted since it is quite similar to the proof of Lemma 13.

Lemma 30. *Let $R > 0$, $M > 0$, and $T > 0$. There exists $C_S(R,M,T)$ such that, for every $G : L^2(0,1) \to \mathbb{R}$ a (stationary) feedback law satisfying*

$$|G(y) - G(z)| \leqslant M\|y - z\|_{L^2(0,1)}, \ \ \forall y \in L^2(0,1), \ \ \forall z \in L^2(0,1) \text{ and } G(0) = 0, \tag{4.46}$$

$$|G(y)| \leqslant M, \ \ \forall y \in L^2(0,1), \tag{4.47}$$

for every $y_0^{\pm} \in L^2(0,1)$ satisfying

$$\|y_0^{\pm}\|_{L^2(0,1)} \leqslant R, \tag{4.48}$$

the solutions $y^{\pm}$ to the Cauchy problem (4.45) satisfy

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant C_S(R,M,T)\|y_0^+ - y_0^-\|_{L^2(0,1)}. \tag{4.49}$$

Until the end of the proof of Theorem 3 our feedback law F is defined by $A := 0$, $U_1 := 0$, and (4.38)-(4.40). Let us recall that the time-varying feedback law in Section 4.2 is piecewisely (with respect to time) given by the stationary feedback laws (4.41), where K_λ is designed in Section 4.1. Let us point out that, for every $\lambda \in [1, +\infty)$,

$$|\mathcal{K}_\lambda(y)| \leqslant 1, \ \ \forall y \in L^2(0,1), \text{ and } \mathcal{K}_\lambda(0) = 0 \tag{4.50}$$

and there exists $M_\lambda > 0$ such that

$$|\mathcal{K}_\lambda(y^1) - \mathcal{K}_\lambda(y^2)| \leqslant M_\lambda\|y^1 - y^2\|_{L^2(0,1)}, \ \ \forall (y^1, y^2) \in L^2(0,1) \times L^2(0,1). \tag{4.51}$$

Hence, by Lemma 29, these stationary feedback laws are proper on $(-\infty, +\infty)$. In particular, the time-varying feedback law F is proper on $[0, s_2]$ for every $s_2 \in (0,T)$. Hence, for every $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ and $t_1 \in [0,T)$ we get the existence and the uniqueness of $y : [t_1, T) \to L^2(0,1)$ and $a : [t_1, T) \to \mathbb{R}$ such that, for every $t_2 \in (t_1, T)$ the restriction of (y, a) to $[t_1, t_2]$ is the solution on $[t_1, t_2]$ to the Cauchy problem of the closed-loop system (1.4) with initial data (y_0, a_0) at time t_1 (in the sense of Definition 16).

In order to get the properness of the feedback law F_2 (defined in (1.24)), it suffices to show that

$$\lim_{t\to T^-} y(t) \text{ exists in } L^2(0,1). \tag{4.52}$$

In order to prove (4.52), we check that $\{y(t)\}(t\to T^-)$ is a Cauchy sequence in $L^2(0,1)$. Let us point out that

$$y\in L^\infty(t_1,T;L^2(0,1))\cap L^2(t_1,T;L^\infty(0,1)). \tag{4.53}$$

Indeed, (4.53) follows from the maximum principle (Lemma 12), Proposition 10 (applied with $f=0$, $\beta=0$, $\gamma=\pm1$), and (4.42). Let

$$f:=-(1/2)(y^2)_x. \tag{4.54}$$

By (4.53) and (4.54),

$$f\in L^2(t_1,T;H^{-1}(0,1)). \tag{4.55}$$

Let $t_2\in(t_1,T)$. Let $y^{\pm}_{t_2}$ be the solutions of

$$\begin{cases} (y^{\pm}_{t_2})_t-(y^{\pm}_{t_2})_{xx}=f & \text{for } (t,x)\in(t_2,T)\times(0,1),\\ (y^{\pm}_{t_2})(t,0)=0 & \text{for } t\in(t_2,T),\\ (y^{\pm}_{t_2})(t,1)=\pm1 & \text{for } t\in(t_2,T),\\ (y^{\pm}_{t_2})(t_2,\cdot)=y(t_2). \end{cases} \tag{4.56}$$

Let us define $w_{t_2}:=(y_{t_2})^+-(y_{t_2})^-$. Then

$$\begin{cases} (w_{t_2})_t-(w_{t_2})_{xx}=0 & \text{for } (t,x)\in(t_2,T)\times(0,1),\\ (w_{t_2})(t,0)=0 & \text{for } t\in(t_2,T),\\ (w_{t_2})(t,1)=2 & \text{for } t\in(t_2,T),\\ (w_{t_2})(t_2,\cdot)=0. \end{cases} \tag{4.57}$$

Let $\varepsilon>0$. From Proposition 7, (4.54), and (4.57), there exists $t_2\in(t_1,T)$ such that

$$\|w_{t_2}\|_{C^0([t_2,T];L^2(0,1))}\leqslant\varepsilon/4. \tag{4.58}$$

Moreover, from the maximum principle in the linear case (see Proposition 6), (4.42), (4.45), and (4.56), we know that

$$y^-_{t_2}(t)\leqslant y(t)\leqslant y^+_{t_2}(t),\quad \forall t\in[t_2,T), \tag{4.59}$$

which, together with (4.58), implies that

$$\|y^+_{t_2}-y\|_{C^0([t_2,T];L^2(0,1)}\leqslant\varepsilon/4. \tag{4.60}$$

Since $y^+_{t_2}$ is in $C^0([t_2,T];L^2(0,1))$, there exists $\tilde t_2\in[t_2,T)$ such that

$$\|y^+_{t_2}(t)-y^+_{t_2}(T)\|_{L^2}\leqslant\varepsilon/4,\quad \forall t\in[\tilde t_2,T]. \tag{4.61}$$

From (4.60) and (4.61),

$$\|y(t)-y(t')\|_{L^2}\leqslant\varepsilon,\quad \forall t,t'\in[\tilde t_2,T). \tag{4.62}$$

This implies (4.52) and concludes the proof of the properness of the feedback law F_2.

Now we are ready to prove Theorem 3. Since "a" does not change (see (1.24)), it suffices to only consider y. About property *(i)*, as we saw in Section 4.1, $\|y(t)\|_{L^2}$ decays rapidly on $[t_n, t_{n+1})$ provided that $y(t_n)$ is small enough in $L^2(0,1)$. Our idea is to set $\|y(0)\|_{L^2}$ sufficiently small so that the flow will decay exponentially (in $L^2(0,1)$) with rate $\lambda_0/2$ on $[t_0, t_1)$; then at time t_1, the energy $y(t_1)$ is already small enough to have an exponential decay with rate $\lambda_1/2$ on $[t_1, t_2)$. Continuing this way one may expect that, at the end, $y(T) = 0$. In order to have an exponential decay with rate $\lambda_n/2$ on $[t_n, t_{n+1})$, it is sufficient to have

$$\|y(t_n)\|_{L^2} \leqslant e^{-2C_3\sqrt{\lambda_n}}. \tag{4.63}$$

These exponential decay rates on $[t_n, t_{n+1})$ for every $n \in \mathbb{N}$ can be achieved for $\|y(0)\|_{L^2}$ sufficiently small if there exists $c > 0$ such that

$$c|\Pi_{\lambda_n}||\Pi_{\lambda_n}^{-1}| \prod_{k=n_0}^{n-1} \left(|\Pi_{\lambda_k}||\Pi_{\lambda_k}^{-1}|e^{-(t_{k+1}-t_k)\lambda_k/2}\right) \leqslant e^{-2C_3 n^4}, \text{ for all } n \in \mathbb{N}. \tag{4.64}$$

Hence, it suffices to find $c > 0$ such that

$$c\left(\prod_{k=n_0}^{n-1} e^{-k^5}\right)\left(\prod_{k=n_0}^{n} e^{3C_1k^4}\right) \leqslant e^{-2C_3n^4} \text{ holds for every } n \in \mathbb{N}. \tag{4.65}$$

Such c obviously exist, and one can find similar computations in [65].

Actually, the above proof also shows the following lemma.

Lemma 31. *Let $\varepsilon > 0$. Let $0 < T_0 < T$. There exists a constant $\eta > 0$ such that*

$$(\|y_0, a_0)\|_V \leqslant \eta) \Rightarrow \left(\|\Phi_2(t, t'; y_0, a_0)\|_V \leqslant \varepsilon, \;\; \forall 0 \leqslant t' \leqslant t \leqslant T_0\right). \tag{4.66}$$

The second part, *(ii)*, of Theorem 3 is then a consequence of the following lemma.

Lemma 32. *Let $\varepsilon > 0$. There exists $0 < T_1 < T$ such that*

$$(\|y_0, a_0)\|_V \leqslant \varepsilon) \Rightarrow \left(\|\Phi_2(t, t'; y_0, a_0)\|_V \leqslant 2\varepsilon, \;\; \forall T_1 \leqslant t' \leqslant t \leqslant T\right). \tag{4.67}$$

Property (4.67) is a consequence of Proposition 10 and (4.42). This completes the proof of Theorem 3.

5 Proper feedback laws for system (1.2)

Finally, we are now in position to define our proper feedback law $F = (A, U_1, U_2)$ for system (1.2). We define a $2T$-periodic feedback law which leads to the approximate stabilization in the first stage ($[0,T]$) and then "stabilizes" (y, a) to 0 in the second stage ($[T, 2T)$). Our feedback law F is defined as follows.

$$A(t; y, a) := \begin{cases} A_0(y,a), & \text{if } t \in [0, T/2), \\ A_1(a), & \text{if } t \in [T/2, T), \\ 0, & \text{if } t \in [T, 2T), \end{cases} \tag{5.1}$$

$$U_1(t;y,a) := \begin{cases} a, & \text{if } t \in [0,T/2), \\ a, & \text{if } t \in [T/2,T), \\ 0, & \text{if } t \in [T,2T), \end{cases} \tag{5.2}$$

$$U_2(t;y,a) := \begin{cases} a, & \text{if } t \in [0,T/2), \\ a, & \text{if } t \in [T/2,T), \\ \mathcal{K}_{\lambda_n}(y), & \text{if } t \in [T+t_n, T+t_{n+1}), \end{cases} \tag{5.3}$$

where λ_n and t_n are defined in (4.35) and (4.36), $\mathcal{K}_\lambda$ is defined in (4.41), A_0 is defined in (3.22), and A_1 is defined in (3.25).

Thanks to Section 3.2 and Section 4.3, the feedback law (5.2)–(5.1) is proper (in the sense of Definition 17).

6 Small-time global stabilization

The small-time global stabilization (Theorem 1) contains two parts, *(i)* and *(ii)*. Let us first consider *(i)*. Let us denote by Φ the flow associated to the feedback law F. From (1.16) and (1.25) we get that

$$\Phi(2T,0;y,a) = (0,0). \tag{6.1}$$

Let $t \in [0,2T)$. Then

$$\Phi(4T,t;y,a) = \Phi(4T,2T;\Phi(2T,t;y,a)) = (0,0), \tag{6.2}$$

which shows that *(i)* holds. Property *(ii)* follows directly from (1.15), (1.26), and (6.1). This completes the proof of Theorem 1.

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A Appendix: Proofs of Proposition 7, of Proposition 10, of Lemma 12, and of Lemma 13

This appendix is devoted to the proof of two propositions and of two lemma that we stated in Section 2: Proposition 7, Proposition 10, Lemma 12, and Lemma 13.

Let us start with the proof of Proposition 7. Without loss of generality we may assume that $t_1 = 0$. Let $T_0 > 0$ and let $t_2 = T \leq T_0$. We consider the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ y(0,\cdot) = 0. \end{cases} \tag{A.1}$$

If $\gamma \in L^\infty(0,T)$, then the solution is in $C^0([0,T];H^{-1}(0,1))$. By the maximum principle (Lemma 6), one knows that y is also in $L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;L^\infty(0,1))$. Let us now assume that $\gamma \in C^1$. Then that solution is in $C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1(0,1))$. In order to give estimates on y in that space, let us define

$$z := y - x^n\gamma \text{ with } n \in \mathbb{N} \setminus \{0,1\} \text{ to be chosen later.} \tag{A.2}$$

Hence

$$\begin{cases} z_t - z_{xx} = -x^n\gamma_t + n(n-1)x^{n-2}\gamma & \text{for } (t,x)\in(0,T)\times(0,1),\\ z(t,0)=0 & \text{for } t\in(0,T),\\ z(t,1)=0 & \text{for } t\in(0,T),\\ z(0,\cdot)=-x^n\gamma(0). \end{cases} \tag{A.3}$$

Then, by Proposition 5, we have

$$\begin{aligned}\|z\|_{C^0L^2\cap L^2H_0^1} &\leqslant 2C_1\|-x^n\gamma_t+n(n-1)x^{n-2}\gamma\|_{L^1L^2}+2|\gamma(0)|\|x^n\|_{L^2}\\ &\leqslant 2C_1\left((\|\gamma_t\|_{L^1}+2|\gamma(0)|)\|x^n\|_{L^2}+n(n-1)\|\gamma\|_{L^1}\|x^{n-2}\|_{L^2}\right).\end{aligned} \tag{A.4}$$

For $y\in H^1(0,1)$ such that $y(0)=0$, let us define the $\dot H(0,1)$-norm of y by

$$\|y\|_{\dot H(0,1)} := \|y_x\|_{L^2(0,1)}. \tag{A.5}$$

By direct calculations, we know that

$$\|x^n\gamma\|_{C^0L^2\cap L^2\dot H^1}\leqslant \|x^n\|_{L^2}\|\gamma\|_{C^0}+n\|x^{n-1}\|_{L^2}\|\gamma\|_{L^2}. \tag{A.6}$$

Let $\eta\in(0,1/2)$. Taking n large enough, we get the existence of $C_\eta>0$, which is independent of $T\leqslant T_0$ and of γ, such that

$$\|y\|_{C^0L^2\cap L^2\dot H^1}\leqslant \eta(\|\gamma_t\|_{L^1}+\|\gamma\|_{C^0})+C_\eta\|\gamma\|_{L^2}. \tag{A.7}$$

Now, suppose that $\gamma\in L^\infty$. Let us consider the solution $y^\pm$ of

$$\begin{cases} y^\pm_t - y^\pm_{xx} = 0 & \text{for } (t,x)\in(0,T)\times(0,1),\\ y^\pm(t,0)=0 & \text{for } t\in(0,T),\\ y^\pm(t,1)=\pm v & \text{for } t\in(0,T),\\ y^\pm(0,\cdot)=0, \end{cases} \tag{A.8}$$

with $v:=\|\gamma\|_{L^\infty}\in[0,+\infty)$. Thanks to (A.7), we have

$$\|y^\pm\|_{C^0L^2\cap L^2\dot H^1}\leqslant \eta v + C_\eta T^{1/2}v = (\eta+C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{A.9}$$

By direct computations, there exists $C>0$ such that, for every $\varphi\in H^1(0,1)$ with $\varphi(0)=0$,

$$\|\varphi\|_{L^\infty(0,1)}\leqslant C\|\varphi\|^{1/2}_{L^2(0,1)}\|\varphi_x\|^{1/2}_{L^2(0,1)}, \tag{A.10}$$

Actually, since $\varphi(0)=0$, we have

$$\varphi^2(x) = 2\int_0^x \varphi(s)\varphi'(s)ds \tag{A.11}$$

which leads to inequality (A.10). It is also a simple case of Gagliardo–Nirenberg interpolation inequality. From (A.10) one gets that, for every $T>0$ and for every $\varphi\in L^\infty(0,T;L^2(0,1))\cap L^2(0,T;\dot H^1(0,1))$ such that $\varphi(\cdot,0)=0\in L^2(0,T)$,

$$\|\varphi\|_{L^2(0,T;L^\infty(0,1))}\leqslant CT^{1/4}\|\varphi\|^{1/2}_{L^\infty(0,T;L^2(0,1))}\|\varphi\|^{1/2}_{L^2(0,T;\dot H^1(0,1))}. \tag{A.12}$$

Hence we have,

$$\|y^{\pm}\|_{C^0L^2} \leqslant (\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{A.13}$$

$$\|y^{\pm}\|_{L^2L^\infty} \leqslant CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{A.14}$$

Since $-v \leqslant \gamma \leqslant +v$, by the maximum principle (Lemma 6), we have

$$y^- \leqslant y \leqslant y^+, \text{ for all } t \in [0, T], \tag{A.15}$$

which, together with (A.13) and (A.14), implies that

$$\|y\|_{L^\infty L^2} \leqslant 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{A.16}$$

$$\|y\|_{L^2L^\infty} \leqslant 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{A.17}$$

Let us now prove that if $\gamma \in C^0([0, T])$ then the solution y is in $C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; L^\infty(0, 1))$. Suppose that $\gamma \in C^0([0, T])$ is given, then there exists

$$\{\gamma_n\}_{n\in\mathbb{N}}, \text{ a sequence of } C^1([0, T]) \text{ functions which uniformly converges to } \gamma. \tag{A.18}$$

Let us denote by $\{y_n\}_{n\in\mathbb{N}}$ the sequence of solutions of (A.1) with controls given by $\{\gamma_n\}_{n\in\mathbb{N}}$. Thanks to (A.18), for any $\varepsilon > 0$, there exists N such that when $m, n > N$, we have

$$\|\gamma_m - \gamma_n\|_{C^0([0,T])} \leqslant \varepsilon. \tag{A.19}$$

Hence, by (A.16) and (A.17),

$$\|y_m - y_n\|_{L^2L^\infty \cap L^\infty L^2} \leqslant C(\eta + C_\eta T^{1/2})\|\gamma_m - \gamma_n\|_{L^\infty}. \tag{A.20}$$

Since $\gamma_n \in C^1([0, T])$, from Proposition 5 we have $y_n \in C^0([0, T]; L^2(0, 1))$. Hence

$$\|y_m - y_n\|_{L^2L^\infty \cap C^0L^2} \leqslant C(\eta + C_\eta T^{1/2})\|\gamma_m - \gamma_n\|_{L^\infty}, \tag{A.21}$$

which means that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0, T]; L^2(0, 1))$. Hence

$$y \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; L^\infty(0, 1)). \tag{A.22}$$

Letting also n to infinity in (A.16) and (A.17) for y_n and γ_n, we get again (A.16) and (A.17).

Let us finally consider the case where $\gamma \in L^\infty(0, T)$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}^*}$ of functions in $C^0([0, T])$ such that

$$\|\gamma_n\|_{L^\infty(0,T)} \leq \|\gamma\|_{L^\infty(0,T)}, \quad \forall n \in \mathbb{N}^*, \tag{A.23}$$

$$\lim_{n\to+\infty} \gamma_n(t) = \gamma(t) \text{ for almost every t } \in (0, T). \tag{A.24}$$

One can, for example, take

$$\gamma_n(t) := \frac{1}{n}\int_{\max(0,t-(1/n))}^{t} \gamma(s)ds. \tag{A.25}$$

Let us denote by $\{y_n\}_{n\in\mathbb{N}^*}$ the sequence of solutions of (A.1) with control given by $\{\gamma_n\}_{n\in\mathbb{N}^*}$. Then $\{y_n\}_{n\in\mathbb{N}^*}$ is bounded in $L^2((0, T) \times (0, 1))$. Then there exists a subsequence of the $\{y_n\}_{n\in\mathbb{N}^*}$, that one also denotes by $\{y_n\}_{n\in\mathbb{N}^*}$, and $y \in L^2((0, T) \times (0, 1)$ such that

$$y_n \rightharpoonup y \text{ weakly in } L^2((0, T) \times (0, 1)). \tag{A.26}$$

Then one easily checks that y is a solution of (A.1) with control given by γ and that y satisfies (A.16) and (A.17).

All these calculations are based on the assumption $\beta = 0$. If $\gamma = 0$ and $\beta \neq 0$, similar estimates hold. By linearity, one then gets Proposition 7.

Remark 33. *One can also get the continuity of $y : [0, T] \to L^2(0, 1)$ when γ is in $BV([0, 1]) \cap L^\infty(0, T)$. The idea is to use directly (A.7).*

Remark 34. *Multiplying (A.1) by $(1-x)y$ and integration by parts show that $(1-x)y_x^2 \in L^1L^1$ if $\beta = 0$.*

We are now ready to prove Proposition 10 and Lemma 12, the proof is given by 4 steps.

Step 1. Local existence and uniqueness of the solution,

Step 2. Continuity of the solution with respect to the initial data and the boundary conditions,

Step 3. Maximum principle (Lemma 12),

Step 4. Global existence of the solution.

Step 1. Local existence and uniqueness of the solution. In this step we prove the second part of the statement of Proposition 10. Again, for simplicity we only treat the case where $\beta = 0$. To simplify the notations we let $t_1 = 0$ and $T := t_2$, i.e. we consider the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) + yy_x(t,x) = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ y(0,\cdot) = y_0. \end{cases} \tag{A.27}$$

We use the standard Banach fixed point theorem to get the local existence and uniqueness. We consider the space

$$X := L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; L^\infty(0,1)) \tag{A.28}$$

with the norm, adapted to deal with (2.16) and (2.17),

$$\|y\|_{X_\mu} := \|y\|_{L^\infty L^2} + \frac{1}{\mu}\|y\|_{L^2 L^\infty}, \tag{A.29}$$

with $\mu > 0$ to be chosen later. We denote by X_μ the space X with the norm $\|\cdot\|_{X_\mu}$, which is a Banach space. The choice of the norm $\|\cdot\|_{X_\mu}$ is based on the observation that $\|y\|_{L^2L^\infty}$ can be sufficiently small once we set time small enough.

We consider the following map $\Gamma : X_\mu \to X_\mu$, where $\Gamma(y)$ is the unique solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -\frac{1}{2}(y^2)_x(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0. \end{cases} \tag{A.30}$$

This map is well defined thanks to Proposition 5, (A.16) and (A.17). A function y is solution to (A.27) if and only if it is a fixed point of Γ. The function $\Gamma(y)$ can be decomposed as follows

$$\Gamma(y) = z^1 + z^2 + z^3, \tag{A.31}$$

where z^1, z^2, and z^3 are the solutions to the following Cauchy problems

$$\begin{cases} z^1_t(t,x) - z^1_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^1(t,0) = 0 & \text{for } t \in (0,T),\\ z^1(t,1) = 0 & \text{for } t \in (0,T),\\ z^1(0,\cdot) = y_0 \end{cases} \tag{A.32}$$

$$\begin{cases} z^2_t(t,x) - z^2_{xx}(t,x) = -\frac{1}{2}(y^2)_x(t,x) & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^2(t,0) = 0 & \text{for } t \in (0,T),\\ z^2(t,1) = 0 & \text{for } t \in (0,T),\\ z^2(0,\cdot) = 0, \end{cases} \tag{A.33}$$

$$\begin{cases} z^3_t(t,x) - z^3_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^3(t,0) = 0 & \text{for } t \in (0,T),\\ z^3(t,1) = \gamma(t) & \text{for } t \in (0,T),\\ z^3(0,\cdot) = 0. \end{cases} \tag{A.34}$$

From Proposition 5, (A.16), and (A.17), one gets

$$\|z^1\|_{C^0L^2} \leqslant \|y_0\|_{L^2} \text{ and } \|z^1\|_{L^2H^1_0} \leqslant \|y_0\|_{L^2}, \tag{A.35}$$

$$\|z^2\|_{C^0L^2} + \|z^2\|_{L^2H^1} \leqslant C_1\|yy_x\|_{L^2H^{-1}} \leqslant \frac{C_1}{2}\|y^2\|_{L^2L^2} \leqslant \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}, \tag{A.36}$$

$$\|z^3\|_{L^\infty L^2} \leqslant 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{A.37}$$

$$\|z^3\|_{L^2L^\infty} \leqslant 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{A.38}$$

From (A.36), (A.10), and (A.12), we get further

$$\|z^1\|_{L^2L^\infty} \leqslant CT^{1/4}\|y_0\|_{L^2}, \tag{A.39}$$

$$\|z^2\|_{C^0L^2} \leqslant \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}, \tag{A.40}$$

$$\|z^2\|_{L^2L^\infty} \leqslant CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}. \tag{A.41}$$

Since $\|y_0\|_{L^2} \leqslant R$ and $\|\gamma\|_{L^\infty} \leqslant r$, let us choose the ball

$$B_R := \{y \in X : \|y\|_{X_\mu} \leqslant 2R\}. \tag{A.42}$$

Then, from (A.29) and (A.31)–(A.42), one knows that

$$\begin{aligned} \|\Gamma(y)\|_{L^\infty L^2} &\leqslant \|y_0\|_{L^2} + \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2} + 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}\\ &\leqslant \|y_0\|_{L^2} + 2C_1\mu R^2 + 2(\eta + C_\eta T^{1/2})r, \end{aligned} \tag{A.43}$$

and

$$\begin{aligned} \|\Gamma(y)\|_{L^2L^\infty} &\leqslant CT^{1/4}\|y_0\|_{L^2} + CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2} + 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}\\ &\leqslant CT^{1/4}\|y_0\|_{L^2} + 2CT^{1/4}C_1\mu R^2 + 2CT^{1/4}(\eta + C_\eta T^{1/2})r. \end{aligned} \tag{A.44}$$

Hence

$$\|\Gamma(y)\|_{X_\mu} \leqslant (1+CT^{1/4}/\mu)R + 2C_1\mu(1+CT^{1/4}/\mu)R^2 + 2(1+CT^{1/4}/\mu)(\eta + C_\eta T^{1/2})r, \quad \text{(A.45)}$$

and we can successively choose η, μ, and T such that

$$\|\Gamma(y)\|_{X_\mu} \leqslant (3/2)R. \quad \text{(A.46)}$$

Hence

$$\Gamma(B_R) \subset B_R. \quad \text{(A.47)}$$

Let us now prove that Γ is a contraction in B_R. We perform similar computations. Let us assume that

$$y_1 \text{ and } y_2 \in B_R. \quad \text{(A.48)}$$

Then $w := \Gamma(y_1) - \Gamma(y_2)$ is the solution of

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) = -\frac{1}{2}\left((y_1^2)_x - (y_2^2)_x\right)(t,x) & \text{for } (t,x) \in (0,T)\times(0,1), \\ w(t,0) = 0 & \text{for } t \in (0,T), \\ w(t,1) = 0 & \text{for } t \in (0,T), \\ w(0,\cdot) = 0. \end{cases} \quad \text{(A.49)}$$

Hence, by Proposition 5,

$$\begin{aligned} \|w\|_{C^0L^2\cap L^2H_0^1} &\leqslant \frac{C_1}{2}\|y_1+y_2\|_{L^2L^\infty}\|y_1-y_2\|_{L^\infty L^2} \\ &\leqslant C_1R\mu\|y_1-y_2\|_{L^\infty L^2}. \end{aligned} \quad \text{(A.50)}$$

Thus

$$\|w\|_{L^2L^\infty} \leqslant C_1CT^{1/4}R\mu\|y_1-y_2\|_{L^\infty L^2}. \quad \text{(A.51)}$$

When μ and T are small enough, we have

$$\|\Gamma(y_1)-\Gamma(y_2)\|_{X_\mu} \leqslant (1/2)\|y_1-y_2\|_{X_\mu}. \quad \text{(A.52)}$$

Hence we get the existence of a unique solution in B_R. Let us now prove the uniqueness of solution in X_μ. It suffice to show the uniqueness of solution in X_μ for small time. Let $\|y_0\|_{L^2} \leqslant R$, $\|\gamma\|_{L^\infty} \leqslant r$. Let $y \in X_\mu$ be a solution to (A.27). One can always find $0 < T_s < T$ such that

$$\|y\|_{L^\infty(0,T_s;L^2(0,1))} + \|y\|_{L^2(0,T_s;L^\infty(0,1))} \leqslant 2R, \quad \text{(A.53)}$$

which implies the uniqueness of the solution in time $(0,T_s)$.

The above proof gives the local existence and uniqueness of $L^\infty L^2 \cap L^2 L^\infty$ solution. When $\gamma \in C^0$, Proposition 7 shows that the solution is also in $C^0L^2 \cap L^2L^\infty$.

Step 2. Continuity of the solution with respect to the data y_0, β and γ. More precisely, in this step, we prove the following lemma.

Lemma 35. *For every $R > 0$, $r > 0$, and $\varepsilon > 0$ such that $4\varepsilon RC_1 < 1$, there exists $0 < \tilde{T}^\varepsilon_{R,r} < T^\varepsilon_{R,r}$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + \tilde{T}^\varepsilon_{R,r}$, for every $y_0^\pm \in L^2(0,1)$, for every $\beta^\pm \in L^\infty(t_1,t_2)$, and for every $\gamma^\pm \in L^\infty(t_1,t_2)$ such that*

$$\|y_0^\pm\|_{L^2} \leqslant R \text{ and } \|\beta^\pm\|_{L^\infty} + \|\gamma^\pm\|_{L^\infty} \leqslant r, \quad \text{(A.54)}$$

the solutions $y^{\pm}$ to the Cauchy problem

$$\begin{cases} y_t^{\pm}(t,x) - y_{xx}^{\pm}(t,x) + y^{\pm} y_x^{\pm} = 0 & \text{for } (t,x) \in (t_1,t_2) \times (0,1), \\ y^{\pm}(t,0) = \beta^{\pm}(t) & \text{for } t \in (t_1,t_2), \\ y^{\pm}(t,1) = \gamma^{\pm}(t) & \text{for } t \in (t_1,t_2), \\ y^{\pm}(t_1,\cdot) = y_0^{\pm}, \end{cases} \tag{A.55}$$

satisfy

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1)}\leqslant 2(\|y_0^+ - y_0^-\|_{L^2}+\|\beta^+ - \beta^-\|_{L^\infty}+\|\gamma^+ - \gamma^-\|_{L^\infty}). \tag{A.56}$$

Proof of Lemma 35. Let us first point out that the existence of $y^{\pm}$ (on $[t_1,t_2]$) follows from Step 1 and (A.54). We also only treat the case where $\beta = 0$ in order to simplify the notations. From Step 1 we also know that

$$\|y^{\pm}\|_{L^\infty(t_1,t_2;L^2(0,1))}\leqslant 2R, \tag{A.57}$$

$$\|y^{\pm}\|_{L^2(t_1,t_2;L^\infty(0,1))}\leqslant \varepsilon R. \tag{A.58}$$

Let us denote $z := y^+ - y^-$ as the solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -\frac{1}{2}\left(\left((y^+ - y^-)(y^+ + y^1)\right)_x\right)(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = \gamma^+ - \gamma^- & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0^+ - y_0^-. \end{cases} \tag{A.59}$$

Hence by using the same estimates as in Step 1, we get

$$\begin{aligned} \|z\|_{L^\infty L^2} &\leqslant C_1\|z(y^+ + y^-)\|_{L^2L^2}+\|y_0^+ - y_0^-\|_{L^2}+(\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ &\leqslant 2\varepsilon C_1 R\|z\|_{L^\infty L^2}+\|y_0^+ - y_0^-\|_{L^2}+(\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ &\leqslant 1/2\|z\|_{L^\infty L^2}+\|y_0^+ - y_0^-\|_{L^2}+(\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty}, \end{aligned} \tag{A.60}$$

where $T := t_2 - t_1$. Hence

$$\begin{aligned} \|z\|_{L^\infty L^2} &\leqslant 2\|y_0^+ - y_0^-\|_{L^2}+2(\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ & \quad 2\big(\|y_0^+ - y_0^-\|_{L^2}+\|\gamma^+ - \gamma^-\|_{L^\infty}\big), \end{aligned} \tag{A.61}$$

by choosing $0 < \tilde{T}^{\varepsilon}_{R,r} < T^{\varepsilon}_{R,r}$ small enough such that $\eta + C_\eta(\tilde{T}^{\varepsilon}_{R,r})^{1/2} < 1$, which concludes the proof of Lemma 35. □

Remark 36. *We observe from (A.61) that*

$$\|y^+ - y^-\|_{L^\infty L^2}\to 0, \tag{A.62}$$

if

$$\|y_0^+ - y_0^-\|_{L^2}\to 0,\ \|\beta^+ - \beta^-\|_{L^\infty}\to 0,\ \text{and } \|\gamma^+ - \gamma^-\|_{L^\infty}\to 0. \tag{A.63}$$

Step 3. Maximum principle: nonlinear case (Lemma 12). Let us first point out that Lemma 12 is a consequence of the following local version of Lemma 12.

Lemma 37 (Local maximum principle: nonlinear case)**.** *Let $R > 0$, $r > 0$, and $\varepsilon > 0$ be given such that $4\varepsilon RC_1 < 1$. Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2 < t_1 + \tilde{T}^{\varepsilon}_{R,r}$. Let $y_0^{\pm} \in L^2(0,1)$, $\beta^{\pm} \in L^{\infty}(t_1,t_3)$ be piecewise continuous, and $\gamma^{\pm} \in L^{\infty}(t_1,t_2)$ be piecewise continuous such that (2.19) holds and*

$$\|y_0^{\pm}\|_{L^2} \leq R \text{ and } \|\beta^{\pm}\|_{L^{\infty}} + \|\gamma^{\pm}\|_{L^{\infty}} \leq r. \tag{A.64}$$

Then the solutions $y^{\pm}$ to the Cauchy problem (A.55) satisfy (2.20).

Indeed, using this lemma and arguing by contradiction, one easily gets that, under the assumptions of Lemma 12,

$$\max\{\tau \in [0,T];\, y^-(t,\cdot) \leq y^+(t,\cdot),\ \ \forall t \in [0,\tau]\} = T. \tag{A.65}$$

Proof of Lemma 37. Under the extra assumption that β and γ are in $H^{1/4}(t_1,t_2)$, property (2.20) follows from [57, Lemma 1]. The general case follows from this special case by using a density argument and Lemma 35 (or Remark 36). Indeed, using the fact that $\beta^{\pm}$ and $\gamma^{\pm}$ are piecewise continuous, there are sequences $\beta^{n\pm} \in H^{1/4}(t_1,t_2) \cap L^{\infty}(t_1,t_2)$ and $\gamma^{n\pm} \in H^{1/4}(t_1,t_2) \cap L^{\infty}(t_1,t_2)$ such that

$$\beta^{n\pm} \to \beta^{\pm} \text{ in } L^{\infty}(t_1,t_2) \text{ and } \gamma^{n\pm} \to \gamma^{\pm} \text{ in } L^{\infty}(t_1,t_2). \tag{A.66}$$

Moreover, using (2.19) and (A.64), we may also impose that

$$\|\beta^{n\pm}\|_{L^{\infty}} + \|\gamma^{n\pm}\|_{L^{\infty}} \leq r,\ \ \forall n \in \mathbb{N}, \tag{A.67}$$

$$\beta^{n-} \leqslant \beta^{n+} \text{ and } \gamma^{n-} \leqslant \gamma^{n+},\ \ \forall n \in \mathbb{N}. \tag{A.68}$$

Let $y^{n\pm}$ be the solutions to the Cauchy problem (A.55) for $\beta^{\pm} := \beta^{n\pm}$ and $\gamma^{\pm} := \gamma^{n\pm}$. From [57, Lemma 1]

$$y^{n-}(t,\cdot) \leqslant y^{n+}(t,\cdot),\ \ \forall t \in [t_1,t_2]. \tag{A.69}$$

By Lemma 35 and (A.66),

$$\|y^{n+} - y^+\|_{L^{\infty}L^2} \to 0 \text{ and } \|y^{n-} - y^-\|_{L^{\infty}L^2} \to 0 \text{ as } n \to +\infty. \tag{A.70}$$

Property (2.20) follows from (A.69) and (A.70). This concludes the proof of Lemma 37. □

Step 4. It only remains to prove the global existence of the solution to (2.12) with $f = 0$. Let

$$B := \|\beta\|_{L^{\infty}(0,T)} + \|\gamma\|_{L^{\infty}(0,T)}, \tag{A.71}$$

$$R_M := 2\|y_0\|_{L^2} + 4B,\ r_M := B,\ \varepsilon_M := \frac{1}{8C_1R_M}, \text{ and } T_M := T^{\varepsilon_M}_{R_M,r_M}. \tag{A.72}$$

Note that

$$\|y_0\|_{L^2} \leq R_M \|\beta\|_{L^{\infty}(0,T)} + \|\gamma\|_{L^{\infty}(0,T)} \leq r_M. \tag{A.73}$$

By (A.73) and the second part of Proposition 10 (Step 1) the solution y of (A.27) is defined at least on $[0, \min\{T, T_M\}]$. Hence we may assume that $T > T_M$.

Let $y^{\pm}:(0,\tau_{\pm})\to L^2(0,1)$ be the (maximal) solution to the Cauchy problems

$$\begin{cases} y^{\pm}_t(t,x)-y^{\pm}_{xx}(t,x)+y^{\pm}y^{\pm}_x(t,x)=0 & \text{for } (t,x)\in(0,\tau_{\pm})\times(0,1),\\ y^{\pm}(t,0)=\pm B & \text{for } t\in(0,\tau_{\pm}),\\ y^{\pm}(t,1)=\pm B & \text{for } t\in(0,\tau_{\pm}),\\ y^{\pm}(0,\cdot)=y_0. \end{cases} \tag{A.74}$$

As for y, we have $\tau_{\pm}\le T_M$. Let $z^{\pm}:[0,T_M]\to L^2(0,1)$ be defined by

$$z^{\pm}:=y^{\pm}\mp B. \tag{A.75}$$

Then $z^{\pm}$ is a solution of

$$\begin{cases} z^{\pm}_t(t,x)-z^{\pm}_{xx}(t,x)\pm Bz^{\pm}_x(t,x)+z^{\pm}z^{\pm}_x(t,x)=0 & \text{for } (t,x)\in(0,T_M)\times(0,1),\\ z^{\pm}(t,0)=0 & \text{for } t\in(0,T_M),\\ z^{\pm}(t,1)=0 & \text{for } t\in(0,T_M),\\ z^{\pm}(0,\cdot)=y_0\mp B, \end{cases} \tag{A.76}$$

from which we get that

$$\frac{d}{dt}\int_0^1 (z^{\pm})^2dx\le 0 \text{ in } \mathcal{D}'(0,T_M). \tag{A.77}$$

Hence

$$\|y^{\pm}(t,\cdot)\|_{L^2(0,1)}\leqslant \|y_0\|_{L^2}+2B. \tag{A.78}$$

Moreover, thanks to the maximum principle (Lemma 12), we have

$$y^-\le y\le y^+,\ \ \forall t\in[0,T_M]. \tag{A.79}$$

In particular, using (A.78),

$$\|y(T_M)\|_{L^2}\le R_M,\ \|y^-(T_M)\|_{L^2}\le R_M,\ \text{ and } \|y^+(T_M)\|_{L^2}\le R_M. \tag{A.80}$$

This allows to redo the above procedure with the initial time T_M and the initial data $y(T_M)$, $y^-(T_M)$, and $y^+(T_M)$. Let us emphasize that the initial data for $y^{\pm}$ at time T_M is not $y(T_M)$ but $y^{\pm}(T_M)$ (which is given by the definition of $y^{\pm}$ on $[0,T_M]$). In particular y, y^-, and y^+ are defined on $[0,\min\{T,2T_M\}]$. So we may assume that $T>2T_M$. Moreover, using once more the maximum principle,

$$y^-(t)\le y(t)\le y^+(t),\ \ \forall t\in[T_M,2T_M]. \tag{A.81}$$

Property (A.77) and therefore also (A.78) hold on $[T_M,2T_M]$. Together with (A.81) this implies that

$$\|y(2T_M)\|_{L^2}\le R_M,\ \|y^-(2T_M)\|_{L^2}\le R_M,\ \text{ and } \|y^+(2T_M)\|_{L^2}\le R_M. \tag{A.82}$$

We keep going and using an induction argument get that, for every integer $n>0$, y is defined on $[0,\min\{nT_M,T\}]$. This concludes the proof of Proposition 10.

At last, Lemma 13 follows directly from Lemma 35 and Step 4.

B Appendix: Proof of Lemma 24

Let us start the proof of Lemma 24 by proving the following lemma, which deals with the well-posedness for small time of the Cauchy problem

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (0,T)\times(0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \\ a_t = \tilde{A}_0(z,a) & \text{for } t \in (0,T), \\ z(0,\cdot) = z_0 \quad \text{and} \quad a(0) = a_0, \end{cases} \tag{B.1}$$

with

$$\tilde{A}_0(z,a) = A_0(z+a,a). \tag{B.2}$$

Lemma 38. *Let $A_0(y,a)$ be given by (3.22). Let $R > 0$. There exists $T_R > 0$ such that, for every $(z,a) \in L^2(0,1)\times\mathbb{R}$ such that $\|(z,a)\|_V \leqslant R$, there exists one and only one $z : [0,T_R]\times(0,L)$ such that*

$$z \in C^0([0,T_R]; L^2(0,1)) \cap L^2(0,T_R; H^1(0,1)), \tag{B.3}$$

$$a \in C^0([0,T_R]),\ a_t = \alpha \text{ in the distribution sense, and } a(0) = a_0, \tag{B.4}$$

and such that $y := z$ is a solution on $[0,T_R]$ to the Cauchy problem (2.1), in the sense of Definition 4, for $f := -zz_x - a(t)z_x$, $\beta := 0$, $\gamma := 0$, and $y_0 := z_0$.

Let us define

$$W := \{(z,a) : z \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)), a \in C^0([0,T])\}. \tag{B.5}$$

We introduce the W_η-norm on W by

$$\|(z,a)\|_{W_\eta} := \|z\|_{C^0L^2\cap L^2H_0^1} + \eta\|a\|_{C^0}, \tag{B.6}$$

with $\eta < 1$. Hence (z,a) is as requested in Lemma 38 if and only if it is a fixed point of $\Lambda : W \to W$, $(z,a) \mapsto W(z,a) =: (w,b)$ where w is the unique solution to the Cauchy problem (in the sense of Definition 4)

$$\begin{cases} w_t - w_{xx} = -zz_x - a(t)z_x & \text{for } (t,x) \in (0,T)\times(0,1), \\ w(t,0) = 0 & \text{for } t \in (0,T), \\ w(t,1) = 0 & \text{for } t \in (0,T), w(0,\cdot) = z_0 \textit{ and } b(0) = a_0 \end{cases} \tag{B.7}$$

and

$$b(t) := a_0 + \int_0^t \tilde{A}_0(z(\tau),a(\tau))d\tau \ \text{ for } t \in [0,T]. \tag{B.8}$$

It follows from Proposition 5 that Λ is well defined. For $\|(z_0,a_0)\|_V \leqslant R$, we try to find a fixed point of Λ on

$$\tilde{B}_R := \{(z,a) \in W : \|(z,a)\|_{W_\eta} \leqslant 3R\}. \tag{B.9}$$

For every $(z,a) \in \tilde{B}_R$, by Proposition 5 and (A.10)–(A.12), we have

$$\begin{aligned} \|w\|_{C^0L^2\cap L^2H_0^1} &\leqslant 2\|zz_x + az_x\|_{L^1L^2} + 2\|z_0\|_{L^2} \\ &\leqslant CT^{1/4}\|z\|^2_{C^0L^2\cap L^2H_0^1} + 2T^{1/2}\|a\|_{C^0}\|z\|_{L^2H_0^1} + 2R \\ &\leqslant 2R + 9CR^2T^{1/4} + 9R^2T^{1/2}/\eta. \end{aligned} \tag{B.10}$$

Moreover

$$
\begin{aligned}
\|b\|_{C^0} &\leqslant |a_0| + \|\tilde{A}_0(z,a)\|_{L^1} \\
&\leqslant R + \lambda \| -2\int_0^1 z_x^2 e^{-x} dx + (1-a)\, V_1(z) - \frac{2}{3}\int_0^1 z^3 e^{-x} dx \|_{L^1} \\
&\qquad + \| \frac{V_1}{2} - \frac{1}{2}\lambda (a - \lambda V_1(z))^3 \|_{L^1} \\
&\leqslant R + 2\lambda \|z\|^2_{L^2 H^1_0} + \lambda T \|z\|^2_{C^0 L^2} + \lambda T \|a\|_{C^0} \|z\|^2_{C^0 L^2} + \lambda T^{1/2} \|z\|_{L^2 H^1_0} \|z\|^2_{C^0 L^2} \\
&\qquad + T\|z\|^2_{C^0 L^2} + \lambda T (\|a\|_{C^0} + \lambda \|z\|^2_{C^0 L^2})^3 \\
&\leqslant R + 18\lambda R^2 + 9\lambda T R^2 + 27 \lambda T R^3/\eta + 27\lambda T^{1/2} R^3 + 9TR^2 + \lambda T (3R/\eta + 9\lambda R^2)^3.
\end{aligned} \tag{B.11}
$$

Hence

$$
\begin{aligned}
\|(w,b)\|_{W_\eta} &\leqslant 2R + 9CR^2 T^{1/4} + 9R^2 T^{1/2}/\eta + \eta(R + 18\lambda R^2) \\
&\qquad + \eta(9\lambda T R^2 + 27\lambda T R^3/\eta + 27 \lambda T^{1/2} R^3 + 9TR^2 + \lambda T(3R/\eta + 9\lambda R^2)^3).
\end{aligned} \tag{B.12}
$$

We can successively choose η and T so that the right hand side of (B.12) is less or equal than $3R$, leading to

$$
\|(w,b)\|_{W_\eta} \leqslant 3R, \tag{B.13}
$$

which implies that

$$
\Lambda(\tilde{B}_R) \subset \tilde{B}_R. \tag{B.14}
$$

It remains to get the contraction property. Suppose that $(w_i, b_i) := \Lambda((z_i, a_i))$ with $i \in \{1,2\}$, then by using Proposition 5 one gets

$$
\begin{aligned}
\|w_1 - w_2\|_{C^0 L^2 \cap L^2 H^1_0} &\leqslant 2\|z_1 (z_1)_x - z_2 (z_2)_x + a_1 (z_1)_x - a_2 (z_2)_x\|_{L^1 L^2} \\
&\leqslant 2CT^{1/4} \left(\|z_1\|_{C^0 L^2 \cap L^2 H^1_0} + \|z_2\|_{C^0 L^2 \cap L^2 H^1_0} \right) \|z_1 - z_2\|_{C^0 L^2 \cap L^2 H^1_0} \\
&\qquad + 2T^{1/2} \|a_1 - a_2\|_{C^0} \|z_2\|_{L^2 H^1} + 2T^{1/2} \|a_1\|_{C^0} \|z_1 - z_2\|_{L^2 H^1} \\
&\leqslant \left(12 CRT^{1/4} + 12 RT^{1/2}/\eta \right) \|(z_1, a_1) - (z_2, a_2)\|_{W_\eta}
\end{aligned} \tag{B.15}
$$

and

$$
\begin{aligned}
\|b_1 - b_2\|_{C^0} &\leqslant \|\tilde{A}_0(z_1, a_1) - \tilde{A}_0(z_2, a_2)\|_{L^1} \\
&\leqslant \lambda \| -2 \int_0^1 \left((z_1)_x^2 - (z_2)_x^2 \right) e^{-x} dx - \frac{2}{3}\int_0^1 (z_1^3 - z_2^3) e^{-x} dx \|_{L^1} + \| \frac{V_1(z_1) - V_1(z_2)}{2} \|_{L^1} \\
&\qquad + \frac{1}{2}\lambda \|(a_1 - \lambda V_1(z_1))^3 - (a_2 - V_1(z_2))^3\|_{L^1} + \lambda \|(1-a_1)\, V_1(z_1) - (1-a_2)\, V_1(z_2)\|_{L^1} \\
&\leqslant 12\lambda R \|z_1 - z_2\|_{L^2 H^1_0} + 24 \lambda R^2 T^{1/2} \|z_1 - z_2\|_{L^2 H^1_0} \\
&\qquad + T(3R + 6\lambda R)\|z_1 - z_2\|_{C^0 L^2} + (27 T R^2/\eta) \|(z_1, a_1) - (z_2, a_2)\|_{W_\eta} \\
&\qquad + \lambda T \left(\|a_1 - a_2\|_{C^0} + \lambda \|z_1 - z_2\|_{C^0 L^2} \|z_1 + z_2\|_{C^0 L^2} \right) \cdot 4(3R/\eta + \lambda 9 R^2)^2 \\
&\leqslant \left(12\lambda R + 24\lambda R^2 T^{1/2} + 3RT + 6\lambda R T + 27 T R^2/\eta + 36 \lambda T R^2 (1/\eta + 6\lambda R)^3 \right) \\
&\qquad\qquad \|(z_1, a_1) - (z_2, a_2)\|_{W_\eta}.
\end{aligned} \tag{B.16}
$$

Hence one gets the contraction property of the map Λ on $\tilde{B}_R$ when η and T are well chosen, which implies the existence of a unique solution in $\tilde{B}_R$. Then, proceeding as in the proof of uniqueness part of Proposition 10, we can further get the uniqueness of solution in W_η. This completes the proof of Lemma 38.

In order to end the proof of Lemma 24, it only remains to prove the existence of the solution (z, a) for large time. For this existence in large time, it suffices to check that $\|z(t)\|_{L^2}$ remains bounded. This can be done by using the maximum principle for the nonlinear Burgers equation (Lemma 12) as in the proof of Proposition 10 (Step 4).

C Appendix: Proof of Lemma 29

This section is devoted to the proof of Lemma 29. Let us start our proof of this lemma with a proof of the following lemma.

Lemma 39. *Let $M > 0$ and let $G : L^2(0,1) \to \mathbb{R}$ be a (stationary) feedback law satisfying (4.43). Let $R > 0$ and $\varepsilon > 0$. There exists $T_R^\varepsilon > 0$ such that, for every $y_0 \in L^2(0,1)$ satisfying*

$$\|y_0\|_{L^2} \leqslant R, \tag{C.1}$$

the Cauchy problem (4.45) has a unique solution

$$y \in C^0([0, T_R^\varepsilon]; L^2(0,1)) \cap L^2(0, T_R^\varepsilon; L^\infty(0,1)), \tag{C.2}$$

and moreover this solution satisfies

$$\|y\|_{C^0([0,T_R^\varepsilon];L^2(0,1))} \leqslant 2R, \tag{C.3}$$

$$\|y\|_{L^2(0,T_R^\varepsilon;L^\infty(0,1))} \leqslant \varepsilon R. \tag{C.4}$$

This lemma is quite similar to Proposition 10. Therefore we use the same strategy to get the proof of this lemma. Let us consider the space

$$Y := C^0([0,T]; L^2(0,1)) \cap L^2(0,T; L^\infty(0,1)) \tag{C.5}$$

and the norm

$$\|y\|_{Y_\mu} := \|y\|_{C^0L^2} + \frac{1}{\mu}\|y\|_{L^2L^\infty}, \tag{C.6}$$

with $\mu > 0$ to be chosen later. Then we consider the following map $\Gamma : Y_\mu \to Y_\mu$, where $\Gamma(y)$ is the unique solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -(yy_x)(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = G(y) & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0. \end{cases} \tag{C.7}$$

Again, this map is well defined (one only need to notice the $L^\infty L^2$ estimate can be replaced by C^0L^2 estimate since $G(y(t))$ is continuous). As in the proof of Proposition10, it suffices to find the unique fixed point in the following ball

$$B_R^1 := \{y \in Y : \|y\|_{Y_\mu} \leqslant 2R\}. \tag{C.8}$$

From (A.31)–(A.41) and (A.43)–(A.44), we get

$$\begin{aligned}\|\Gamma(y)\|_{C^0L^2} &\leqslant \|y_0\|_{L^2}+\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}+2(\eta+C_\eta T^{1/2})\|G(y)\|_{C^0}\\ &\leqslant \|y_0\|_{L^2}+2C_1\mu R^2+4(\eta+C_\eta T^{1/2})MR,\end{aligned}\tag{C.9}$$

and

$$\begin{aligned}\|\Gamma(y)\|_{L^2L^\infty} &\leqslant CT^{1/4}\|y_0\|_{L^2}+CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}+2CT^{1/4}(\eta+C_\eta T^{1/2})\|G(y)\|_{C^0}\\ &\leqslant CT^{1/4}\|y_0\|_{L^2}+2CT^{1/4}C_1\mu R^2+4CT^{1/4}(\eta+C_\eta T^{1/2})MR.\end{aligned}\tag{C.10}$$

With a good choice of η,μ and T, Γ is from B^1_R to B^1_R. By using similar estimates (see also the proofs of (A.50) and (A.51)), we have

$$\begin{aligned}\|\Gamma(y_1)-\Gamma(y_2)\|_{C^0L^2} &\leqslant \frac{C_1}{2}\|y_1+y_2\|_{L^2L^\infty}\|y_1-y_2\|_{L^\infty L^2}+2(\eta+C_\eta T^{1/2})\|G(y_1)-G(y_2)\|_{C^0}\\ &\leqslant C_1R\mu\|y_1-y_2\|_{C^0L^2}+2(\eta+C_\eta T^{1/2})M\|y_1-y_2\|_{C^0L^2},\end{aligned}$$

and

$$\begin{aligned}\|\Gamma(y_1)-\Gamma(y_2)\|_{L^2L^\infty} &\leqslant CT^{1/4}\frac{C_1}{2}\|y_1+y_2\|_{L^2L^\infty}\|y_1-y_2\|_{L^\infty L^2}\\ &\qquad +2CT^{1/4}(\eta+C_\eta T^{1/2})\|G(y_1)-G(y_2)\|_{C^0}\\ &\leqslant CT^{1/4}C_1R\mu\|y_1-y_2\|_{C^0L^2}+2CT^{1/4}(\eta+C_\eta T^{1/2})M\|y_1-y_2\|_{C^0L^2}.\end{aligned}$$

Hence a good choice of η, μ, and T, makes Γ a contraction map. This concludes the proof of Lemma 39.

Remark 40. *If we replace C^0L^2 by $L^\infty L^2$, we get the local well-posedness in $L^\infty L^2\cap L^2L^\infty$.*

So far, we get the local existence and uniqueness of the solution of (4.45). In order to get the global existence statement of Lemma 29 it suffices to control the L^2-norm of $y(t)$. This control follows from (4.44), which leads to (A.78) with $B:=M$. This concludes the proof of Lemma 29.

D Appendix: Proof of Lemma 27

The proof is to consider an equation of $z(x):=\Pi_\lambda(y(x))$ instead of equation (4.15) (see (4.16)), this gives the advantage that $z(t,0)=z(t,1)=0$. The local existence and uniqueness of the solution z is given by a standard procedure (by considering the nonlinear term I as a force term and using Banach fixed point theorem).

Proof of Lemma 27. In this proof, the constant C may change from line to line, but it is independent of $0<T<1$ and of R. From (4.16) and (4.18)

$$I(z)=-yy_x+\int_0^x k_\lambda(x,v)(yy_x)(v)dv.\tag{D.1}$$

We notice that

$$\|I(z)\|_{L^2(0,1)}\leqslant C\|yy_x\|_{L^2}\leqslant C\|y\|_{L^\infty}\|y_x\|_{L^2}\leqslant C\|y\|^{1/2}_{L^2}\|y\|^{3/2}_{H^1_0}\leqslant C\|z\|^{1/2}_{L^2}\|z\|^{3/2}_{H^1_0},\tag{D.2}$$

and that

$$\begin{aligned}
\|I(z_1)-I(z_2)\|_{L^2(0,1)} &\leqslant C\|y_1-y_2\|_{L^\infty}\|y_{1x}+y_{2x}\|_{L^2}+C\|y_1+y_2\|_{L^\infty}\|y_{1x}-y_{2x}\|_{L^2}\\
&\leqslant C\|y_1-y_2\|_{L^2}^{1/2}\|y_1-y_2\|_{H_0^1}^{1/2}\|y_1+y_2\|_{H_0^1}\\
&\quad + C\|y_1+y_2\|_{L^2}^{1/2}\|y_1+y_2\|_{H_0^1}^{1/2}\|y_1-y_2\|_{H_0^1}\\
&\leqslant C\|z_1-z_2\|_{L^2}^{1/2}\left(\|z_1-z_2\|_{H_0^1}^{1/2}+\|z_1-z_2\|_{L^2}^{1/2}\right)\left(\|z_1+z_2\|_{H_0^1}+\|z_1+z_2\|_{L^2}\right)\\
&\quad + C\|z_1+z_2\|_{L^2}^{1/2}\left(\|z_1+z_2\|_{H_0^1}^{1/2}+\|z_1+z_2\|_{L^2}^{1/2}\right)\left(\|z_1-z_2\|_{H_0^1}+\|z_1-z_2\|_{L^2}\right)\\
&\leqslant C\|z_1-z_2\|_{L^2}^{1/2}\|z_1-z_2\|_{H_0^1}^{1/2}\|z_1+z_2\|_{H_0^1}\\
&\quad + C\|z_1+z_2\|_{L^2}^{1/2}\|z_1+z_2\|_{H_0^1}^{1/2}\|z_1-z_2\|_{H_0^1}. \qquad \text{(D.3)}
\end{aligned}$$

Regarding the linear Cauchy problem

$$\begin{cases}
z_t - z_{xx} + \lambda z = f & \text{for } (t,x)\in(s,s+T)\times(0,1),\\
z(t,0)=0 & \text{for } t\in(s,s+T),\\
z(t,1)=0 & \text{for } t\in(s,s+T),\\
z(0,\cdot)=z_0,
\end{cases} \qquad \text{(D.4)}$$

similar to Proposition 5, we have

$$\|z\|_{C^0L^2}\leqslant \|z_0\|_{L^2}+\|f\|_{L^1L^2}, \qquad \text{(D.5)}$$

$$\|z\|_{L^2H_0^1}\leqslant \|z_0\|_{L^2}+\|f\|_{L^1L^2}. \qquad \text{(D.6)}$$

As normal, let us denote the space $C^0([s,s+T];L^2(0,1))\cap L^2(s,s+T;H_0^1(0,1))$ endowed with norm $\|\cdot\|_{C^0L^2}+\|\cdot\|_{L^2H_0^1}$ by $\mathcal{H}$ (or $\mathcal{H}_T$ if necessary).

For y_0 with $\|y_0\|_{L^2}\leqslant R$ given, we have $\|z_0\|_{L^2}\leqslant e^{3/2C_1\sqrt{\lambda}}\|y_0\|_{L^2}\leqslant e^{3/2C_1\sqrt{\lambda}}R$. Let us define

$$\mathcal{B} := \{z\in\mathcal{H} : \|z\|_{\mathcal{H}}\leqslant 3e^{3/2C_1\sqrt{\lambda}}R\}. \qquad \text{(D.7)}$$

We consider the map $\Gamma:\mathcal{H}\to\mathcal{H}$, $z\mapsto w$ where w is the unique solution of

$$\begin{cases}
w_t - w_{xx} + \lambda w = I(z) & \text{for } (t,x)\in(s,s+T)\times(0,1),\\
w(t,0)=0 & \text{for } t\in(s,s+T),\\
w(t,1)=0 & \text{for } t\in(s,s+T),\\
w(0,\cdot)=z_0.
\end{cases} \qquad \text{(D.8)}$$

From (4.12), (D.5), and (D.6), we know that

$$\|\Gamma(z)\|_{\mathcal{H}}\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + 2\|I(z)\|_{L^1L^2}. \qquad \text{(D.9)}$$

Hence, for every $z\in\mathcal{B}$, by (D.2) we have

$$\begin{aligned}
\|\Gamma(z)\|_{\mathcal{H}} &\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + 2\|I(z)\|_{L^1L^2}\\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + C\left\|\|z\|_{L^2}^{1/2}\|z\|_{H_0^1}^{3/2}\right\|_{L^1(s,s+T)}\\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + CT^{1/4}\|z\|_{C^0L^2}^{1/2}\|z\|_{L^2H_0^1}^{3/2}\\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + 9CT^{1/4}e^{3C_1\sqrt{\lambda}}R^2. \qquad \text{(D.10)}
\end{aligned}$$

For every z_1 and $z_2 \in \mathcal{B}$, we have

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{H}} \leqslant 2\|I(z_1) - I(z_2)\|_{L^1 L^2}, \tag{D.11}$$

Above estimate together with (D.3) give

$$\begin{aligned}
\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{H}} &\leqslant C\|z_1 - z_2\|_{L^2}^{1/2}\|z_1 - z_2\|_{H_0^1}^{1/2}\|z_1 + z_2\|_{H_0^1} \\
&\quad + C\|z_1 + z_2\|_{L^2}^{1/2}\|z_1 + z_2\|_{H_0^1}^{1/2}\|z_1 - z_2\|_{H_0^1} \\
&\leqslant CT^{1/4}\|z_1 - z_2\|_{C^0 L^2}^{1/2}\|z_1 - z_2\|_{L^2 H_0^1}^{1/2}\|z_1 + z_2\|_{L^2 H_0^1} \\
&\quad + CT^{1/4}\|z_1 + z_2\|_{C^0 L^2}^{1/2}\|z_1 + z_2\|_{L^2 H_0^1}^{1/2}\|z_1 - z_2\|_{L^2 H_0^1} \\
&\leqslant T^{1/4} 3 e^{3/2 C_1 \sqrt{\lambda}} RC\|z_1 - z_2\|_{\mathcal{H}}.
\end{aligned} \tag{D.12}$$

From (D.10) and (D.12), we get the existence of T_R^{tr} which completes the proof. □

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