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# The Optimal Duration of Unemployment Benefits\*

Gilles Joseph<sup>†</sup>, Paul-Emile Maingé<sup>‡</sup>

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**Abstract:** This paper studies the optimal duration of unemployment insurance (UI) benefits in a basic job search model where a risk neutral UI agency can not monitor the search effort of risk-averse workers. Social assistance payments are taken as exogenous by the unemployment agency which chooses optimally the level of UI benefits, the date of their exhaustion and the level of the financing tax. So, due to possible finite values of the duration of unemployment benefits, the resulting agency's problem brings nonstationarity complexities that are usually deemed intractable in models where utility and search costs functions are nonlinear. We then propose a new strategy, based on the study of the geometric properties of the set of constraints, and explicit formal conditions, with very general utility and search costs functions, for obtaining a zero, positive or infinite optimal duration of UI.

**Keywords:** Moral hazard, Job search, Potential benefits duration, Unemployment insurance.

**JEL Classification:** D83, C61, J64, J65

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# 1 Introduction

UI programs are generally limited in time even if the range of their potential duration is strikingly large across countries. For example, UI benefits last only 6 months in the United States, whereas in Europe they can vary from 6 (in United Kingdom) to 38 months (in the Netherlands). Besides the insurance system, there is often a less generous assistance or solidarity system that is not subject to prior contributive participations and that gives some monetary support to those who are no longer eligible for UI benefits.<sup>1</sup> As a suitable instrument for containing moral hazard problem created by weak monitoring of search efforts, the maximum duration of UI benefits thus defines a time sequence of insurance and assistance policies that provides incentives to tail off the unemployment period.<sup>2</sup>

Davidson and Woodbury (1997) were the first who really question the optimal termination of UI benefits in a discrete time search and matching framework where wages are exogenous and the number of jobs is fixed. In their model, the utility function is linear but they argue that risk aversion comes from the assumption that search costs are convex in search effort. As the potential duration of UI benefits is finite, the model is nonstationary. With the help of numerical exercises, they conclude that the optimal duration of unemployment should always be infinite, which renders suboptimal almost all existing UI programs. As noticed by Fredriksson and Holmlund (2006), they do not examine the optimal time sequence of UI benefits since they only compare a situation where UI benefits are unlimited to a situation where benefits are limited then arbitrarily set to zero. Fredrikson and Holmlund (2001) also suggest that the assumption of an exogenous social payment is the cause of the optimal infinite duration in the paper of Davidson and Woodbury (1997). We will see that this argument doesn't hold. The paper of Fredriksson and Holmlund (2001) considers the optimal duration of UI benefits in a Pissarides (2000) framework where search efforts and wages are endogenous but stationary, that renders the analysis mathematically tractable. Stationarity comes from the assumption that the necessary waiting duration to change state is a random variable that follows a Poisson process. UI is then on average duration limited. Using numerical simulations, they find that the optimal profile of UI is decreasing or, equivalently, that the optimal duration of UI benefits is finite.

The issue we particularly address in this paper is how to reconcile analytically the numerical result of Davidson and Woodbury (1997) with the one of Fredrikson and Holmlund (2001) in a same framework. We then start by integrating in a Mortensen (1977)'s model the fact that insurance and social assistance roughly describe a decreasing profile

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<sup>1</sup>Few countries provide UI benefits indefinitely: Australia, Belgium and New Zealand (see e.g. Tatsiramos and van Ours (2014), for a recent survey on the different existing UI designs and their potential labor market effects).

<sup>2</sup>There is a large theoretical literature on the optimal design of UI that suggests that a decreasing sequence gives better incentives than a non-increasing profile (see e.g. Shavell and Weiss 1979; Hopenhayn and Nicolini 1997, 2009; Cahuc and Lehmann, 2001, Fredriksson and Holmlund, 1998). There are also strong empirical results that extending eligibility duration has a negative effect on unemployment duration (see e.g. Lalive, (2007, 2008) ; Rothstein, 2011).

of unemployment payments insofar as the latter program is less generous. This pattern of unemployment transfers indeed constitutes a common feature in most OECD countries. We also consider social assistance as a given policy for an unemployment agency which has to optimally choose the level of UI benefits, their duration and the financing tax. Risk-averse workers receive an exogenous wage and pay tax to finance benefits. Search efforts are unobservable so the unemployment agency never stops inducing the unemployed to search, whatever the length of unemployment spell or unemployment status. As in Mortensen (1977), we consider here only the finite termination of UI benefits as the source of nonstationarity of search efforts, while UI benefits and wage taxes are constant over time (see van der Berg, 1990 for other sources of nonstationarity in search models). Nonstationarity usually brings analytical difficulties deemed intractable in models where utility and search costs functions are nonlinear. Despite this, we will give a precise formulation of the optimal duration of UI benefits.

Another important feature of our model is that the budget constraint is supposed to be balanced in average but not at every time. So, in the considered setting, we aim at maximizing the initial utility value  $U(0)$  of an eligible unemployed, with respect to  $T$  (maximum duration of UI benefits), the tax  $\tau$  and the level of UI benefits  $b$ , under the budget constraint  $B(0) = 0$ . The model under consideration will be referred to as "the agency's problem" and can be regarded as a dynamic optimization problem where the objective function (i.e. the expected utility of an eligible unemployed) as well as the inter-temporal budget constraint, are derived from solutions of differential equations. The main contribution of our paper is twofold. First, we provide a clear formalization of a finite optimal duration  $\bar{T}$  of UI benefits by exploiting the geometric properties of the constraints. Secondly, we show that the special cases when the optimal duration  $\bar{T}$  is zero or infinity are obtained as limiting cases of our strategy.

The paper unrolls as follows. In Section 2, we discuss the worker's behavior and the agency's behavior. We also set up the agency's problem as a dynamic optimization problem with respect to the variables  $\{b, \tau, T, U_0\}$ , where  $U_0$  denotes the initial utility value. Section 3 remodels the agency's problem. In particular, Theorem 3.1 establishes that solutions  $(\bar{b}, \bar{\tau}, \bar{T}, \bar{U}_0)$  to the agency's problem can be obtained from some formulation of  $\bar{T}$  depending on any element  $(\bar{b}, \bar{\tau}, \bar{U}_0)$  that solves a classical three dimensional optimization problem (with respect to the unknowns  $\{b, \tau, U_0\}$ ). In Section 4, we prove that the agency's problem can be recast as a two dimensional optimization problem over a nonempty and bounded set  $\Gamma \subset \mathbb{R}^2$  with a possibly non-convex geometric form that depends on the fixed parameters. Specifically, Theorem 4.1 states for some real-valued mapping  $J$  that an optimal contract  $(\bar{b}, \bar{\tau}, \bar{T}, \bar{U}_0)$  can be alternatively obtained from an element  $(\bar{b}, \bar{\tau}) \in \operatorname{argmin}_{\Gamma} J$  (namely  $(\bar{b}, \bar{\tau}) \in \Gamma$  and  $J(\bar{b}, \bar{\tau}) = \min_{\Gamma} J$ ), together with a positive optimal duration  $\bar{T} = \bar{T}(\bar{b}, \bar{\tau})$  (depending on  $\bar{b}$  and  $\bar{\tau}$ ) and some value  $\bar{U}_0 = \bar{U}_0(\bar{b}, \bar{\tau})$ . Finally, Theorem 4.2 puts out the following two possibilities: either there exists a minimizer of  $J$  over  $\Gamma$ , which ensures the existence of optimal values  $(\bar{b}, \bar{\tau})$  associated with a finite and positive optimal duration  $\bar{T}$ ; or there exists no minimizer of  $J$  over  $\Gamma$ . This later case is related to any situation in which the optimal duration is zero or infinite. In section 5, we give a numerical illustration of the model and its qualitative properties.

Section 6 concludes.

## 2 The model

We consider a standard continuous job search model *à la* Mortensen (1977) where mortal risk-averse workers have no access to financial markets and so cannot save or borrow.<sup>3</sup> We focus on search decisions, so that the gross wage, denoted by  $w$ , is set at an exogenous level. We also consider a two-tiered unemployment compensation system and so two types of job-seekers : those eligible to UI benefits  $b$  and those who after a maximum period of eligibility  $T$  are still jobless. The latter benefit from welfare or assistance system that is generally less generous. After  $T$  periods, the environment becomes stationary as assistance payments for noneligible unemployed,  $z$ , are unlimited. We assume, eventually, that UI benefits  $b$  and the employment tax  $\tau$  are constant over time.

### 2.1 Worker's behavior

Each worker maximizes his expected discounted utility, which writes

$$E_0 \int_0^{\infty} e^{-rt} [u(x) - v(\varepsilon_t)] dt,$$

where  $r \in (0, 1)$  is the discount rate,  $x$  is the consumption level,  $E_0$  is the expectation operator and the other parameters are described as follows:

- The instantaneous utility function of consumption  $u(\cdot)$  has standard properties namely continuous, increasing and concave over  $]0, +\infty[$ .
- The search cost function  $v(\varepsilon_t)$ , where  $\varepsilon_t$  denotes the search effort, is positive, convex and has the usual standard form:

$$v(\varepsilon_t) = \frac{c}{1+\alpha} \varepsilon_t^{1+\alpha}, \text{ with } \alpha > 0.$$

In our partial model, an employee can not search for another job and be laid off but, similarly to an unemployed worker, can die according to a Poisson process with arrival rate  $\mu$ . So for an employee, we always have  $v(0) = 0$  and his expected utility  $E$  is simply :

$$E = \frac{u(w - \tau)}{\eta}, \text{ where } \eta = r + \mu. \tag{2.1}$$

An unemployed worker chooses his search effort level  $\varepsilon$  and exit out of unemployment at rate  $\lambda(\varepsilon)$ . For the sake of simplicity, we assume that  $\lambda(\varepsilon) = \varepsilon$ . The expected utility of

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<sup>3</sup>A recent branch of the research on UI shed some light on how savings and borrowing behaviors can affect the optimal timing of UI benefits by using numerical simulations (e.g. Kacherlakota, 2004; Shimer and Werning, 2008; Mitchell and Zhang 2010; Rendahl 2012). Card et al. (2007) and Chetty (2008) also give some evidences that the exit rate from unemployment is affected by the agent's wealth.

an eligible unemployed, denoted by  $U(t)$ , satisfies the following Bellman equation

$$\begin{aligned} rU(t) - \dot{U}(t) &= \max_{\varepsilon_t} \{u(b) - v(\varepsilon_t) + \varepsilon_t [E - U(t)] - \mu U(t)\} \\ U(T) &= U_a, \end{aligned} \quad (2.2)$$

where  $U_a$  is the expected utility of a noneligible jobless that we will describe below. The flow value of being an eligible unemployed at  $t$  equals the flow payoff  $u(b) - v(\varepsilon_t)$  augmented by the average gain of status change  $\varepsilon_t [E - U(t)]$  and lowered by the loss caused by death, which occurs at rate  $\mu$ . At the exhaustion of the insurance benefits period  $T$ , an eligible jobless becomes a noneligible one. Therefore his expected utility is  $U_a$ , what is formalized by the terminal condition  $U(T) = U_a$ . The first order condition for optimal search effort at  $t$  is given by

$$\varepsilon(t) := \varepsilon_t = \left( \frac{E - U(t)}{c} \right)^{\frac{1}{\alpha}}. \quad (2.3)$$

It is straightforward to verify that if  $E = U(t)$  then  $\varepsilon(t) = 0$  as there is no gain to search for a job. Moreover, the profile of  $U(t)$  determines completely, in an opposite way, the evolution of  $\varepsilon(t)$ . A noneligible unemployed has a stationary environment since the assistance payment  $z$  is unlimited over time. The discounted expected utility  $U_a$  thus satisfies the following Bellman equation :

$$rU_a = \max_{\varepsilon_a} \{u(z) - v(\varepsilon_a) + \varepsilon_a (E - U_a) - \mu U_a\} \quad (2.4)$$

and the first order condition for optimal search effort in this case is given by

$$\varepsilon_a = \left( \frac{E - U_a}{c} \right)^{\frac{1}{\alpha}}. \quad (2.5)$$

Rearrange (2.2), taking into account the optimal search effort (2.3), and setting  $\beta = \frac{\alpha}{1+\alpha} c^{\frac{-1}{\alpha}}$  and  $q = \frac{1}{\alpha}$ , gives us the following differential equation

$$\dot{U}(t) = \eta U(t) - \beta [E - U(t)]^{1+q} - u(b). \quad (2.6)$$

Observe that  $U_a$  is defined through (2.4) (also using (2.5)), that can be rewritten as

$$\eta U_a - \beta (E - U_a)^{1+q} - u(z) = 0. \quad (2.7)$$

## 2.2 Agency's behavior

Let  $B(t)$ ,  $B_a$ ,  $B_e$  be the values of the costs associated with insurance, assistance and employment, respectively. The actuarially fair relation between benefits and taxes for an unemployed eligible worker is represented by the following Bellman equation

$$\begin{aligned} rB(t) &= b + \varepsilon(t) [B_e - B(t)] - \mu B(t) + \dot{B}(t), \\ B(T) &= B_a, \end{aligned} \quad (2.8)$$

where  $B_e$  and  $B_a$  satisfy

$$B_e = \frac{-\tau}{\eta}, \quad (2.9a)$$

$$rB_a = \varepsilon_a [B_e - B_a] - \mu B_a. \quad (2.9b)$$

From an easy computation we get

$$B_a = -\frac{\tau\varepsilon_a}{\eta(\eta + \varepsilon_a)}. \quad (2.10)$$

The expected gain from an employee is given by (2.9a) and the gain from an current noneligible unemployed is given by (2.9b). Let us notice that the unemployment agency does not finance and does not provide the welfare payment  $z$  but captures the tax employment from a former noneligible unemployed. Then the value cost of insurance  $B$  satisfies the following first-order differential equation

$$\dot{B}(t) = (\eta + \varepsilon(t))B(t) + \frac{\tau}{\eta}\varepsilon(t) - b. \quad (2.11)$$

This last relation is, from the point of view of the UI agency, the financial counterpart of the worker's position on the labor market as the flow of taxes and benefits are contingent to the worker's job status. The unemployment agency collects employment taxes  $\tau$  and distributes UI benefits  $b$  such as the net cost of an eligible unemployed at date 0 is nil. We do not impose a balanced budget at each date.

### 2.3 The optimal contract

As search is a private information, the agency then chooses  $b$ ,  $\tau$  and  $T$  so as to maximize worker's utility under the budget constraint. More precisely, we are then interested in maximizing the expected utility  $U(0)$  obtained from (2.6) with respect to nonnegative values of  $b$  and  $\tau$  and positive values of  $T$  verifying the set of constraints (s1)-(s3) given below:

(s1) The utility and cost functions  $U$  and  $B$  satisfy the ordinary differential equations (2.6) and (2.11), respectively, and are such that

$$U(T) = U_a, \quad U(0) > U_a, \quad (2.12a)$$

$$B(T) = B_a, \quad B(0) = 0. \quad (2.12b)$$

(s2) The following additional conditions on the parameters are needed:

$$\tau \geq 0, \quad w - \tau \geq b, \quad (2.13a)$$

$$b \geq z. \quad (2.13b)$$

(s3) The discounted utility  $U_a$  at time  $T$  is implicitly given by (2.7) while the value cost  $B_a$  at time  $T$  is defined in (2.10).

Let us discuss briefly some of the conditions above. In our principal-agent framework, we suppose that the unemployment agency wants to propose to each agent that enters unemployment a contract that defines  $\{b, \tau, T\}$  such as the present value of insured unemployment is higher than the present value of social assistance, subject to an appropriate budget constraint. Constraints (2.12a) formalize this assumption and overall assure that a positive date of exhaustion  $T$  is a relevant parameter of our problem. Indeed, if  $U(t) = U_a$ , it comes that  $\varepsilon(t) = \varepsilon_a$  and the environment of the model becomes stationary. Recall that for the sake of tractability, we presume a two-tier UI scheme where unemployment payments and tax rate are constant over time. We then start by proceeding with the assumption that  $b \geq z$  and let the analysis assert whether a strict inequality is optimal or not. The interpretation of the additional conditions (s2) is obvious insofar as there is no rational incentive that drives a worker to accept a contract that enforces him to pay a tax in order to receive less than what he would have with social assistance.

From a mathematical viewpoint, the optimal contract is solution of the following problem:

$$\max_{(T, b, \tau, U_0) \in \mathcal{Q}} U_0, \quad (2.14)$$

where  $\mathcal{Q}$  is the set of elements  $(T, b, \tau, U_0)$  such that :

- (b1) condition (2.13) is fulfilled (by  $b$  and  $\tau$ ) ;
- (b2) there exist mappings  $U : [0, T] \rightarrow \mathbb{R}$  and  $B : [0, T] \rightarrow \mathbb{R}$  verifying (2.6), (2.11) and (2.12) with  $U(0) = U_0$ .

An alternative characterization of elements  $(T, b, \tau, U_0) \in \mathcal{Q}$  can be obtained by introducing the following sets:

$$S(T, b, \tau) := \{U : [0, T] \rightarrow \mathbb{R} \text{ s.t. (2.6) holds}\}, \quad (2.15a)$$

$$\mathcal{G}_U(T, b, \tau) := \{B : [0, T] \rightarrow \mathbb{R} \text{ s.t. (2.11), (2.12b) hold with } U : [0, T] \rightarrow \mathbb{R}\}, \quad (2.15b)$$

$$G(T, b, \tau) := \{U \in S(T, b, \tau) \text{ s.t. } \mathcal{G}_U(b, \tau, T) \neq \emptyset\}. \quad (2.15c)$$

So, we readily observe that  $\mathcal{Q}$  can be written as the set of elements  $(T, b, \tau, U_0)$  verifying (2.13) and such that :

$$G(T, b, \tau) \cap \{U : [0, T] \rightarrow \mathbb{R}; U(0) = U_0, U(0) > U_a\} \neq \emptyset. \quad (2.16)$$

It is worth noticing that a natural approach to investigating problem (2.14) (at first sight) would consist in computing explicitly  $U(0)$ , which is not a simple task. So, we use a different methodology.

### 3 Remodeling the agency's problem

The objective of this section is to provide a more tractable formulation of the maximization problem (2.14), by exploiting specific properties obtained from the analysis of



the utility function  $U$  and also by computing the cost function  $B$  with respect to  $U$ . As an important part of our methodology, we will henceforth work with the two differences  $W(t) = E - U(t)$  and  $W_a = E - U_a$  instead of the expected utilities themselves. We also deal here with only finite positive values of  $T$ . It will clearly appear in the next section that this approach is not restrictive at all since any (possible) optimal values  $T = +\infty$  and  $T = 0$  should necessarily appear as limiting cases of our strategy.

### 3.1 A transformed utility equation

Instead of using the formulation (2.6), we focus our analysis on a simpler and transformed equation. Clearly, setting  $W = E - U$  and recalling that  $\eta E = u(w - \tau)$  (according to (2.1)), we readily see that (2.6) becomes

$$\dot{W}(t) = \eta W(t) + \beta [W(t)]^{1+q} - u(w - \tau) + u(b). \quad (3.1)$$

Let us introduce the two mappings  $F$  and  $H$  defined for  $s \in \mathbb{R}_+$  by

$$F(s) = \eta s + \beta s^{1+q}, \quad (3.2a)$$

$$H(s) = F(s) - \gamma, \quad \text{where } \gamma = u(w - \tau) - u(b). \quad (3.2b)$$

Then it is immediately seen that (3.1) is equivalently written as

$$\dot{W}(t) = H(W(t)). \quad (3.3)$$

This last formulation will be very useful regarding the analysis of the utility function as well as for the computation of the cost function. Moreover, introducing the new variable  $W_a = E - U_a$ , we observe that (2.12a) can be rewritten as

$$W(T) = W_a, \quad (3.4a)$$

$$W(0) < W_a. \quad (3.4b)$$

Now, with regards to our methodology, we consider some quantity  $W_*$  that verifies  $H(W_*) = 0$ , or equivalently

$$F(W_*) = \gamma, \quad \text{where } \gamma = u(w - \tau) - u(b), \quad (3.5)$$

while it can be furthermore checked from (2.7) that  $W_a$  is implicitly given by the following equality

$$F(W_a) = \kappa, \quad \text{where } \kappa = u(w - \tau) - u(z). \quad (3.6)$$

It is of importance to see that  $W_*$  and  $W_a$  are well-defined and nonnegative. Indeed, we observe that  $\gamma$  and  $\kappa$  (introduced in (3.2b) and (3.6)) are nonnegative parameters (under conditions (2.13)), since  $u$  is assumed to be increasing on  $(0, +\infty)$ . Consequently, as  $F$  is continuous and increasing  $[0, +\infty)$ , together with  $F(0) = 0$ , we deduce that there exists a unique couple of nonnegative values  $(W_*, W_a)$  such that  $F(W_*) = \gamma$  (namely  $H(W_*) = 0$ )

and  $F(W_a) = \kappa$ . These two specific values  $W_*$  (depending on  $b$  and  $\tau$ ) and  $W_a$  (only depending on  $\tau$ ) are obviously defined by

$$W_* = F^{-1}(u(w - \tau) - u(b)), \quad W_a = F^{-1}(u(w - \tau) - u(z)). \quad (3.7)$$

A very more interesting formulation of (2.14) is proposed farther, through the computation of the initial cost value (by using (3.3) and the cost equation) and the analysis of the utility function, with respect to the values  $W_a$  and  $W_*$ .

From now on, the values  $W_a$  and  $W_*$  (given in (3.7)) are sometimes denoted by  $W_a(\tau)$  and  $W_*(b, \tau)$ , respectively, so as to be more precise. We also consider sometimes the more precise notations  $H_{(b,\tau)}(s)$  and  $\gamma(b, \tau)$  (instead of  $H(s)$  and  $\gamma$ ) given for some values  $(b, \tau, s)$  by  $H_{(b,\tau)}(s) = F(s) - \gamma(b, \tau)$ , where  $\gamma(b, \tau) = u(w - \tau) - u(b)$  and  $F(s) = \eta s + \beta s^{1+q}$  (introduced in (3.2)).

### 3.2 Analysis of the utility function

In this section we study the behavior of solutions  $W$  to (3.3) with respect to the knowledge of  $W(T)$ . More precisely, given  $T > 0$  and  $W_a > 0$ , we are interested in solutions  $W$  of the following system

$$\dot{W} = H(W), \quad \text{with } W(T) = W_a, \quad (3.8)$$

so as to give us a more precise characterization of the set of constraints  $\mathcal{Q}$  arising in (2.14). In particular, we establish that solutions to (3.8) can be described by using the following system (3.9) that we first investigate:

$$\dot{V} = -H(V), \quad \text{with } V(0) = W_a. \quad (3.9)$$

Some nice properties of the function  $H(\cdot)$  are exploited for investigating existence, uniqueness and regularity results (thanks to the well-known Cauchy-Lipschitz theorem).

**Lemma 3.1** *For any  $W_a > 0$ , there exists a unique global solution to the system (3.9). Moreover,  $V \in C^2(\mathbb{R}_+)$  and, for  $t \geq 0$ ,  $V(t)$  can be written as*

$$V(t) = W_* + (V(0) - W_*)e^{-\phi(t)}, \quad (3.10)$$

$$\text{where } \phi(t) := \eta t + \int_0^t P(s)ds \quad (P \text{ being some nonnegative mapping}). \quad (3.11)$$

Proof. See Appendix.

The next lemma shows us that the profile of solutions  $W$  to (3.3) on  $[0, T]$  depends on the relative values of the quantities  $W_a$  and  $W_*$  defined in (3.7) (see Figure 1).

**Lemma 3.2** *Let  $W_a > 0$  and  $W_* \geq 0$  be given by (3.7). Then, for any  $T > 0$ , problem (3.8) admits a unique classical solution  $W$  on  $[0, T]$ , defined by*

$$W(t) = V(T - t), \quad \text{for } t \in [0, T], \quad (3.12)$$

where  $V$  satisfies (3.9). So,  $W \in C^2([0, T])$  and the following statements are reached:

- (i1)  $W_a > W_*$   $\Rightarrow$   $W$  is increasing and, for  $t \in [0, T]$ , we have  $W_* < W(t) \leq W_a$ .
- (i2)  $W_a = W_*$   $\Rightarrow$   $W \equiv W_*$  on  $[0, T]$ .
- (i3)  $W_a < W_*$   $\Rightarrow$   $W$  is decreasing and, for  $t \in [0, T]$ , we have  $W_a \leq W(t) < W_*$ .

Proof. See Appendix.

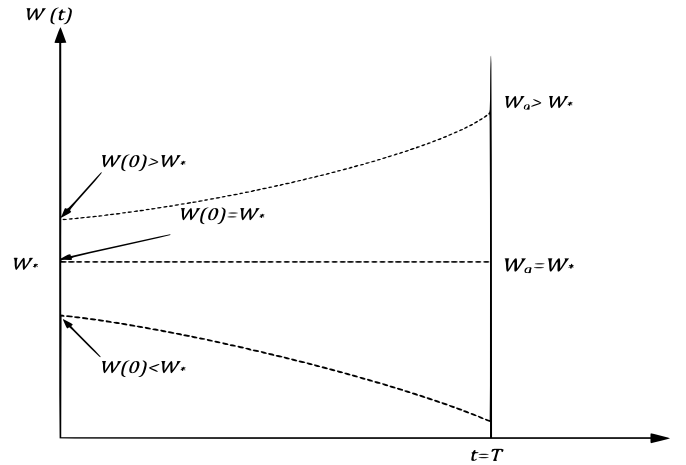


Figure 1: The evolution of  $W(t)$  and the duration of unemployment.

In light of Lemma 3.2, we observe that the expected utility of an insured unemployed  $U(t)$  is decreasing, constant or increasing if the difference  $W_a - W_*$  is respectively positive, null or negative. A decreasing expected utility  $U(t)$  (obtained for  $W_a > W_*$ ) means that UI benefits  $b$  is larger than the assistance payments  $z$ . When  $W_a = W_*$  (namely  $b = z$ ), we have  $U(t) = U_a$  and so the expected utility of an insured jobless is independent of  $T$ ; so it can be verify from (2.11) that the budget constraint is also independent of  $T$  and that it simply gives the level of tax rate  $\tau$  with respect to the search effort  $\varepsilon$ . These results are very intuitive insofar as the only difference between the expected incomes of an insured unemployed and those of a non-insured jobless comes precisely from the gap between the level of UI benefits  $b$  and the assistance payments  $z$ .

The first statement (i1) illustrates the case highlighted by Mortensen (1977) where the expected utility  $U(t)$  declines and the search effort  $\varepsilon_t$  rises during insured unemployment, until the maximum benefit duration  $T$ . Once the UI benefits are exhausted, the search behavior as the gain from unemployment are stationary, respectively equal to  $\varepsilon_a$  and  $U_a$ .

Note that the two statements (i2) and (i3) in Lemma 3.2 are not compatible with condition (2.12a), which writes  $W(0) < W_a$  and  $W(T) = W_a$ , where the function  $W = E - U$  satisfies (3.3). So, from Lemma 3.2 (also see Fig. 1), condition (2.12a) requires to assume that  $W$  satisfies  $W_* < W(0) < W_a$ . Furthermore, these last inequalities obviously entail that  $W_* < W_a$ , which is equivalent to  $b > z$  (according to the definitions of  $W_a$  and  $W_*$  given in (3.7)). It turns out that condition (2.12a) makes sense provided that  $b > z$

and that  $W_* < E - U(0) < W_a$ .

As a result, we readily deduce the more precise definition of the set of constraints  $\mathcal{Q}$  (arising in (2.14)) given by the following lemma.

**Lemma 3.3** *The set  $\mathcal{Q}$  involved in (2.14) can be rewritten as the set of elements  $(T, b, \tau, U_0)$  such that:*

$$(b, \tau) \in \Omega, \quad G(T, b, \tau) \cap \mathcal{K}(T, b, \tau, U_0) \neq \emptyset, \quad (3.13)$$

where  $G(T, b, \tau)$  is given in (2.15), while  $\Omega$ , illustrated in Fig. (2), is, defined by

$$\Omega := \{(b, \tau) \in \mathbb{R}^2; \tau > 0, \quad w - \tau \geq b, \quad b > z\}, \quad (3.14)$$

together with  $\mathcal{K}(T, b, \tau, U_0)$  defined by

$$\begin{aligned} \mathcal{K}(T, b, \tau, U_0) \\ := \{U : [0, T] \rightarrow \mathbb{R}; U(0) = U_0, E - U(T) = W_a, W_* < E - U_0 < W_a\}. \end{aligned} \quad (3.15)$$

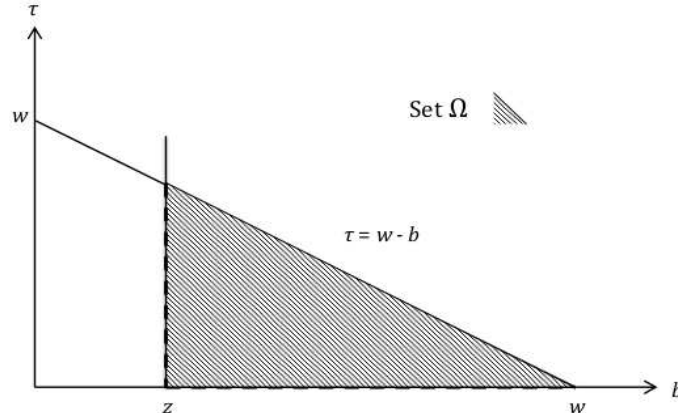


Figure 2: The set  $\Omega$ .

It can be noticed that  $\Omega = \emptyset$  if  $w \leq z$ . So, as a standing assumption in the sequel of the paper, we set

$$z \in ]0, w[. \quad (3.16)$$

A relevant simplification of the set  $G(T, b, \tau) \cap \mathcal{K}(T, b, \tau, U_0)$  (involved in (3.13)) is given in the next section.

### 3.3 A first reduction step of the model with exact formulation of the cost function

#### 3.3.1 Computation the cost function

The real issue of this section is to refined the formulation of  $\mathcal{Q}$  given in Lemma 3.3. For this purpose, we compute  $B(0)$  with respect to some utility function  $U$  and some

$(T, b, \tau, U_0)$ , under the condition  $B(T) = B_a$  (from (2.12b)) and the following standing assumptions:

$$(b, \tau) \in \Omega \text{ and } U \in S(T, b, \tau) \cap \mathcal{K}(T, b, \tau, U_0). \quad (3.17)$$

Let us stress that (3.17) guarantees that  $0 \leq W_* < W_a$  (since  $(b, \tau) \in \Omega$ ), besides  $W = E - U$  satisfies (3.3) with  $W(0) = E - U_0$ ,  $W(T) = W_a$  and  $W_* < E - U_0 < W_a$ .

The following lemma shows us that  $B(0)$  can be formulated as a first step with respect to the quantity  $\psi$  defined for  $t \geq 0$  by  $\psi(t) = e^{-\eta t - \int_0^t \varepsilon(s) ds}$ , where  $\eta = r + \mu$  and  $\varepsilon$  denotes the search effort.

**Lemma 3.4** *The cost function  $B$  at time  $t = 0$  is given by*

$$B(0) = \left( B(T) + \frac{\tau}{\eta} \right) \psi(T) + (b + \tau) \int_0^T \psi(s) ds - \frac{\tau}{\eta}. \quad (3.18)$$

Proof. See Appendix. •

The first term of equation (3.18) can be interpreted as the net cost of an non-insured unemployed. The second term is the net unemployment subsidy to an insured jobless as he does not pay tax, whereas the last term is the potential gain from an entirely life of employment, which lasts in average  $1/\eta$  unit of time.

Some quantities occurring in the above formulation of  $B(0)$  are given by the next lemma.

**Lemma 3.5** *The following statements are obtained:*

$$\psi(T) = \frac{H(W(0))}{H(W(T))}, \text{ where } W = E - U, \quad (3.19a)$$

$$\int_0^T \psi(t) dt = H(W(0)) \int_{W(0)}^{W(T)} \frac{ds}{H^2(s)}, \quad (3.19b)$$

$$B_a + \frac{\tau}{\eta} = \frac{\tau}{F'(W_a)}. \quad (3.19c)$$

Proof. See Appendix.

Substituting the results of Lemma 3.5 into (3.18) with  $B(T) = B_a$  amounts to the following lemma.

**Lemma 3.6** *The differential cost equation (2.11) with  $B(T) = B_a$  admits a unique classical solution on  $[0, T]$  such that*

$$B(0) = \tau \frac{H(W(0))}{\rho(b)F'(W_a)} + (b + \tau)H(W(0)) \int_{W(0)}^{W_a} \frac{ds}{H^2(s)} - \frac{\tau}{\eta}, \quad (3.20)$$

where  $H$  and  $F$  are the mappings given in (3.2),  $W(0) = E - U(0)$ , while  $\rho(b)$  is a positive real number defined by

$$\rho(b) = u(b) - u(z) \quad (= H(W_a(\tau))). \quad (3.21)$$

Lemma 3.6 provides an explicit expression of the cost function with respect to the differences  $W(0)$  and  $W_a$ . It especially establishes that  $B(0)$  is depending on  $T$  only through the value of  $U(T)$ .

### 3.3.2 A model with a simplified cost constraint

Clearly, under the assumption (3.17) and by Lemma 3.6, we observe that the condition  $\mathcal{G}_U(T, b, \tau) \neq \emptyset$  (where  $\mathcal{G}_U(T, b, \tau)$  is given in (2.15)) can be reduced to

$$\Phi(b, \tau, W_a, W(0)) = 0, \quad (3.22)$$

where  $W(0) = E - U_0$  and where  $\Phi$  is defined for  $(b, \tau, W_a, W(0))$  by

$$\begin{aligned} \Phi(b, \tau, W_a, W(0)) \\ = \tau H(W(0)) + \rho(b) F'(W_a) \left( (b + \tau) H(W(0)) \int_{W(0)}^{W_a} \frac{ds}{H^2(s)} - \frac{\tau}{\eta} \right). \end{aligned} \quad (3.23)$$

As a straightforward consequence, we conclude that the second condition in (3.13) (i.e.,  $\mathcal{K}(T, b, \tau, U_0) \cap G(T, b, \tau) \neq \emptyset$ ) reduces to

$$\mathcal{K}(T, b, \tau, U_0) \cap \{U \in S(T, b, \tau) \text{ s.t. } \Phi(b, \tau, W_a, E - U_0) = 0\} \neq \emptyset, \quad (3.24)$$

where  $S(T, b, \tau)$  is given in (2.15). It is further readily checked that (3.24) is equivalent to

$$\mathcal{K}(T, b, \tau, U_0) \cap S(T, b, \tau) \neq \emptyset \quad \text{and} \quad \Phi(b, \tau, W_a, E - U_0) = 0. \quad (3.25)$$

So, by Lemma 3.3, we immediately obtain the following result.

**Lemma 3.7** *The set of constrains  $\mathcal{Q}$  of the worker's problem (2.14) can be equivalently expressed as*

$$\mathcal{Q} = \mathcal{C} \cap \{(T, b, \tau, U_0); \quad \Phi(b, \tau, W_a, E - U_0) = 0\} \quad (3.26)$$

where  $\mathcal{C}$  of is the set of elements  $(b, \tau, U_0, T)$  that satisfy the following (i1)-(j1):

$$(i1): (b, \tau) \in \Omega, \quad (j1): \mathcal{K}(T, b, \tau, U_0) \cap S(T, b, \tau) \neq \emptyset. \quad (3.27)$$

The set  $\mathcal{C}$  plays a crucial role in our methodology and its formulation will be simplified in the next section.

### 3.4 A second reduction step of the model based upon a separate formulation of the optimal duration

The next lemma is helpful for a key transformation of the set  $\mathcal{C}$  involved in (3.26).

**Lemma 3.8** *For any  $W_0$  verifying  $W_* < W_0 < W_a$ , where  $W_*$  and  $W_a$  are nonnegative values given by (3.7), there exists a unique positive time, given by*

$$T = \int_{W_0}^{W_a} \frac{1}{H(s)} ds, \quad (3.28)$$

with a unique  $W \in C^\infty([0, T])$  that satisfies (3.3) with  $W(0) = W_0$  and  $W(T) = W_a$ .

Proof. See Appendix. •

Now, we give a fruitful simplification of  $\mathcal{C}$ .

**Lemma 3.9** *The set of constraints  $\mathcal{C}$  arising in problem (3.26) is nothing but the set of elements  $(b, \tau, U_0, T)$  that satisfy the following (i1)-(i3):*

$$(i1) : (b, \tau) \in \Omega, \quad (i2) : W_* < E - U_0 < W_a, \quad (i3) : T = \int_{E-U_0}^{W_a} \frac{1}{H(s)} ds. \quad (3.29)$$

Proof. See Appendix.

The next theorem is reached as an immediate consequence of Lemmas 3.7 and 3.9.

**Theorem 3.1** *Solutions  $(\bar{T}, \bar{b}, \bar{\tau}, \bar{U}_0)$  to the agency's problem (2.14) are obtained from solutions  $(\bar{b}, \bar{\tau}, \bar{U}_0)$  of the following optimization problem*

$$\max_{b, \tau, U_0} U_0 \quad (3.30a)$$

$$\text{subject to : } \Phi(b, \tau, W_a(\tau), E(\tau) - U_0) = 0, \quad (3.30b)$$

$$W_*(b, \tau) < E(\tau) - U_0 < W_a(\tau), \quad (3.30c)$$

$$(b, \tau) \in \Omega, \quad (3.30d)$$

where  $(W_*, W_a)$  is given in (3.7),  $\Omega$  is the bounded set given in (3.14) and  $\Phi$  is defined in (3.23), together with the corresponding optimal duration  $\bar{T}$  given by

$$\bar{T} = \int_{E(\bar{\tau}) - \bar{U}_0}^{W_a(\bar{\tau})} \frac{1}{H(\bar{b}, \bar{\tau})(s)} ds. \quad (3.31)$$

Let us underline that the use of standard Lagrangian techniques does not seem appropriate for investigating the constrained optimization problem (3.30). In the next section, we then use another methodology that first consist in simplifying the above formulation of the model and more specifically its set of constraints.

## 4 Characterization of optimal durations through a two dimensional model

In this section , we show that the agency's problem can be reduced to minimizing some continuous function of variables  $b$  and  $\tau$  over a closed and bounded set of  $\mathbb{R}^2$ . It is also obtained that the specific optimal durations  $\bar{T} = 0$  and  $\bar{T} = +\infty$  are associated only with optimal values  $(\bar{b}, \bar{\tau})$  that are located on some parts of the boundary of the set of constraints.

### 4.1 The agency's problem as an implicit two-dimensional minimization problem

We use the same notations as in the previous section and show that (3.30) can be rewritten as a more exploitable two-dimensional minimization problem. This strategy essentially brings the numbers of relevant variables down to only two, namely  $b$  and  $\tau$ .

In order to reach such a transformation, we begin with introducing the new variables  $y = E - U_0$  and  $\nu = E - U_a$  (namely,  $y = W_0$  and  $\nu = W_a$ ). Clearly, we have  $U_0 = E - y = (1/\eta)u(w - \tau) - y$ .

So model (3.30) can be alternatively considered through the following formulation

$$\min_{b, \tau, y} y - (1/\eta)u(w - \tau) \quad (4.1a)$$

$$\text{subject to : } \Phi(b, \tau, y, \nu(\tau)) = 0, \quad (4.1b)$$

$$W_*(b, \tau) < y < \nu(\tau), \quad (4.1c)$$

$$(b, \tau) \in \Omega, \quad (4.1d)$$

where  $\Omega$  is given in (3.14) and the other quantities are defined by

$$W_*(b, \tau) := F^{-1}(u(w - \tau) - u(b)), \quad (4.2a)$$

$$\nu(\tau) := F^{-1}(u(w - \tau) - u(z)), \quad (4.2b)$$

$$\Phi(b, \tau, y, \nu) := \tau H_{(b, \tau)}(y) + \rho(b) H'_{(b, \tau)}(\nu) \left( (b + \tau) H_{(b, \tau)}(y) \int_y^\nu \frac{ds}{H_{(b, \tau)}^2(s)} - \frac{\tau}{\eta} \right), \quad (4.2c)$$

together with  $\rho(b) := u(b) - u(z)$  ( $= H_{(b, \tau)}(\nu(\tau))$ ), while  $F$  and  $H_{(b, \tau)}$  are given in (3.2).

As a consequence, the following result follows immediately from Theorem 3.1.

**Lemma 4.1** *Any solution  $(\bar{T}, \bar{b}, \bar{\tau}, \bar{U}_0)$  of the agency's problem (2.14) is obtained from any element of the form  $(\bar{b}, \bar{\tau}, \bar{y})$  that solves (4.1), together with*

$$\bar{U}_0 = E(\bar{\tau}) - \bar{y} \text{ and } \bar{T} = \int_{\bar{y}}^{\nu(\bar{\tau})} \frac{1}{H_{(\bar{b}, \bar{\tau})}(s)} ds. \quad (4.3)$$



Now, we focus on simplifying the set of constraints involved in (4.1). More precisely, we prove that the constraints related to the parameters  $(b, \tau, y)$  can be reduced to  $(b, \tau) \in \Gamma$  (for some subset  $\Gamma$  of  $\Omega$ ) together with some unique value  $y = y(b, \tau)$  (that only depends on the pair  $(b, \tau)$ ).

Specifically, for  $(b, \tau) \in \Omega$ , we recall that  $\nu(\tau) > W_*(b, \tau)$  (since  $b > z$ ), and we discuss the existence and uniqueness of some real value  $y(b, \tau) \in [W_*(b, \tau), \nu(\tau)]$  such that  $\Phi(b, \tau, y, \nu(\tau)) = 0$ .

**Lemma 4.2** *Let  $(b, \tau) \in \Omega$ , set  $I_{(b, \tau)} = [W_*(b, \tau), \nu(\tau)]$  and consider the mapping  $\Phi_{(b, \tau)}$  defined for  $s \in I_{(b, \tau)}$  by*

$$\Phi_{(b, \tau)}(s) = \Phi(b, \tau, s, \nu(\tau)). \quad (4.4)$$

*Then  $\Phi_{(b, \tau)}$  is continuous and decreasing on  $I_{(b, \tau)}$ . Moreover,  $\Phi_{(b, \tau)}$  has a zero  $y(b, \tau)$  on  $I_{(b, \tau)}$  (which is unique) iff the pair  $(b, \tau)$  also satisfies*

$$\zeta(b, \tau) \geq 0, \quad \text{where } \zeta(b, \tau) := \frac{b+\tau}{F'(W_*(b, \tau))} - \frac{\tau}{\eta}. \quad (4.5)$$

*In particular, under condition (4.5), the following statements are reached:*

- if  $\zeta(b, \tau) > 0$  then  $y(b, \tau) \in ]W_*(b, \tau), \nu(\tau)[$ ;
- if  $\zeta(b, \tau) = 0$  then  $y(b, \tau) = W_*(b, \tau) < \nu(\tau)$ .

Proof. See Appendix. •

Lemma 4.2 gives us general conditions that guarantee that the budget constraint (4.1b) is binding. More specifically, the first statement in Lemma 4.2 ensures a positive value of the potential duration of unemployment  $T$  with positive values of  $b$  and  $\tau$  such that  $U_0 > U_a$ . The second statement stipulates that the budget constraint can be also satisfied with some positive values of  $b$  and  $\tau$  such that  $U_0 = U < U_a$  where  $U$  is constant. This last result implies according to (3.3) that the duration of unemployment  $T$  must be infinite.

Finally, we show that (4.1) can be transformed into a two-dimensional minimization problem over the set  $\Gamma$  defined as follows:

$$\Gamma = \{(b, \tau) \in \Omega; \zeta(b, \tau) > 0\}, \quad \text{where } \zeta \text{ is given in (4.5)}. \quad (4.6)$$

The next result is obviously deduced from Lemma 4.2.

**Lemma 4.3** *There exists a single-valued mapping  $y : \Gamma \rightarrow (0, +\infty)$  that is implicitly defined for  $(b, \tau) \in \Gamma$  by*

$$y(b, \tau) \in (W_*(b, \tau), \nu(\tau)), \quad \Phi(b, \tau, y(b, \tau), \nu(\tau)) = 0. \quad (4.7)$$

It is interesting to see from Lemma 4.2 that, for  $(b, \tau) \in \Gamma$ , the value of  $y(b, \tau)$  can be computed (numerically) by applying a dichotomy method to the mapping  $\Phi_{(b, \tau)}$  on  $[W_*(b, \tau), \nu(\tau)]$ .

The following theorem is readily deduced from Lemmas 4.1 and 4.3. It provides a more exploitable formulation of (4.1) that involves the set  $\Gamma$  introduced in (4.6) and that will be useful with regards to both theoretical and numerical viewpoints.

**Theorem 4.1** *Solutions  $(\bar{T}, \bar{b}, \bar{\tau}, \bar{U}_0)$  to the agency's problem (2.14) are obtained from solutions  $(\bar{b}, \bar{\tau})$  of the following optimization problem*

$$\min_{b, \tau} J(b, \tau) := y(b, \tau) - (1/\eta)u(w - \tau), \quad (4.8a)$$

$$\text{subject to : } (b, \tau) \in \Gamma, \quad (4.8b)$$

where  $y(b, \tau)$  is given in (4.7). Moreover, we have  $\bar{U}_0 = -J(\bar{b}, \bar{\tau})$  together with a finite positive optimal duration

$$\bar{T} = \int_{y(\bar{b}, \bar{\tau})}^{\nu(\bar{\tau})} \frac{1}{H_{(\bar{b}, \bar{\tau})}(s)} ds. \quad (4.9)$$

Theorem 4.1 establishes that the optimal duration  $\bar{T}$  is non-null and finite whenever (4.8) admits a solution belonging to  $\Gamma$ . Note also that  $J$  is bounded below over  $\Gamma$ , as the mapping  $y(\cdot, \cdot)$  is positive on  $\Gamma$  and as the utility function  $u$  is assumed to be continuous on  $(0, +\infty[$  (together with  $w - \tau \geq z > 0$  for  $(b, \tau) \in \Omega$ ). This latter fact does not guarantee the existence of a solution to (4.8), even if  $\Gamma$  is nonempty and  $J$  is continuous over  $\Gamma$ . Let us underly that  $\Gamma$  is bounded but not necessarily closed if nonempty. So any existing minimizer of the objective function  $J$  over  $\Gamma$  does not necessarily belong to  $\Gamma$  but to its closure (denoted by  $\bar{\Gamma}$ ). These issues will be investigated in the next section. However we do not prove uniqueness of a solution for the agency's problem.

## 4.2 Description of the feasible set and continuity of the objective function

As we cannot ensure the existence of a minimizer of  $J$  (occurring in Theorem 4.1) over  $\Gamma$  (introduced in (4.6)), we intend to guarantee the existence of a minimizer of  $J$  over the closure of  $\Gamma$ . For this purpose, we give a precise description of the geometric form of  $\Gamma$  and we establish the continuity of  $J$  over the closure of  $\Gamma$ . These informations will be helpful for discussing some specific cases of optimal duration.

**Lemma 4.4** *For  $(b, \tau) \in \Omega$ , we have the following equivalencies:*

$$\zeta(b, \tau) = 0 \Leftrightarrow \xi(b, \tau) = 0;$$

$$\zeta(b, \tau) > 0 \Leftrightarrow \xi(b, \tau) > 0,$$

where  $\zeta$  is defined in (4.5) and  $\xi$  is the function which writes

$$\xi(b, \tau) = F\left(c\left(\frac{\eta b}{\tau}\right)^\alpha\right) + u(b) - u(w - \tau). \quad (4.10)$$

So the set  $\Gamma$  can be alternatively defined by

$$\Gamma = \{(b, \tau) \in \Omega; \xi(b, \tau) > 0\}. \quad (4.11)$$

Proof. See appendix. •

Note that  $\Gamma$  is nothing but the set of elements of  $\Omega$  that do not belong to the set  $\Sigma$  defined by  $\Sigma = \{(b, \tau) \in \mathbb{R}^2; b > 0, \xi(b, \tau) \leq 0\}$ .

Resorting to the implicit function theorem, and using (4.11), we can provide a precise description of  $\Gamma$  (as illustrated on Figure 3).

**Lemma 4.5** *There exists a real-valued mapping  $g$  such that  $g$  is continuous on  $[0, w]$  and  $g$  is of class  $C^1$  on  $]0, w[$ , and such that for any  $(b, \tau) \in ]0, w[ \times ]0, w[$ ,*

$$(i1) \xi(b, \tau) = 0 \Leftrightarrow b = g(\tau);$$

$$(i2) \xi(b, \tau) > 0 \Leftrightarrow b > g(\tau).$$

Moreover, we have  $g(0) = g(w) = 0$  and for any  $\tau \in ]0, w[$  we have  $g(\tau) \in ]0, w - \tau[$ , so that  $\Gamma = \{(b, \tau) \in \Omega; b > g(\tau)\} \neq \emptyset$  and  $\Gamma$  can be expressed as

$$\Gamma = \{(b, \tau) \in \mathbb{R}^2; 0 < \tau < w - z, \max\{z, g(\tau)\} < b \leq w - \tau\}. \quad (4.12)$$

Proof. See Appendix. •

In light of Lemma 4.5 we obviously reach the following result.

**Lemma 4.6** *The set  $\Gamma$  is nonempty, not closed and  $\bar{\Gamma}$  (the closure of  $\Gamma$ ) is the closed set verifying  $\Gamma \subset \bar{\Gamma}$  and defined by*

$$\bar{\Gamma} = \{(b, \tau) \in \mathbb{R}^2; 0 \leq \tau \leq w - z, \max\{z, g(\tau)\} \leq b \leq w - \tau\}. \quad (4.13)$$

Now, we are in position to deal with the continuity of the objective function  $J$  over  $\bar{\Gamma}$ .

**Lemma 4.7** *The mapping  $y(\cdot, \cdot)$  (defined in (4.3)) is continuous over  $\Gamma$  and it can be extended to a continuous function over  $\bar{\Gamma}$  such that:*

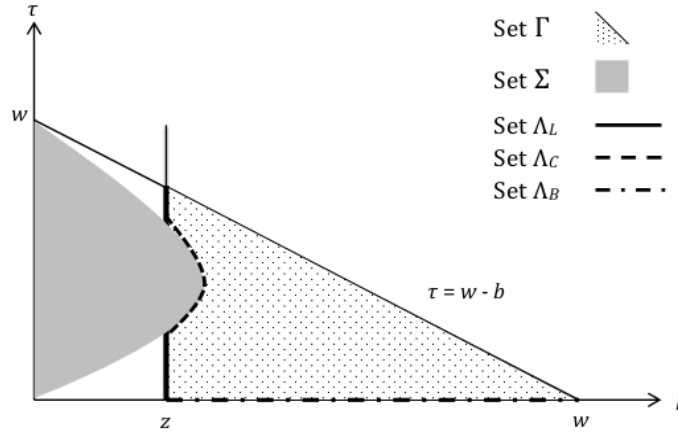
- if  $\tau = 0$  and  $z < b \leq w$ , then  $y(b, \tau) = \nu(0)$ ;
- if  $b = z$ , then  $y(b, \tau) = \nu(\tau)$ ;
- if  $b > z$  and  $b = g(\tau)$  then  $y(b, \tau) = W_*(b, \tau)$ .

So the objective  $J : (b, \tau) \rightarrow y(b, \tau) - 1/\eta u(w - \tau)$  can be regarded as a continuous mapping over  $\bar{\Gamma}$ .

Proof. See Appendix. •

### 4.3 Optimal durations linked with possible minimizers

Let us recall that  $\Gamma$  is a bounded but not a closed set, and so, by continuity of  $J$  over  $\bar{\Gamma}$  (from Lemma 4.7), we know that there exists  $(\bar{b}, \bar{\tau})$  belonging to  $\bar{\Gamma}$  (but not necessarily


 Figure 3: The set  $\Gamma$ .

belonging to  $\Gamma$ ) and such that  $J(\bar{b}, \bar{\tau}) = \min_{\Gamma} J$ . Such an element  $(\bar{b}, \bar{\tau})$  equivalently solves the following well-posed minimization problem

$$\min_{(b, \tau) \in \bar{\Gamma}} J(b, \tau). \quad (4.14)$$

It is also clear that  $\operatorname{argmin}_{\Gamma} J$ , the set of (feasible) solutions to (4.8), is nothing but the set of solutions to (4.14) that also belong to  $\Gamma$ . So the set of solutions to (4.14) can be divided into two parts  $S_1$  and  $S_2$  that are described below:

1) The first part  $S_1$  corresponds to the set of solutions to (4.14) that also belong to  $\Gamma$ , which is nothing but  $\operatorname{argmin}_{\Gamma} J$ , namely the set of (feasible) solutions to (4.8). In particular, each element  $(\bar{b}, \bar{\tau}) \in S_1$  is linked with some non-zero and finite optimal duration for the agency's problem given by  $T(\bar{b}, \bar{\tau}) = \int_{y(\bar{b}, \bar{\tau})}^{\nu(\bar{\tau})} \frac{1}{H_{(\bar{b}, \bar{\tau})}(s)} ds$  (see, Theorem 4.1), hence  $0 < T(b, \tau) < +\infty$ .

2) The second part  $S_2$  corresponds to the set of solutions to (4.14) that do not belong to  $\Gamma$ , which is nothing but the set of elements  $(\bar{b}, \bar{\tau}) \in \bar{\Gamma} \setminus \Gamma$  verifying  $J(\bar{b}, \bar{\tau}) = \min_{\Gamma} J$ . In other words,  $S_2$  is the set of solutions to (4.14) that belong to one of the sets  $\Lambda_B$ ,  $\Lambda_L$  and  $\Lambda_C$  defined by:

$$\Lambda_B = \{(b, \tau) : \tau = 0, z < b \leq w\}, \quad (4.15)$$

$$\Lambda_L = \{(b, \tau) : b = z, 0 \leq \tau \leq w - z, \zeta(z, \tau) > 0\}, \quad (4.16)$$

$$\Lambda_C = \{(b, \tau) : z < b < w, 0 < \tau < w - b, \zeta(b, \tau) = 0\}. \quad (4.17)$$

Specifically,  $S_2$  can be regarded as the set of infeasible solutions to (4.8). Any element  $(\bar{b}, \bar{\tau})$  of  $S_2$  does not solve the agency's problem (4.8) but can be approximated by an infinite sequence  $((b_n, \tau_n)) \subset \Gamma$  (also referred to as a minimizing sequence of  $J$  over  $\Gamma$ ) such that  $J(b_n, \tau_n)$  decreases and converges to  $J(\bar{b}, \bar{\tau})$  (as  $n \rightarrow +\infty$ ). It is also obvious that each element  $(b_n, \tau_n)$  is associated with some non-zero and finite duration of UI benefits  $T(b_n, \tau_n) = \int_{y(b_n, \tau_n)}^{\nu(\tau_n)} \frac{1}{H_{(b_n, \tau_n)}(s)} ds$ . So an optimal duration for the (4.8) linked with  $(\bar{b}, \bar{\tau})$

can be regarded as the value given by  $\bar{T} = \lim_{n \rightarrow +\infty} T(b_n, \tau_n)$ , whenever this limit exists. In particular, we have  $\bar{T} = T(\bar{b}, \bar{\tau})$ , provided that the mapping  $T(., .)$  is continuous over  $\bar{\Gamma}$ .

In addition, for  $(b, \tau) \in \bar{\Gamma}$ , we know that  $W_*(b, \tau)$ ,  $\nu(\tau)$  and  $y(b, \tau)$  are well-defined values (see, Lemma 4.7) that satisfies  $W_*(b, \tau) \leq y(b, \tau) \leq \nu(\tau)$ . It also is easily seen that  $1/H_{(b, \tau)}$  is continuous over  $(W_*(b, \tau), +\infty)$  and that the following property holds:

$$\frac{1}{H_{(b, \tau)}(s)} \sim \frac{1}{(F^{(1)}(W_*(b, \tau)))(s - W_*)} \text{ as } s \rightarrow W_*(b, \tau). \quad (4.18)$$

It follows from the limit comparison test (for improper integrals) that a sufficient condition for having  $T(b, \tau) = +\infty$  is given by  $y(b, \tau) = W_*(b, \tau)$  and  $y(b, \tau) < \nu(\tau)$ .

Therefore the following observations can be done in light of Lemma 4.7.

**Lemma 4.8** For any element  $(b, \tau) \in \bar{\Gamma}$ , we have the following results:

- $(b, \tau) \in \Lambda_B \cup \Lambda_L \Rightarrow y(b, \tau) = \nu(\tau) \Rightarrow T(b, \tau) = 0$ ;
- $(b, \tau) \in \Lambda_C \Rightarrow y(b, \tau) = W_*(b, \tau)$  and  $y(b, \tau) < \nu(\tau) \Rightarrow T(b, \tau) = +\infty$ .

The following theorem is immediately deduced from the previous arguments.

**Theorem 4.2** Let  $S := \operatorname{argmin}_{\bar{\Gamma}} J$  be the set of solutions of (4.14). An optimal duration for the agency's problem can be considered through the following situations related to (4.8):

- If  $S \cap \Gamma \neq \emptyset$ , then any element  $(\bar{b}, \bar{\tau}) \in S \cap \Gamma$  is linked with a positive and finite optimal duration  $\bar{T}$  for the worker's problem, given by  $\bar{T} = T(\bar{b}, \bar{\tau})$ .
- If  $S \cap \Gamma = \emptyset$ , then there exists some element  $(\bar{b}, \bar{\tau}) \in S \cap (\Lambda_B \cup \Lambda_L \cup \Lambda_C)$ . Such an element is linked with a optimal duration  $\bar{T}$  for the worker's problem, given by

$$\bar{T} = T(\bar{b}, \bar{\tau}) = \begin{cases} 0, & \text{if } (\bar{b}, \bar{\tau}) \in \Lambda_B \cup \Lambda_L, \\ +\infty, & \text{if } (\bar{b}, \bar{\tau}) \in \Lambda_C, \end{cases} \quad (4.19)$$

provided that the mapping  $T(., .)$  is continuous over  $\bar{\Gamma}$

If the minimizer of the function  $J$  over  $\bar{\Gamma}$  belongs to the set  $\Lambda_C$ , then it is always possible to find a pair  $(b, \tau) \in \Gamma$  more and more closer to  $(\bar{b}, \bar{\tau})$  and linked with a more and more longer potential duration of UI benefits  $T$ . When  $(b, \tau)$  is close enough of  $(\bar{b}, \bar{\tau})$ , the optimal insurance system is then characterized by a flat profile of unemployment payments since the assistance system is never reached.

If the minimizer of the function  $J$  over  $\bar{\Gamma}$  belongs to the set  $\Lambda_B \cup \Lambda_L$ , then, according to Theorem 4.2, there also always exists a couple  $(b, \tau) \in \Gamma$  that tends to  $(\bar{b}, \bar{\tau})$  and improves the value of  $J$  with a compatible shorter potential duration of UI benefits  $T$ . If  $(b, \tau) \in \Gamma$  is sufficiently near to  $(\bar{b}, \bar{\tau})$ , the optimal UI then meets the assistance system.

## 5 Numerical illustrations

In this section we give an illustrative numerical example of our model based on the American economy. We first roughly calibrate some unknown parameters so as to fit some keys characteristics of the American labor market.

### 5.1 Calibration of the model

Let us consider the very usual CRRA function  $u$  defined by

$$u(x) = \frac{1}{1-\lambda}x^{1-\lambda} \text{ with } \lambda \neq 1. \quad (5.20)$$

The aversion risk parameter  $\lambda$  is calibrated to be 2 but we discuss later of its impact on the numerical results. The time unit is one month. The interest rate is therefore set to  $r = 0.00327$  which corresponds to an annual rate of 4%. In the United-Sates, life expectancy at birth is somehow above 78 years whereas the minimum legal age at which people can work is 15 years (OECD, 2014). The life expectancy in our model is set at 63 years since we only consider the working age population, which makes the death monthly rate  $\mu = 0.0013$ . The duration of UI benefits  $T$  is set at 6 months which is the duration of regular UI benefits. To calibrate the UI program, we follow Nakajima (2012), who takes into account both monetary and non-monetary benefits of unemployment. Eligible UI benefits  $b$  and non-eligible unemployment benefits  $z$  are then respectively set to \$1393 and \$961 with a mean wage equals to \$3202.

Parameters		Value
$\alpha$	Curvative parameter of disutility from search	0.827
$c$	Level parameter of disutility from search	0.019
$\mu$	Death rate	0.0013
$r$	Monthly interest rate	0.00327
$\lambda$	Coefficient of relative risk aversion	2
$T$	Maximum period of UI eligibility in months	6
$w$	Gross wage	3202
$b$	Unemployment insurance benefits	1393
$z$	Unemployment assistance benefits	961
$1/\varepsilon_0$	Mean duration of unemployment for eligible unemployed at 0	5
$1/\varepsilon_a$	Mean duration of unemployment for non eligible unemployed	4

Table 1: Calibration

It is also necessary to assign a value to the level parameter of disutility of search  $c$  and a value to the curvature parameter of disutility from search  $\alpha$ , for which there are no available information. The values of  $\alpha$  and  $c$  are chosen such that the mean duration of unemployment for an eligible and an non-eligible unemployed are respectively 4 and 5 months. Then we uncover the values  $\{c, \alpha, y, \tau\}$  by solving the system composed of the following constraints: the first order conditions for optimal search effort (2.3) and (2.5), the equality (2.7) that gives  $U_a$ , the budget constraint (4.7) and the expression of the potential duration of UI benefits (3.31) from Theorem 3.1.

## 5.2 Simulations results

Using the values of parameters given in the previous section, we begin with illustrating the evolution in time of the search effort and the gains from employment. As expected from Lemma 3.2, search effort increases monotonously over time (Fig. 4) while the expected utility of an eligible jobless,  $U(t)$ , decreases (Fig. 5).

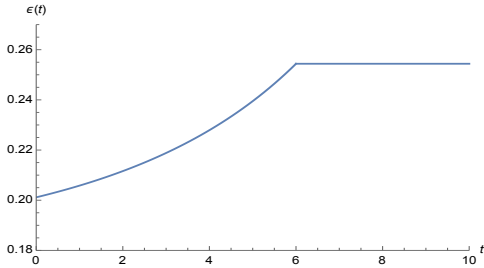


Figure 4: The evolution of  $\varepsilon(t)$  and the duration of unemployment.

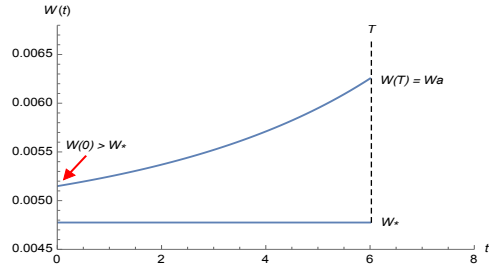


Figure 5: Evolution in time of the gains from employment  $W(t)$ .

The computation of the optimal contract is described in the second row of Table 2 (see also Fig. 6 and 7). The optimal duration of UI benefits is 12.68 months, double the potential duration of the baseline model with more generous UI benefits but also a higher tax rate.<sup>4</sup> The latter was equal to 21.9 in the baseline model *versus* 78.63 for the optimal contract. The greater generosity of the optimal contract reduces incentives to find a job since search efforts decrease significantly for the eligible unemployed to  $\varepsilon_0 = 0.1$  (at  $t = 0$ ) and, to a lesser extent, for the non-insured unemployed to  $\varepsilon_a = 0.249$ .

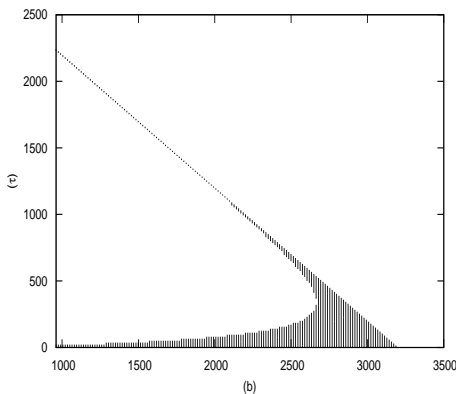


Figure 6: Geometric form of the domain  $\Gamma$ .

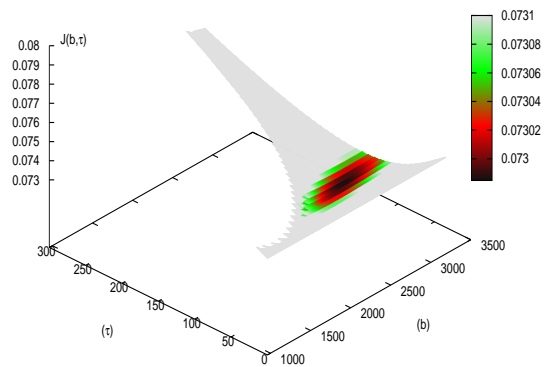


Figure 7: 3D view of the profile of  $J$  near the solution  $(\bar{b}, \bar{\tau}) = (2653.52, 78.63)$

<sup>4</sup>The optimal duration of UI seems very high for the American economy. Let us underline that these simulations are only for illustrative purposes. Taking into account, for example, transitions between employment and unemployment or eligibility criteria would certainly affect our numerical results.

Finally, we consider how the results can be affected when the risk aversion parameter  $\lambda$  is modified. Intuitively, the risk aversion parameter should play an important role in the definition of the optimal contract since a higher value of risk aversion makes both consumption smoothing and insurance more important for the worker.

Rows 3-7 of Table 2 reports the results obtained when the risk aversion is gradually

$\lambda$	$b$	$\bar{\tau}$	$T$	$U_0$
2	2653.26	78.63	12.68	-0.073
2.1	2611.11	121.23	19.72	-0.031
2.2	2561.14	184.962	29.88	-0.013
2.3	2486.27	276.693	43.88	-0.0062
2.4	2382.94	402.69	61.60	-0.0034
2.5	2253.0	564.82	81.11	-0.0015

Table 2: Optimal unemployment insurance

raised to 2.5. We can notice the striking increase of the optimal expected utility  $U_0$  with higher values of  $\lambda$  and the trade-off between the level of UI benefits and the duration of their paiement.

However, it is obvious that with such degrees of risk aversion, the optimal contract gives much lower incentives for employment than the baseline program. In our framework and for the parameters chosen, a later exhaustion date of UI benefits and longer duration of unemployment, as  $\lambda$  grows, do not compensate the financial gains from the fall in benefits, that leads to an increasing tax rate.

## 6 Conclusion

If a significant work has been done to determine the optimal profile of UI benefits, it was generally with the implicit assumption that their potential duration was infinite. It is true that a fixed and predictable date of exhaustion of benefits often brings inextricable or for the less hardly extricable technical difficulties since one has to deal with time dependency of search effort (see Mortensen, 1977; van der Berg, 1990). To stride over these drawbacks, some models assume a given Poisson process for a random duration of benefits and/or run numerical exercises to determine the optimal profile. In so doing, they can find an infinite (e.g. Davidson and Woodbury, 1997) or a finite (e.g. Fredriksson and Holmlund, 2001) expected optimal duration.

In this paper we propose a new methodology based on the study of the geometric properties of the set of constraints to deal with nonstationarity complexities. This strategy permits us to formally characterize the optimal duration of UI benefits. However, the model used here is very basic and many extensions can be contemplated by lifting some restrictions. In particular, we assume constant benefits and wage taxes over the spell of unemployment and an exogenous assistance payment. Maybe a decreasing sequence of benefits with an assistance payment optimally chosen could substantially influence the trade-off between the level and the duration of UI benefits. Another important feature of our model is the absence of financial markets. In a sequential job search model, Shimer and Werning (2008) show how savings and borrowing behaviors can play a significant role



in the definition of an optimal insurance unemployment system. It is probably the case in our model. These questions will be the subject of future investigations.

## 7 Appendix

### 7.1 Proof of Lemma 3.1

We begin with proving the existence and uniqueness of a global classical solution to (3.9). Let us recall that the mapping  $H$  involved in (3.9) is defined for  $s \geq 0$  by :

$$H(s) = \eta s + \beta s^{1+q} - \gamma \quad (\text{for some values } \eta > 0, \beta > 0, \gamma \geq 0 \text{ and } q > 0),$$

hence its derivative  $H'$  is defined for  $s \geq 0$  by :

$$H'(s) = \eta + \beta(q+1)s^q.$$

Clearly,  $H$  is of class  $C^1$  on the open set  $(0, +\infty)$ , hence  $H$  is locally Lipschitz on  $(0, +\infty)$ . Consequently, for  $W_a \in (0, +\infty)$ , by the Cauchy-Lipschitz theorem, we know that there exists a maximal positive time  $T_m$  (finite or infinite) that ensures existence and uniqueness of a classical solution  $V$  to (3.9) on  $[0, T_m)$  such that

$$V(t) \in (0, +\infty) \text{ for } t \in [0, T_m).$$

We also know that there exists some value  $W_* \geq 0$  such that  $H(W_*) = 0$ . Then, given any  $t \in [0, T_m)$  and applying the Taylor formula to  $H$  between  $V(t)$  and  $W_*$  gives us:

$$H(V(t)) = H(W_*) + (V(t) - W_*)H'(\chi(t)) = (V(t) - W_*)H'(\chi(t)),$$

where  $\chi(t)$  is some value between  $V(t)$  and  $W_*$  (hence  $\chi(t)$  is nonnegative). So, for  $t \in [0, T_m)$ , by (3.9),  $H(W_*) = 0$  and setting  $Z(\cdot) = V(\cdot) - W_*$ , we equivalently obtain  $\dot{Z}(t) = -Z(t)H'(\chi(t))$ , namely,

$$\frac{d}{dt} \left( Z(t)e^{\phi(t)} \right) (t) = 0, \quad (7.1)$$

together with  $\phi(t) := \int_0^t H'(\chi(s))ds$ , namely

$$\phi(t) = \eta t + \int_0^t P(s)ds, \text{ where } P(s) = \beta(q+1)(\chi(s))^q.$$

Hence, given  $t \in [0, T_m)$ , by integrating (7.1) between 0 and  $t$  we get  $Z(t) = Z(0)e^{-\phi(t)}$ , that is

$$V(t) - V(0) = (V(0) - W_*)e^{-\eta t - \int_0^t P(s)ds}. \quad (7.2)$$

As a result, we immediately deduce that  $V(T_m^-)$  ( $:= \lim_{t \rightarrow T_m^-} V(t)$ ) is well-defined and positive (namely,  $V(T_m^-) \in (0, +\infty)$ ), which entails that  $T_m = +\infty$  (otherwise,  $T_m$  cannot be the maximal existence time of  $V$ ). This ensures the existence and uniqueness of a global classical solution  $V \in C^2(\mathbb{R}_+)$  to (3.9), given by (7.2) for  $t \in \mathbb{R}_+$ . This leads to the desired result. •

### 7.2 Proof of Lemma 3.2

Given  $T > 0$  and setting  $W(t) = V(T - t)$ , where  $V$  is the global classical solution to (3.9) (hence  $V \in C^2(\mathbb{R}_+)$  according to Lemma 3.1), we clearly have

$$W \in C^2([0, T]) \text{ and } W(T) = V(0) = W_a,$$

while for  $t \in [0, T]$  and by (3.9) we get

$$\dot{W}(t) = -\dot{V}(T - t) = H(V(T - t)) = H(W(t)).$$

This entails the existence of a classical solution on  $[0, T]$  to the following system:

$$\dot{W} = H(W), \text{ with } W(T) = W_a. \quad (7.3)$$

Conversly, we set  $V(t) = W(T - t)$ , where  $W$  is a classical solution to (7.3) on  $[0, T]$ . It is clear that  $V \in C^1([0, T])$  and, for  $t \in [0, T]$ , by (7.3) we obtain

$$\dot{V}(t) = -\dot{W}(T - t) = -H(W(T - t)) = -H(V(t)),$$

together with  $V(0) = W(T) = W_a$ .

Then, observing that such a function  $V$  is uniquely defined on  $[0, T]$  (according to Lemma 3.1), we deduce that (7.3) has at most one classical solution on  $[0, T]$  and that this unique solution is given for  $t \in [0, T]$  by

$$W(t) = V(T - t), \tag{7.4}$$

where  $V$  is the global classical solution to (3.9). Now, from Lemma 3.1, we have

$$V(t) = W_* + (V(0) - W_*)e^{-\phi(t)}.$$

As the mapping  $\phi$  is increasing on  $\mathbb{R}_+$  and satisfies  $\phi(0) = 0$  and  $\lim_{t \rightarrow +\infty} \phi(t) = +\infty$ , we readily deduce the following statements:

(k1)  $V(0) > W_* \Rightarrow V$  is decreasing and  $W_* < V \leq V(0)$  on  $\mathbb{R}_+$ ;

(k2)  $V(0) = W_* \Rightarrow V \equiv W_*$  on  $\mathbb{R}_+$ ;

(k3)  $V(0) < W_* \Rightarrow V$  is increasing and  $V(0) \leq V < W_*$  on  $\mathbb{R}_+$ .

The rest of the proof follows from the above results (k1)-(k3) together with (7.4). •

### 7.3 Proof of Lemma 3.4

Proof. Let us recall that  $B$  satisfies on  $[0, T]$  the cost differential equation (2.11) that can be rewritten as

$$\dot{B}(t) = a(t)B(t) + f(t), \text{ where } a(t) = \eta + \varepsilon(t) \text{ (with } \eta = r + \mu) \text{ and } f(t) = \frac{\tau}{\eta}\varepsilon(t) - b.$$

From this equation and setting  $G_T(t) = \int_t^T a(s)ds$ , we equivalently have

$$\frac{d}{dt} \left( B(t)e^{G_T(t)} \right) = f(t)e^{G_T(t)}.$$

Now, integrating on  $[t, T]$  each side of this last equality amounts to

$$B(t)e^{G_T(t)} - B(T) = - \int_t^T f(\rho)e^{G_T(\rho)} d\rho. \tag{7.5}$$

From the definitions of  $f$  and  $a$  and observing that  $G_T'(t) = -a(t)$  we also have

$$f(t) = \frac{\tau}{\eta}a(t) - (b + \tau) = -\frac{\tau}{\eta}G_T'(t) - (b + \tau),$$

hence,

$$f(t)e^{G_T(t)} = -\frac{\tau}{\eta} \frac{d}{dt} \left( e^{G_T(t)} \right) - (b + \tau)e^{G_T(t)}.$$

So, by (7.5) and noticing that  $G_T(T) = 0$ , we get

$$B(t)e^{G_T(t)} - B(T) = \frac{\tau}{\eta}(1 - e^{G_T(t)}) + (b + \tau) \int_t^T e^{G_T(\rho)} d\rho. \tag{7.6}$$

Checking that  $G_T(\rho) = G_T(t) - G_\rho(t)$  and multiplying equality (7.6) by  $e^{-G_T(t)}$ , we deduce

$$B(t) = (B(T) + \frac{\tau}{\eta})e^{-G_T(t)} + (b + \tau) \int_t^T e^{-G_\rho(t)} d\rho - \frac{\tau}{\eta},$$

which entails the desired value of  $B(0)$  (with respect to  $T$  and  $B(T)$ ). •

## 7.4 Proof of Lemma 3.5

We recall that  $W = E - U$  satisfies on  $[0, T]$  the utility equation (3.3) which writes

$$\dot{W} = \eta W + \beta W^{1+1/\alpha} - \gamma, \quad (\text{with constants } \eta > 0, \gamma \geq 0 \text{ and } \beta = c^{-1/\alpha} \frac{\alpha}{\alpha+1}).$$

So differentiating the above equation on  $[0, T]$  yields

$$\ddot{W} = \eta \dot{W} + c^{-1/\alpha} W^{1/\alpha} \dot{W}. \quad (7.7)$$

In light of Lemma 3.2, we also know that  $W$  is increasing and satisfies  $W_* < W$  on  $[0, T]$ , hence, by  $\dot{W}(t) = H(W(t))$  (from (3.3)), we can see that  $\dot{W}$  is positive on  $[0, T]$  (as  $H$  is increasing on  $\mathbb{R}_+$  and  $H(W_*) = 0$ ). Consequently, recalling that  $\epsilon = c^{-1/\alpha} W^{1/\alpha}$  (from (2.3)), by (7.7) we deduce that  $\epsilon = \frac{\dot{W}}{W} - \eta$ . Next, by integrating on  $[0, t]$ , for  $t \in [0, T]$ , we get  $\eta t + \int_0^t \epsilon(s) ds = -\ln \frac{\dot{W}(0)}{\dot{W}(t)}$ , which by  $\psi(t) = e^{-\eta t - \int_0^t \epsilon(s) ds}$  (namely the definition of  $\psi$ ) amounts to

$$\psi(t) = \frac{\dot{W}(0)}{\dot{W}(t)}. \quad (7.8)$$

So, recalling that  $\dot{W}(t) = H(W(t))$  (from (3.3)), we immediately deduce (3.19a). Now, from (7.8) and (3.3), we readily have

$$\int_0^T \psi(x) dx = H(W(0)) \int_0^T \frac{dt}{\dot{W}(t)},$$

hence, by the change of variable  $s = W(t)$  (hence  $ds = \dot{W} dt$ ), we obtain

$$\begin{aligned} \int_0^T \psi(x) dx &= H(W(0)) \int_{W(0)}^{W(T)} \frac{ds}{(\dot{W}(t))^2} = H(W(0)) \int_{W(0)}^{W(T)} \frac{ds}{(H(W(t)))^2} \\ &= H(W(0)) \int_{W(0)}^{W(T)} \frac{ds}{H^2(s)}, \end{aligned}$$

that is (3.19b). Finally, taking into account (2.10) and (2.5), we recall that  $B_a = -\frac{\tau \epsilon^a}{\eta(\eta + \epsilon^a)}$ ,

where  $\eta = r + \mu$  and  $\epsilon^a = c^{-1/\alpha} W_a^{1/\alpha}$ . Consequently, we obtain

$$B_a + \frac{\tau}{\eta} = \frac{\tau}{\eta} \left( -\frac{\epsilon^a}{\eta + \epsilon^a} + 1 \right) = \frac{\tau}{\eta + \epsilon^a}.$$

Then it can be seen that  $B_a + \frac{\tau}{\eta} = \frac{\tau}{F'(W_a)}$ , that is (3.19c). •

## 7.5 Proof of Lemma 3.8

Let  $W_0$  be such that  $0 \leq W_* < W_0 < W_a$ . Then, it is not difficult to see from the formulation of  $V$  given in Lemma 3.1 that there exists a unique positive time  $T$  such that  $V(T) = W_0$ . Moreover, by Lemma 3.2, we deduce that there exists a unique mapping  $W$  verifying (3.3) with  $W(T) = W_a$ . Also recalling from Lemma 3.2 that  $W(t) = V(T - t)$  for  $t \in [0, T]$ , we obtain  $W(0) = V(T) = W_0$ .

Furthermore, we know by Lemma 3.2 that  $W$  is strictly increasing on  $[0, T]$ , while it is obvious that  $T = \int_0^T ds$ . Consequently, by the change of variable  $u = W(s)$  (hence  $du = \dot{W}(s) ds$ ), and recalling that  $\dot{W}(s) = H(W(s))$ , we obtain

$$T = \int_{W(0)}^{W(T)} \frac{1}{H(W(s))} du = \int_{W_0}^{W_a} \frac{1}{H(u)} du.$$

This ends the proof. •

## 7.6 Proof of Lemma 3.9

Let  $\mathcal{C}_1$  denote the set of elements  $(b, \tau, U_0, T)$  verifying conditions (i1) to (i3). The proof can be divided into two parts:

- Given  $(b, \tau, U_0, T) \in \mathcal{C}$ , by definition of  $\mathcal{C}$ , we have  $(b, \tau) \in \Omega$  and there exists some  $U \in \mathcal{K}(T, b, \tau, U_0) \cap \mathcal{S}(T, b, \tau) \neq \emptyset$ . Consequently, setting  $W = E - U$ , we know that  $W$  satisfies (3.3) on  $[0, T]$ , together with  $W(0) = E - U_0$ ,  $W(T) = W_a$  and  $W_* < W(0) < W_a$ . As a result, invoking Lemma 3.8, we conclude that  $T$  is nothing but the positive quantity given by  $T = \int_{W(0)}^{W_a} \frac{1}{H(s)} ds$ . So we have  $(b, \tau, U_0, T) \in \mathcal{C}_1$  and it follows that  $\mathcal{C} \subset \mathcal{C}_1$ .

- Conversely, taking  $(b, \tau, U_0, T) \in \mathcal{C}_1$ , we obviously have  $(b, \tau) \in \Omega$  together with  $W_* < E - U_0 < W_a$  and  $T = \int_{E-U_0}^{W_a} \frac{1}{H(s)} ds$ . Applying Lemma 3.8 yields that there exists a unique classical solution  $W$  to (3.3) on  $[0, T]$  with  $W(0) = E - U_0$  and  $W(T) = W_a$ . So, setting  $U = E - W$ , we can easily check that  $U$  satisfies the differential equation (2.6) on  $[0, T]$ . Then it is readily seen that  $U \in \mathcal{K}(T, b, \tau, U_0) \cap \mathcal{S}(T, b, \tau)$ . As a result we get  $(b, \tau, U_0, T) \in \mathcal{C}$  and we deduce that  $\mathcal{C}_1 \subset \mathcal{C}$ .

The conclusion  $\mathcal{C} = \mathcal{C}_1$  follows immediately. •

## 7.7 Proof of Lemma 4.2

We begin with stating a preliminary estimate that will help us to compute the value  $\Phi(b, \tau, y, \nu)$  defined (4.2c) in special cases when  $y \rightarrow W_*(b, \tau)$ . For the sake of legibility we sometimes omit the parameters  $b$  and  $\tau$  in the formulations of  $\nu(\tau)$  and  $W_*(b, \tau)$ .

**Lemma 7.1** *Let  $(b, \tau) \in \Omega$  and consider real values  $y$  and  $\nu$  verifying  $W_*(b, \tau) < y < \nu$ , and set  $\Psi(b, \tau, y, \nu) := H(y) \int_y^\nu \frac{ds}{H^2(s)}$ . Then the following estimate is reached:*

$$\left| \Psi(b, \tau, y, \nu) - \frac{H'(l_y)(\nu - y)}{(H'(W_*))^2(\nu - W_*)} \right| \leq (2c^{-q}\eta^{-3})D(b, \tau, y, \nu), \quad (7.9)$$

for some  $l_y \in (W_*, y)$  together with

$$D(b, \tau, y, \nu) = \begin{cases} \frac{1}{1-q} H'(l_y) \left( (y - W_*)^q - \frac{y - W_*}{(\nu - W_*)^{1-q}} \right), & \text{if } q < 1, \\ -q\nu^{q-1} H'(l_y)(y - W_*) \ln \left( \frac{y - W_*}{\nu - W_*} \right), & \text{otherwise.} \end{cases} \quad (7.10)$$

Proof. Given  $s \in (y, \nu)$ , applying Taylor's formula to  $H$  between  $s$  and  $W_*(b, \tau)$ , and recalling that  $H(W_*) = 0$ , we have  $H(s) = (s - W_*)H'(c_s)$ , for some  $c_s \in (W_*, s)$ , with  $H'(c_s) = \eta + c^{-q}c_s^q$ . So we get

$$\begin{aligned} \frac{1}{(H(s))^2} - \frac{1}{(s - W_*)^2 (H'(W_*))^2} &= \frac{1}{(s - W_*)^2} \left( \frac{1}{(H'(c_s))^2} - \frac{1}{(H'(W_*))^2} \right) \\ &= \frac{(H'(W_*) - H'(c_s))}{(s - W_*)^2} \left( \frac{H'(W_*) + H'(c_s)}{(H'(c_s))^2 (H'(W_*))^2} \right), \end{aligned}$$

namely

$$\begin{aligned} \frac{1}{(H(s))^2} - \frac{1}{(s - W_*)^2 (H'(W_*))^2} &= -r(s), \\ \text{where } r(s) &:= \frac{(H'(c_s) - H'(W_*))}{(s - W_*)^2 (H'(W_*))^2} \left( \frac{H'(W_*)}{(H'(c_s))^2} + \frac{1}{H'(c_s)} \right). \end{aligned} \quad (7.11)$$

So integrating this last equality between  $y$  and  $\nu$  amounts to

$$\int_y^\nu \frac{1}{(H(s))^2} ds - \frac{(\nu - y)}{(H'(W_*))^2 (y - W_*) (\nu - W_*)} = - \int_y^\nu r(s) ds.$$

Moreover, again by Taylor's formula, we have

$$H(y) = (y - W_*)H'(l_y), \text{ for some } l_y \in (W_*, y).$$

Consequently, recalling that  $\Psi(b, \tau, y, \nu) := H(y) \int_y^\nu \frac{1}{(H(s))^2} ds$  and using the previous two equalities, we obtain

$$\begin{aligned} \Psi(b, \tau, y, \nu) &= P - Q, \\ \text{where } P &= \frac{(\nu - y)H'(l_y)}{(H'(W_*))^2(\nu - W_*)}, \quad Q = (y - W_*)H'(l_y) \int_y^\nu r(s) ds, \end{aligned}$$

which gives us

$$|\Psi(b, \tau, y, \nu) - P| = |Q| = Q. \quad (7.12)$$

Moreover, on the one hand, we have

$$0 \leq H'(c_s) - H'(W_*) = c^{-q}(c_s^q - W_*^q) \leq c^{-q}(s^q - W_*^q).$$

On the other hand, noticing that  $0 < H'(c_s) \leq H'(W_*)$ , we have

$$0 \leq \frac{H'(W_*)}{(H'(c_s))^2} + \frac{1}{H'(c_s)} \leq \frac{H'(W_*)}{(H'(W_*))^2} + \frac{1}{H'(W_*)} = \frac{2}{H'(W_*)}.$$

Hence by (7.11) we deduce that

$$0 \leq r(s) \leq \delta \frac{s^q - W_*^q}{(s - W_*)^2}, \text{ where } \delta := 2c^{-q}\eta^{-3}.$$

It is also a classical matter to check for  $W_* \leq s \leq \nu$  that:

$$s^q - W_*^q \leq (s - W_*)^q, \quad \text{if } q < 1, \quad (7.13a)$$

$$s^q - W_*^q \leq q\nu^{q-1}(s - W_*), \quad \text{if } q \geq 1. \quad (7.13b)$$

As a result, for  $q < 1$ , we obtain

$$\begin{aligned} 0 \leq Q &\leq \delta(y - W_*)H'(l_y) \int_y^\nu (s - W_*)^{q-2} ds \\ &= \frac{\delta}{q-1}(y - W_*)H'(l_y) ((\nu - W_*)^{q-1} - (y - W_*)^{q-1}), \end{aligned}$$

namely

$$0 \leq Q \leq \frac{\delta}{1-q}H'(l_y) \left( (y - W_*)^q - \frac{y - W_*}{(\nu - W_*)^{1-q}} \right) \quad (\text{for } q < 1). \quad (7.14)$$

In a similar way, for  $q \geq 1$ , we obtain

$$0 \leq Q \leq \delta q \nu^{q-1} (y - W_*) H'(l_y) \int_y^\nu \frac{1}{(s - W_*)} ds,$$

namely

$$0 \leq Q \leq -\delta q \nu^{q-1} H'(l_y) (y - W_*) \ln \left( \frac{y - W_*}{\nu - W_*} \right) \quad (\text{for } q \geq 1). \quad (7.15)$$

This desired result follows from (7.12) together with (7.14) and (7.15). •

Now we are in position to prove Lemma 4.2. Given  $(b, \tau) \in \Omega$ , we readily observe that the mapping  $\Phi_{(b, \tau)}$  is of class  $C^2$  over  $(W_*(b, \tau), \nu(\tau)]$ . In addition, by definition of  $\Phi_{(b, \tau)}$  and by  $H'(\nu) = \eta + c^{-q}\nu^q$ , we obtain

$$\Phi_{(b, \tau)}(\nu(\tau)) = \tau H(\nu(\tau)) (1 - (1/\eta)H'(\nu(\tau))) \quad (7.16)$$

$$= -\frac{c^{-q}(\nu(\tau))^q}{\eta} \tau H(\nu(\tau)). \quad (7.17)$$

So it is easily checked that  $\Phi_{(b, \tau)}(\nu(\tau))$  is negative. Furthermore, for  $y \in (W_*(b, \tau), \nu(\tau)]$  we obtain the following derivatives:

$$\Phi_{(b, \tau)}^{(1)}(y) = \tau H'(y) - H(\nu(\tau))H'(\nu(\tau))(b + \tau) \left( H'(y) \int_{\nu(\tau)}^y \frac{ds}{H^2(s)} + \frac{1}{H(y)} \right), \quad (7.18a)$$

$$\Phi_{(b, \tau)}^{(2)}(y) = \tau H^{(2)}(y) + H(\nu(\tau))H'(\nu(\tau))(b + \tau)H^{(2)}(y) \int_y^{\nu(\tau)} \frac{ds}{H^2(s)}. \quad (7.18b)$$

Clearly,  $\Phi_{(b,\tau)}^{(2)}$  is nonnegative, so that  $\Phi_{(b,\tau)}^{(1)}$  is nondecreasing on  $(W_*(b,\tau), \nu(\tau)]$ . Furthermore, we readily have

$$\Phi_{(b,\tau)}^{(1)}(\nu(\tau)) = -bH'(\nu(\tau)) < 0. \quad (7.19)$$

It is then immediate that  $\Phi_{(b,\tau)}^{(1)}$  is negative on  $(W_*(b,\tau), \nu(\tau)]$ . So we deduce that  $\Phi_{(b,\tau)}$  is decreasing on  $(W_*(b,\tau), \nu(\tau)]$ . Moreover, Lemma 7.1 obviously gives us

$$\lim_{y \rightarrow W_*(b,\tau)} H(y) \int_y^{\nu(\tau)} \frac{ds}{H^2(s)} = \frac{1}{F'(W_*(b,\tau))}, \quad (7.20)$$

which (from the definition of  $\Phi_{(b,\tau)}$ ) amounts to

$$\lim_{y \rightarrow W_*(b,\tau)} \Phi_{(b,\tau)}(y) = H(\nu(\tau))F'(\nu(\tau)) \left( \frac{b+\tau}{F'(W_*(b,\tau))} - \frac{\tau}{\eta} \right). \quad (7.21)$$

This shows us that  $\Phi_{(b,\tau)}$  can be extended to a continuous mapping over  $[W_*(b,\tau), \nu(\tau)]$  by setting  $\Phi_{(b,\tau)}(W_*(b,\tau)) = \lim_{y \rightarrow W_*(b,\tau)} \Phi_{(b,\tau)}(y)$ . The rest of the proof follows immediately from the previous arguments. •

## 7.8 Proof of Lemma 4.4

Given  $(b,\tau) \in \Omega$ , by (4.5) we have

$$\begin{aligned} \zeta(b,\tau) \geq 0 &\Leftrightarrow \frac{b+\tau}{F'(W_*(b,\tau))} \geq \frac{\tau}{\eta} \\ &\Leftrightarrow \frac{\eta}{\tau}(b+\tau) \geq F'(W_*) \quad (\text{where } F'(W_*) = \eta + c^{-q}W_*^q) \\ &\Leftrightarrow \frac{\eta b}{\tau} \geq c^{-q}W_*^q \quad (\text{where } q = 1/\alpha) \\ &\Leftrightarrow c \left( \frac{\eta b}{\tau} \right)^\alpha \geq W_* \quad (\text{where } W_* = F^{-1}(u(w-\tau) - u(b))) \\ &\Leftrightarrow F \left( c \left( \frac{\eta b}{\tau} \right)^\alpha \right) \geq u(w-\tau) - u(b) \quad (\text{since } F \text{ is increasing on } \mathbb{R}_+). \end{aligned}$$

This leads to the desired result. •

## 7.9 Proof of Lemma 4.5

Note that  $\xi$  is of class  $C^1$  over the open set  $]0, w[$ . It is also readily checked for any  $\tau \in ]0, w[$  that the mapping  $\xi(\cdot, \tau)$  is continuous and increasing on  $]0, w[$  with  $\xi(0^+, \tau) < 0$  and  $\xi(w, \tau) > 0$ . So there exists some (unique) value  $b \in ]0, w[$  such that  $\xi(b, \tau) = 0$ . Furthermore, for any  $(b, \tau) \in ]0, w[$ , we have  $\partial_b \xi(b, \tau) > 0$ . Clearly, by the implicit function theorem, we deduce the existence of some positive mapping  $g$  of class  $C^1$  on  $]0, w[$  that satisfies the properties (i1) and (i2) of Lemma 4.5.

Next, we prove for any  $\tau \in ]0, w[$  that  $g(\tau) < w - \tau$ . Indeed, given  $\tau \in ]0, w[$  and setting  $b = w - \tau$  (so  $b \in ]0, w[$ ) we have  $W_*(b, \tau) = 0$ , hence  $F'(W_*(b, \tau)) = \eta$ . It follows from the definition of  $\zeta$  that  $\zeta(b, \tau) = b/\eta > 0$ . Then by (i2) of Lemma 4.5 we deduce that  $b > g(\tau)$ , or equivalently  $w - \tau > g(\tau)$ , that is the desired result.

Finally we prove that  $\lim_{\tau \rightarrow w^-} g(\tau) = 0$  and  $\lim_{\tau \rightarrow 0^+} g(\tau) = 0$ .

On the one hand, by recalling that  $g(\tau) \in ]0, w - \tau[$  for  $\tau \in ]0, w[$  we readily obtain  $\lim_{\tau \rightarrow w^-} g(\tau) = 0$ .

On the other hand, as  $g$  is positive and bounded on  $]0, w[$ , we can consider a cluster point of  $g$  at  $0^+$  denoted by  $l$ . So, we clearly have  $l \geq 0$  and there exists some sequence  $(\tau_n) \subset ]0, w[$  such that  $\lim_{n \rightarrow +\infty} \tau_n = 0$  and  $\lim_{n \rightarrow +\infty} g(\tau_n) = l$ . Then we have  $\xi(g(\tau_n), \tau_n) = 0$ , namely

$$F\left(c\left(\frac{\eta g(\tau_n)}{\tau_n}\right)^\alpha\right) + u(g(\tau_n)) - u(w - \tau_n) = 0,$$

where  $F$  is defined for  $s \geq 0$  by  $F(s) = \eta s + \beta s^{1+q}$  (from (3.2a)) with  $q = 1/\alpha$ . So the previous equality can be rewritten as

$$\eta c\left(\frac{\eta g(\tau_n)}{\tau_n}\right)^\alpha + \beta\left(c\left(\frac{\eta g(\tau_n)}{\tau_n}\right)^\alpha\right)^{1+q} + u(g(\tau_n)) - u(w - \tau_n) = 0,$$

hence multiplying this last equality by  $\tau_n^{\alpha(1+q)}$  entails

$$\eta c \tau_n (\eta g(\tau_n))^\alpha + \beta (c (\eta g(\tau_n))^\alpha)^{1+q} + \tau_n^{\alpha(1+q)} (u(g(\tau_n)) - u(w - \tau_n)) = 0.$$

Now, assume that  $l > 0$ . So, by continuity of  $u$  on  $]0, \infty[$ , passing to the limit as  $n \rightarrow +\infty$  in the latter equality amounts to  $l = 0$ , which contradicts our assumption. Consequently, we deduce that  $l = 0$  is the unique cluster point of  $g$  at  $0^+$ , and so  $\lim_{\tau \rightarrow 0^+} g(\tau) = 0$ . This ensures the continuity of  $g$  over  $[0, w]$  together with  $g(0) = g(w) = 0$ . •

## 7.10 Proof of Lemma 4.7

Consider the open set  $\text{int}(\Gamma)$  defined by

$$\text{int}(\Gamma) := \{(b, \tau) \in \mathbb{R}^2; \ 0 < \tau < w - z, \ \max\{z, g(\tau)\} < b < w - \tau\}.$$

It is readily seen that  $\text{int}(\Gamma) \subset \Gamma \subset \bar{\Gamma}$  and that any element of  $\bar{\Gamma}$  can be reached as the limit of a sequence of elements belonging to  $\text{int}(\Gamma)$ . It is also clear from Lemma 4.5 that  $\text{int}(\Gamma) \neq \emptyset$  and that  $y$  is well-defined over  $\text{int}(\Gamma)$  (from Lemma 4.2). Now, let  $(b, \tau) \in \bar{\Gamma}$  and let  $((b_n, \tau_n)) \subset \text{int}(\Gamma)$  such that  $\lim_{n \rightarrow +\infty} (b_n, \tau_n) = (b, \tau)$ . It is immediate from Lemma 4.2 that

$$\Phi(b_n, \tau_n, y(b_n, \tau_n), \nu(\tau_n)) = 0, \tag{7.22a}$$

$$W_*(b_n, \tau_n) < y(b_n, \tau_n) < \nu(\tau_n). \tag{7.22b}$$

Observing that  $W_*(b_n, \tau_n) \rightarrow W_*(b, \tau)$  and that  $\nu(\tau_n) \rightarrow \nu(\tau)$ , we deduce from the last two inequalities that  $(y(b_n, \tau_n))$  is a bounded sequence. Therefore any cluster point  $y$  of  $(y(b_n, \tau_n))$  satisfies

$$W_*(b, \tau) \leq y \leq \nu(\tau). \tag{7.23}$$

Two additional results are established in the following parts (A1) and (A2):

A1) Let us prove that  $y \neq W_*(b, \tau)$  whenever  $b > z$  and  $b > g(\tau)$ . Indeed, assuming that  $y = W_*(b, \tau)$ , by Lemma 7.1 we can check that

$$\lim_{n \rightarrow +\infty} \Psi(b_n, \tau_n, y(b_n, \tau_n)) = \frac{1}{F'(W_*(b, \tau))},$$

while (7.22a) and (4.2c) yield

$$\begin{aligned} 0 &= \Phi(b_n, \tau_n, y(b_n, \tau_n), \nu(\tau_n)) \\ &= \tau_n H_{(b_n, \tau_n)}(y(b_n, \tau_n)) + \rho(b_n) F'(\nu(\tau_n)) \left( (b_n + \tau_n) \Psi(b_n, \tau_n, y(b_n, \tau_n)) - \frac{\tau_n}{\eta} \right); \end{aligned}$$

so passing to the limit as  $n \rightarrow +\infty$  in this last equality amounts to

$$0 = \tau H_{(b, \tau)}(W_*(b, \tau)) + \rho(b) F'(\nu(\tau)) \left( \frac{b + \tau}{F'(W_*(b, \tau))} - \frac{\tau}{\eta} \right),$$

hence, by  $H_{(b, \tau)}(W_*(b, \tau)) = 0$  and  $F'(\nu(\tau)) \neq 0$ , we are led to

$$0 = \rho(b) \left( \frac{b + \tau}{F'(W_*(b, \tau))} - \frac{\tau}{\eta} \right).$$

It is then deduced that either  $\rho(b) := u(b) - u(z) = 0$  (that is  $b = z$ ) or  $\zeta(b, \tau) = 0$  (that is  $b = g(\tau)$ ), which is absurd according to our assumption.



A2) We provide some useful equality when it is assumed that  $y \neq W_*(b, \tau)$ ; if so, by (7.23) we have

$$W_*(b, \tau) < y = \lim_{n \rightarrow +\infty} y(b_n, \tau_n) \leq \nu(\tau).$$

Then, in light of (4.2c), passing to the limit as  $n \rightarrow +\infty$  in (7.22a) and invoking classical continuity arguments, we obtain

$$0 = \Phi(b, \tau, y, \nu(\tau)) = \tau H(y) + \rho(b) H'(\nu(\tau)) \left( (b + \tau) H(y) \int_y^{\nu(\tau)} \frac{ds}{H^2(s)} - \frac{\tau}{\eta} \right). \quad (7.24)$$

The rest of the proof can be divide into the following parts:

B1) Taking  $(b, \tau) \in \bar{\Gamma}$  such that  $b > z$  and  $b > g(\tau)$ , by (A1) and (A2) we obtain

$$W_*(b, \tau) < y = \lim_{n \rightarrow +\infty} y(b_n, \tau_n) \leq \nu(\tau)$$

together with equality (7.24). So we consider the following two cases:

- if  $\tau > 0$  (hence  $(b, \tau) \in \Gamma$ ), by Lemma 4.2, we deduce that  $y = y(b, \tau)$ , and so

$$\lim_{n \rightarrow +\infty} y(b_n, \tau_n) = y(b, \tau);$$

- if  $\tau = 0$ , then (7.24) reduces to

$$H(y) \int_y^{\nu(\tau)} \frac{ds}{H^2(s)} = 0,$$

which readily gives us  $y = \nu(\tau)$  (since  $H(y) \neq 0$ ), and so  $\lim_{n \rightarrow +\infty} y(b_n, \tau_n) = \nu(\tau)$ .

B2) Now, let  $(b, \tau) \in \bar{\Gamma}$  verify  $b = z$  or  $b = g(\tau)$ :

- if  $b = z$ , we immediately have  $W_*(b, \tau) = \nu(\tau)$ , which by (7.23) gives us  $y = \nu(\tau)$ , and so  $\lim_{n \rightarrow +\infty} y(b_n, \tau_n) = \nu(\tau)$ .

- if  $b > z$  and  $b = g(\tau)$  (hence,  $\tau \neq 0$  from Lemma 4.5) then we prove by contradiction that  $y = W_*(b, \tau)$ . Otherwise, by (A2), we obtain  $0 = \Phi(b, \tau, y, \nu(\tau))$ , while we can observe that  $(b, \tau) \in \Omega$  (since  $\tau \neq 0$ ) and  $\zeta(b, \tau) = 0$  (since  $b = g(\tau)$ ), so by Lemma 4.2 we get  $y = W_*(b, \tau)$ , which contradicts our assumption. It results that  $\lim_{n \rightarrow +\infty} y(b_n, \tau_n) = W_*(b, \tau)$ .

This concludes the proof. •

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