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A quasi lower bound model for porous materials with incompressible plastic matrix

Long Cheng^a, Djimedo Kondo^b, Géry De Saxcé^a

^a Laboratoire de Mécanique de Lille, UMR CNRS 8107, Université de Lille 1, F59655 Villeneuve d'Ascq, France

^b Institut Jean Le Rond d'Alembert, UMR CNRS 7190, Université Pierre et Marie Curie, 4 place Jussieu, F75005 Paris, France

Résumé — The aim of this work is to obtain by homogenization techniques a macroscopic plastic model for porous materials with von Mises matrix using the hollow sphere model and a trial stress field in internal equilibrium, which is composed of the exact solution for the pure hydrostatic loading and a uniform deviatoric field. Considering Hill's variational principle for rigid plastic materials and relaxing the stress condition on the void boundary, simultaneously using a lagrangean multiplier, we satisfy the plastic criterion on average. On this ground, We obtain a closed form yield function, which could be seemed as a quasi lower macroscopic criterion against the Gurson's upper one. The theoretical result is confirmed by numerical simulations.

Mots clés — Porous material, Limit analysis, Homogenization, von Mises model

1 Introduction

The present work can be considered as an important step to propose a macroscopic plastic model for "Porous non associated Drucker-Prager"-type materials, using homogenization techniques and the hollow sphere model. In order to solve the non associated problem, one could use bipotential concept [5, 6] as the main tool to deduce the corresponding macroscopic model ([2], in preparation) with both dual variables : stress and velocity fields, while the later one has been given in the recent paper [11]. Moreover, following Gurson's footsteps and [11], the incompressible porous materials can be seemed with the von Mises type matrix, which is a particular case of the pressure-sensitive dilatant ones (Drucker-Prager model when friction angle is equal to zero). Several extensions of Gurson's model have been further proposed in the literature, the probably most important developments being those accounting for void shape effects [9, 8, 15]. Plastic anisotropy was treated by [1, 16]. all of these extension works are based on the determination of a suitable trial velocity field. On the other hand, no attempts have been made to develop a dual stress model excepted the pioneering paper of [10] and more recently [19] who developed a semi-analytical approach. The aim of this paper is to propose a stress model leading to a closed analytical expression of the macroscopic criterion.

The limit analysis is a general method to determine the plastic collapse of structures under proportional loading [18]. The variational formulation of the lower bound theorem is based on Hill's functional [13] of which we present a specialized version adapted to the homogenization techniques by applying it to the hollow sphere model. The lower bound is conserved only if the trial stress field is statically and plastically admissible. This condition is very difficult to satisfy strictly for three dimensional fields in a hollow sphere. In order to obtain a full analytical model, the key idea is to satisfy only the internal equilibrium equations, relaxing the stress boundary condition and the plastic criterion.

2 Variational formulation

For the purpose of porous material homogenization, let us consider a reference elementary volume or macro-element V composed of a void V_f and matrix V_M made of a plastic material with a yield criterion :

$$F(\boldsymbol{\sigma}) \leq 0, \quad (1)$$

and the normality law :

$$\mathbf{d} = \dot{\varepsilon}^p \frac{\partial F}{\partial \boldsymbol{\sigma}}, \quad (2)$$

The macro-element V is enclosed by surface S and the void V_f by S_f . The macroscopic stress $\boldsymbol{\Sigma}$ and strain rate \mathbf{D} are defined as volume averages of their microscopic counterparts $\boldsymbol{\sigma}$ and \mathbf{d} :

$$\boldsymbol{\Sigma} = V^{-1} \int_V \boldsymbol{\sigma} dV, \quad \mathbf{D} = V^{-1} \int_V \mathbf{d} dV.$$

The set of kinematical admissible velocity fields is defined in the following sense :

$$\mathcal{K}_a = \{ \mathbf{v} \text{ s.t. } \mathbf{v}(\mathbf{x}) = \mathbf{D} \cdot \mathbf{x} \text{ on } S \}.$$

The associated strain rate field is $\mathbf{d}(\mathbf{v}) = \text{grad}_s \mathbf{v}$. The set of statically admissible stress fields is :

$$\mathcal{S}_a = \{ \boldsymbol{\sigma} \text{ s.t. } \text{div } \boldsymbol{\sigma} = 0 \text{ in } V_M, \quad \boldsymbol{\sigma} \mathbf{n} = 0 \text{ on } S_f, \quad \boldsymbol{\sigma} = 0 \text{ in } V_f \}.$$

The homogenization problem consists in determining the macroscopic stress for which there exists at least a couple $(\mathbf{v}, \boldsymbol{\sigma}) \in \mathcal{K}_a \times \mathcal{S}_a$ of admissible couples satisfying anywhere in the matrix the yield criterion (1) and the normality rule (2). Taking into account its strong non linear nature, it has in general no closed analytical solution. We present now an equivalent variational formulation, more appropriate for simple approximations, thanks to relevant choice of trial fields and minimization procedure. Let us consider an admissible couple $(\mathbf{v}, \boldsymbol{\sigma})$. Thus, by Hill's lemma, one has :

$$\mathbf{D} : \boldsymbol{\Sigma} = V^{-1} \int_V \mathbf{d}(\mathbf{v}) : \boldsymbol{\sigma} dV = V^{-1} \int_{V_M} \mathbf{d}(\mathbf{v}) : \boldsymbol{\sigma} dV,$$

Besides, the plastic material is characterized by the stress potential $\boldsymbol{\sigma} \mapsto \psi(\boldsymbol{\sigma})$ equal to 0 when $F(\boldsymbol{\sigma}) \leq 0$ and to $+\infty$ otherwise. This suggests to state Hill's variational principle for the hollow sphere :

$$\min \left(\Pi(\boldsymbol{\sigma}) = V^{-1} \int_{V_M} \psi(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right),$$

or equivalently :

$$\min_{\boldsymbol{\sigma} \in \mathcal{S}_l} (-\mathbf{D} : \boldsymbol{\Sigma}),$$

where the set of licit stress fields is defined by :

$$\mathcal{S}_l = \{ \boldsymbol{\sigma} \in \mathcal{S}_a \text{ s.t. } F(\boldsymbol{\sigma}) \leq 0 \text{ in } V_M \}.$$

The limit analysis approach consists in finding non trivial solutions qualified as collapse mechanisms. It is expected that they exist only under an equality condition on $\boldsymbol{\Sigma}$ that can be interpreted as the equation of the yielding surface in the macroscopic model.

It is worth to remark that if a stress field is in internal equilibrium (see for instance [7]) :

$$\boldsymbol{\Sigma}^{void} = V^{-1} \int_{V_f} \boldsymbol{\sigma} dV = V^{-1} \int_{S_f} (\boldsymbol{\sigma} \mathbf{n}) \otimes \mathbf{x} dS,$$

then, considering the mean stress $\boldsymbol{\sigma}_m = \text{Tr}(\boldsymbol{\sigma})/3$:

$$V^{-1} \int_{V_M} \boldsymbol{\sigma}_m dV = (3V)^{-1} \int_{S_f} \mathbf{x} \cdot (\boldsymbol{\sigma} \mathbf{n}) dS.$$

As the continuity condition :

$$\boldsymbol{\sigma} \mathbf{n} = 0 \text{ on } S_f, \quad (3)$$

is difficult to satisfy for simple fields, we relax it as follows :

$$\int_{S_f} \mathbf{x} \cdot (\boldsymbol{\sigma} \mathbf{n}) dS = 0, \quad (4)$$

that leads to the relaxed variational principle :

$$\min_{\boldsymbol{\sigma} \in \mathcal{S}_r} (-\mathbf{D} : \boldsymbol{\Sigma}) ,$$

where the modified set of licit stress fields is defined by :

$$\mathcal{S}_r = \{ \boldsymbol{\sigma} \text{ s.t. } \operatorname{div} \boldsymbol{\sigma} = 0, \quad F(\boldsymbol{\sigma}) \leq 0 \text{ in } V_M \quad \boldsymbol{\sigma} = 0 \text{ in } V_f \text{ and } \int_{S_f} \mathbf{x} \cdot (\boldsymbol{\sigma} \mathbf{n}) dS = 0 \} . \quad (5)$$

In the same spirit as the work of [11], the second approximation consists in relaxing the yield criterion (1). Introducing Lagrange's multiplier field $\mathbf{x} \mapsto \lambda(\mathbf{x})$, this constrained minimization problem is transformed into an equivalent saddle-point problem

$$\max_{\lambda \geq 0} \min_{\boldsymbol{\sigma} \in \mathcal{S}_r} \left(\mathcal{L}(\boldsymbol{\sigma}, \lambda) = V^{-1} \int_{V_M} \lambda F(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right) .$$

We perform a new approximation by imposing Lagrange's multiplier field to be uniform in V_M :

$$\max_{\lambda \geq 0} \min_{\boldsymbol{\sigma} \in \mathcal{S}_r} \left(\mathcal{L}(\boldsymbol{\sigma}, \lambda) = \lambda V^{-1} \int_{V_M} F(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right) .$$

that is equivalent to minimize the bifunctional Π under the relaxed condition :

$$V^{-1} \int_{V_M} F(\boldsymbol{\sigma}) dV = 0 . \quad (6)$$

Satisfying the conditions (1) and (3) only in an average sense but not locally anywhere in V_M is a strong approximation but leading to easier calculations. The minimum principle allows obtaining the "better" solution within the framework imposed by the approximations.

3 Hollow sphere model

3.1 Macroscopic criteria : closed form and approximate ones

We consider a hollow sphere made up of a spherical void embedded in a homothetic cell of a rigid-plastic isotropic and homogeneous material with von Mises model :

$$F(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_e(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_0 ,$$

where $\boldsymbol{\sigma}_e$ is the comparison stress and $\boldsymbol{\sigma}_0$ the yield stress. The inner and outer radii are respectively denoted a and b , giving the void volume fraction $f = (a/b)^3 < 1$. Accounting for the central symmetry of the problem, the spherical coordinates (r, θ, ϕ) and the cylindrical ones (ρ, ϕ, z) are used, ρ and ϕ being the polar radius and angle, z the height with respect to the Oxy plane and $r = \sqrt{\rho^2 + z^2}$.

In order to limit the errors due to approximations, we hope the macroscopic model to be exact at least for the pure hydrostatic case, that leads to consider the following stress field in spherical coordinates :

$$\boldsymbol{\sigma}^{(1)} = -A_0 \left(\ln \left(\frac{a}{r} \right) \mathbf{1} - \frac{1}{2} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi) \right) , \quad (7)$$

where $\mathbf{1}$ is the unit tensor. $\boldsymbol{\sigma}^{(1)}$ is completed by two linear terms to capture the shear effect, introducing the additional uniform stress field in cylindrical coordinates :

$$\boldsymbol{\sigma}^{(2)} = A_1 (\mathbf{e}_\rho \otimes \mathbf{e}_\rho + \mathbf{e}_\phi \otimes \mathbf{e}_\phi) + A_2 \mathbf{e}_z \otimes \mathbf{e}_z , \quad (8)$$

The resultant trial stress field is defined in the matrix V_M as :

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} ,$$

and vanishes in the void V_f .

Next, we would like to satisfy condition (4). The contribution of $\boldsymbol{\sigma}^{(1)}$ is null because, on the void boundary S_f of radius a , the stress vector vanishes. Hence a straightforward calculation gives :

$$\Sigma_m^{void} = f \frac{2A_1 + A_2}{3} = f \boldsymbol{\sigma}_m^{(2)} = 0, \quad (9)$$

Then the stress field $\boldsymbol{\sigma}^{(2)}$ is deviatoric. Similarly, we have :

$$\boldsymbol{\Sigma} = V^{-1} \int_S (\boldsymbol{\sigma} \mathbf{n}) \otimes \mathbf{x} dS - V^{-1} \int_{S_f} (\boldsymbol{\sigma} \mathbf{n}) \otimes \mathbf{x} dS,$$

Eliminating A_2 by (9), we obtain :

$$\boldsymbol{\Sigma} = -\frac{A_0 \ln f}{3} \mathbf{1} + (1-f) A_1 (\mathbf{e}_\rho \otimes \mathbf{e}_\rho + \mathbf{e}_\phi \otimes \mathbf{e}_\phi - 2\mathbf{e}_z \otimes \mathbf{e}_z),$$

that gives the macroscopic mean stress, comparison stress and third invariant of the deviator :

$$\Sigma_m = -\frac{A_0 \ln f}{3}, \quad \Sigma_e = 3(1-f) |A_1|, \quad \Sigma_3 = -2(1-f)^3 A_1^3. \quad (10)$$

Expressing A_0 and A_1 with respect to Σ_m and Σ_e into (7) and (8) by means of (10) and introducing the effective stresses :

$$\tilde{\Sigma}_e = \frac{\Sigma_e}{1-f}, \quad \tilde{\Sigma}_m = -\frac{3\Sigma_m}{2 \ln f}, \quad \tilde{\Sigma}_3 = \frac{\Sigma_3}{(1-f)^3}, \quad (11)$$

the comparison stress depends only on θ through :

$$\sigma_e = \sqrt{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2 + \eta \tilde{\Sigma}_e \tilde{\Sigma}_m (3 \cos^2 \theta - 1)},$$

where :

$$\eta = \text{sign}(A_1) = \frac{27 \tilde{\Sigma}_3}{2 \tilde{\Sigma}_e^3}.$$

The yield condition :

$$V^{-1} \int_{V_M} \lambda F(\boldsymbol{\sigma}) dV = V^{-1} 2\pi \int_a^b r^2 dr \int_0^\pi \sigma_e \sin \theta d\theta - (1-f) \sigma_0 = 0$$

is reduced to :

$$\frac{1}{2} \int_0^\pi \sqrt{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2 - \eta \tilde{\Sigma}_e \tilde{\Sigma}_m (3 \cos^2 \theta - 1)} \sin \theta d\theta = \sigma_0$$

Hence the closed form macroscopic yield criterion reads :

$$\sqrt{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2} \mathcal{J} \left(27 \frac{\tilde{\Sigma}_3}{\tilde{\Sigma}_e^3} \frac{\tilde{\Sigma}_e \tilde{\Sigma}_m}{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2} \right) = \sigma_0, \quad (12)$$

where :

$$\mathcal{J}(\zeta) = \frac{1}{2} \int_0^\pi \sqrt{1 + \frac{1}{2}(3 \cos^2 \theta - 1)\zeta} \sin \theta d\theta.$$

In the interval $[-1, 1]$ of variation of the adimensional coupling parameter :

$$\zeta = 27 \frac{\tilde{\Sigma}_3}{\tilde{\Sigma}_e^3} \frac{\tilde{\Sigma}_e \tilde{\Sigma}_m}{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2} \quad (13)$$

this function is defined for $-1 \leq \zeta \leq 0$ by :

$$\mathcal{J}(\zeta) = \frac{1}{2} \left(\sqrt{1 + \zeta} + \frac{2 - \zeta}{\sqrt{6|\zeta|}} \sin^{-1} \sqrt{\frac{3|\zeta|}{2 - \zeta}} \right),$$

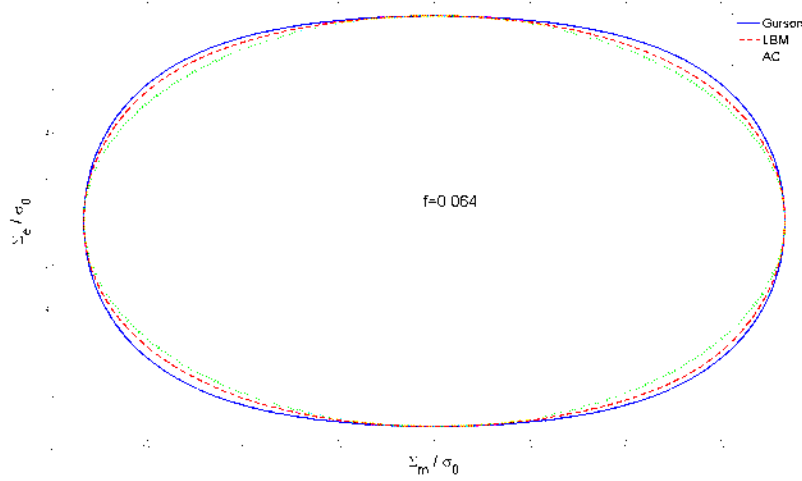


Fig. 1 – Comparison between the closed-form yield surface (LBM) and the Gurson one with full-field solutions, the asymmetry with respect to the third invariant of the deviator depicted in section 3.2. Porosity : 0.064

and for $0 \leq \zeta \leq 1$ by :

$$\mathcal{J}(\zeta) = \frac{1}{2} \left(\sqrt{1+\zeta} + \frac{2-\zeta}{\sqrt{6\zeta}} \ln \frac{\sqrt{3\zeta} + \sqrt{2(1+\zeta)}}{2-\zeta} + \frac{\sqrt{6}}{12} \frac{(2-\zeta) \ln(2-\zeta)}{\sqrt{\zeta}} \right).$$

It is smooth over $[-1, 1]$ with extreme values $\mathcal{J}_{max} = \mathcal{J}(0) = 1$, $\mathcal{J}_{min} = \mathcal{J}(1) = 0.976$ over $[0, 1]$ and $\mathcal{J}_{min} = \mathcal{J}(-1) = 0.962$ over $[-1, 0]$. Because of the occurrence of the third invariant in the coupling parameter (13), the criterion is slightly asymmetrical with respect not only to the sign change of Σ_m but also to the one of the shear component, as already remarked by [4, 20, 17].

Following [12], this function may be taken equal to unity, that reduces the macroscopic yield criterion (12) to an simple approximation, denoted AC :

$$\sqrt{\tilde{\Sigma}_e^2 + \tilde{\Sigma}_m^2} = \sigma_0, \quad (14)$$

or explicitly with respect to Σ_e , Σ_m and f :

$$\left(\frac{\Sigma_e}{1-f} \right)^2 + \left(\frac{3\Sigma_m}{2 \ln f} \right)^2 = \sigma_0^2.$$

Both expressions (12) and (14) gives the same value for the two particular cases :

- **Pure hydrostatic case** : $\Sigma_e = 0$, $\Sigma_m = -\frac{2\sigma_0 \ln f}{3}$ which is exact,
- **Pure shear case** : $\Sigma_e = (1-f)\sigma_0$, $\Sigma_m = 0$.

3.2 Analytical validation of established criteria

We provide in this subsection the analytical validation of the closed form quasi-lower bound model (LBM), approximate one (AC) by comparing with the Gurson's one. One case with the value of porosity $f = 0.064$ is firstly defined, whose surfaces are illustrated in Fig.1. An agreement at the point $\Sigma_e = 0$ (pure hydrostatic loading) and the one $\Sigma_m = 0$ (pure shear loading) is obtained, while the solution at the pure hydrostatic one is virtually exact. Due to the Hill's variational principle, even the relaxed stress fields (8) was adopted, the yield surfaces from LBM and AC are strictly below the Gurson one except for the above two particular cases. Furthermore, the LBM is slightly asymmetric about the origin ($\Sigma_m = \Sigma_e = 0$) with respect to the third invariant of the deviator as detailed in subsection 3.1, while the AC is symmetric due to the approximation of function $\mathcal{J}(\zeta)$.

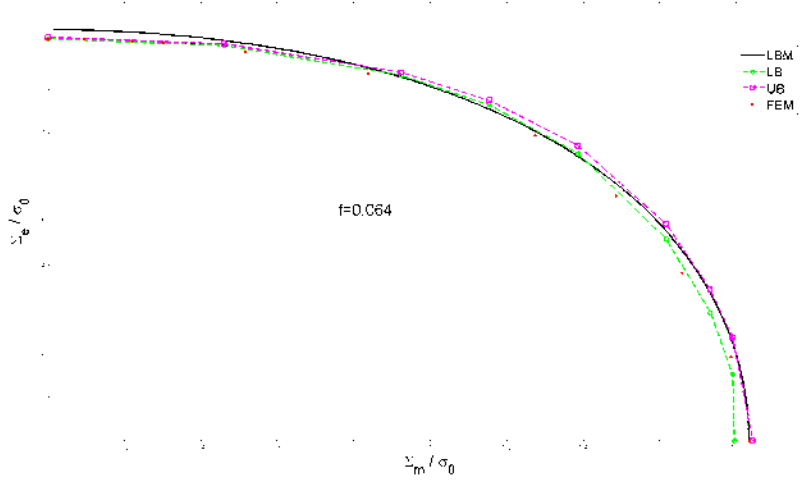


Fig. 2 – Comparison between the closed-form yield surface (LBM), the numerical bounds and the FEM solution with boundary condition of MPC, Porosity : 0.064

4 Numerical simulations and comparison

In this section, the yield surface prediction from the quasi Lower Bound Model (LBM) detailed in Section 3 will be compared with the numerical solutions obtained from the Finite Element Method (FEM). The numerical analysis is carried out by means of an axisymmetric model of the spherical shell. Taking advantage of its symmetry, a quarter of the model is considered by using 1500 axisymmetric elements. Moreover, an incremental analysis of elastoplastic materials in small deformations is adopted. However, the analytical formulation detailed in section 2 and 3 is corresponding to a perfectly rigid-plastic one. Hence, in order to overcome the influence of the elastic phase iterations, we wish that the load could be applied under the condition of a constant stress triaxiality Σ_p / Σ_z or Σ_m / Σ_e . This can be accomplished by means of ABAQUS/Standard and a user subroutine MPC (Multi-Points Constraints), whose implementation has been described in [3]. Moreover, it should be emphasized that the numerical solutions reported by Trillat and Pastor [21] give the reliable upper and lower bound surfaces, respectively denoted UB and LB. For this reason, we wish to evaluate and validate the new criterion by comparing not only with the FEM solution, but also these numerical bounds.

Fig.2 displays the comparisons between the LBM, the numerical lower bound (LB) and the upper one (UB) for $f = 0.064$. In this group of comparison, an agreement is obtained between the analytical predictions and the FEM solution, and they are both almost between the numerical bounds especially for the big values of the stress triaxiality $t = |\Sigma_p / \Sigma_z|$, while $\Sigma_p / \Sigma_z \in [-\frac{1}{2}, 1]$ separately for the general traction ($\Sigma_m > 0$) and general compression ($\Sigma_m < 0$). Furthermore, the FEM solution of pure hydrostatic loading coincides almost exactly with the analytical one of LBM. While for pure shear loading, it can be observed that the FEM result is smaller than the LBM one. Finally, other two cases are contributed to study the influence of porosity to the numerical validation of LBM. For the lower one $f = 0.01$, It should be paid attention to the fact that, as illustrated in Fig.3, even the FEM solution is wholly between the numerical bounds, the yield surface of LBM is outside of LB for the small stress triaxiality loadings. As discussed in Section 2, the difference between the reference analytical solution given by LBM and the finite element ones can be attributed to the adoption of the relax stress fields from Eq.5, which gives a relatively remarkable impact for this fact, while the influence of numerical errors is rather smaller. Nevertheless, for the bigger porosity $f = 0.1$, it is shown in Fig.4 that, LBM has a good agreement with UB, and the corresponding value of Σ_e / σ_0 is bigger than the FEM solution.

5 Conclusion

In this study, a stress variational model for the von Mises type porous materials has been derived by applying the homogenization and limit analysis theories. Due to the Hill's variational principle and the

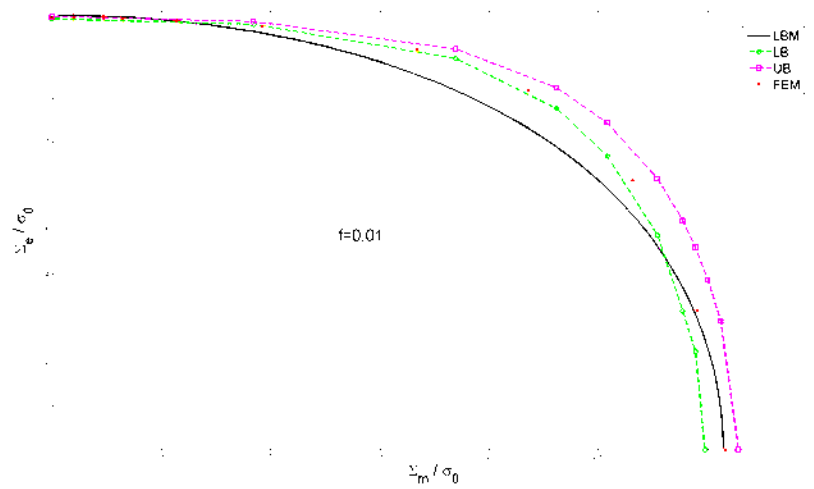


Fig. 3 – Comparison between the closed-form yield surface (LBM), the numerical bounds and the FEM solution with boundary condition of MPC, Porosity : 0.01

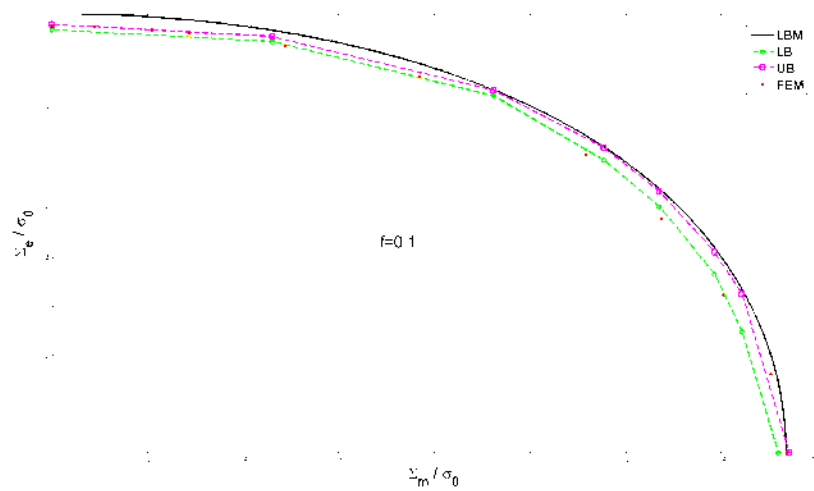


Fig. 4 – Comparison between the closed-form yield surface (LBM), the numerical bounds and the FEM solution with boundary condition of MPC, Porosity : 0.1

relax licit stress fields, it could be seemed as a quasi-lower bound model against the Gurson's upper one. While the LBM are expressed as only through suitable effective stresses and an adimensional coupling parameter, as a result, it is asymmetric with respect not only to the sign of Σ_m , but also to the one of shear component Σ_e .

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