On the steady solution of linear advection problems in steady recirculating flows

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Numerical modelling of non-Newtonian flows typically involves the coupling between equations of motion characterized by an elliptic behaviour, and the fluid constitutive equation, which is an advection equation linked to the fluid history. In this paper we prove that linear steady advection problems in steady recirculating flows have only one solution when the kinematics differs from a rigid motion. We also give a numerical procedure to determine this steady solution. We will describe this numerical procedure for two linear models the first will be the SFRT flow model and the second will be a simplified linear formulation of the Pom–Pom viscoelastic model.

Keywords: Linear advection; Steady recirculating flows; Non-Newtonian fluid mechanics

1. Introduction

Numerical modelling of non-Newtonian flows usually involves the coupling between equations of motion, which define an elliptic problem, and the fluid constitutive equation, which introduces an advection problem related with the fluid history. For example, in a short fiber-reinforced thermoplastic (SFRT) model, the extra-stress tensor depends on the fiber orientation whose evolution is given by an advection equation. In some viscoelastic models, the extra-stress tensor evolution is governed by an advection equation. For example, in the recent Pom–Pom viscoelastic model, the extra-stress tensor depends on both the molecular orientation and the molecular extension, whose evolutions are also given by two advection equations. Thus, generally, such a model is defined from the equations of motion, which define an elliptic problem, and from the extra-stress tensor evolution, which defines a hyperbolic problem.

Usually the solution of the coupled model is found by an uncoupled fixed-point strategy. In this technique, the resolving process starts with an arbitrarily chosen extra-stress tensorial field from which the

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velocity and pressure fields are obtained by solving the equations of motion. Then, as we know the kinematics of the flow, the associated extra-stress tensorial field is obtained from integration of its governing equation. The process must be continued until convergence.

The numerical discretisation of the equations of motion does not introduce great difficulties. However, the discretisation of the advection equation is more complex. It can be carried out either by integrating its Lagrangian description by means of the method of characteristics [1–6], or by using the Eulerian discretisation of its variational formulation: streamline upwind (SU) or streamline upwind Petrov–Galerkin (SUPG) finite elements, [7–11], discontinuous finite elements [12] or discontinuous finite volumes [11]. However, for the orientation equations as encountered in SFRT or Pom–Pom models among others, the interpolation of the orientation tensors (which is required if standard Eulerian discretisation techniques are used) introduces non-physical orientation effects [6]. Many different general differential viscoelastic models exist, some of them involving advection problems in the constitutive equation. A great number of references about the discretisation of such models can be found, for example, in [13,14].

Such numerical modellings are made in order to simulate real industrial processes and to predict resulting mechanical properties of conformed pieces afterwards. Many of the experimental and industrial flows show recirculating areas or recirculate themselves. As an example, in a lid-driven cavity problem, the flow recirculates under the influence of a moving plate. In a contraction or in an expansion flow (as encountered, for example, in extrusion processes), various recirculating areas are observed [15–17]. Actually many rheometric devices involve this type of flows. Most of these phenomena are associated with a steady state of the flow. This steady state of the flow introduces some additional difficulties in the numerical simulation. Actually, the advection equation is supposed to have a steady solution in these steady recirculating flows but neither boundary conditions nor initial conditions are known in such flows. General steady solution of the linear advection equations in non-Newtonian steady recirculating flows have not been found so far.

In this paper, we prove that linear steady advection problems in steady recirculating flows have only one solution when the kinematics differs from a rigid motion. We also give a numerical procedure to determine this steady solution, based on the fact that steady solution must be periodic along the closed trajectories of the flow. This exact solution may be used in the context of steady simulations giving the steady solution in the steady recirculating parts of the flow. It also may be used to check the accuracy of other discretisation techniques, which are solving the evolution model and for which initial conditions are given in the fluid domain.

However, even if this result concerns only linear advection equations, it will be very useful to solve non-linear advection equations. In this last case the solution is searched by an iterative algorithm, which at each iteration solves a linear advection problem. In this way, the result concerning the existence and unicity of solution for linear advection equations in steady recirculating flows and the numerical procedure to obtain this solution (given in Section 3) allow us to develop accurate and efficient solvers for non-linear advection equations [18].

This work generalizes the theoretical result given in [6] which proves that, in a general recirculating flow, for fibers with a quasi-infinite aspect ratio, the only stable solution for the fiber orientation is the local alignment of the fibers with the flow.

In order to simplify the description of the proposed technique, we will consider two linear models: the first one concerns the short fiber-reinforced thermoplastics (SFRT) flow and the second one consists in a simplified linear formulation of the Pom–Pom viscoelastic model. Both models will be introduced in the next section. These are only two examples among many others. In general non-Newtonian flow

models involve constitutive laws described with linear and non-linear advection equations. Thus, if we consider, for example, the upper convected Maxwell viscoelastic model (UCM) then the extra-stress tensor evolution is given by the solution of an advection equation. In the case of steady recirculating flows we need to solve this advection equation without initial or boundary conditions. In this way the results and numerical procedures presented in this paper can be applied to treat many problems in non-Newtonian fluid mechanics.

From now, and without losing generality, we will restrict our discussion to the 2D case; however, all the results can be easily generalized to the 3D case.

The following notation will be used hereafter.

- We will denote a vector by an underline, e.g. $\underline{v}$, with components v_i.
- In the same way, the components of a second-order tensor $\underline{\underline{a}}$ will be expressed as a_{ij}.
- The unit vector associated with the mth coordinate axis is $\underline{e}_m$.
- With the symbol $\otimes$ we indicate the tensorial product, e.g. $\underline{u} \otimes \underline{v}$, with components $u_i v_j$.
- With Tr we denote the trace of a tensor.

2. Advection problems in polymer processing modelling

2.1. A first example: the short fiber-reinforced thermoplastic flow

The flow model associated with a short fibers reinforced thermoplastic model is defined by the following equations [19–25]:

- The momentum balance equations, without inertia and mass terms:

$$\mathrm{Div}\,\underline{\underline{\sigma}} = \underline{0}, \tag{1}$$

  where $\underline{\underline{\sigma}}$ is the stress tensor.
- The incompressibility condition:

$$\mathrm{Div}\,\underline{v} = 0, \tag{2}$$

  where $\underline{v}$ represents the velocity field.
- The constitutive equation, with a quadratic closure relation for the fourth-order orientation tensor and other simplifying assumptions [26], is

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\{\underline{\underline{D}} + N_p\,\mathrm{Tr}(\underline{\underline{s}}\,\underline{\underline{D}})\underline{\underline{s}}\}, \tag{3}$$

  where p denotes the pressure $\underline{\underline{I}}$ the unit tensor, μ the equivalent suspension viscosity (μ depends on the chosen model [23]), $\underline{\underline{D}}$ the strain rate tensor, N_p a scalar parameter depending on both the fiber concentration and the fiber aspect ratio and $\underline{\underline{s}}$ the second-order orientation tensor defined by

$$\underline{\underline{s}} = \oint \underline{\rho} \otimes \underline{\rho}\Psi(\underline{\rho})\,\mathrm{d}\underline{\rho}, \tag{4}$$

  where $\underline{\rho}$ is the unit vector associated with the direction of the fiber axis, and $\Psi(\underline{\rho})$ the orientation distribution function, satisfying

$$\oint \Psi(\underline{\rho})\,\mathrm{d}\underline{\rho} = 1. \tag{5}$$

If $\Psi(\underline{\rho}) = \delta(\underline{\rho} - \underline{\hat{\rho}})$ with $\delta()$ the Dirac delta function, all the orientation probability is concentrated in the direction defined by $\underline{\hat{\rho}}$, and the corresponding orientation tensor is $\underline{\underline{\hat{s}}} = \underline{\hat{\rho}} \otimes \underline{\hat{\rho}}$. In a planar case, the isotropic orientation state is defined by the uniform distribution function:

$$\Psi(\underline{\rho}) = \frac{1}{2\pi}. \tag{6}$$

The orientation tensor related to a planar isotropic orientation state is then

$$\underline{\underline{s}} = \frac{\underline{\underline{I}}}{2}. \tag{7}$$

From a physical point of view, we can consider that the eigenvalues of the second-order orientation tensor ($\underline{\underline{s}}$) represent the probability of finding the fiber in the direction of the corresponding eigenvectors.

- With a quadratic closure relation for the fourth-order orientation tensor, the orientation equation is expressed as

$$\frac{\mathrm{d}\underline{\underline{s}}}{\mathrm{d}t} = \underline{\underline{\Omega}}\,\underline{\underline{s}} - \underline{\underline{s}}\,\underline{\underline{\Omega}} + k(\underline{\underline{D}}\,\underline{\underline{s}} + \underline{\underline{s}}\,\underline{\underline{D}} - 2\mathrm{Tr}(\underline{\underline{s}}\,\underline{\underline{D}})\underline{\underline{s}}) + D_r\left(\underline{\underline{s}} - \frac{\underline{\underline{I}}}{2}\right), \tag{8}$$

where $\underline{\underline{s}}$ satisfies

$$\begin{cases} \underline{\underline{s}} = \underline{\underline{s}}^t, \\ \mathrm{Tr}(\underline{\underline{s}}) = 1. \end{cases} \tag{9}$$

$\underline{\underline{D}}$ and $\underline{\underline{\Omega}}$ are the symmetric and skew-symmetric components of Grad$v$, k is a constant that depends on the fiber aspect ratio r : $k = (r^2-1)/(r^2+1)$, with $r = L/\Phi$ ratio of the length L and the diameter Φ of the fiber, and D_r a diffusion coefficient.

The non-linear term in Eq. (8), $-2\mathrm{Tr}(\underline{\underline{s}}\,\underline{\underline{D}})\underline{\underline{s}}$, assures the unit trace of tensor $\underline{\underline{s}}$, i.e. $\mathrm{Tr}(\underline{\underline{s}}) = 1$.

Slender-body theories [24,25] consider fibers with quasi-infinite aspect ratio (i.e. $k = 1$), the viscosity of the solvent and a diffusion coefficient depending on the equivalent strain rate ($D_r = G_I\sqrt{2\mathrm{Tr}(\underline{\underline{D}}\,\underline{\underline{D}})}$) [26]. Other authors propose to introduce the apparent increase of the shear viscosity by the fiber rotation, considering $k \neq 1$ [23]. In this case it is assumed that the fibers have a spheroidal shape and the introduction of the third rheological parameter C_I seems no longer necessary ($C_I = 0$).

For both models, slender-body theories ($k = 1$, $C_I \neq 0$) and non-spherical particles suspensions ($k \neq 1$, $C_I = 0$), Eq. (8) keeps its non-linear character.

The orientation equation (Eq. (8)) can be written in the form

$$\frac{\mathrm{d}\underline{\underline{a}}}{\mathrm{d}t} = \underline{\underline{\Omega}}\,\underline{\underline{a}} - \underline{\underline{a}}\,\underline{\underline{\Omega}} + k(\underline{\underline{D}}\,\underline{\underline{a}} + \underline{\underline{a}}\,\underline{\underline{D}}) + D_r^*\left(\underline{\underline{a}} - \frac{\underline{\underline{I}}}{2}\right), \tag{10}$$

with $D_r = D_r^*/\mathrm{Tr}(\underline{\underline{a}})$. The unit trace orientation tensor $\underline{\underline{s}}$ becomes

$$\underline{\underline{s}} = \frac{\underline{\underline{a}}}{\mathrm{Tr}(\underline{\underline{a}})}. \tag{11}$$

Eq. (10) is also non-linear due to the dependence of the diffusion parameter on the orientation tensor

$$D_r^* = D_r\,\mathrm{Tr}(\underline{\underline{a}}).$$

However, when diffusion effects are neglected ($D_r = 0$), Eq. (10) becomes linear.

In the following, we may consider the general expression of the orientation evolution, Eq. (10), but we will limit our discussion to the linear case, because in Section 3 a result concerning the existence and unicity of solution for linear advection equations in steady recirculating flows will be given.

The flow model is defined in $\Omega = R^2$. On the boundary $\Gamma = \partial\Omega$, either the velocity or the traction is imposed.

$$\underline{v}(\underline{x} \in \Gamma_1) = \underline{v}_g, \tag{12}$$

and

$$\underline{\underline{\sigma}}\ \underline{n}(x_- \in \Gamma_2) = \underline{F}_d, \tag{13}$$

with $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$, and $\underline{n}(\underline{x})$ is the outward unit vector, normal to the boundary at the point $\underline{x}$. The inflow boundary will be denoted by Γ_-

$$\Gamma_- = \{\underline{x} \in \Gamma_1, \underline{v}^t(\underline{x})\underline{n}(\underline{x}) < 0\}. \tag{14}$$

As the orientation equation has a hyperbolic character, the integration of the orientation equation only requires an orientation boundary condition on the inflow boundary in a steady-state flow simulation

$$\underline{\underline{s}}(\underline{x} \in \Gamma_-) = \underline{\underline{s}}_0. \tag{15}$$

In this way, in order to obtain the steady solution of the flow problem, we need to solve Eqs. (1)–(3), (10) and (11) with the corresponding boundary conditions. If we use an uncoupled strategy, then the solution is reached by means of an iterative schema. For example, a fixed point strategy allows us to solve the kinematics of the flow, Eqs. (1)–(3), from the fiber orientation calculated at the previous iteration. Now, the fiber orientation can be actualized solving Eqs. (10) and (11) from the flow kinematics which we have just evaluated. Thus, at each iteration, the steady solution of the advection Eq. (10) must be obtained, and for steady recirculating flows the numerical procedure proposed in Section 3 can be applied with accuracy.

2.2. A second example: a simplified linear formulation of the Pom–Pom viscoelastic model

In this section we consider a simplified formulation of the Pom–Pom viscoelastic model proposed by McLeish and Larson [27]. In this case the constitutive equation is given by

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \beta\lambda^2\underline{\underline{s}}, \tag{16}$$

where λ represents the molecular extension, and $\underline{\underline{s}}$ the molecular orientation.

The evolution of the molecular extension field and the evolution of the molecular orientation field are given by two advection equations. In a first approximation, we consider these two equations as linear:

$$\frac{\partial \underline{\underline{a}}}{\partial t} + (\underline{v}\mathrm{Grad})\underline{\underline{a}} - \underline{\underline{\Omega}}\ \underline{\underline{a}} - \underline{\underline{a}}\ \underline{\underline{\Omega}} + \underline{\underline{D}}\ \underline{\underline{a}} + \underline{\underline{a}}\ \underline{\underline{D}} - \frac{1}{\tau_b}\left(\underline{\underline{a}} - \frac{\underline{\underline{I}}}{2}\right), \tag{17}$$

with

$$\underline{\underline{s}} = \frac{\underline{\underline{a}}}{\mathrm{Tr}(\underline{\underline{a}})} \tag{18}$$

and

$$\frac{\partial \lambda}{\partial t} + (\underline{v}\mathrm{Grad})\lambda = \lambda \mathrm{Tr}(\underline{\underline{s}}\ \underline{\underline{D}}) - \frac{1}{\tau_s}(\lambda - 1), \tag{19}$$

where τ_b and τ_s are two relaxation times that we will consider constants.

We can notice that the advection problem governing molecular orientation is very close to the one governing inextensible short fiber orientation. The model defined by Eqs. (17) and (18) may be rewritten in the equivalent form:

$$\frac{\mathrm{d}\underline{\underline{s}}}{\mathrm{d}t} = \underline{\underline{\Omega}}\ \underline{\underline{s}} - \underline{\underline{s}}\ \underline{\underline{\Omega}} + \underline{\underline{D}}\ \underline{\underline{s}} + \underline{\underline{s}}\ \underline{\underline{D}} - 2\mathrm{Tr}(\underline{\underline{s}}\ \underline{\underline{D}})\underline{\underline{s}} - \frac{1}{\tau_b^*}\left(\underline{\underline{s}} - \frac{\underline{\underline{I}}}{2}\right), \tag{20}$$

where the relation between the two relaxation times depends on the molecular extension ($\tau_b^* = \tau_b \mathrm{Tr}(\underline{\underline{a}}) = \tau_b \lambda^2$) [28]. This fact introduces a first non-linearity. Moreover, in order to impose the unit trace of tensor $\underline{\underline{s}}$ another non-linear term $-2\,\mathrm{Tr}(\underline{\underline{s}}\ \underline{\underline{D}})\underline{\underline{s}}$ is introduced. We focus our attention on the first formulation (Eqs. (17) and (18)) where the advection problem (Eq. (17)) remains linear if the corresponding relaxation time is considered as constant.

3. Steady solution of linear advection equations

3.1. Introduction

Eqs. (10) or (17) can be written taking into account the linearity and symmetry of the orientation tensors in the Lagrangian vectorial form:

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = \underline{\underline{A}}(\underline{x}(t))\underline{a} + \underline{B}(\underline{x}(t)), \tag{21}$$

where

$$\underline{a} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \end{pmatrix}. \tag{22}$$

The matrix $\underline{\underline{A}}$ and the vector $\underline{B}$ depend only on the flow kinematics or on the material properties, and in consequence on the particle position given by the integration of

$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}t} = \underline{v}(\underline{x}(t)). \tag{23}$$

The differential system (21) is linear and in consequence its general solution may be written as the addition of the general solution of the homogeneous differential system:

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = \underline{\underline{A}}(\underline{x}(t))\underline{a}, \tag{24}$$

and a particular solution of the non-homogeneous differential system (Eq. (21)).

3.2. General solution of the homogeneous differential system

It is well known that for a linear homogeneous differential system, the linear combination of solutions is also a solution of the differential system. Thus, if we denote by $\underline{a}_h^{(11)}(t)$, $\underline{a}_h^{(12)}(t)$ and $\underline{a}_h^{(22)}(t)$, the solutions of the differential system (24) associated with the initial conditions given by $\underline{e}_1 \otimes \underline{e}_1$, $\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1$ and $\underline{e}_2 \otimes \underline{e}_2$:

$$\underline{a}_h^{(11)}(t=0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{25}$$

$$\underline{a}_h^{(12)}(t=0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{26}$$

$$\underline{a}_h^{(22)}(t=0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{27}$$

Then, the solution of (24) associated with the initial condition

$$\underline{a}_h(t=0) = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix}, \tag{28}$$

with $(\alpha_{11}, \alpha_{12}, \alpha_{22}) \in \mathbb{R}^3$, will be given by

$$\underline{a}_h(t) = \alpha_{11}\underline{a}_h^{(11)}(t) + \alpha_{12}\underline{a}_h^{(12)}(t) + \alpha_{22}\underline{a}_h^{(22)}(t). \tag{29}$$

3.3. Particular solution of the non-homogeneous differential system

Since we are looking for a particular solution, we proceed by integrating Eq. (21) with an arbitrary initial condition given by

$$\underline{a}_c(t=0) = \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{22} \end{pmatrix}, \tag{30}$$

from which the particular solution $\underline{a}_c(t)$ is obtained.

Remark. In our simulations we have considered the isotropic form $\beta_{11} = 0.5$, $\beta_{12} = 0.0$ and $\beta_{22} = 0.5$, but the final solution does not depend on this arbitrary choice.

The general solution of the system (21), $\underline{a}(t)$, is then

$$\underline{a}(t) = \alpha_{11}\underline{a}_h^{(11)}(t) + \alpha_{12}\underline{a}_h^{(12)}(t) + \alpha_{22}\underline{a}_h^{(22)}(t) + \underline{a}_c(t), \tag{31}$$

and its initial value is

$$\underline{a}(t=0) = \begin{pmatrix} \alpha_{11} + \beta_{11} \\ \alpha_{12} + \beta_{12} \\ \alpha_{22} + \beta_{22} \end{pmatrix}. \tag{32}$$

3.4. Solutions in steady recirculating flows

A necessary condition that the steady solution of the advection equation must verify is its periodicity along the closed streamlines. Thus, the solution after a spatial period T (complete rotation) must be the initial value, therefore

$$\underline{a}(t=0) = \underline{a}(t=T). \tag{33}$$

This condition implies

$$\underline{a}(t=0) = \alpha_{11}\underline{a}_h^{(11)}(T) + \alpha_{12}\underline{a}_h^{(12)}(T) + \alpha_{22}\underline{a}_h^{(22)}(T) + \underline{a}_c(T). \tag{34}$$

This defines the linear system which can be denoted by

$$\underline{\underline{C}}\,\underline{\alpha} = \underline{D}. \tag{35}$$

From this system, we can obtain the values of the three components of the vector $\underline{\alpha}$: α_{11}, α_{12} and α_{22}. Thus, the steady solution of the orientation equation on the closed streamlines is defined by Eq. (31). This general solution is unique if $\det \underline{\underline{C}} \neq 0$. For rigid motion kinematics $\det \underline{\underline{C}} \neq 0$ and the solution along a closed streamline depends only on an arbitrary orientation considered at a point on the streamline, from which the orientation is integrated.

We have just found that for general steady recirculating flows there is only one solution at the considered point verifying the periodicity relation, Eq. (33). Thus, at this point the solution must be constant, and from the fact that the solution at each point on the trajectory can be obtained from this initial value, we prove that there is only one steady solution which can be obtained by imposing the periodicity condition given by Eq. (33).

For the considered examples, the periodicity is only required in the constitutive equation, which implies the periodicity of tensor $\underline{\underline{s}}$, but not necessarily the periodicity of the tensorial field $\underline{\underline{a}}$. In this case, we can replace the condition

$$\underline{\underline{a}}(t=T) = \underline{\underline{a}}(t=0), \tag{36}$$

by

$$m\,\underline{\underline{a}}(t=T) = \underline{\underline{a}}(t=0), \quad m \in \mathbb{R}, \tag{37}$$

without destroying the periodicity of $\underline{\underline{s}}$, since

$$\underline{\underline{s}}(t=0) = \frac{\underline{\underline{a}}(t=0)}{\mathrm{Tr}(\underline{\underline{a}}(t=0))} = \frac{m\underline{\underline{a}}(t=T)}{m\mathrm{Tr}(\underline{\underline{a}}(t=T))} = \underline{\underline{s}}(t=T). \tag{38}$$

For the molecular extension, in the second example, the advection problem (19) is scalar, and an exact solution is given by

$$\lambda(t) = \mathrm{e}^{\gamma(t)} \left\{ \lambda_0 + \int_0^t \frac{\mathrm{e}^{-\gamma(t')}}{\tau_s} \, \mathrm{d}t' \right\}, \quad \lambda_0 \in \mathbb{R}, \tag{39}$$

with

$$\gamma(t) = \int_0^t \left(\mathrm{Tr}(\underline{\underline{s}} \, \underline{\underline{D}}) - \frac{1}{\tau_s} \right) \mathrm{d}', \tag{40}$$

where $\underline{\underline{s}}$ and $\underline{\underline{D}}$ depend on the Lagrangian time through the characteristic equation: $\underline{\underline{s}}(\underline{x}(t))$ and $\underline{\underline{D}}(\underline{x}(t))$.

The associated initial condition $\lambda(t = 0)$ is

$$\lambda(t = 0) = \lambda_0, \tag{41}$$

and the periodicity condition $\lambda(t = 0) = \lambda(t = T)$ is written as

$$\lambda_0 = \mathrm{e}^{\gamma(T)} \left\{ \lambda_0 + \int_0^T \frac{\mathrm{e}^{-\gamma(t)}}{\tau_s} \, \mathrm{d}t \right\}. \tag{42}$$

Thus, we can obtain the value of the initial extension λ_0 from that equation.

4. Numerical examples

In order to verify the numerical procedure we are going to analyze different simple recirculating flows. In the integration of differential equations a fourth-order Runge–Kutta schema has been used.

4.1. SFRT flow without diffusion effects

In this first example we consider the model given in Eq. (10) with the diffusion coefficient equal to zero. In this case, and for fibers with a quasi-infinite aspect ratio in a general flow, the only solution is given by local alignment of the fibers with the streamlines [6].

4.1.1. Rotation of a rigid body

We now consider the rigid motion given by the velocity field

$$\underline{v}^T = (-y, x). \tag{43}$$

This equation describes the rigid body rotation around the centre point (0,0). The velocity is constant on a circle trajectory centred in this point. The magnitude of the velocity depends linearly on the radius. We solve the problem for the point $\underline{x}^T = (0, 1)$. The matrix of coefficients given in (35) is singular. This means that any solution may be considered at the chosen point as a periodic initial solution.

4.1.2. Planar and circular flow

We will consider here the planar flow given by the following velocity field

$$\underline{v}^T = (-y\sqrt{x^2 + y^2}, \, x\sqrt{x^2 + y^2}). \tag{44}$$

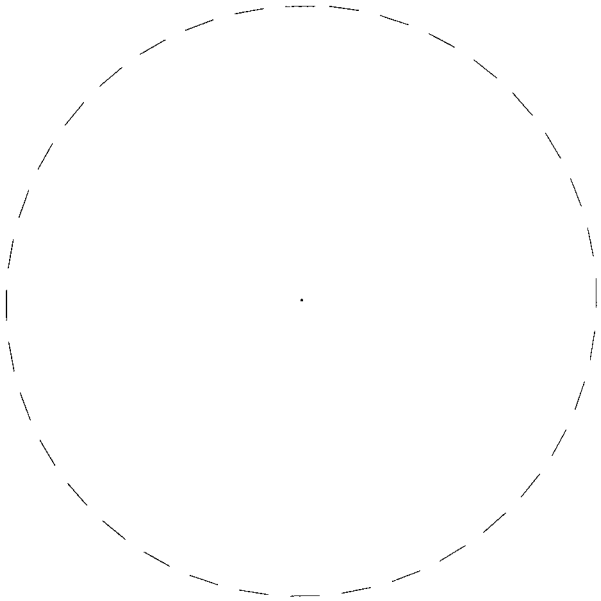

Fig. 1. Steady orientation along a closed streamline for fibers with infinite aspect ratio.

This flow is circular and the velocity is constant on a circle trajectory centred in the point (0,0). The magnitude of the velocity is the radius squared. We are looking for a periodic orientation solution with a unit trace at the point $\underline{x}^T = (0, 1)$ with the strategy described above. The solution found, as depicted in Fig. 1, corresponds with the local alignment of the fibers with the flow, i.e.

$$\underline{\underline{a}}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{45}$$

We represent the orientation tensor by means of an ellipse. The ellipse axes directions correspond to the eigenvectors of the orientation tensor and the half of the length of the axes correspond to the eigenvalues. In Fig. 1 we see that the ellipses are resumed to unit segments oriented in the flow direction. This result shows us that all the fibers are aligned in the flow direction all along the streamline. When inextensible fibers with a finite aspect ratio are considered, the local alignment of the fibers with the flow is not a solution, and moreover in [29] it is proved that for general flows the periodic solution can not have a unit eigenvalue. We can verify that result using our strategy.

If we are looking for a periodic fiber orientation in the flow defined by (44) for fibers with $k = 0.6$, then the solution found at the point $\underline{x}^T = (0, 1)$ is given by

$$\underline{\underline{a}}_{(0,1)} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix}. \tag{46}$$

We increase the isotropic behaviour with the k parameter, until we reach the isotropic orientation $\underline{\underline{a}} = \underline{\underline{I}}/2$ for circular particles $k = 0$.

The periodic solution associated with the initial condition given by Eq. (46) along the streamline is depicted in Fig. 2. We can see, in this figure, that ellipses have a certain thickness corresponding of a

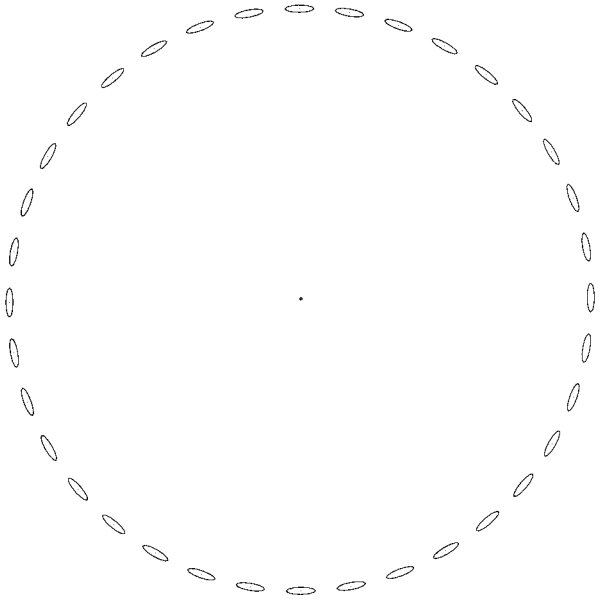

Fig. 2. Steady orientation for fibers with finite aspect ratio.

non-null eigenvalue in the radial direction. That means that the most probable direction of a fiber is the direction of the streamline but there is a light misalignment. As the orientation tensor represents the orientation probability we only obtain a solution in an averaged sense.

4.1.3. Lid-driven cavity flow

Finally we consider the lid-driven cavity flow problem, with the flow kinematics given by the solution of the Stokes problem defined by

$$\begin{aligned} &\nabla p = \mu\, \Delta \underline{v}, \quad \text{in}]0,1[\times]0,1[, \qquad \underline{v} = \underline{0} \quad \text{on } x = 0; \quad x = 1; \quad y = 0, \\ &\underline{v}^T = (\tfrac{2}{5}x(x-1), 0) \text{ on } y = 1. \end{aligned} \tag{47}$$

In order to get accurate incompressible solutions, we use a stream function formulation and solve the resulting biharmonic equation with a pseudospectral Chebyshev technique, neglecting the fiber orientation effects on the kinematics of the flow. Figs. 3 and 4 show the steady orientation solution along some closed trajectories for fibers with quasi-infinite and finite ($k = 0.9$) aspect ratios, respectively.

In the first case (Fig. 3), the local alignment of the fibers with the flow is identified, except in the neighbourhood of the centre of rotation, where kinematics very close to rigid motion is found. However, even in these areas, the local alignment may be considered as a steady orientation solution.

In the second case (Fig. 4), as indicated in [6], high gradients near the corners and elongational flow both induce high deviations from the local alignment in some regions of the fluid domain. In order to emphasize this deviation, we have superimposed the orientation field with the unit vectors in the streamline direction.

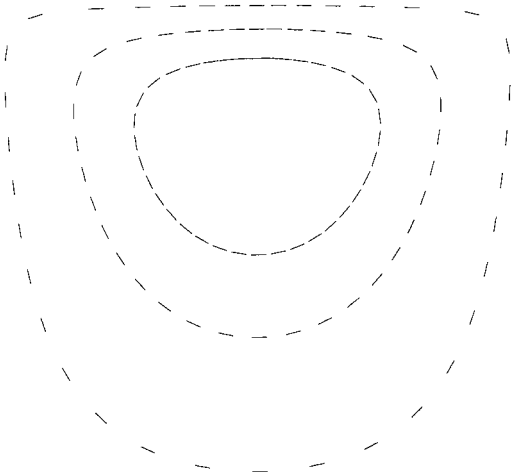

Fig. 3. Steady orientation in a cavity flow for fibers with infinite aspect ratio.

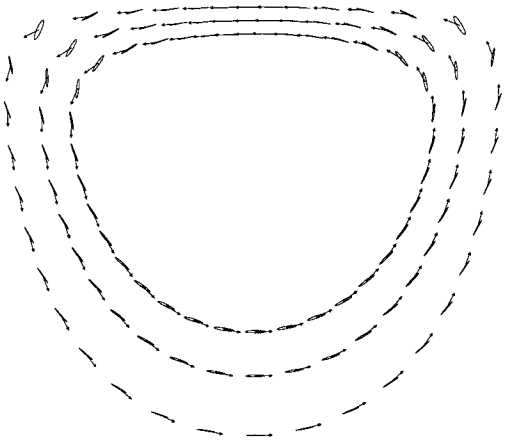

Fig. 4. Steady orientation in a cavity flow for fibers with finite aspect ratio.

4.2. The coupling between orientation and extension

Now, we are going to consider the linear formulation of the Pom–Pom viscoelastic model.

4.2.1. Planar and circular flow

We can expect that for relaxation times much longer than the period time, the steady orientation solution in the flow defined by (Eq. (44)) will be the local alignment, and due to the fact that this flow is a simple shear flow we do not expect any molecular extension.

In applying our strategy we obtain a steady orientation at the point $\underline{x}^T = (0, 1)$ that approaches

$$\underline{S}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{48}$$

without molecular extension.

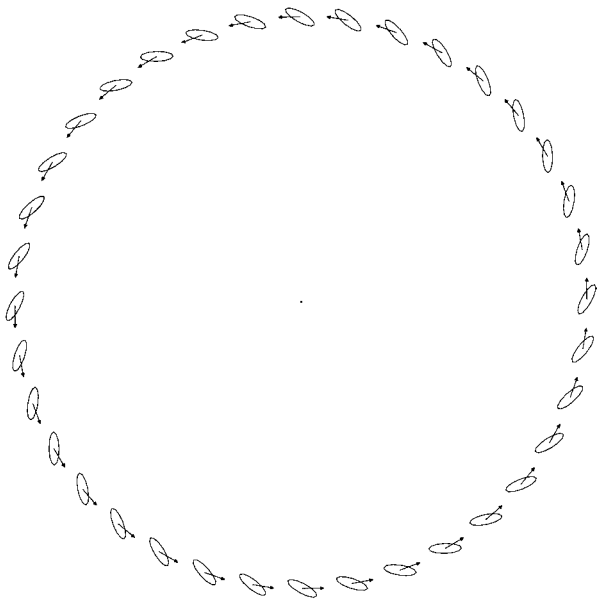

Fig. 5. Steady molecular orientation in a non-rigid planar circular flow, with a relaxation time $\tau_b = T/10$.

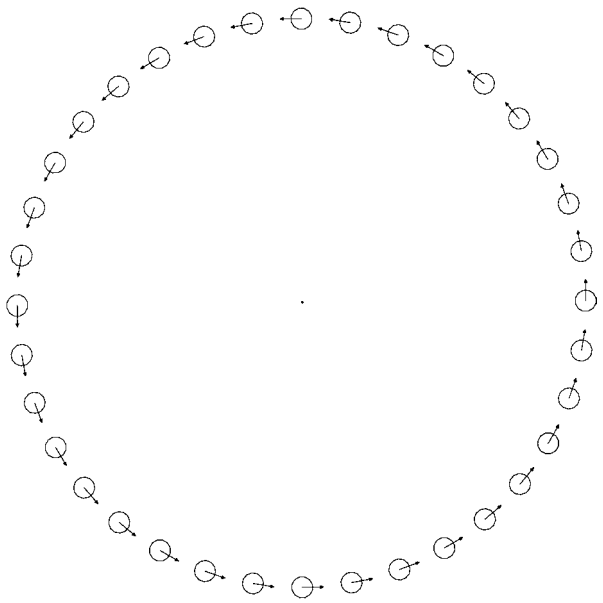

Fig. 6. Steady molecular orientation in a non-rigid planar circular flow, with a relaxation time $\tau_b = \ll T$.

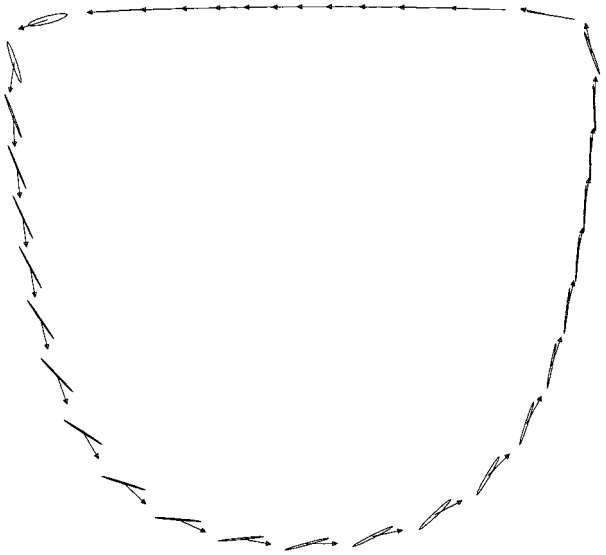

Fig. 7. Steady molecular orientation in a lid-driven cavity flow, with a relaxation time $\tau_b = T/10$.

With relaxation times $\tau_b = \tau_s = T/10$ (where T is the period), another steady solution is found as depicted in Fig. 5.

The isotropy increases as the relaxation time of the orientation process decreases. When the relaxation time is much shorter than the period, the steady orientation tends to the isotropic tensor, as we can see in Fig. 6.

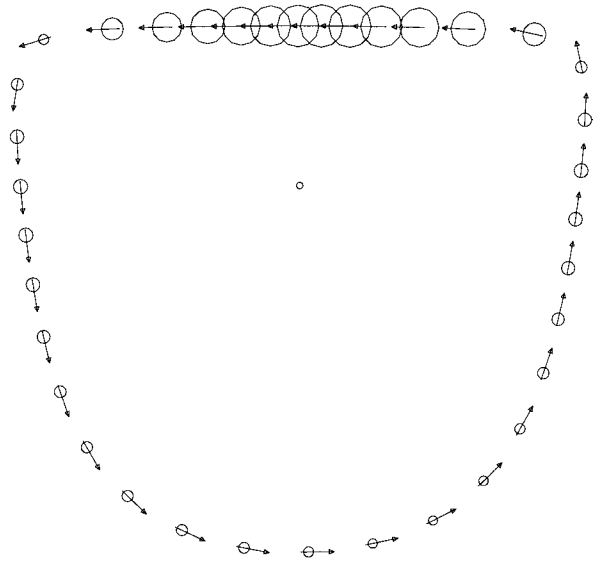

Fig. 8. Steady molecular extension in a lid-driven cavity flow, with a relaxation time $\tau_b = \tau_s = T/10$.

4.2.2. Lid-driven cavity flow

We now consider the cavity problem (see Eq. (47)). The steady orientation along the streamline which passes through the point $\underline{x}^T = (0.5, 0.99)$ and with $\tau_b = \tau_s = T/10$ is given in Fig. 7. In this case we observe a high molecular extension. In Fig. 8, we represent the extension along the streamline associated with the orientation depicted in Fig. 7. The extension magnitude is related to the circle diameter, with the natural extension $\lambda = 1$ given by the circle located at the centre of rotation. In this case the steady extension found at the point $\underline{x}^T = (0.5, 0.99)$ is $\lambda(0.5, 0.99) \approx 6$. This result shows that a steady-state value of the extension is reached, and corresponds to the steady state of the flow in the recirculating area.

5. Conclusions

In this paper, we have proved that a steady state solution exists for linear advection equations defined in steady recirculating flows. We also show that this solution is unique in most cases. We have presented a numerical procedure to evaluate steady solutions of linear advection problems defined in steady recirculating flows as encountered in non-Newtonian flow modelling (contraction flows, rotary rheometric devices). This strategy may be used in uncoupled algorithms where a steady solution of the advection equation in a recirculating part of the flow is required at each step, or to verify the accuracy of other numerical techniques. It may also be used to evaluate the accuracy of the hypothesis eventually introduced (as for example the local alignment of the fibers with the flow in SFRT simulations).

Non-linear advection equations which are usually involved in non-Newtonian flow modelling do not represent a strong handicap for the proposed technique, because as discussed in the introduction of this paper, the resolution of a non-linear problem is transformed in the resolution of a sequence of linear ones, for which the proposed strategy is adapted.

Acknowledgements

We want to thank Professor Ken Walters FRS from the Department of Mathematics of the University of Wales, Aberystwyth. First, he gave us the chance to work in his research team, where most of this work was done. Then, he offered us his time and his knowledge during compelling discussions.

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