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To cite this version:

Shengquan Xiang. Small-time local stabilization for a Korteweg-de Vries equation. Systems and Control Letters, 2018, Volume 111, pp.Pages 64-69. hal-01723178

HAL Id: hal-01723178

https://hal.science/hal-01723178

Submitted on 5 Mar 2018

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Small-time local stabilization for a Korteweg-de Vries equation

Shengquan Xiang[1]

Abstract

This paper focuses on the (local) small-time stabilization of a Korteweg-de Vries equation on bounded interval, thanks to a time-varying Dirichlet feedback law on the left boundary. Recently, backstepping approach has been successfully used to prove the null controllability of the corresponding linearized system, instead of Carleman inequalities. We use the "adding an integrator" technique to gain regularity on boundary control term which clears the difficulty from getting stabilization in small-time.

Keywords: Korteweg-de Vries, backstepping, small-time stabilization, adding an integrator.
American Subject Classification: 35Q53, 34H05, 35P10.

1. Introduction

We consider the stabilization problem of the Korteweg-de Vries equation

$$y_t + y_x + y_{xxx} + yy_x = 0 \tag{1.1}$$

posed on a bounded domain $[0, L]$. This system requires three boundary conditions including both left and right end-point (see [26], the system fails to be well-posed when all boundary conditions are given at only one end-point), among which the most studied case is

$$y(t, 0),\ y(t, L),\ y_x(t, L). \tag{1.2}$$

When there is only one control term y_x, *i.e.* $y(t, 0) = y(t, L) = 0$, the phenomenon becomes quite mysterious: starting from the linearized KdV equation, Lionel Rosier [23] found that the system is controllable if and only if the length of the interval satisfies

$$L \notin \mathcal{N} := \Big\{2\pi\sqrt{\frac{l^2 + lk + k^2}{3}};\, l, k \in \mathbb{N}^*\Big\}. \tag{1.3}$$

It allows us to decompose the $L^2(0, L)$ space into the controllable state and the uncontrollable state (for the linearized KdV equation). To get the controllability of the KdV equations, "Power Series Expansion" method was introduced in [1, 3, 13], which turned out to be a classical example of getting controllability by using nonlinear terms. The stabilization problem is even more interesting, as we need to investigate a closed-loop system which involves more difficulties (even for the well-posedness). Several works [5, 6, 15, 18, 22, 25] have been made in this special KdV control system, here we refer [18] where the authors used the nonlinear term (and also uncontrollable part) to find a time-varying feedback law which stabilize the system exponentially. As we can see, in this model we really used the nonlinear term, by "fixing" the uncontrollable part, to reach the goal of controlling and stabilizing. However, at the same time, the results are only local and the stabilization is only exponential. To get the same results in global sense, is a challenging and interesting problem, since one may need to find the other techniques for the nonlinear term.

In this paper, we will focus on the control acting on $y(t, 0)$, with $y(t, L) = y_x(t, L) = 0$. This system has an advantage of being locally controllable, see [2, 19, 24] for discussions on this system. From the stabilization point of view, the final aim should be (local) small-time stabilization, especially by means of time-varying feedback laws (inspired by [8] for the finite dimensional case). Recently in [16], Jean-Michel Coron and Hoai-Minh Nguyen made a first step, they used a piece-wise backstepping control to get the null controllability and the semi-global small-time stabilization for the heat equation. Here we refer to [12, 14, 20, 21] for the history, explanation and development of backstepping method. As the backstepping method has already been used for a rapid stabilization for this KdV system, one may naturally expect the small-time stabilization. In the recent paper [26], the author used this technique to give a new proof of null controllability of the linearized KdV equation. There came a difficulty of lacking regularity on the control term $y(t, 0)$, where *a priori* a $H^{1/3}$ regularity on $y(t, 0)$ is needed, but one can hardly ensure the feedback to be more than C^0 with respect to time.

A technique called "adding an integrator" is going to

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solve our problem. Usually used to avoid the offset in the stabilization problem, this technique also has the advantage of gaining regularity. Indeed, if we "add" another term, $a(t)$, as

$$y(t,0)=a(t), \quad a_t(t)=u(t), \tag{1.4}$$

where $u(t)$ is the control, then we will have $a(t)\in H^1(0,T)$ if $u(t)\in L^2(0,T)$. One can see the paper of Jean-Michel Coron [10], where this method is used for the stabilization of Euler equations. Let us also point out that the control system with the additional integral term has controllability and stabilizability properties which are related but may be different than the ones of the initial control system; see, in particular [11, Proposition 3.30 and Section 12.5] and [17] for finite dimensional control systems.

We consider in this paper the following system:

$$\begin{aligned} y_t+y_x+y_{xxx}+yy_x&=0, && (1.5)\\ y(t,L)=y_x(t,L)&=0, && (1.6)\\ y(t,0)&=a(t), && (1.7)\\ a_t(t)+y_{xx}(t,0)+\frac{1}{2}a(t)+\frac{1}{3}a^2(t)&=u(t), && (1.8)\end{aligned}$$

in the interval $[0,1]$ (we only consider only the case when $L=1$ to simplify the notations). We notice the extra $y_{xx}(t,0)$ term in (1.8). It naturally comes from (1.7) and helps to ensure the well-posedness of our new system. As for terms $(1/2)a$ and $(1/3)a^2$, which could be put in the control term, we let them here to make the dissipative nature visible. It is a control system where the state is $(y(x),a)$, but with only one control u. Let us set

$$V:=L^2(0,1)\times\mathbb{R} \text{ and } \|(y,a)\|_V^2:=\|y\|^2_{L^2(0,1)}+a^2. \tag{1.9}$$

Then easy calculations show that the "flow" of system (1.5)–(1.8) satisfies:

$$\begin{aligned}\frac{d}{dt}\|(y,a)\|_V^2&=2\langle y_t,y\rangle_{L^2}+2a_ta\\ &=2\langle -y_x-y_{xxx}-yy_x,y\rangle_{L^2}+2a_ta\\ &=a^2+\frac{2}{3}a^3-y_x(0)^2+2ay_{xx}(0)+2a_ta\\ &\leqslant 2au. \end{aligned} \tag{1.10}$$

In order to get the well-posedness of the nonlinear system, we may need some smoothing effects. We first consider the linearized system of (1.5)–(1.8). By multiplying xy the linearized part of (1.5), we get the Kato smoothing effect $y\in L^2(0,T;H^1(0,1))$. This together with (1.10) and some fixed point argument, allows us to get the well-posedness of the nonlinear system (1.5)–(1.8) in the transposition sense (with initial data $(y_0,a_0)\in V$ and control $u(t)\in L^1(0,T)$), the solution being in

$$\left(C^0([0,T];L^2(0,1))\cap L^2(0,T;H^1(0,1))\right)\times C^0([0,T];\mathbb{R}).$$

Here, we are not going to reconstruct the whole theory of transposition solutions, which is already well explained in the book [11] (one can also see similar cases in [16, 18]). Based on the method introduced in [16] and the estimates given in [26], we are able to stabilize system (1.5)–(1.8) in small time. More precisely, for every $T>0$, we will construct time-varying feedback laws $U(t;y,a):\mathbb{R}\times L^2(0,1)\times\mathbb{R}\mapsto\mathbb{R}$, satisfying the following three properties:

($\mathcal{P}_1$) The feedback law U is T-periodic with respect to time:

$$U(t;y,a)=U(T+t;y,a). \tag{1.11}$$

($\mathcal{P}_2$) There exists an increasing sequence $\{t_n\}$ of real numbers such that

$$\begin{aligned} t_0&=0, && (1.12)\\ \lim_{n\to\infty}t_n&=T, && (1.13)\\ \text{U is of class } C^1 \text{ in } [t_n,t_{n+1})&\times L^2(0,1)\times\mathbb{R}. && (1.14)\end{aligned}$$

($\mathcal{P}_3$) The feedback law U vanishes on $\mathbb{R}\times\{0\}\times\{0\}$. There exists a continuous function $M:[0,T)\to[0,+\infty)$ such that

$$\begin{aligned} &|U(t;y_1,a_1)\,|-|U(t;y_2,a_2)\,| && (1.15)\\ &\leqslant M(t)(\|y_1-y_2\|_{L^2}+\,|\,a_1-a_2\,|), && (1.16)\end{aligned}$$

for $\forall t\in[0,T)$.

($\mathcal{P}_4$) For all $(t;y,a)\in\mathbb{R}\times L^2(0,1)\times\mathbb{R}$, we have

$$|\,U(t;y,a)\,|<1. \tag{1.17}$$

($\mathcal{P}_5$) $\|(y,a)\|_V\geqslant 1\Rightarrow U(t;y,a)=0$, for all $t\in\mathbb{R}$.

From now on, let us consider the Cauchy problem of the closed-loop system (1.5)–(1.8)

$$\begin{cases} y_t+y_x+y_{xxx}+yy_x=0,\\ y(t,1)=y_x(t,1)=0,\\ y(t,0)=a(t),\\ a_t+y_{xx}(0)+\frac{1}{2}a+\frac{1}{3}a^2=u,\\ y(s,x)=y_0,\\ a(s)=a_0,\\ u:=U(t;y,a)\end{cases} \tag{1.18}$$

with $(t,x)\in(s,+\infty)\times(0,1)$. For this Cauchy problem, from properties ($\mathcal{P}_1$)–($\mathcal{P}_3$) we have the existence and uniqueness of solution in small-time. A solution $(y_1,a_1):[s,\tau_1)\to V$ to the Cauchy problem is maximal, if there is no solution $(y_2,a_2):[s,\tau_2)\to V$ such that $\tau_2>\tau_1$ and $(y_1,a_1)=(y_2,a_2)$ in $[s,\tau_1]$. From the uniqueness of solution, let us denote $\Phi(t,s;y_0,a_0)$ with $t\in[s,\iota(s;y_0,a_0)]$ the unique maximal solution with initial data (y_0,a_0), we will call this solution the flow of the Cauchy system (1.18). Properties ($\mathcal{P}_4$)–($\mathcal{P}_5$) let every maximal solution to be defined on $[s,+\infty)$, *i.e.* $\iota(s;y_0,a_0)=+\infty$.

The main purpose of this paper is to prove the following theorem:

Theorem 1. *Let $T>0$. There exists $\varepsilon>0$ and a time-varying feedback law $U(t;y,a)$ satisfying properties ($\mathcal{P}_1$)–($\mathcal{P}_5$) such that following properties hold:*

(i) $\iota(s;y_0,a_0)=+\infty$, *for every* $(s;y_0,a_0)\in\mathbb{R}\times V$.

(ii) $\Phi(t+2T,t;y_0,a_0)=0$, *if* $\|(y_0,a_0)\|_V\leqslant\varepsilon$.

(iii) (Uniform stability property) For $\forall\delta>0,\exists\eta>0$ *such that*

$$(\|(y_0,a_0)\|_V\leqslant\eta)\Rightarrow(\|\Phi(t,t';y_0,a_0)\|_V\leqslant\delta,\forall t\geqslant t'). \tag{1.19}$$

This paper is organized as follows. In Section 2, we give a stationary feedback law F_λ which can locally exponentially stabilize the system with decay rate λ. Section 3 contains the construction of the time-varying feedback law, which leads to the local small-time stabilization that we will prove in Section 4.

2. Rapid stabilization

This section is based on the rapid stabilization of a KdV system proved in [2] and estimate given in [26]. Let us start from the linearized system

$$\begin{cases} y_t+y_x+y_{xxx}=0,\\ y(t,1)=y_x(t,1)=0,\\ y(t,0)=u(t). \end{cases} \tag{2.1}$$

It is proved in [2] that for any given positive λ, there is a kernel k_λ defined in the triangle $\mathcal{T}:=\{(x,v):x\in(0,1),v\in(x,1)\}$ such that if we perform the transformation $\Pi_\lambda:L^2(0,1)\longrightarrow L^2(0,1)$

$$z(x)=\Pi_\lambda(y(x)):=y(x)-\int_x^1 k_\lambda(x,v)y(v)dv, \tag{2.2}$$

then the solution y of system (2.2) with feedback law

$$u(t):=\int_0^1 k_\lambda(0,v)y(t,v)dv \tag{2.3}$$

is mapped to the solution z of the system

$$\begin{cases} z_t+z_x+z_{xxx}+\lambda z=0,\\ z(t,1)=z_x(t,1)=0,\\ z(t,0)=0. \end{cases} \tag{2.4}$$

Therefore we have the exponential stabilization:

$$\|z(t)\|_{L^2(0,1)}\leqslant e^{-\lambda t}\|z(0)\|_{L^2(0,1)}, \tag{2.5}$$

hence exponentially decay for the solution $y(t,\cdot)$ thanks to the invertibility of the transformation Π_λ.

As for the kernel, the following result is given in [2]:

Lemma 1.

(1) The kernel k_λ satisfies equation

$$\begin{cases} k_{xxx}+k_{vvv}+k_x+k_v+\lambda k=0 & \text{in } \mathcal{T},\\ k(x,1)=0 & \text{in } [0,1],\\ k(x,x)=0 & \text{in } [0,1],\\ k_x(x,x)=\frac{\lambda}{3}(1-x) & \text{in } [0,1]. \end{cases} \tag{2.6}$$

(2) The inverse of transformation Π_λ, Π_λ^{-1}, is given by

$$y(x)=\Pi_\lambda^{-1}(z(x)):=z(x)+\int_x^1 l_\lambda(x,v)z(v)dv. \tag{2.7}$$

And the kernel l_λ satisfies

$$\begin{cases} l_{xxx}+l_v+l_x+l_v-\lambda l=0 & \text{in } \mathcal{T},\\ l(x,1)=0 & \text{in } [0,1],\\ l(x,x)=0 & \text{in } [0,1],\\ l_x(x,x)=\frac{\lambda}{3}(1-x) & \text{in } [0,1]. \end{cases} \tag{2.8}$$

Later, in [26], more information on k_λ is given:

Lemma 2. *The equation (2.6) has a unique solution. There exists a constant C_1, which is independent of $\lambda>1$, such that*

$$\|k_\lambda\|_{C^3(\mathcal{T})}\leqslant e^{C_1\sqrt{\lambda}},\ \text{ and } \|l_\lambda\|_{C^3(\mathcal{T})}\leqslant e^{C_1\sqrt{\lambda}}. \tag{2.9}$$

Remark 1. *In [26], the estimate is only given for the C^1-norm, but one can easily get similar C^3 estimates by using the same method. As stated in [26], it will be a challenging and interesting problem to know whether the right hand side of (2.9) can be replaced by exp($C\lambda^{1/3}$).*

Now, we consider the linearized system of (1.18):

$$\begin{cases} y_t+y_x+y_{xxx}=0,\\ y(t,1)=y_x(t,1)=0,\\ y(t,0)=a(t),\\ a_t+y_{xx}(0)+\frac{1}{2}a=u. \end{cases} \tag{2.10}$$

We look for a transformation Ξ_λ with $(z,b)=\Xi_\lambda((y,a))$ of the form:

$$\begin{cases} z:=\Pi_\lambda(y),\\ b:=a+F_\lambda(y), \end{cases} \tag{2.11}$$

and a feedback law of the form

$$u(t):=K_\lambda(y)+L_\lambda a. \tag{2.12}$$

We want (z,b) to satisfy the following target system:

$$\begin{cases} z_t+z_x+z_{xxx}+\lambda z=0,\\ z(t,1)=z_x(t,1)=0,\\ z(t,0)=b(t),\\ b_t+z_{xx}(0)+\frac{1}{2}b+\mu b=w, \end{cases} \tag{2.13}$$

with μ and w to be chosen later.

Actually, performing the same calculation as in [2, page 1690] with (2.6), we get

$$z_t + z_x + z_{xxx} + \lambda z = 0. \tag{2.14}$$

Besides, we have

$$z(t,1) = y(t,1) = 0, \tag{2.15}$$
$$z_x(t,1) = y_x(t,1) + k_\lambda(1,1)y(t,1) = 0, \tag{2.16}$$
$$z(t,0) = a - \int_0^1 k_\lambda(0,v)y(t,v)dv. \tag{2.17}$$

Hence, $F_\lambda(y)$ should be

$$F_\lambda(y) := -\int_0^1 k_\lambda(0,v)y(v)dv. \tag{2.18}$$

At last, let us calculate w:

$$\begin{aligned}
w &= b_t + z_{xx}(0) + \frac{1}{2}b + \mu b \\
&= (a + F_\lambda(y))_t + (\mu + \frac{1}{2})(a + F_\lambda(y)) + y_{xx}(0) \\
&\quad - \Big(\int_x^1 k_\lambda(x,v)y(v)dv\Big)_{xx}(0) \\
&= \Big(a_t + y_{xx}(0) + \frac{1}{2}a\Big) + (F_\lambda(y))_t + \frac{1}{2}F_\lambda(y) \\
&\quad - \Big(\int_0^1 k_{\lambda,xx}(0,v)y(v)dv - k_{\lambda,x}(0,0)a\Big) + \mu(a + F_\lambda(y)).
\end{aligned} \tag{2.19}$$

Since

$$\begin{aligned}
(F_\lambda(y))_t &= -\int_0^1 k_\lambda(0,v)y_t(t,v)dv \\
&= \int_0^1 k_\lambda(0,v)\big(y_x(t,v) + y_{xxx}(t,v)\big)dv \\
&= -\int_0^1 k_{\lambda,v}(0,v)y(t,v)dv + k_\lambda(0,v)y(t,v)\Big|_0^1 \\
&\quad - \int_0^1 k_{\lambda,vvv}(0,v)y(t,v)dv + k_\lambda(0,v)y_{xx}(t,v)\Big|_0^1 \\
&\quad - k_{\lambda,v}(0,v)y_x(t,v)\Big|_0^1 + k_{\lambda,vv}(0,v)y(t,v)\Big|_0^1 \\
&= -\int_0^1 \big(k_{\lambda,v}(0,v) + k_{\lambda,vvv}(0,v)\big)y(t,v)dv \\
&\quad + k_{\lambda,v}(0,0)y_x(t,0) - k_{\lambda,vv}(0,0)a,
\end{aligned} \tag{2.20}$$

from (2.10)–(2.20) we get

$$\begin{aligned}
w &= u - \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\
&\qquad + \mu k_\lambda + k_{\lambda,xx}\Big)(0,v)y(t,v)dv \\
&\quad - \big(-k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu\big)a + k_{\lambda,v}(0,0)y_x(t,0) \\
&= u - \big(-k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu\big)a + k_{\lambda,v}(0,0)z_x(t,0) \\
&\quad - \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\
&\qquad + \mu k_\lambda + k_{\lambda,xx} + \frac{\lambda}{3}k_{\lambda,x}\Big)(0,v)y(t,v)dv,
\end{aligned}$$

where we used the fact that

$$z_x(t,0) - y_x(t,0) = -\int_0^1 k_{\lambda,x}(0,v)y(v)dv. \tag{2.21}$$

Hence we define the feedback, $u(t) = K_\lambda(y) + L_\lambda a$, by

$$\begin{cases} K_\lambda(y) := \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\ \qquad + \mu k_\lambda + k_{\lambda,xx} + \frac{\lambda}{3}k_{\lambda,x}\Big)(0,v)y(v)dv, \\ L_\lambda := -k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu, \end{cases} \tag{2.22}$$

which leads to

$$w = k_{\lambda,v}(0,0)z_x(t,0) = -\frac{\lambda}{3}z_x(t,0). \tag{2.23}$$

Let us choose

$$\mu := \lambda^2 + \lambda. \tag{2.24}$$

From (2.13), (2.23) and (2.24), we get

$$\begin{aligned}
\frac{d}{dt}\|(z,b)\|_V^2 &= -z_x(0)^2 - 2\lambda\|z\|_{L^2}^2 - 2(\lambda^2+\lambda)b^2 + 2wb \\
&\leqslant -2\lambda\|(z,b)\|_V^2,
\end{aligned} \tag{2.25}$$

which leads to the (global) exponential decay with rate λ to the target system (2.13). In order to get exponential decay to the system (2.10), we need to point that both $\Xi_\lambda, \Xi_\lambda^{-1}: V \to V$:

$$\Xi_\lambda : \begin{pmatrix} y \\ a \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_\lambda & 0 \\ F_\lambda & 1 \end{pmatrix} \begin{pmatrix} y \\ a \end{pmatrix},$$
$$\Xi_\lambda^{-1} : \begin{pmatrix} z \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_\lambda^{-1} & 0 \\ -F_\lambda\Pi_\lambda^{-1} & 1 \end{pmatrix} \begin{pmatrix} z \\ b \end{pmatrix},$$

are bounded.

From (2.2), (2.18), (2.22) and Lemma 2, there exists C_2 independent of $\lambda > 1$ such that following estimates on the norm of operators hold

$$|L_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \ \ |K_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \ \ |F_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \tag{2.26}$$
$$|\Pi_\lambda| \leqslant e^{C_2\sqrt{\lambda}} \text{ and } \left|\Pi_\lambda^{-1}\right| \leqslant e^{C_2\sqrt{\lambda}}. \tag{2.27}$$

Hence

$$|\Xi_\lambda| \leqslant 2e^{2C_2\sqrt{\lambda}} \text{ and } \left|\Xi_\lambda^{-1}\right| \leqslant 2e^{2C_2\sqrt{\lambda}}. \tag{2.28}$$

Let us consider now the stability of nonlinear system (1.18) with feedback law u given by (2.12) and (2.22). Suppose that $(y,a)(t)$ is a solution of (1.18) with (2.12), then $(z,b) := \Xi_\lambda(y,a)$ satisfies

$$\begin{aligned}
&z_t(t,x) + z_x(t,x) + z_{xxx}(t,x) + \lambda z(t,x) \\
&= -\Big(z(t,x) + \int_x^1 l_\lambda(x,v)z(t,v)dv\Big) \\
&\qquad \cdot \Big(z_x(t,x) + \int_x^1 l_{\lambda,x}(x,v)z(t,v)dv\Big) \\
&\quad - \frac{1}{2}\int_x^1 k_{\lambda,v}(x,v)\big(\Pi_\lambda^{-1}z\big)^2(t,v)dv \ = I,
\end{aligned} \tag{2.29}$$

$$z(t,1)=0, z_x(t,1)=0, z(t,0)=b, \tag{2.30}$$

$$\begin{aligned} & b_t + z_{xx}(0) + \frac{1}{2}b + \mu b + \frac{\lambda}{3} z_x(t,0) \\ =& -\frac{1}{3}a^2 - \frac{1}{2}\int_0^1 k_{\lambda,v}(0,v) y^2(t,v)dv \\ =& -\frac{1}{3}(F_\lambda \Pi_\lambda^{-1} z - b)^2 - \frac{1}{2}\int_0^1 k_{\lambda,v}(0,v)\big(\Pi_\lambda^{-1} z\big)^2(t,v)dv \\ =& J. \end{aligned} \tag{2.31}$$

Hence, together with (2.24), the flow (z,b) satisfy

$$\begin{aligned} \frac{d}{dt}\|(z,b)\|_V^2 =& -z_x(0)^2 - 2\lambda\|z\|_{L^2}^2 - 2(\lambda^2+\lambda)b^2 - \frac{2\lambda}{3}z_x(0)b \\ & + 2\langle z, I\rangle_{L^2} + 2bJ \\ \leqslant & -2\lambda\|(z,b)\|_V^2 + 2\langle z, I\rangle_{L^2} + 2bJ. \end{aligned} \tag{2.32}$$

Performing the same calculation as in [2, page 1692], we get

$$|2\langle z, I\rangle_{L^2}| + 2|bJ| \leqslant e^{C_3\sqrt{\lambda}}\|(z,b)\|_V^3, \tag{2.33}$$

with $C_3 > 3C_2$ independent of $\lambda > 1$.

Hence if the initial state (z_0, b_0) satisfies

$$\|(z_0,b_0)\|_V \leqslant e^{-C_3\sqrt{\lambda}} (i.e.\ \ \|(y_0,a_0)\|_V \leqslant e^{-2C_3\sqrt{\lambda}}), \tag{2.34}$$

the solution (z,b) will have the exponential decay

$$\|(z,b)(t)\|_V \leqslant e^{-\frac{\lambda}{2}t}\|(z_0,b_0)\|_V. \tag{2.35}$$

3. Control design

This section is devoted to the construction of the feedback law. As what is done in [26], we will find a piecewise continuous feedback on time $[0,T)$ such that properties $(\mathcal{P}_2)$–$(\mathcal{P}_5)$ holds. Actually, once we find this feedback on $[0,T)$, we can prolong it periodically to get a feedback law fulfills $(\mathcal{P}_1)$–$(\mathcal{P}_5)$. Since the feedback law (2.12) given in Section 2 is Lipschitz in V, it is not difficult to design such piecewise feedback laws.

The difficult part is the choice of $\{\lambda_n\}$ (increasing positive numbers that tend to infinity) and $\{t_n\}$ (increasing numbers with $\mathbf{t_0=0}$ that tend to $\mathbf{T}$ as $\mathbf{n}$ tends to infinity), such that Theorem 1 holds.

For any piece $[t_n, t_{n+1})$, let us define

$$u(t;y,a)\Big|_{t\in[t_n,t_{n+1})} := \varphi_{\lambda_n}(\|(y,a)\|_V)\Big(K_{\lambda_n}(y) + L_{\lambda_n}a\Big), \tag{3.1}$$

where $\varphi_{\lambda_n} := \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$\begin{cases} 1, & \text{if } x\in[0, e^{-C_2\sqrt{\lambda_n}}/5], \\ 2-5e^{C_2\sqrt{\lambda_n}}x, & \text{if } x\in[e^{-C_2\sqrt{\lambda_n}}/5, 2e^{-C_2\sqrt{\lambda_n}}/5] \\ 0, & \text{if } x\in[2e^{-C_2\sqrt{\lambda_n}}/5, +\infty). \end{cases}$$

Actually, one can easily verify that properties $(\mathcal{P}_2)$–$(\mathcal{P}_5)$ hold for proper choice of $\{t_n\}$. Let us directly choose

$$t_n := 0, \lambda_n := 0 \text{ for } n < n_0 := 1 + [\frac{2}{\sqrt{T}}], \tag{3.2}$$

$$t_n := T - 1/n^2, \lambda_n := 2n^8, \text{ for } n \geqslant n_0 := 1 + [\frac{2}{\sqrt{T}}].$$

We prove Theorem 1 with this feedback law in the next section.

4. Small-time stabilization

The proof of Theorem 1 is divided into three parts:

(1) The solution exists in arbitrary time.

(2) There exists $\varepsilon > 0$ such that, $\Phi(T,0;y_0,a_0) = 0$, if $\|(y_0,a_0)\|_V \leqslant \varepsilon$.

(3) Uniform stability property, see (1.19).

In fact, (1) equals to (i), (3) equals to (iii), and (2)–(3) imply (ii).

Let us start by (1). By classical fixed point argument, for every $R > 0$, we know the existence of T_R such that for every initial state $\|(y,a)\|_V < R$, the solution exists on $(0, T_R)$. We only need to verify that the solution will never blow-up. Following the simple calculation in (1.10) with the help of $(\mathcal{P}_4)$–$(\mathcal{P}_5)$, we can control the V-norm of the solution in arbitrary time. As the time-varying feedback law is bounded at every time except $t = T$, we also need to prove that for $\forall s \in [0,T)$, following limit

$$\lim_{t\to T^-} \Phi(t,s;y_0,a_0) \tag{4.1}$$

exists. This can be proved by using the same method given in [16, page 22]. We omit it here.

The next and the most important step is to prove (2). On time $[t_n, t_{n+1})$, the feedback u is given by

$$\varphi_{\lambda_n}(\|(y,a)\|_V)\Big(K_{\lambda_n}(y) + L_{\lambda_n}a\Big).$$

We observe that if $\varphi_{\lambda_n} \neq 1$, the exponential decay no longer holds. The idea is to prove that

$$\varphi_{\lambda_n}\big(\|(y,a)(t))\|_V\big) \equiv 1, \text{ for } t\in[t_n,t_{n+1})$$

which is equivalent to have that

$$\|(y,a)(t))\|_V \leqslant e^{-C_2\sqrt{\lambda_n}/5}, \text{ for } t\in[t_n,t_{n+1}). \tag{4.2}$$

As we have seen in Section 2, in order to get exponential stabilization of our nonlinear system (1.18), the following condition on the "initial state"

$$\|(y,a)(t_n)\|_V \leqslant e^{-2C_3\sqrt{\lambda_n}} \tag{4.3}$$

is sufficient. One can simply verify the following lemma:

Lemma 3. *For every $n \geqslant 1$. For every*

$$\|(y,a)(t_n)\|_V \leqslant e^{-4C_3\sqrt{\lambda_n}}, \tag{4.4}$$

conditions (4.2)–(4.3) hold for $t \in [t_n, t_{n+1})$.

If both (4.2) and (4.4) are fulfilled, the solution $(y,a)(t)$ is controlled by the following estimate:
For $t \in [0, T-1/n_0^2]$, we have $\|(y,a)(t)\|_V \leqslant \|(y_0,a_0)\|_V$.
For $t \in [t_n, t_{n+1})$ with $n \geqslant n_0$, we have

$$\begin{aligned}&\|(y,a)(t)\|_V/\|(y_0,a_0)\|_V\\ \leqslant&|\Xi_{\lambda_n}||\Xi_{\lambda_n}^{-1}|\prod_{k=n_0}^{n-1}\left(|\Xi_{\lambda_k}||\Xi_{\lambda_k}^{-1}|e^{-(t_{k+1}-t_k)\lambda_k/2}\right)\\ \leqslant&\left(\prod_{k=n_0}^{n-1}e^{-ck^5}\right)\left(\prod_{k=n_0}^{n}e^{5C_2k^4}\right).\end{aligned}$$

In order to ensure the conditions (4.2) and (4.4), and to get the stabilization to 0 on time T, we only need to find $\varepsilon > 0$ such that

$$\varepsilon\left(\prod_{k=n_0}^{n-1}e^{-ck^5}\right)\left(\prod_{k=n_0}^{n}e^{5C_2k^4}\right)\leqslant e^{-4(C_2+C_3)n^4} \tag{4.5}$$

for all $n \geqslant n_0$. Such ε obviously exists.

At last, it remains to prove (3), the uniform stability property. On the one hand, observe from (1.10) and $(\mathcal{P}_4)$ that, for $\forall \delta_0 > 0$, there exists $T_0 \in [0,T)$ such that

$$\begin{aligned}&(\|(y_0,a_0)\|_V \leqslant \delta_0/2, t_0 \in [T_0,T))\\ \Rightarrow&(\|\Phi(t,t_0;y_0,a_0)\|_V \leqslant \delta_0, \forall t \in [t_0,T)).\end{aligned} \tag{4.6}$$

On the other hand, from $(\mathcal{P}_3)$ we can find a M such that

$$u(t;y,a) \leqslant M\|(y,a)\|_V, \text{ for } t \in [0,T_0], \tag{4.7}$$

which concludes the existence of C such that

$$\|\Phi(t,s;y_0,a_0)\|_V \leqslant C\|(y_0,a_0)\|_V, \forall 0 \leqslant s \leqslant t \leqslant T_0. \tag{4.8}$$

Estimates (4.6) and (4.8) together with (2) give the uniform stability property (3), which completes the proof.

Remark 2. *As we have seen, the main idea is to use the "kernel" (linear part), which forces our results to be local. From the controllability point of view, one can use the return method to get global control results (even in small time), see [4, 7, 9]. From local stabilization to some global result, there still exists a big gap, especially for small-time.*

Acknowledgments. The author would like to thank Jean-Michel Coron for having attracted his attention to this problem, for his constant support, and for fruitful discussions. He also thanks Amaury Hayat, Peipei Shang and Christophe Zhang for discussions on this problem.

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