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# Nonlinear Step-Stress Accelerated Degradation Modelling considering Three Sources of Variability

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**Abstract:** In the absence of enough run-to-failure data, step-stress accelerated degradation testing (SSADT) is often an attractive alternative way to evaluate the reliability of a product, with the advantage of requiring small sample size and short test time. However, the development of a statistical SSADT model for reliability assessment should take into account different sources of variability in the degradation process that generate uncertainty: 1) temporal variability determining the inherent variability of degradation process over time; 2) unit-to-unit variability in three aspects: degradation rates, initial degradation values, time-points of elevating stress levels; and 3) measurement errors in both covariates and degradation performance. As a contribution towards this aim, a new nonlinear Wiener-process-based SSADT model considering simultaneously nonlinearity and three sources of variability is proposed. Using the proposed SSADT model, the lifetime law of the tested product under normal conditions is derived based on the concept of first hitting time (FHT) of a predetermined failure threshold. Following an approach based on genetic algorithms (GA), a modified simulation and extrapolation method, called GA-SIMEX, is also developed for the model parameter estimation. Finally, a simulation study of fatigue crack length growth is presented to illustrate the implementation of the proposed SSADT model.

**Keywords:** Step-Stress Accelerated Degradation Testing; Nonlinearity; Temporal variability; Unit-to-unit variability; Measurement errors; GA-SIMEX.

## Acronyms

**SSADT** step-stress accelerated degradation test

**pdf** probability density function

**FHT** first hitting time

**GA** genetic algorithm

**SIMEX** simulation and extrapolation method

**ALT** accelerated life tests

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**ADT** accelerated degradation test  
**CSADT** constant stress accelerated degradation test

## Nomenclature

$K$	Number of accelerated stress levels in the SSADT
$N$	Number of test units
$S = \{S_1, \dots, S_K\}$	Predetermined accelerated stress levels in the SSADT
$T$	Terminal time of the SSADT
$\tau_{i,1}, \dots, \tau_{i,K-1}$	Time-points of elevating stress levels
$H_1, \dots, H_{K-1}$	Predetermined threshold levels of elevating stress levels
$X(t)$	Theoretical degradation performance at time $t$
$B(t)$	Standard Wiener process
$\lambda(t; \gamma, \beta)$	Nonlinear drift of the degradation model
$\sigma_B$	Diffusion parameter of the degradation model
$X(0)$	Initial degradation value
$\varepsilon$	Measurement errors in degradation performance
$\phi$	Measurement errors in covariates
$Y(t)$	Actual degradation measurement at time $t$
$S'$	Measurement error-prone covariates
$S_0$	Normal stress level of the products
$T_0$	Lifetime of products under normal stress level
$f_0(t)$	Pdf for exponential nonlinear Wiener process with random effects
$f_1(t)$	Pdf for exponential nonlinear Wiener process with random effects and covariates
$f(t)$	Pdf for exponential nonlinear Wiener process with random effects, covariates and random initial degradation values
$\mathbf{t}_{i,k} = (t_{i,1,k}, t_{i,2,k}, \dots, t_{i,J_{i,k},k})^T$	Observation times of the $i$ th product under the $k$ th stress level $S_k$
$\mathbf{T}_{i,k}$	Nonlinear transformation of $\mathbf{t}_{i,k}$
$\mathbf{X}_{i,k} = (x_{i,1,k}, x_{i,2,k}, \dots, x_{i,J_{i,k},k})^T$	Theoretical degradation performance of the $i$ th product under the $k$ th stress level
$\mathbf{Y}_{i,k} = (y_{i,1,k}, y_{i,2,k}, \dots, y_{i,J_{i,k},k})^T$	Actual degradation measurement of the $i$ th product under the $k$ th stress level $S_k$
$\Gamma_{i,k}$	The $k$ th time point of elevated stress level of the $i$ th product
$\mathbf{I}_{J_{i,k}}$	Unit matrix with dimension $J_{i,k}$

# 1 Introduction

Continuous improvements in modern design and manufacturing technologies, coupled with increasing social requirements for high quality products, have promoted the production of goods with extremely high reliability and long life. Since reliability tests of such products, even accelerated life tests (ALTs), often fail to generate sufficient lifetime data, analysis of degradation data turns to be an alternative way to evaluate product reliability. It assumes that the degradation of a product quality characteristic is related to its reliability, and collects degradation data under severer-than-normal stresses to predict the product reliability under normal condition, which is called an accelerated degradation test (ADT) [1].

In industry applications, ADTs can be conducted in mainly two ways due to the different settings of stress loadings, i.e., constant stress accelerated degradation test (CSADT) [2] and step-stress accelerated degradation test (SSADT) [3]. SSADT has an advantage over CSADT since it allows a comparable assessment accuracy with a lower sample size [4]. Therefore, the last few decades have witnessed wide application of SSADT in reliability tests for LEDs [5], electrical connectors [6], missile tanks [7] and transistors [8]. The purpose of this study is to model the SSADT process in a more practical way, and the emphasis is to handle with various sources of variability simultaneously.

Generally, several sources of variability contribute to the uncertainty of degradation processes under normal conditions, which have attracted much attention recently. Si et al. [9] developed a Wiener process based degradation model considering three-source variability, including temporal variability, unit-to-unit variability of degradation rates and degradation performance measurement variability. Tang et al. [10] presented a methodology for managing prognostic uncertainty, which consists of modeling uncertainty, sensor measurement uncertainty, future load profile uncertainty and fault detection (diagnostics) uncertainty. Baraldi et al. [11] classified the sources of uncertainty that contribute to the RUL prediction of degradation processes into three categories: randomness in the future degradation of equipment, modeling error and uncertainty in current and past equipment degradation data.

Just as under normal conditions considered in the above mentioned works, degradation processes under severe conditions (i.e. in SSADT procedures) also witness several sources of variability, including temporal variability, unit-to-unit variability of degradation rates and so on. Furthermore, the complexity and new characteristics of SSADTs lead to the existence of new sources of variability, such as the heterogeneity of time-points of elevating stress levels among test units. In addition, nonlinearity is another feature commonly encountered in SSADTs conditions. This study intends to develop a more practical SSADT model by integrating nonlinearity and three sources of variability, including temporal variability, various kinds of unit-to-unit variability and measurement errors in both degradation performance and covariates.

First, temporal variability determines the inherent variability of degradation process over time. Pandey et al. [12] compared the common used random variable degradation model with stochastic process model, and pointed out that the stochastic process model can incorporate the temporal variability associated with the evolution of degradation. Recent advances in applying stochastic process model, including Wiener process, Gamma process and Inverse Gaussian Process, in SSADTs can be found in many papers. For example, Liao and Tseng [13] used a stochastic diffusion process to model the SSADT and derived the optimal test plan by searching optimization

algorithm. Tseng, Balakrishnan and Tsai [14] designed an efficient SSADT plan based on Gamma process for degradation processes with a monotone increasing pattern. Wang et al. [15] proposed a cumulative exposure model based on Inverse Gaussian degradation process for the typical SSADT problem and provided an optimal test plan.

Second, unit-to-unit variability in SSADTs results from the heterogeneity among test units, including the variability in degradation rates, initial degradation values as well as time-points of elevating stress levels. First of all, a random effect model [16] is often adapted to capture the conventional unit-to-unit variability referring to the heterogeneity among degradation rates of units. Besides that, the variability in initial degradation values [17] of different units also leads to diversity in reliability analysis of such degradation processes. Furthermore, the traditional SSADTs always elevate the test stress to a higher level at some predetermined time-points for all units, which causes a problem that some units may experience little or even no degradation within a specified test duration. Confronted with this, Pan and Balakrishnan [18] presented another stress level elevating scheme, i.e., elevating stress levels when the degradation path of a unit firstly reaches pre-specified values. This test scheme results in randomness in time-points of elevating stress levels between units. To be more practical and economical, Amini, Shemehsavar and Pan [19] extended the previous degradation-value-based stress level elevating scheme from continuous inspections to discrete ones.

Last, measurement errors can be found in most measurement processes of both covariates (test stresses) and degradation performance, due to the limitations of measurement equipment's performance, artificial deviations in the reading of analogue measuring instruments and approximation of measurement procedures. For measurement errors in covariates, He, Yi and Xiong [20] presented a novel accelerated failure time model with measurement error-prone covariates to study the impact of ignoring covariates measurement error, and proposed a corresponding simulation and extrapolation (SIMEX) parameter estimation method. Based on the longitudinal observations with covariates subject to Berkson-type measurement errors, Zhang and Zhang [21] developed an ADT model and derived the minimum distance estimator for the covariates measurement error model parameters. For error-prone degradation performance measurement, extensive researches can be referred to in [22–25]. For example, Lu, Pandey and Xie [22] adopted a Genz transform and quasi-Monte Carlo method to filter out the effect of sizing errors from the in-service and non-destructive measured degradation data. Ye and Xie [23] provided a summary of stochastic process models for degradation processes contaminated by random white noises.

Besides the different kinds of variability mentioned above, nonlinearity is another feature that should be considered in SSADT modelling process. In fact, many degradation processes are not linear and extensive studies have been devoted to it. Lei et al. [26] constructed a novel nonlinear degradation model considering the temporal variability, unit-to-unit variability in degradation rates, measurement variability and nonlinear variability. Huang et al. [27] employed a state-space based method to characterize nonlinear heterogeneous degradation processes. Zheng et al. [28] presented a general degradation model considering nonlinearity as well as three important sources of variability, i.e., temporal variability, unit-to-unit variability in degradation rates, and measurement variability in degradation performance.

All the above introductory remarks lead to the conclusion that the deterioration nonlinearity and the existence of three sources of variability, including temporal variability, unit-to-unit variability and measurement errors, are all important features to be taken into consideration in the development of SSADT models. However, most works on

SSADT modelling are limited to the case of considering only one or two of the above-mentioned features. Besides, researches concerning variability in initial degradation values and measurement errors in covariates are extremely limited. This paper precisely addresses this problem, and makes a threefold contribution:

- First, a new nonlinear Wiener-process-based SSADT model is developed, which simultaneously considers nonlinearity, temporal variability, various kinds of unit-to-unit variability and measurement errors in both degradation performance and covariates.
- Secondly, based on the concept of first hitting time (FHT) of a failure threshold, and using the proposed SSADT model, the reliability of the tested item under normal conditions is assessed, which can be useful for establishing more economical and efficient design improvement methods and maintenance strategies.
- Finally, the inference issue is studied for the proposed model, and by combining the methods of SIMEX and genetic algorithm (GA), a modified GA-SIMEX method is proposed to estimate the model parameters based on SSADT data.

The rest of the article is organized as follows. Section 2 gives the description of the nonlinear SSADT model considering three sources of variability. In Section 3, reliability assessment methods under normal conditions are derived. And Section 4 develops a modified GA-SIMEX method to estimate the model parameters. Section 5 provides numerical experiments on fatigue crack length simulated data to illustrate the implementation and the use of the proposed model. Finally, some conclusions are given in Section 6.

## 2 Model description

We consider a  $K$ -steps step-stress accelerated degradation test with  $N$  units operating independently. The SSADT starts at a lower stress level  $S_1$  at fixed time 0, and ends at fixed time  $T$ . Each unit will experience a predetermined stress loading process  $S = \{S_1, S_2, \dots, S_K\}$ . And the stress level for a unit is elevated when its degradation performance reaches for the first time a predetermined threshold level  $H_j$  ( $j = 1, 2, \dots, K-1$ ). Denote  $\tau_{i,0} = 0$ ,  $\tau_{i,K} = T$  and  $\tau_{i,j}$  ( $j = 1, 2, \dots, K-1$ ) are the time points of elevating stress levels for the  $i$ th unit. Therefore, its testing stress of the  $K$ -steps SSADT can be expressed in Equation (1) and shown in Figure 1.

$$S_i = \begin{cases} S_1, \tau_{i,0} \leq t < \tau_{i,1} \\ \vdots & \vdots \\ S_K, \tau_{i,K-1} \leq t < \tau_{i,K} \end{cases} \quad (1)$$

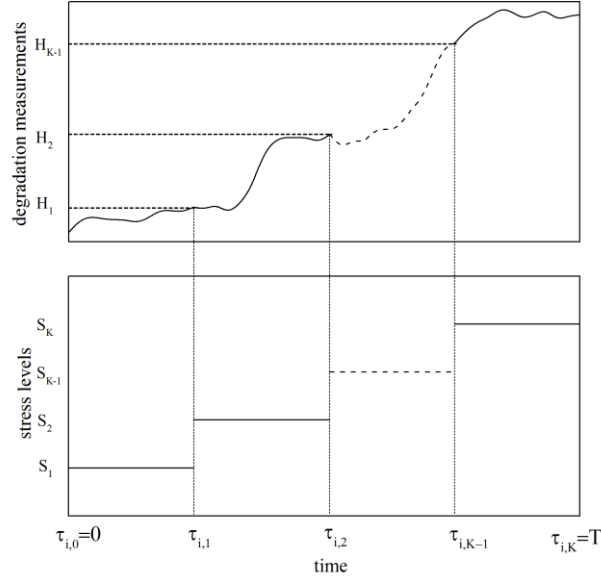


Figure 1 The stress loading process for the  $i$ th unit in the step-stress accelerated degradation test

Denote the theoretical degradation at time  $t$  under stress level  $S$  by  $X(t)$ , which is characterized by a standard Wiener process  $B(t)$  and a nonlinear drift  $\lambda(t; \gamma, \beta)$  along with the diffusion  $\sigma_B$ , i.e.,

$$X(t) = X(0) + \lambda(t; \gamma, \beta) + \sigma_B B(t) \quad (2)$$

The initial degradation value  $X(0)$  is assumed to be a random variable representing the heterogeneity and randomness in product manufacturing processes. The nonlinear drift  $\lambda(t; \gamma, \beta)$  is assumed to satisfy the regularity condition with the special cases of  $\lambda(t; \gamma, \beta) = \gamma(e^{\beta t} - 1)$ ,  $\lambda(t; \gamma, \beta) = \gamma t^\beta$ , etc., where  $\gamma$  is dependent on the acceleration variable  $S$ ,  $\gamma = \varphi(S; a, b)$ , and the random variable  $a$  represents the unit-to-unit variability, while  $b$  is the fixed effect for all units. Furthermore, measurement errors in degradation performance and covariates can be respectively characterized by random variables  $\varepsilon$  and  $\phi$ , which makes that the actually measured degradation level  $Y(t)$  is expressed as

$$\begin{cases} S' = S + \phi; \\ Y(t) = X(t; S') + \varepsilon = X(0) + \lambda(t; \gamma, \beta, S') + \sigma_B B(t) + \varepsilon \end{cases} \quad (3)$$

For a specific product, the distribution of the initial degradation value  $X(0)$  can be selected and tested based on the factory testing or incoming inspection for batches of products. In the case that prior knowledge is available or the degradation physics of the considered products is known, the form of  $\lambda(t; \gamma, \beta)$  can be obtained easily. Otherwise, a statistical method can be used for the determination of the form of the nonlinear function, by fitting the test data with possible time-transformed forms, such as the linear function, binomial function, power law function and exponential function, and a best fitting is determined by some indicators like correlation coefficient  $R^2$ . Besides, the model for covariates  $\gamma = \varphi(S; a, b)$  is often determined based on engineering experience, like different types of accelerated test for the same product or the same type of accelerated test for similar products. For

the distributions of measurement errors  $\varepsilon$  and  $\phi$ , they can be obtained through measurement system analysis, such as the uncertainty analysis of a measurement system.

For a better illustration of the proposed model without loss of generality, and following classical assumptions in relevant papers [9, 16, 17, 20, 23, 30], the rest of this article will respectively take  $X(0) \sim N(\mu_0^2, \sigma_0^2)$ ,  $\lambda(t; \gamma, b) = \gamma(e^{\beta t} - 1)$ ,  $\gamma = ae^{-b/S}$ ,  $a \sim N(\mu_a, \sigma_a^2)$ ,  $\varepsilon \sim N(0, \sigma_\varepsilon^2)$  and  $\phi \sim N(0, \sigma_\phi^2)$  as examples of the distribution of the initial degradation value, nonlinear drift, model for covariate and distributions of measurement errors. Under these assumptions, the actually measured degradation level  $Y(t)$  is given by

$$\begin{cases} S' = S + \phi; \\ Y(t) = X(t; S') + \varepsilon = X(0) + ae^{-b/S'}(e^{\beta t} - 1) + \sigma_B B(t) + \varepsilon \end{cases} \quad (4)$$

It should be noted that, because of measurement errors,  $S'$  is the stress level that is actually applied to the system, even if the intended stress level is  $S$ . Consequently, the actually measured degradation level  $Y(t)$  depends on the measurement error-prone covariates - but unknown -  $S'$ , rather than on the measured covariates  $S$ , because generally accelerated degradation tests are assumed to be conducted under specified stresses  $S$ , which are measured and controlled to specified level in the test. But the existence of measurement error makes that the products are actually degrading under unknown test stresses  $S'$ .

The basic principles for the proposed SSADT model considering nonlinearity and three sources of variability are shown in Figure 2.

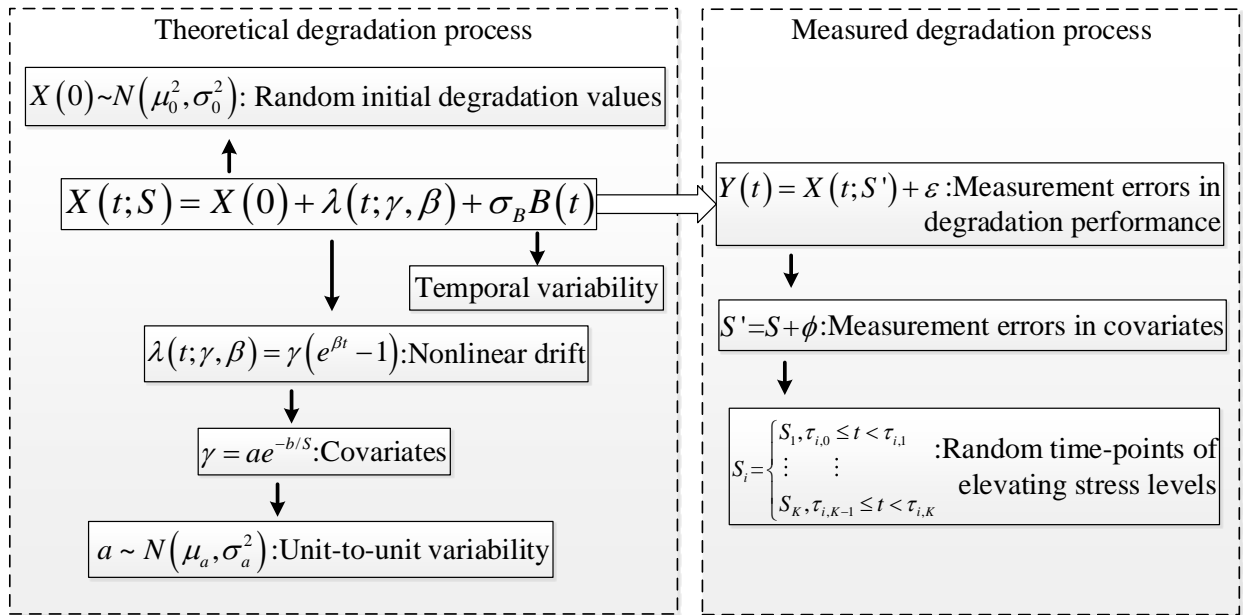


Figure 2 Basic principles for the proposed SSADT model considering nonlinearity and three sources of variability

### 3 Reliability assessment

Since the purpose of an ADT, both CSADT and SSADT, is to assess the reliability of products under normal stress level  $S_0$ , this section derives the reliability assessment method based on the proposed SSADT model. Although the reliability assessment results derived in the following are based on a specific example with



$X(0) \sim N(\mu_0^2, \sigma_0^2)$ ,  $\lambda(t; \gamma, b) = \gamma(e^{\beta t} - 1)$ ,  $\gamma = ae^{-b/s}$ ,  $a \sim N(\mu_a, \sigma_a^2)$ ,  $\varepsilon \sim N(0, \sigma_\varepsilon^2)$  and  $\phi \sim N(0, \sigma_\phi^2)$ , the reliability assessment procedure and the obtained results are also applicable for other models with different nonlinear drift and covariates.

Generally, the lifetime  $T_0$  in degradation analysis is defined as the first hitting time (FHT) of a predetermined threshold level  $H$ , i.e.,

$$T_0 = \inf \{t : X(t) \geq H\} \quad (5)$$

Then the corresponding reliability function  $R_0(t)$  is:

$$R_0(t) = P\{T_0 \geq t\} = P\{M(t) \leq H\} \quad (6)$$

where  $M(t) = \max_{0 \leq s \leq t} X(s)$  is the Wiener maximum Process [29].

For the most original case, i.e., the linear Wiener process without random effects or random initial degradation values, it is known that the lifetime follows an inverse Gaussian distribution [30]. Then based on some assumptions and the law of total probability, the probability density function (pdf) of the FHT for an exponential nonlinear Wiener process with random effects is obtained by [31]:

$$f_0(t) = \frac{H - (e^{\beta t} - \beta t e^{\beta t} - 1) \frac{H \sigma_\gamma^2 (e^{\beta t} - 1) + \mu_\gamma \sigma_B^2 t}{\sigma_\gamma^2 (e^{\beta t} - 1)^2 + \sigma_B^2 t}}{\sqrt{2\pi t^2 [\sigma_\gamma^2 (e^{\beta t} - 1)^2 + \sigma_B^2 t]}} \times \exp \left\{ -\frac{[H - \mu_\gamma (e^{\beta t} - 1)]^2}{2[\sigma_\gamma^2 (e^{\beta t} - 1)^2 + \sigma_B^2 t]} \right\} \quad (7)$$

Furthermore, when introducing covariates  $\gamma = ae^{-b/s}$ ,  $a \sim N(\mu_a, \sigma_a^2)$  into the model, the lifetime pdf can be derived by replacing  $\gamma \sim N(\mu_\gamma, \sigma_\gamma^2)$  with  $\gamma \sim N(\mu_a e^{-b/S_0}, \sigma_a^2 e^{-2b/S_0})$ :

$$f_1(t) = \frac{H - (e^{\beta t} - \beta t e^{\beta t} - 1) \frac{H \sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1) + \mu_a e^{-b/S_0} \sigma_B^2 t}{\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t}}{\sqrt{2\pi t^2 [\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t]}} \times \exp \left\{ -\frac{[H - \mu_a e^{-b/S_0} (e^{\beta t} - 1)]^2}{2[\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t]} \right\} \quad (8)$$

Introducing random initial degradation values into the model, the products lifetime turns to be:

$$\begin{aligned} T_0 &= \inf \{t : X(0) + ae^{-b/S_0} (e^{\beta t} - 1) + \sigma_B B(t) \geq H\} \\ &= \inf \{t : ae^{-b/S_0} (e^{\beta t} - 1) + \sigma_B B(t) \geq H - X(0)\} \end{aligned} \quad (9)$$

which means that the lifetime is equivalent to the FHT of a random threshold level  $H - X(0)$  with normal distribution  $H - X(0) \sim N(H - \mu_0, \sigma_0^2)$ , and its pdf is denoted by  $f_{H-X(0)}(h)$ .

By the law of total probability, the reliability function can be expressed as follows:

$$\begin{aligned} R(t) &= P\{T_0 \geq t\} = P\{M(t) \leq H - X(0)\} \\ &= \int_{-\infty}^{+\infty} P\{M(t) \leq h\} f_{H-X(0)}(h) dh \\ &= \int_{-\infty}^{+\infty} \int_t^{+\infty} f_1(s; h) f_{H-X(0)}(h) ds dh \end{aligned} \quad (10)$$

Differentiate the above equation, and the pdf of lifetime  $T$  can be obtained:

$$f(t) = \int_{-\infty}^{+\infty} f_1(t; h) f_{H-X(0)}(h) dh \quad (11)$$

where  $f_1(t;h)$  represents the pdf of lifetime in Equation (8), where the threshold level is equal to  $h$ .  
Before the derivation of the pdf, a lemma is given in advance.

**Lemma 1:** If  $Z$  is a normal distributed variable  $Z \sim N(\mu, \sigma^2)$ , and  $B, C, D, E \in R$ , then

$$E_Z \left[ (BZ - C) e^{-\frac{(Z-D)^2}{2E}} \right] = \sqrt{\frac{E}{\sigma^2 + E}} \left( \frac{D\sigma^2 + \mu E}{\sigma^2 + E} B - C \right) e^{-\frac{(\mu-D)^2}{2(\sigma^2 + E)}} \quad (12)$$

Proof:

Since  $Z \sim N(\mu, \sigma^2)$  and  $B, C, D, E \in R$ , the pdf of  $Z$  is formulated by:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \quad (13)$$

then the expectation of  $(BZ - C) e^{-\frac{(Z-D)^2}{2E}}$  is

$$E_Z \left[ (BZ - C) e^{-\frac{(Z-D)^2}{2E}} \right] = B E_Z \left[ Z e^{-\frac{(Z-D)^2}{2E}} \right] - C E_Z \left[ e^{-\frac{(Z-D)^2}{2E}} \right] \quad (14)$$

where

$$\begin{aligned} E_Z \left[ Z e^{-\frac{(Z-D)^2}{2E}} \right] &= \int_{-\infty}^{+\infty} z e^{-\frac{(z-D)^2}{2E}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} z e^{-\frac{\sigma^2(z-D)^2 + E(z-\mu)^2}{2E\sigma^2}} dz = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} z e^{-\frac{\left( z - \frac{D\sigma^2 + \mu E}{\sigma^2 + E} \right)^2 + \frac{E\sigma^2(\mu-D)^2}{(\sigma^2 + E)^2}}{2E\sigma^2/(\sigma^2 + E)} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu-D)^2}{2(\sigma^2 + E)}} \int_{-\infty}^{+\infty} z e^{-\frac{\left( z - \frac{D\sigma^2 + \mu E}{\sigma^2 + E} \right)^2}{2E\sigma^2/(\sigma^2 + E)}} dz = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu-D)^2}{2(\sigma^2 + E)}} \sqrt{2\pi \frac{E\sigma^2}{\sigma^2 + E} \frac{D\sigma^2 + \mu E}{\sigma^2 + E}} \\ &= \sqrt{\frac{E}{\sigma^2 + E}} \frac{D\sigma^2 + \mu E}{\sigma^2 + E} e^{-\frac{(\mu-D)^2}{2(\sigma^2 + E)}} \end{aligned} \quad (15)$$

And the second part of Equation (14) can be derived in a similar way:

$$E_Z \left[ e^{-\frac{(Z-D)^2}{2E}} \right] = \sqrt{\frac{E}{\sigma^2 + E}} e^{-\frac{(\mu-D)^2}{2(\sigma^2 + E)}} \quad (16)$$

Therefore Equations (14), (15) and (16) jointly completes the proof of Lemma 1.

Based on Lemma 1, the pdf of  $T$  can be formulated as:

$$\begin{aligned}
f(t) &= \int_{-\infty}^{+\infty} f_1(t;h)f_{H-X(0)}(h)dh = E_{H-X(0)}[f_1(t;h)] \\
&= E_{H-X(0)} \left[ \frac{h - (e^{\beta t} - \beta t e^{\beta t} - 1) \frac{h\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1) + \mu_a e^{-b/S_0} \sigma_B^2 t}{\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t}}{\sqrt{2\pi t^2 [\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t]}} \times \exp \left\{ -\frac{[h - \mu_a e^{-b/S_0} (e^{\beta t} - 1)]^2}{2[\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t]} \right\} \right] \\
&= AE_{H-X(0)} \left[ (Bh - C) \exp \left\{ -\frac{[h - D]^2}{2E} \right\} \right] \\
&= A \sqrt{\frac{E}{\sigma_0^2 + E}} \left( \frac{D\sigma_0^2 + (H - \mu_0)E}{\sigma_0^2 + E} B - C \right) e^{-\frac{(H - \mu_0 - D)^2}{2(\sigma_0^2 + E)}}
\end{aligned} \tag{17}$$

where

$$A = \frac{1}{\sqrt{2\pi t^2 [\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t]}}, \quad B = 1 - \frac{\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1) (e^{\beta t} - \beta t e^{\beta t} - 1)}{\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t},$$

$$C = \frac{\mu_a e^{-b/S_0} \sigma_B^2 t (e^{\beta t} - \beta t e^{\beta t} - 1)}{\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t}, \quad D = \mu_a e^{-b/S_0} (e^{\beta t} - 1) \quad \text{and} \quad E = [\sigma_a^2 e^{-2b/S_0} (e^{\beta t} - 1)^2 + \sigma_B^2 t].$$

After obtaining the pdf of lifetime  $T$ , the reliability curve can also be obtained through formula  $R(t_0) = 1 - \int_0^{t_0} f(t)dt$ . Since the analytical form of  $R(t)$  can be hard to derive, a numerical integration method can be utilized in the computation process.

## 4 Parameter estimation

Considering that the test stresses, such as temperature, humidity and so on, are usually measured by commonly used instruments, it is feasible to estimate the parameter  $\sigma_\phi^2$  from independent samples. Therefore  $\sigma_\phi^2$  are assumed to be known in this article. Besides, since the distribution of the random initial degradation value,  $X(0) \sim N(\mu_0^2, \sigma_0^2)$ , can be fitted based on initial measurements of the test samples, the parameters  $\{\mu_0, \sigma_0^2\}$  are easy to be estimated by distribution fitting method and logically independent of the other unknown parameters  $\theta = \{b, \beta, \mu_a, \sigma_a^2, \sigma_B^2, \sigma_\varepsilon^2\}$ , whose estimators are obtained in the rest of this section.

Suppose that the test stress levels are  $S = \{S_1, S_2, \dots, S_K\}$ , and under the  $k$ th stress level  $S_k$ ,  $J_{i,k}$  measurements for the  $i$ th product are observed at times  $t_{i,j,k}$ , where  $i = 1, 2, \dots, N; k = 1, 2, \dots, K; j = 1, 2, \dots, J_{i,k}$ . The corresponding actual degradation measurements and theoretical degradation performance are respectively denoted by  $y_{i,j,k}$  and  $x_{i,j,k}$ . Denote the observation times vectors  $\mathbf{t}_{i,k} = (t_{i,1,k}, t_{i,2,k}, \dots, t_{i,J_{i,k},k})^T$ ,  $\mathbf{T}_{i,k} = (e^{\beta t_{i,1,k}} - 1, e^{\beta t_{i,2,k}} - 1, \dots, e^{\beta t_{i,J_{i,k},k}} - 1)^T$ , and the degradation measurements vectors  $\mathbf{X}_{i,k} = (x_{i,1,k}, x_{i,2,k}, \dots, x_{i,J_{i,k},k})^T$ ,  $\mathbf{Y}_{i,k} = (y_{i,1,k}, y_{i,2,k}, \dots, y_{i,J_{i,k},k})^T$ . Besides, the  $k$ th time point of elevated stress level of the  $i$ th product is  $\Gamma_{i,k}$ , and  $\Gamma_i = \{\Gamma_{i,1}, \Gamma_{i,2}, \dots, \Gamma_{i,K-1}\}$ ,  $\Delta\Gamma_i = \{\Delta\Gamma_{i,1}, \Delta\Gamma_{i,2}, \dots, \Delta\Gamma_{i,K-1}\} = \{\Gamma_{i,1}, \Gamma_{i,2} - \Gamma_{i,1}, \dots, \Gamma_{i,K-1} - \Gamma_{i,K-2}\}$ .

#### 4.1 Maximum likelihood estimation without measurement error in covariates

Based on the degradation measurement model (4) in Section 2 and the above denotations, and ignoring the measurement error of covariates  $\phi$ , the likelihood function can be derived by [18]

$$L(\boldsymbol{\theta}; \mathbf{S}) = \prod_{i=1}^N \left\{ g_{r_i}(\boldsymbol{\gamma}_i; \boldsymbol{\theta}, \mathbf{S}) \prod_{k=1}^K f_{Y_{ik}}(y_{i,k}; \boldsymbol{\theta}, \mathbf{S}) \right\} \quad (18)$$

where  $g_{r_i}(t; \boldsymbol{\theta}, \mathbf{S})$  and  $f_{Y_{ik}}(y; \boldsymbol{\theta}, \mathbf{S})$  respectively denote the pdf of  $\Gamma_i$  and  $\mathbf{Y}_{i,k}(t)$  with measurement error-free covariates  $\mathbf{S}$ .

From the system description and Equation (4), it can be derived that  $\mathbf{Y}_{i,k}(t)$  with measurement error-free covariates  $\mathbf{S}$  follows a multivariate normal distribution with mean and covariance as follows:

$$\begin{aligned} \boldsymbol{\mu}_{Y_{ik}} &= \mu_0 \mathbf{I}_{J_{ik}} + \mu_a e^{-b/S_k} \mathbf{T}_{i,k} \\ \boldsymbol{\Sigma}_{i,k} &= e^{-2b/S_k} \sigma_a^2 \mathbf{T}_{i,k} \mathbf{T}_{i,k}^T + (\sigma_0^2 + \sigma_\varepsilon^2) \mathbf{I}_{J_{ik}} + \sigma_B^2 \begin{bmatrix} t_{i,1} & t_{i,1} & t_{i,1} & \cdots & t_{i,1} \\ t_{i,1} & t_{i,2} & t_{i,2} & \cdots & t_{i,2} \\ t_{i,1} & t_{i,2} & t_{i,3} & \cdots & t_{i,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{i,1} & t_{i,2} & t_{i,3} & \cdots & t_{i,N} \end{bmatrix} \end{aligned} \quad (19) \quad (20)$$

where  $\mathbf{I}_{J_{ik}}$  is an unit matrix with dimension  $J_{i,k}$ .

Therefore,  $Y(t)$  with measurement error-free covariates  $\mathbf{S}$  is distributed by:

$$f_{Y_{ik}}(\mathbf{y}; \boldsymbol{\theta}, \mathbf{S}) = \frac{1}{(2\pi)^{J_{i,k}/2} \sqrt{|\boldsymbol{\Sigma}_{i,k}|}} \exp \left[ -\frac{1}{2} (\mathbf{y} - \boldsymbol{\mu}_{Y_{ik}})^T \boldsymbol{\Sigma}_{i,k}^{-1} (\mathbf{y} - \boldsymbol{\mu}_{Y_{ik}}) \right] \quad (21)$$

As for  $g(\Gamma_i; \boldsymbol{\theta}, \mathbf{S})$ , since  $\Gamma_{i,j}$  ( $j=1, 2, \dots, K-1$ ) is the time point when the degradation performance firstly reaches the independently predetermined threshold level  $H_j$  ( $j=1, 2, \dots, K-1$ ),  $\Delta\Gamma_{i,1}, \Delta\Gamma_{i,2}, \dots, \Delta\Gamma_{i,K-1}$  are statistically dependent variables due to the characteristic of SSADT. And the randomness in initial degradation values of the products makes that  $\Delta\Gamma_{i,1}$  has a different distribution with  $\Delta\Gamma_{i,2}, \Delta\Gamma_{i,3}, \dots, \Delta\Gamma_{i,K-1}$ .

Based on Section 3 and Equation (17), it can be derived that the pdf of  $\Delta\Gamma_{i,1}$  is:

$$f_{\Delta\Gamma_i}(t) = A \sqrt{\frac{E}{\sigma_0^2 + E}} \left( \frac{D\sigma_0^2 + (H_1 - \mu_0)E}{\sigma_0^2 + E} B - C \right) e^{-\frac{(H_1 - \mu_0 - D)^2}{2(\sigma_0^2 + E)}} \quad (22)$$

$$\text{where } A = \frac{1}{\sqrt{2\pi t^2 \left[ \sigma_a^2 e^{-2b/S_1} (e^{\beta t} - 1)^2 + \sigma_B^2 t \right]}}, \quad B = 1 - \frac{\sigma_a^2 e^{-b/S_1} (e^{\beta t} - 1)(e^{\beta t} - \beta t e^{\beta t} - 1)}{\sigma_a^2 e^{-2b/S_1} (e^{\beta t} - 1)^2 + \sigma_B^2 t}, \quad C = \frac{\mu_a e^{-b/S_1} \sigma_B^2 t (e^{\beta t} - \beta t e^{\beta t} - 1)}{\sigma_a^2 e^{-2b/S_1} (e^{\beta t} - 1)^2 + \sigma_B^2 t},$$

$$D = \mu_a e^{-b/S_1} (e^{\beta t} - 1) \quad \text{and} \quad E = \left[ \sigma_a^2 e^{-2b/S_1} (e^{\beta t} - 1)^2 + \sigma_B^2 t \right].$$

By ignoring the randomness in initial degradation values, i.e.,  $\mu_0 = H_{m-1}$ ,  $\sigma_0^2 = 0$ , the conditional pdf of  $\Delta\Gamma_{i,m}$  ( $m=2, 3, \dots, K-1$ ) can be obtained:

$$f_{\Delta\Gamma_m}(t | \Delta\Gamma_1 = \gamma_1, \dots, \Delta\Gamma_{m-1} = \gamma_{m-1})$$

$$= \frac{(H_m - H_{m-1}) - \left[ \sigma_a^2 e^{-2b/S_m} \left( e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right) (H_m - H_{m-1}) - \mu_a e^{-b/S_m} \sigma_B^2 \left( t + \sum_{j=1}^{m-1} \Delta\gamma_j \right) \right] \cdot Z}{\sqrt{2\pi t^2 \left[ \sigma_a^2 e^{-2b/S_m} \left( e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right)^2 + \sigma_B^2 \left( t + \sum_{j=1}^{m-1} \Delta\gamma_j \right) \right]}} \quad (23)$$

$$\times \exp \left\{ - \frac{\left( H_m - H_{m-1} - \mu_a e^{-b/S_m} \left( e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right) \right)^2}{2 \left[ \sigma_a^2 e^{-2b/S_m} \left( e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right)^2 + \sigma_B^2 \left( t + \sum_{j=1}^{m-1} \Delta\gamma_j \right) \right]} \right\}$$

$$\text{where } Z = \frac{\left[ e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - \beta \left( t + \sum_{j=1}^{m-1} \Delta\gamma_j \right) e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right]}{\sigma_a^2 e^{-2b/S_m} \left( e^{\beta(t + \sum_{j=1}^{m-1} \Delta\gamma_j)} - 1 \right)^2 + \sigma_B^2 \left( t + \sum_{j=1}^{m-1} \Delta\gamma_j \right)}.$$

The Jacobean of the transformation from  $\Gamma_i(\boldsymbol{\theta}, \mathbf{S})$  to  $\Delta\Gamma_i(\boldsymbol{\theta}, \mathbf{S})$  is:

$$J = \begin{vmatrix} \frac{\partial \Delta\Gamma_{i,1}}{\partial \Gamma_{i,1}} & \frac{\partial \Delta\Gamma_{i,1}}{\partial \Gamma_{i,2}} & \dots & \frac{\partial \Delta\Gamma_{i,1}}{\partial \Gamma_{i,K-1}} \\ \frac{\partial \Delta\Gamma_{i,2}}{\partial \Gamma_{i,1}} & \frac{\partial \Delta\Gamma_{i,2}}{\partial \Gamma_{i,2}} & \dots & \frac{\partial \Delta\Gamma_{i,2}}{\partial \Gamma_{i,K-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Delta\Gamma_{i,K-1}}{\partial \Gamma_{i,1}} & \frac{\partial \Delta\Gamma_{i,K-1}}{\partial \Gamma_{i,2}} & \dots & \frac{\partial \Delta\Gamma_{i,K-1}}{\partial \Gamma_{i,K-1}} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{vmatrix}_{(K-1) \times (K-1)} \quad (24)$$

Therefore  $|J|=1$  and

$$g_{\Gamma_i}(\boldsymbol{\gamma}_i; \boldsymbol{\theta}, \mathbf{S}) = g_{\Delta\Gamma_i}(\Delta\boldsymbol{\gamma}_i; \boldsymbol{\theta}, \mathbf{S}) \cdot |J| = f_{\Delta\Gamma_1}(\Delta\boldsymbol{\gamma}_1; \boldsymbol{\theta}, \mathbf{S}) \cdot \prod_{m=2}^{K-1} f_{\Delta\Gamma_m}(\Delta\boldsymbol{\gamma}_m; \boldsymbol{\theta}, \mathbf{S} / \Delta\Gamma_1, \dots, \Delta\Gamma_{m-1}) \quad (25)$$

## 4.2 GA-SIMEX method considering measurement error in covariates

However, the existence of measurement errors in covariates makes the theoretical covariates  $S$  unknown. Therefore, the degradation measurements are actually taken under measurement error-prone covariates  $S' = S + \phi$ , where  $\phi \sim N(0, \mu_\phi^2)$ , and the likelihood function (18) should be updated by introducing the covariate measurement error  $\phi$ , which makes it rather difficult to derive an analytical form of the likelihood function. Confronted with the estimation problem of parametric covariate measurement error models, Cook and Stefanski [33] proposed a SIMEX (simulation and extrapolation) method to adjust for the bias when introducing the covariate measurement error, and its superior performance in nonlinear models has been proven by [34]. Based on this, a modified GA-SIMEX method is proposed in this paper to deal with parameters estimation of nonlinear step-stress accelerated degradation models considering three sources of variability, and its algorithm consists of the following three steps:

### (1) Simulation step

Given an integer  $B$  and a sequence  $\boldsymbol{\Lambda} = \{v_1, v_2, \dots, v_C\}$ , where  $v_1 = 0$ ,  $v_C$  is a given positive number, and

$v_2, \dots, v_{C-1}$  are random numbers generated from  $U(0, v_C)$ . For  $k=1, 2, \dots, K$  and  $b=1, 2, \dots, B$ ,  $K \times B$  samples are generated from normal distribution  $N(0, 1)$  and denoted by  $u_{kb}$ . Therefore setting  $S_k(b, c) = S_k + \sqrt{v_c} \sigma_\phi u_{kb}$ ,  $c=1, 2, \dots, C$  leads to the simulations of  $k$ th stress level of covariates  $S$ .

### (2) GA-based estimation step

Based on the actually measured degradation data vector  $\{\Delta Y_i, \Delta t_i, \tau_i\}$  and the simulated covariates  $S(b, c) = \{S_1(b, c), S_2(b, c), \dots, S_K(b, c)\}$ , the likelihood function (18) turns to be:

$$L[\theta; S(b, v)] = \prod_{i=1}^N \left\{ g_{r_i}[\gamma_i; \theta, S(b, c)] \prod_{k=1}^K f_{\Delta Y_{ik}}[\Delta y_{i,k}; \theta, S(b, c)] \right\} \\ = \prod_{i=1}^N \left\{ \begin{aligned} & f_{\Delta r_i}[\Delta \gamma_{i,1}; \theta, S(b, c)] \cdot \prod_{m=2}^{K-1} f_{\Delta r_m}[\Delta \gamma_{i,m}; \theta, S(b, c)] \\ & \cdot \prod_{k=1}^K \frac{1}{(2\pi)^{J_{i,k}/2} \sqrt{|\Sigma_{i,k}|}} \exp\left[-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_{Y_{i,k}})^T \boldsymbol{\Sigma}_{i,k}^{-1} (\mathbf{y} - \boldsymbol{\mu}_{Y_{i,k}})\right] \end{aligned} \right\} \quad (26)$$

Since unknown parameters have to be estimated, and it is quite complicated to derive analytical estimators from a computational viewpoint, the GA method is employed here to obtain the maximum values of the likelihood function, thus the corresponding MLEs  $\hat{\theta}(b, c)$  are obtained.

### (3) Extrapolation step

For  $b=1, 2, \dots, B$ , define  $\hat{\theta}(c) = \frac{1}{B} \sum_{b=1}^B \hat{\theta}(b, c)$ . Then for sequences  $\{(v_c, \hat{\theta}(c)) : v_c \in \Lambda\}$ , regression analysis is conducted separately for each component of  $\hat{\theta}(c)$ . And the final parameter estimations  $\hat{\theta}$  can be obtained by extrapolating the regression models to  $v = -1$ .

## 5 Numerical experiments and performance evaluation

In this section, the proposed SSADT model and the corresponding parameter estimation method are illustrated by an example of fatigue-crack length growth. A structure made of 2017-T4 aluminum alloy is declared to be failed when its fatigue-crack length exceeds a predefined threshold level. And Si et.al [31] modelled the degradation of the fatigue-crack length by a nonlinear Wiener process. To exemplify the effectiveness of the SSADT model proposed in this article, simulated experiment data sets are generated by extending the constant stress degradation test in [31] to a corresponding SSADT.

### 5.1 Simulation of data

According to the model assumptions in Section 2, the nonlinear SSADT model considering three source of variability are as follows:

$$\begin{cases} S' = S + \phi; \\ Y(t) = X(t; S') + \varepsilon = X(0) + ae^{-b/S'}(e^{bt} - 1) + \sigma_B B(t) + \varepsilon \end{cases} \quad (27)$$

Experiment data sets with sample size  $N=10$  are generated based on the model parameters listed in Table I. Based on the initial fatigue-crack lengths of 4 test specimens in [31], parameters  $\mu_0$  and  $\sigma_0$  are estimated under

normal assumption. Parameters  $\mu_a$ ,  $\sigma_a$ ,  $\sigma_B$ ,  $b$  and  $\beta$  are obtained by referring to the estimation results in Table V of [31]. Other parameters concerning the measurement errors in degradation performance and covariates, i.e.,  $\sigma_\varepsilon$  and  $\sigma_\phi$ , are assumed for the purpose of illustration. Considering the degradation test in [31] are conducted under a stress level of 200 MPa, which can be viewed as the normal stress level  $S_0$ , three higher levels are selected here to be the accelerated stress levels:  $S_1 = 240, S_2 = 260, S_3 = 280$ .

For each of the 10 samples, the degradation test is first under stress level  $S_1$ , and the fatigue crack length is measured with intervals  $\Delta M = 3000$  cycles. When the degradation measurements firstly reach the threshold level  $H_1 = 2$ , the stress level is elevated to  $S_2$ , and the time point of elevating the accelerated stress level from  $S_2$  to  $S_3$  is the FHT of  $H_2 = 6$ . The SSADT is terminated at a fixed time  $T = 2.7 \times 10^5$  revolutions. The simulated fatigue-crack growth data are shown graphically in Figure 3. It can be seen that the degradation path of each unit firstly crosses the predetermined threshold levels of  $H_1 = 2$  and  $H_2 = 6$  at different times, therefore the time points of elevating stress levels for them vary slightly. According to the analysis in [31] and the plot of simulated data, it is clear that the degradation of fatigue crack length displays a nonlinear pattern, and an exponential function is appropriate to describe the nonlinearity. In addition, the initial degradation values  $X(0)$  for all 10 units are listed in Table II. And for test of normality, Shapiro-Wilk test [34] in Table III indicates that under confidence level 0.05, it can be assumed that  $X(0)$  follows a normal distribution.

Table I Parameter values of the SSADT model

Parameter	$\mu_0$	$\sigma_0$	$b$	$\beta$	$\mu_a$	$\sigma_a$	$\sigma_B$	$\sigma_\varepsilon$	$\sigma_\phi$
Value	0.27	0.1553	7	4.44	1.50E-4	3.53E-5	7.79E-3	0.1	1

Table II Initial degradation values  $X(0)$  of the simulated 10 units

0.2245	0.3800	0.2311	0.3130	0.1697
0.4349	0.3381	0.3684	0.3025	0.2080

Table III Test of normality for the initial degradation value

sample size	statistic	p value	test result
10	0.96054	0.79197	Can't reject normality

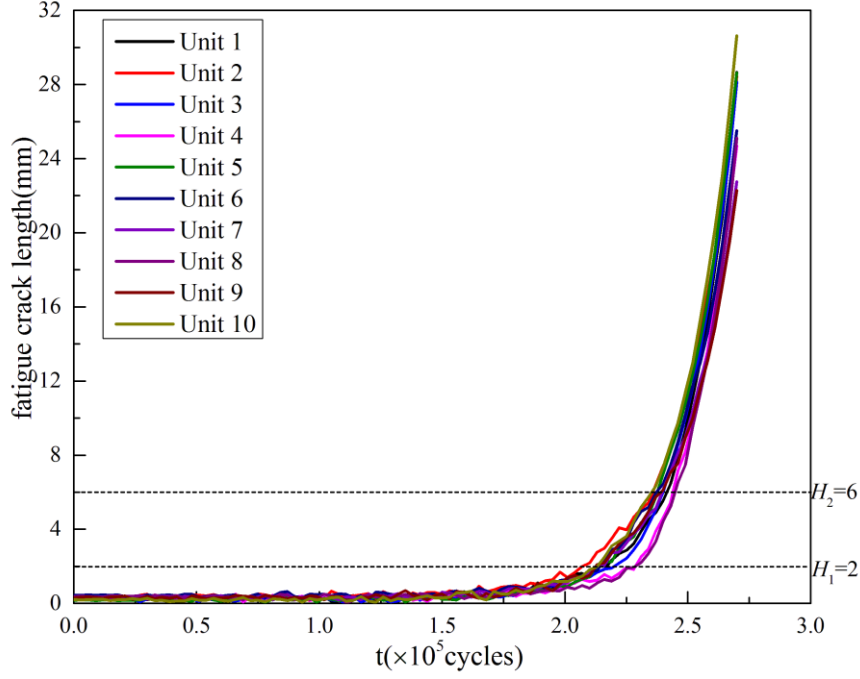


Figure 3 Degradation measurements of fatigue-crack growth data

## 5.2 Reliability assessment

Based on the simulated data, the modified GA-SIMEX method is conducted to estimate the unknown model parameters  $\theta = \{b, \beta, \mu_a, \sigma_a^2, \sigma_B^2, \sigma_\varepsilon^2\}$ , and the estimation results, along with the true values of model parameter, are listed in Table II.

Table IV Comparison between true values and estimation results of model parameters

Parameters	$\mu_0$	$\sigma_0$	$b$	$\beta$	$\mu_a$	$\sigma_a$	$\sigma_B$	$\sigma_\varepsilon$
True value	0.27	0.1553	7	4.44	1.50E-4	3.53E-5	7.79E-3	0.1
Estimation	0.2970	0.0861	6.46	4.46	1.67E-4	3.32E-5	5.29E-3	0.0828

Since the 2017-T4 aluminum alloy fails once the fatigue crack length grows beyond 6mm, the pdf of its lifetime  $f(t)$  under normal stress level, i.e.,  $S_0 = 200Mpa$ , can be calculated based on the parameter estimation results and the derivation in Section 3. Furthermore, the reliability curve can also be obtained through formula  $R(t_0) = 1 - \int_0^{t_0} f(t) dt$ . The assessment results are separately plotted in Figure 4.

It is shown that the reliability curves based on true parameter values and estimated parameter values are following the same pattern, i.e., the reliability of the 2017-T4 aluminum alloy is nearly constant for a period at first, which can be viewed as fatigue crack initiation time [35], and when the fatigue crack begins to grow, its reliability curve presents a sharp decline. Furthermore, the mean relative deviation between the two reliability curves is  $5.6350e-04$ , which is relatively low and indicates the effectiveness of the proposed model as well as the parameter estimation method.



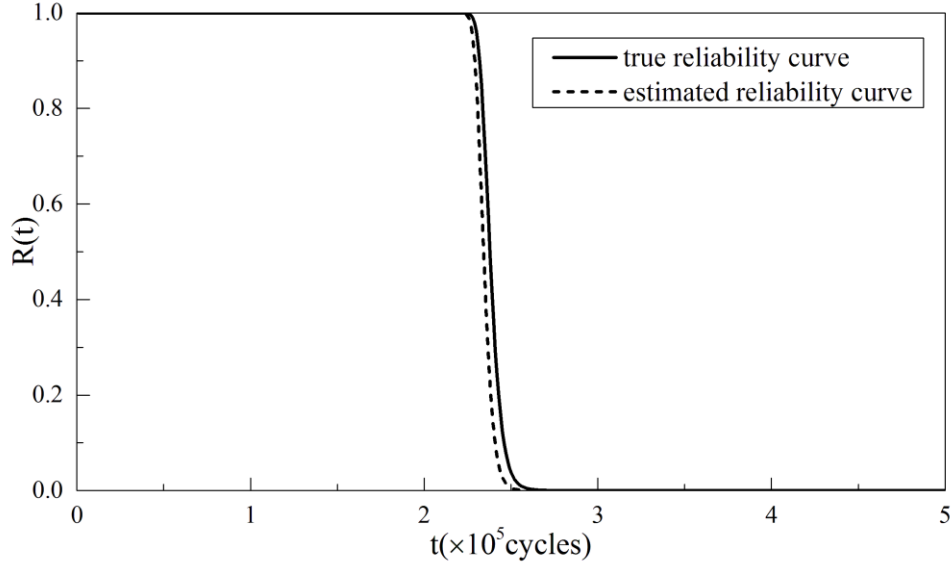


Figure 4 Reliability curves under normal stress level with the fatigue crack length data

### 5.3 Sensitivity analysis

In this section, a sensitivity analysis is performed to investigate the effect of stress-elevating threshold levels, sample size and measurement interval on reliability assessment results, with the purpose of providing some guidance to design an SSADT plan. Since all reliability curves present a nearly constant tendency before  $2 \times 10^5$  revolutions, and tend to 0 after  $3 \times 10^5$  revolutions, the horizontal axis in the sensitivity analysis is set to be  $2 \sim 3 (\times 10^5 \text{ revolutions})$ . Furthermore, a performance indicator of mean relative deviation (MRD) between the

estimated reliability curve and the true reliability curve, i.e.,  $\Delta = \int_2^3 \left| \frac{R_i(t) - R_0(t)}{R_0(t)} \right| dt$ , is utilized to compare the estimation accuracy under different stress-elevating threshold levels.

First, 9 cases in all are studied under different stress-elevating threshold levels, i.e.,  $H_1=2, 2.5, 3$  and  $H_2=6, 6.5, 7$  respectively. Reliability curves for the 9 considered cases are plotted in Figure 5, along with the reliability curve under true model parameter values. Table III lists the values of  $\Delta$  under different  $H_1$  and  $H_2$ . From Figure 5 and Table III, it can be shown that under the proposed SSADT plan, i.e., with random time-points of elevating stress levels, the reliability curves with different  $H_1$  or  $H_2$  are all close to that based on the true parameter values, and the mean relative deviations are all quite small. Therefore it is indicated that under the constraint of total test duration, the accuracy of reliability assessment is not sensitive to the threshold levels of elevating stress levels, which shows that the proposed SSADT plan is superior to the traditional SSADT with pre-determined time duration under each stress level, since for nonlinear degradation processes with limited prior information, it is quite a difficult task to determine in advance appropriate time duration for observing obvious degradation tendency. And by elevating stress levels when the degradation path firstly reaches pre-specified values, tests data under all stress levels will be sufficient to obtain reliability assessment with high accuracy.

Table V MRDs between true reliability curve and estimated reliability curves with different  $H_1, H_2$

Order	$H_1$	$H_2$	Mean relative deviation
1	2	6	5.6530e-04
2	2.5	6	3.6472e-04
3	3	6	1.5823e-04
4	2	6.5	1.8932e-04
5	2.5	6.5	7.6439e-04
6	3	6.5	4.1245e-04
7	2	7	9.6219e-05
8	2.5	7	4.2246e-04
9	3	7	3.9420e-04

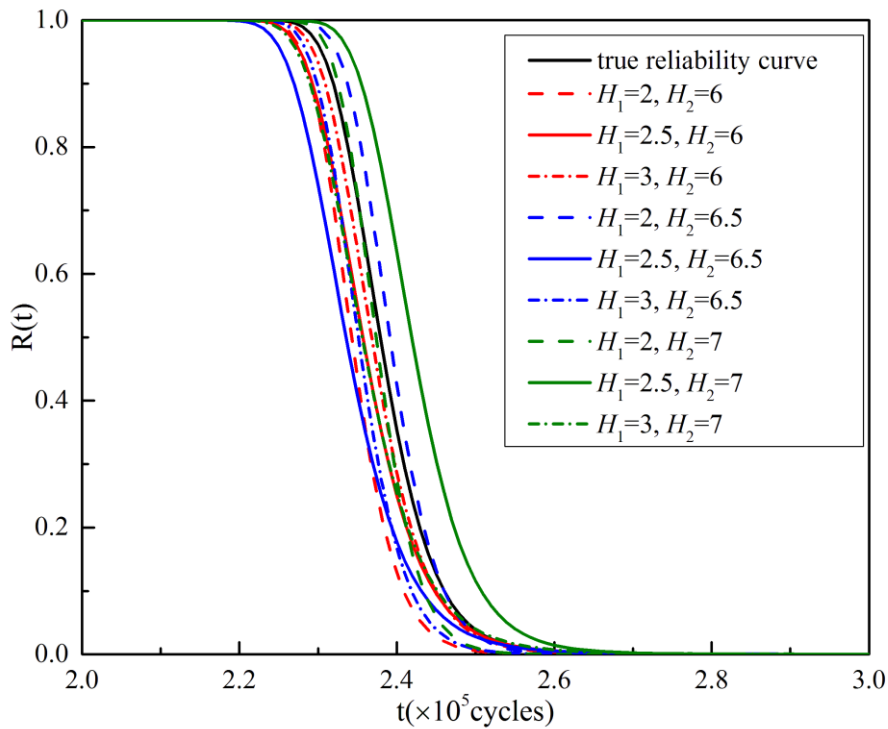


Figure 5 Sensitivity analysis of stress-elevating threshold levels

In addition, we have also investigated the effect of the sample size and the measurement interval. Three cases with different sample sizes, i.e.,  $N=10,20,30$ , and three cases with different measurement intervals, i.e.,  $\Delta M = 1500, 3000, 4500$ , are studied respectively. In each case, the unknown parameters are estimated based on simulated data, and reliability curves are plotted in Figures 6-7, along with the reliability curves under the true model parameter values. Tables VI and VII list the values of  $\Delta$  under different  $N$  and  $\Delta M$ . These figures and tables all indicate that with an increased sample size or a decreased measurement interval, more accurate parameter estimation results and reliability assessment results can be obtained.

Table VI MRDs between true reliability curve and estimated reliability curves with different  $N$

Order	$N$	Mean relative deviation
1	10	5.6530E-04
2	20	3.9798E-04
3	30	2.8046E-04

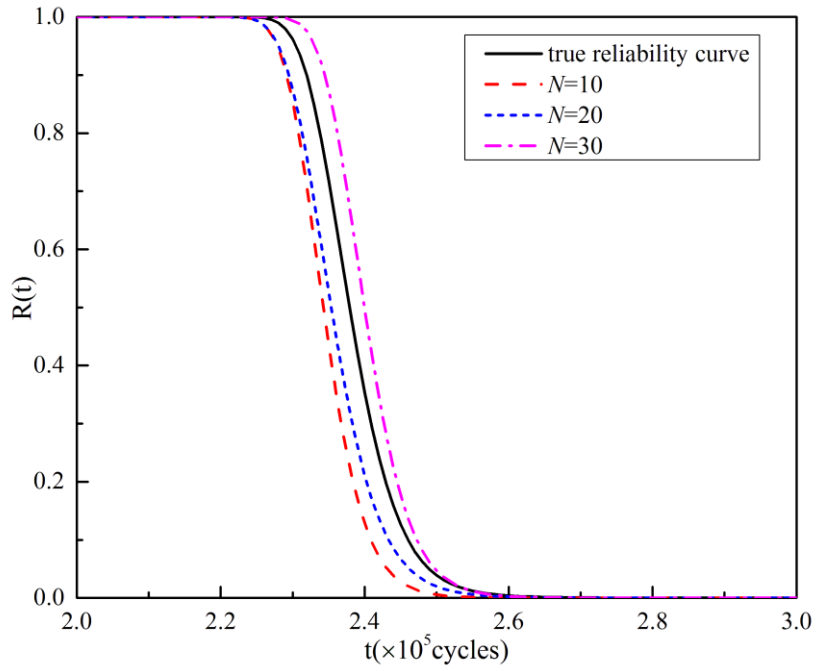


Figure 6 Sensitivity analysis of sample size

Table VII MRDs between true reliability curve and estimated reliability curves with different  $\Delta M$

Order	$\Delta M$	Mean relative deviation
1	0.015	9.3769E-04
2	0.030	5.6530E-04
3	0.045	3.3867E-04

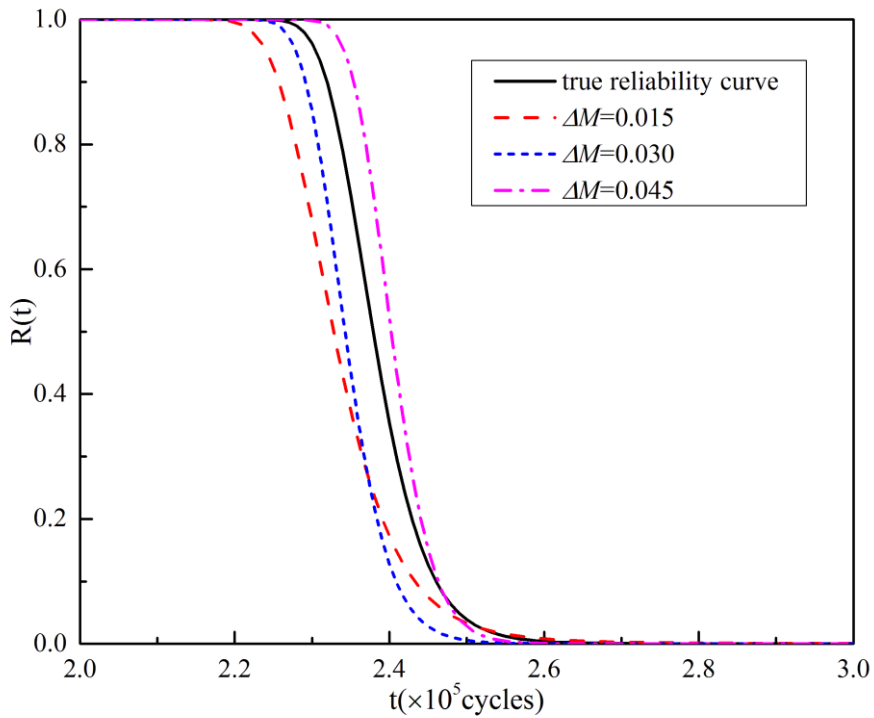


Figure 7 Sensitivity analysis of observation interval

## 6 Conclusion

Confronted with the variability in degradation processes in SSADTs, this paper developed a novel SSADT model simultaneously considering nonlinearity and three sources of variability. A nonlinear Wiener process is adopted to deal with the nonlinearity and temporal variability. Variability in degradation rates, initial degradation values as well as time-points of elevating stress levels are modelled as the heterogeneity among test units. The inevitable measurement errors in both degradation performance and covariates (test stresses) are also considered. Statistical inference including reliability assessment under normal condition and a modified GA-SIMEX parameter estimation method is developed based on the proposed SSADT model. A case study of a simulated SSADT for fatigue crack length growth is conducted to exemplify the proposed model, and the sensitivity analysis indicates that in a SSADT with random time-points of elevating stress levels, the reliability assessment accuracy is not sensitive to the threshold levels of elevating stress levels, which shows its superiority over SSADT with pre-determined time duration under each test level, because in the former way, it is easy to determine robust and appropriate threshold levels of elevating stress levels, and to observe obvious degradation tendency for all stress levels.

For further investigation of this work, an analytical reliability assessment result can be obtained for general distribution forms of initial degradation value or the random effect variable. In addition, more sources of variability can be considered in the SSADT model, such as modelling uncertainty, which can be dealt with by means of model selection, hypothesis testing and mis-specification analysis. Another interesting direction is to design an optimal SSADT plan based on the newly proposed model, with the objective of minimum total experimental budget or maximum estimation accuracy.

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## References

- [1] Nelson W. Analysis of Performance-Degradation Data from Accelerated Tests. *IEEE Trans Reliab* 1981;R-30:149–55. doi:10.1109/TR.1981.5221010.
- [2] Carey MB, Koenig RH. Reliability assessment based on accelerated degradation: a case study. *IEEE Trans Reliab* 1991;40:499–506. doi:10.1109/24.106763.
- [3] Sheng-Tsaing Tseng, Zhi-Chih Wen. Step-Stress Accelerated Degradation Analysis For Highly Reliable Products. *J Qual Technol* 2000;32:209.
- [4] Huang J, Golubović DS, Koh S, Yang D, Li X, Fan X, et al. Lumen degradation modeling of white-light LEDs in step stress accelerated degradation test. *Reliab Eng Syst Saf* 2016;154:152–9. doi:10.1016/j.res.2016.06.002.
- [5] Cai M, Yang D, Tian K, Zhang P, Chen X, Liu L, et al. Step-stress accelerated testing of high-power LED lamps based on subsystem isolation method. *Microelectron Reliab* 2015;55:1784–9. doi:10.1016/j.microrel.2015.06.147.
- [6] Chen W, Liu J, Gao L, Pan J, Lu X. Step-stress accelerated degradation test modeling and statistical analysis methods. *Chin J Mech Eng* 2013;26:1154–9.
- [7] Yao J, Xu M, Zhong W. Research of Step-down Stress Accelerated Degradation Data Assessment Method of a Certain Type of Missile Tank. *Chin J Aeronaut* 2012;25:917–24. doi:10.1016/S1000-9361(11)60462-7.
- [8] Chen W-W, Ma X-H, Hou B, Zhu J-J, Zhang J-C, Hao Y. The degradation mechanism of an AlGaIn/GaN high electron mobility transistor under step-stress. *Chin Phys B* 2013;22:107303. doi:10.1088/1674-1056/22/10/107303.
- [9] Si X-S, Wang W, Hu C-H, Zhou D-H. Estimating Remaining Useful Life With Three-Source Variability in Degradation Modeling. *IEEE Trans Reliab* 2014;63:167–90. doi:10.1109/TR.2014.2299151.
- [10] Tang L, Kacprzynski GJ, Goebel K, Vachtsevanos G. Methodologies for uncertainty management in

prognostics. *Aerosp. Conf.*, 2009, p. 1–12.

[11] Baraldi P, Mangili F, Zio E. Investigation of uncertainty treatment capability of model-based and data-driven prognostic methods using simulated data. *Reliab Eng Syst Saf* 2013;112:94–108. doi:10.1016/j.ress.2012.12.004.

[12] Pandey MD, Yuan X-X, van Noortwijk JM. The influence of temporal uncertainty of deterioration on life-cycle management of structures. *Struct Infrastruct Eng* 2009;5:145–56. doi:10.1080/15732470601012154.

[13] Liao C-M, Tseng S-T. Optimal Design for Step-Stress Accelerated Degradation Tests. *IEEE Trans Reliab* 2006;55:59–66. doi:10.1109/TR.2005.863811.

[14] Sheng-Tsaing Tseng, Balakrishnan N, Chih-Chun Tsai. Optimal Step-Stress Accelerated Degradation Test Plan for Gamma Degradation Processes. *IEEE Trans Reliab* 2009;58:611–8. doi:10.1109/TR.2009.2033734.

[15] Wang H, Wang G, Duan F. Planning of step-stress accelerated degradation test based on the inverse Gaussian process. *Reliab Eng Syst Saf* 2016;154:97–105. doi:10.1016/j.ress.2016.05.018.

[16] Tang S, Guo X, Yu C, Xue H, Zhou Z. Accelerated Degradation Tests Modeling Based on the Nonlinear Wiener Process with Random Effects. *Math Probl Eng* 2014;2014:1–11. doi:10.1155/2014/560726.

[17] Xiao X, Ye Z. Optimal Design for Destructive Degradation Tests With Random Initial Degradation Values Using the Wiener Process. *IEEE Trans Reliab* 2016;65:1327–42. doi:10.1109/TR.2016.2575442.

[18] Pan Z, Balakrishnan\* N. Multiple-Steps Step-Stress Accelerated Degradation Modeling Based on Wiener and Gamma Processes. *Commun Stat - Simul Comput* 2010;39:1384–402. doi:10.1080/03610918.2010.496060.

[19] Amini M, Shemehsavar S, Pan Z. Optimal Design for Step-Stress Accelerated Test with Random Discrete Stress Elevating Times Based on Gamma Degradation Process: Optimal Design for Step-Stress Accelerated Test with Random Discrete Stress Elevating Times Based on Gamma Degradation Process. *Qual Reliab Eng Int* 2016;32:2391–402. doi:10.1002/qre.1943.

[20] He W, Yi GY, Xiong J. Accelerated failure time models with covariates subject to measurement error. *Stat Med* 2007;26:4817–32. doi:10.1002/sim.2892.

[21] Zhang SY, Zhang ZZ. Analysis of accelerated degradation test under constant stress with errors for longitudinal data. *Intell. Secur. Inform. ISI 2011 IEEE Int. Conf. On*, 2011, p. 320–4.

[22] Lu D, Pandey MD, Xie W-C. An efficient method for the estimation of parameters of stochastic gamma process from noisy degradation measurements. *Proc Inst Mech Eng Part O J Risk Reliab* 2013;227:425–33. doi:10.1177/1748006X13477008.

[23] Ye Z-S, Xie M. Stochastic modelling and analysis of degradation for highly reliable products: Z.-S. YE AND M. XIE. *Appl Stoch Models Bus Ind* 2015;31:16–32. doi:10.1002/asmb.2063.

[24] Zhai Q, Ye Z-S, Yang J, Zhao Y. Measurement errors in degradation-based burn-in. *Reliab Eng Syst Saf* 2016;150:126–35. doi:10.1016/j.ress.2016.01.015.

[25] Ye Z-S, Wang Y, Tsui K-L, Pecht M. Degradation Data Analysis Using Wiener Processes With Measurement Errors. *IEEE Trans Reliab* 2013;62:772–80. doi:10.1109/TR.2013.2284733.

[26] Lei Y, Li N, Jia F, Lin J, Xing S. A nonlinear degradation model based method for remaining useful life prediction of rolling element bearings. *Progn. Syst. Health Manag. Conf.*, 2016, p. 1–8.

[27] Huang Z, Xu Z, Wang W, Sun Y. Remaining Useful Life Prediction for a Nonlinear Heterogeneous Wiener Process Model With an Adaptive Drift. *IEEE Trans Reliab* 2015;64:687–700. doi:10.1109/TR.2015.2403433.

[28] Zheng J-F, Si X-S, Hu C-H, Zhang Z-X, Jiang W. A Nonlinear Prognostic Model for Degrading Systems With Three-Source Variability. *IEEE Trans Reliab* 2016;65:736–50. doi:10.1109/TR.2015.2513044.

[29] Singpurwalla ND. On competing risk and degradation processes. *Inst. Math. Stat. Lect. Notes - Monogr. Ser.*, Beachwood, Ohio, USA: Institute of Mathematical Statistics; 2006, p. 229–40. doi:10.1214/074921706000000473.

[30] Cox DR, Miller HD. *The theory of stochastic processes*. John Wiley & Sons, Inc., New York; 1965.

[31] Si X-S, Wang W, Hu C-H, Zhou D-H, Pecht MG. Remaining Useful Life Estimation Based on a Nonlinear Diffusion Degradation Process. *IEEE Trans Reliab* 2012;61:50–67. doi:10.1109/TR.2011.2182221.

[32] Cook JR, Stefanski LA. Simulation-Extrapolation Estimation in Parametric Measurement Error Models. *J Am Stat Assoc* 1994;89:1314–28. doi:10.1080/01621459.1994.10476871.

[33] Carroll RJ, Küchenhoff H, Lombard F, Stefanski LA. Asymptotics for the SIMEX Estimator in Nonlinear Measurement Error Models. *J Am Stat Assoc* 1996;91:242–50. doi:10.1080/01621459.1996.10476682.

[34] Yap BW, Sim CH. Comparisons of various types of normality tests. *J Stat Comput Simul* 2011;81:2141–55. doi:10.1080/00949655.2010.520163.

[35] Paroissin C. Inference for the Wiener Process With Random Initiation Time. *IEEE Trans Reliab* 2016;65:147–57. doi:10.1109/TR.2015.2456056.