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# Aperiodic points in $\mathbb{Z}^2$ -subshifts\*

Anael Grandjean<sup>1</sup>, Benjamin Hellouin de Menibus<sup>2</sup>, and Pascal Vanier<sup>1</sup>

1 Laboratoire d'Algorithmique, Complexité et Logique  
Université de Paris-Est, LACL, UPEC, France

2 Laboratoire de Recherche en Informatique, Université Paris-Sud, CNRS,  
CentraleSupélec, Université Paris-Saclay, France

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## Abstract

We consider the structure of aperiodic points in  $\mathbb{Z}^2$ -subshifts, and in particular the positions at which they fail to be periodic. We prove that if a  $\mathbb{Z}^2$ -subshift contains points whose smallest period is arbitrarily large, then it contains an aperiodic point. This lets us characterise the computational difficulty of deciding if an  $\mathbb{Z}^2$ -subshift of finite type contains an aperiodic point. Another consequence is that  $\mathbb{Z}^2$ -subshifts with no aperiodic point have a very strong dynamical structure and are almost topologically conjugate to some  $\mathbb{Z}$ -subshift. Finally, we use this result to characterize sets of possible slopes of periodicity for  $\mathbb{Z}^3$ -subshifts of finite type.

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A subshift on  $\mathbb{Z}^d$  is a set of colorings of  $\mathbb{Z}^d$  by a finite set of colors avoiding some family of forbidden patterns. When this family is finite, the subshift is called a *subshift of finite type* (SFT). In dimension 2, SFTs are equivalent to sets of tilings by Wang tiles: Wang tiles are unit squares with colored borders that cannot be rotated and may be placed next to each other only if the borders match.

Wang tiles were introduced by Wang in order to study the decidability of some fragments of logic [18, 19]. He thus introduced the Domino Problem: given a set of Wang tiles, do they tile the plane? (in other words, is the corresponding subshift nonempty?) Wang first conjectured that whenever a tileset tiles the plane, it can do so in a periodic manner, which would have implied the decidability of the Domino Problem.

In dimension 1 the problem is decidable. A  $\mathbb{Z}$ -SFT correspond to the set of biinfinite walks on some automaton and it tiles the line if and only if the automaton contains a cycle. Such a cycle provides a periodic point of the SFT, so non-empty  $\mathbb{Z}$ -SFTs always contain a periodic point. The situation is dramatically different in higher dimension. Berger [3] proved that there exists tilesets in dimension 2 that tile the plane only aperiodically, and that the Domino Problem was therefore undecidable.

Thus, from the start, periodicity and aperiodicity have been at the heart of the study of Wang tiles and SFTs, and the main tool in understanding their structural properties and the answer to various decision problems. To give a few examples:

- The presence of a dense set of periodic points is related to the decidability of the problem of deciding whether a given pattern appears in some point of an SFT [12].
- The finite subshifts on  $\mathbb{Z}^d$  are exactly the subshifts containing only periodic configurations with  $d$  non-colinear vectors of periodicity [1, Theorem 3.8]. These configurations can

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be seen as finite configurations. This result has recently been extended to subshifts on groups [15].

- Countable SFTs always contain a finite configuration and if they are not finite, then they contain a configuration with exactly one vector of periodicity [1, Theorem 3.11].
- A subshift always contains a quasiperiodic configuration [4, 6], a configuration in which every finite pattern appears in any window of sufficiently large size depending only on the size of the pattern.

In this article we study the structure of aperiodic points in  $\mathbb{Z}^2$ -SFT, and in particular the repartition of the coordinates where it avoids to be periodic. Our main result is:

► **Theorem 1.** *Let  $X$  be a  $\mathbb{Z}^2$ -subshift. If  $X$  contains an aperiodic configuration, then it contains an aperiodic configuration that avoids every period  $p$  at distance at most  $f(\|p\|)$  from 0, where  $f$  is a computable function.*

This means that aperiodicity can be “organised” in concentric balls around a common center, in such a way that a proof of aperiodicity for any vector may be found near this center. As a consequence, when a subshift does not contain any aperiodic point, it must have a finite number of directions of periodicity:

► **Corollary 2.** *Let  $X$  be a subshift. If for every  $n$ ,  $X$  contains a configuration that avoids all periods  $p$  with  $\|p\| \leq n$ , then  $X$  contains an aperiodic configuration.*

*In other words, for any subshift  $X$  with no aperiodic point, there is a finite set of periods  $P$  such that any configuration of  $X$  is periodic for some period  $p \in P$ .*

This will lead to a further characterization of subshifts containing no aperiodic points in Section 3.2.

[8] proved that for  $d \geq 2$  it is undecidable to know whether an SFT contains a periodic, resp. aperiodic configuration. While it is easy to see that checking whether an SFT contains a periodic configuration is a recursively enumerable problem ( $\Sigma_1^0$  in the arithmetical hierarchy), it remained an open problem whether deciding if an SFT contains an aperiodic configuration was even in the arithmetical hierarchy. One of the consequences of Theorem 1 is that it is  $\Pi_1^0$ .

Periodicity is also a central topic of symbolic dynamics since sets of periods and directions of periodicity constitute conjugacy invariants. For example, we prove that  $\mathbb{Z}^2$ -subshifts with no aperiodic point have a very strong dynamical structure and are essentially equivalent to  $\mathbb{Z}$ -subshifts, and this is true for SFT as well. These sets have also been studied and characterized through computability and complexity theory [9]. [16] recently proved that any  $\Sigma_2^0$  set of  $(\mathbb{Q} \cup \{\infty\})^2$  can be realized as a set of slopes of a  $\mathbb{Z}^3$ -SFT. Another consequence of Theorem 1 is that this becomes a characterization.

The article is organized as follows: Section 1 recalls some definitions and notations, Section 2 is devoted to the proof of Theorem 1, Section 3 is devoted to some of its consequences and Section 4 shows a counter example for  $\mathbb{Z}^d$  subshifts when  $d \geq 3$ .

## 1 Definitions

### 1.1 Subshifts

We provide here standard definitions about subshifts, which may be found in greater detail in [13].

The  $d$ -dimensional full shift is the set  $\Sigma^{\mathbb{Z}^d}$  where  $\Sigma$  is a finite alphabet whose elements are called *letters* or *symbols*. Each element of the full shift may be seen as a coloring of  $\mathbb{Z}^d$  with the letters of  $\Sigma$ . For  $v \in \mathbb{Z}^d$ , the shift function  $\sigma_v : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$  is defined by  $\sigma_v(x_z) = x_{z+v}$ . The full shift equipped with the distance  $d(x, y) = 2^{-\min\{\|v\| \mid v \in \mathbb{Z}^d, x_v \neq y_v\}}$  forms a compact metric space on which the shift functions act as homeomorphisms. A closed shift invariant subset  $X$  of  $\Sigma^{\mathbb{Z}^d}$  is called a *subshift* or *shift*. An element of a subshift  $X$  is called a *configuration* or *point*.

A *pattern* of shape  $P$ , where  $P$  is a finite subset of  $\mathbb{Z}^d$ , is an element of  $\Sigma^P$  or alternatively a function  $p : P \rightarrow \Sigma$ . A configuration  $x$  avoids a pattern  $p$  of shape  $P$  if  $\forall z \in \mathbb{Z}^d, p \neq x|_{z+P}$  and contains it if it does not avoid it.

For a family of forbidden patterns  $\mathcal{F}$ , denote  $\Sigma_{\mathcal{F}}$  the set of configuration that avoid  $\mathcal{F}$ . Then  $\Sigma_{\mathcal{F}}$  is a subshift, and every subshift can be defined in this way. When a subshift can be defined this way by a finite family, it is called a *subshift of finite type*. When a subshift can be defined by a recursively enumerable family of forbidden patterns, it is called an *effectively closed subshift*.

If  $X$  is a subshift, we denote by  $\mathcal{L}(X)$  its *language*, *i.e.* the set of patterns that appear somewhere in one of its points.  $X$  is *irreducible* if there exists a constant  $C$  such that, for any two patterns  $p_1, p_2 \in \mathcal{L}(X)$  of shapes  $P_1$  and  $P_2$  respectively, and assuming that  $d(P_1, P_2) \geq C$ , we have  $p_1 \cup p_2 \in \mathcal{L}(X)$  (where  $p_1 \cup p_2$  is the pattern on  $P_1 \cup P_2$  defined in the intuitive manner).

► **Definition 3 (Periodicity).** A configuration  $x$  is *periodic* of period  $v$  if there exists  $v \in \mathbb{Z}^d \setminus \{(0, 0)\}$  such that  $\forall z \in \mathbb{Z}^d, x_z = x_{z+v}$ . More precisely, a configuration is *k-periodic* if it has exactly  $k$  independent periods. If  $c$  has no period, then it is said to be *aperiodic*. A subshift is *aperiodic* if all its points are aperiodic.

We denote by  $B(x, n)$  the ball of radius  $n$  centered in  $x$ .

Let  $x \in \mathcal{A}^{\mathbb{Z}^d}$  and  $p \in \mathbb{Z}^2$ , if there exists  $z \in \mathbb{Z}^2$  such that  $x_z \neq x_{z+p}$ , we say that  $x$  *avoids* period  $p$ . The pair  $(z, z + p)$  is called an *avoidance* of period  $p$  in configuration  $x$ . We say that a configuration avoids a set of periods  $P$  if it avoids every period in  $P$ .

Let  $P$  be a set of periods. We denote  $P'$  the set obtained from  $P$  by replacing each period  $p$  by the least common multiple of all periods of  $P$  that are colinear to  $p$ . More formally :  $P' = \{\text{lcm}(q \mid q \in P \text{ and } q \text{ and } p \text{ are colinear}) \mid p \in S\}$ . Observe that  $P'$  is a set of pairwise non-colinear periods.

A  $\mathbb{Z}^d$ -*cellular automaton* is a continuous function  $F : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$  that commutes with every shift. It can be defined equivalently by a local rule  $f : \Sigma^{\Gamma} \rightarrow \Sigma$  for a finite pattern  $\Gamma$  by  $F(x)_i = f(x_{i+\Gamma})$ .

Except in the last section, the subshifts we will be considering will implicitly be  $\mathbb{Z}^2$ -subshifts.

## 1.2 Arithmetical hierarchy

We give some basic definitions used in computability theory and in particular about the arithmetical hierarchy. More details may be found in [17].

Usually the arithmetical hierarchy is seen as a classification of sets according to their logical characterization. For our purpose we use an equivalent definition in terms of computability classes and Turing machines with oracles:

- $\Delta_0^0 = \Sigma_0^0 = \Pi_0^0$  is the class of recursive (or computable) problems.
- $\Sigma_n^0$  is the class of recursively enumerable (RE) problems with an oracle  $\Pi_{n-1}^0$ .

- $\Pi_n^0$  the complementary of  $\Sigma_n^0$ , or the class of co-recursively enumerable (coRE) problems with an oracle  $\Sigma_{n-1}^0$ .
- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$  is the class of recursive (R) problems with an oracle  $\Pi_{n-1}^0$ .

In particular,  $\Sigma_1^0$  is the class of recursively enumerable problems and  $\Pi_1^0$  is the class of co-recursively enumerable problems.

## 2 Main result

This whole section is dedicated to the proof of the Theorem 1 and Corollary 2. Given a subshift that contains an aperiodic point, we prove that it contains some aperiodic point where all period avoidances are organised in concentric balls around a common center, in such a way that each period  $p$  is in a ball whose radius only depends on  $\|p\|$ . This result is used in a compactness argument to prove that, if a subshift contains configurations with arbitrarily large smallest period, then it contains an aperiodic point.

Actually, our algorithm can only gather avoidances in a small ball if all the periods are non-colinear. Fortunately we can easily build a set  $P'$ .

► **Lemma 4.** *Let  $P$  be a set of periods. Any configuration avoiding  $P'$  also avoids  $P$ .*

**Proof.** Each period  $p$  in  $P$  has a multiple  $p' \in P'$ . Each avoidance of  $p'$  induces an avoidance of  $p$ . ◀

► **Lemma 5.** *Let  $P$  be a set of pairwise non-colinear periods. Let  $x$  be a configuration avoiding  $P$ . Then there exists a ball of radius  $\sum_{p \in P} \|p\|$  where  $x$  avoids every period of  $P$ .*

**Proof.** We prove the result by induction on the number of periods in  $P$ . When  $P$  is a singleton the case is trivial. Now suppose  $P$  is not a singleton. Denote  $p_0, p_1, \dots, p_n$  the periods in  $P$ . By induction hypothesis, we can find a ball  $B_{n-1}$  of radius  $\sum_{i < n} \|p_i\|$  centered in  $b_{n-1}$  containing avoidances for every period in  $P$  except  $p_n$ . Similarly, we find a ball  $B'_{n-1}$  of radius  $\sum_{i > 0} \|p_i\|$ , centered in  $b'_{n-1}$  that contains avoidances of every period in  $P$  except  $p_0$ . We now show that either an avoidance of  $p_0$  exists near a copy of  $B'_{n-1}$  or an avoidance of  $p_n$  exists near a copy of  $B_{n-1}$ .

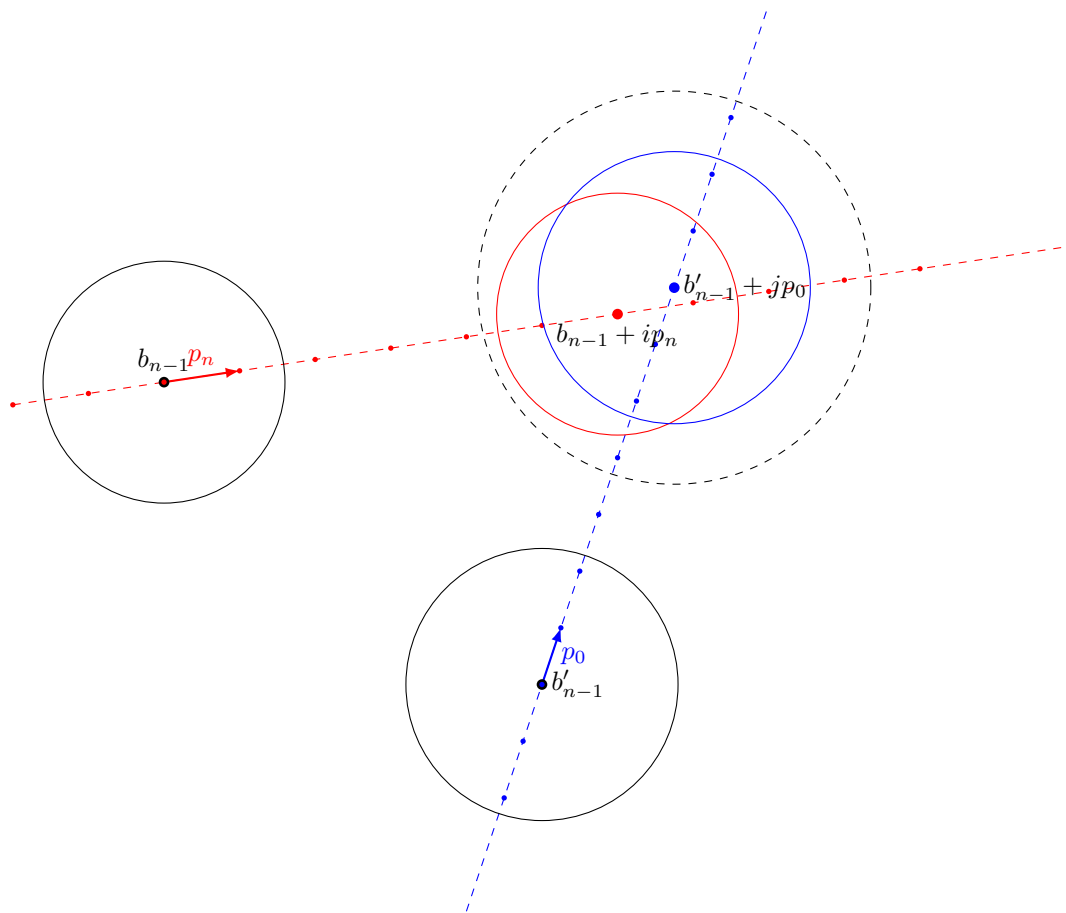
Consider the ball  $\sigma_{p_n} B_{n-1} = B(b_{n-1} + p_n, \sum_{i < n} \|p_i\|)$ , the translated image of  $B_n$  by the vector  $p_n$ . Either  $x|_{B_n} = x|_{\sigma_{p_n} B_{n-1}}$ , i.e. the two balls are filled the same way, or we found an avoidance  $(z, z + p_n)$  with  $z \in B_{n-1}$  and  $z + p_n \in \sigma_{p_n} B'_{n-1}$ . In the latter case, the result is proved.

This process can be iterated for both  $B'_{n-1}$  and  $B_{n-1}$  until either we find the necessary avoidance or the centers of the balls are close to each other: Since  $p_0$  and  $p_n$  are not colinear, and assuming  $\|p_n\| \geq \|p_0\|$ , there exists  $i, j \in \mathbb{Z}$  such that  $\|b_{n-1} + ip_n - b'_{n-1} + jp_0\| \leq \|\frac{p_n}{2}\| + \|\frac{p_0}{2}\| < \|p_n\|$ . We thus found a ball centered in  $b''_{n-1} + jp_0$  and of radius  $\sum_i \|p_i\|$  containing the two balls we translated, and therefore an avoidance of each period in  $P$ . Denote  $B_n$  this new ball and  $b_n$  its center. The process is depicted in Figure 1. ◀

In the previous proof, the distance between  $b_{n-1}$  and  $b_n$  only depends on  $n, p_0, p_n$  and the distance between  $b_{n-1}$  and  $b'_{n-1}$ .

Therefore there is a computable function  $f(P, r)$  such that  $\|b_{n-1} - b_n\| \leq f(P, r)$  assuming that  $\|b_{n-1} - b'_{n-1}\| \leq r$ .

► **Lemma 6.** *Let  $P = \{p_0, \dots, p_n\}$  be a set of noncolinear periods. Define  $f'(P, r)$  recursively as:*



■ **Figure 1** The process of translating  $B_{n-1}$  and  $B'_{n-1}$  close to each other: each translation may uncover the desired avoidance and if not, the two balls next to each other necessarily do so.

- if  $n = 1$ ,  $f'(P, r) = f(P, r)$ ;
- if  $n > 1$ ,  $f'(P, r) = f(P, f'(P \setminus \{p_0\}, r)) + f'(P \setminus \{p_n\}, r)$

Take  $x \in X$ , and assume that  $x$  avoids every period  $p \in P$  in some ball  $B(z, r)$ . Then  $x$  avoids every period  $p \in P$  in some ball  $B(z', \sum_P \|p\|)$ , with  $\|z' - z\| \leq f'(P, r)$ .

**Proof.** We prove the lemma by induction. If  $n = 1$ , then by the lemma hypothesis we find two balls  $B_1 = B(b_1, \|p_0\|)$  and  $B'_1 = B(b'_1, \|p_1\|)$  in  $B(z, r)$  that contain an avoidance of  $p_0$  and  $p_1$ , respectively. Applying Lemma 5 on these balls, we obtain the result.

Now assume  $n > 1$ . By applying the induction hypothesis twice on  $p_0, \dots, p_{n-1}$  and  $p_1, \dots, p_n$ , we find two balls  $B_n = B(b_n, \sum_{i>0} \|p_i\|)$  and  $B'_n = B(b'_n, \sum_{i<n} \|p_i\|)$  such that  $\|b_n - z\| \leq f'(P \setminus \{p_n\}, r)$  and  $\|b'_n - z\| \leq f'(P \setminus \{p_0\}, r)$ . Applying Lemma 5 on these balls, we obtain the desired ball with  $\|b_n - z\| \leq f'(P, r)$  by the triangular inequality. ◀

► **Lemma 7.** Let  $P_n = \{p_0, \dots, p_n\}$  be a set of noncolinear periods. Define recursively a function  $g$  such that

$$g(\{p\}) = \|p\| \quad \text{and} \quad g(P_n) = g(P_{n-1}) + f' \left( P_n', \sum_{P_n'} \|p\| \right) + \sum_{P_n'} \|p\|$$

Take  $x$  an aperiodic point that avoids every  $p \in P_n'$  in a ball  $B(z, \sum_{P_n'} \|p\|)$ . There exist  $z' \in \mathbb{Z}^2$  such that:

- $\|z' - z\| \leq g(P_n)$
- for any  $i \leq n$ , the ball  $B(z', g(P_i))$  contains an avoidance of every period in  $P_i'$ .

Less formally, we can organise avoidances of all periods up to  $p_n$  in concentric balls around a common center, so that the distance of the avoidance of any given period to the center does not depend on  $n$  only on the period itself.

**Proof.** We proceed by induction on  $n$ .

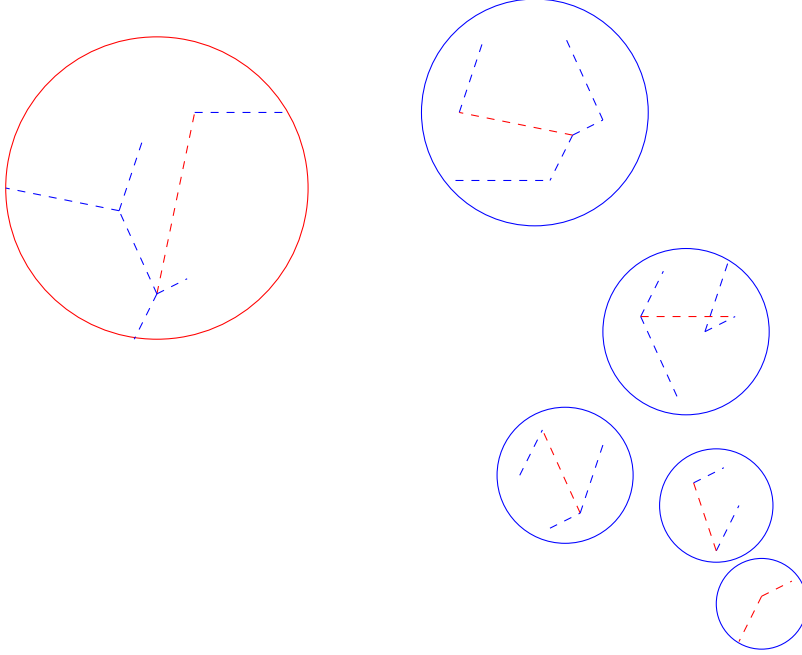
If  $n = 0$ , then since  $B(z, \|p_0\|)$  contains an avoidance of  $P_0' = \{p_0\}$ , taking  $z'$  to be this avoidance satisfies the requisite.

Assume  $n > 0$ . Since  $B(z, \sum_{P_n'} \|p\|)$  contains avoidances of every period in  $P_n'$ , it contains avoidances of every period in  $P_{n-1}'$ . Applying Lemma 6 on  $P_{n-1}'$ , we find a ball  $B(z_0, \sum_{P_{n-1}'} \|p\|)$  that contains avoidances for all periods in  $P_{n-1}'$  and such that  $\|z_0 - z\| \leq f'(P_{n-1}', \sum_{P_n'} \|p\|)$ .

Now apply the induction hypothesis on this ball, obtaining  $z'$  such that  $\|z' - z_0\| \leq g(P_{n-1})$  and for any  $i \leq n$ , the ball  $B(z', g(P_i))$  contains avoidances of every period in  $P_i'$ . Given the inductive nature of Lemma 6, this is a kind of double induction illustrated in Figure 2.

By the triangular inequality,  $\|z' - z\| \leq g(P_{n-1}) + f'(P_n', \sum_{P_n'} \|p\|)$ , so  $B(z', g(P_n))$  contains entirely the ball  $B(z, \sum_{P_n'} \|p\|)$ . Therefore  $B(z', g(P_n))$  contains an avoidance of  $p_n$  and  $\|z' - z\| \leq g(P_n)$ , proving the lemma. ◀

► **Theorem 1.** Let  $X$  be a  $\mathbb{Z}^2$ -subshift. If  $X$  contains an aperiodic configuration, then it contains an aperiodic configuration that avoids every period  $p$  at distance at most  $g(\|p\|)$  from 0, where  $g$  is a computable function.



■ **Figure 2** From a large ball containing avoidances for  $P'_n$  we build a relatively close smaller ball containing avoidances for  $P'_{n-1}$ , and so on inductively.

► **Corollary 2.** *Let  $X$  be a subshift. If for every  $n$ ,  $X$  contains a configuration that avoids all periods  $p$  with  $\|p\| \leq n$ , then  $X$  contains an aperiodic configuration.*

*In other words, for any subshift  $X$  with no aperiodic point, there is a finite set of periods  $P$  such that any configuration of  $X$  is periodic for some period  $p \in P$ .*

**Proof.** Let  $P_n$  be the set of periods of norm  $n$  or less. Let  $x_0, x_1, \dots$  be a sequence of configurations such that  $x_n$  avoids every period in  $P'_n$ . By Lemma 5 and Lemma 7, we can assume (up to replacing each  $x_i$  by a shift of itself) that the ball  $B(0, g(P'_i))$  contains avoidances of all periods in  $P'_i$  for all  $x_k$ ,  $k \geq i$ . Theorem 1 is proved.

Let  $x$  be a limit point of the sequence  $(x_i)_{i \in \mathbb{N}}$ .  $x \in X$  since  $X$  is compact. By the previous remark, for all  $i$ , the ball  $B(0, g(P'_i))$  contains avoidances of all periods in  $P'_i$  for  $x$ . Therefore  $x$  is an aperiodic point. We proved Corollary 2. ◀

### 3 Consequences

#### 3.1 Existence of an aperiodic configuration is $\Pi_1^0$

► **Corollary 8.** *The following problem is  $\Pi_1^0$ -computable:*

**Input** *A finite set of forbidden patterns  $\mathcal{F}$ .*

**Output** *Does the  $\mathbb{Z}^2$ -SFT  $\mathcal{X}_{\mathcal{F}}$  contain an aperiodic configuration?*

**Proof.** Let  $(p_i)_{i \in \mathbb{N}}$  be an enumeration of all possible periods and  $P_n = \{p_0, \dots, p_n\}$ . Theorem 1 gives us a bound on the size of the patterns in which to look for avoidances of each period. For each  $n \in \mathbb{N}$  in order, the algorithm enumerates all patterns on a ball of radius  $g(P_n)$  that do not contain a forbidden pattern, and check if one of them contains avoidances for every period of  $P_k$  ( $k \leq n$ ) in the ball of radius  $g(P_k)$  in its center. If such a pattern does



not exist for some  $n$ , it means that either the SFT is empty or that all its points are periodic for some period  $p$  with  $\|p\| \leq n$ .

Assume the algorithm runs infinitely. For every  $k$ , there exists some  $n \geq qk$  such that if a pattern on the ball of radius  $g(P_n)$  does not contain a forbidden pattern, then the subpattern on the ball of radius  $g(P_k)$  is in the language of  $X$ . Therefore, for each  $P_k$  we find a pattern in  $L(X)$  that avoids all periods of  $P_k$ , and we conclude by Corollary 2.  $\blacktriangleleft$

### 3.2 Structure of subshifts without aperiodic points

► **Definition 9** (Topological conjugacy). For  $i = 1, 2$ , take  $C_i$  a compact set and  $f_i : C_i \rightarrow C_i$  continuous functions.

$(C_1, f_1)$  and  $(C_2, f_2)$  are *topologically conjugate* if there exists a continuous bijection  $\pi : C_1 \rightarrow C_2$  such that  $\pi \circ f_1 = f_2 \circ \pi$ .

$(C_1, f_1)$  and  $(C_2, f_2)$  are *almost topologically conjugate* if there exists  $(C_3, f_3)$  and continuous surjections  $\pi_i : C_3 \rightarrow C_i$  that are bijective almost everywhere<sup>1</sup> such that  $\pi_i \circ f_3 = f_i \circ \pi_i$  for  $i = 1, 2$ .

See [13] or [11] for more information on topological conjugacy and almost conjugacy in the context of symbolic dynamics. We need slightly more general definitions since we consider subshifts of different dimensions.

Notice that we can have  $C_i = C_3$  (and  $f_i = id$ ) in the last definition; this is the case in the next proof.

► **Theorem 10.** *Let  $X$  be a two-dimensional subshift with no aperiodic point. There exists a vector  $v$  and a one-dimensional subshift  $Y$  such that  $(X, \sigma_v)$  is almost topologically conjugate to  $(Y, \sigma)$ .*

*If  $X$  is of finite type, then  $Y$  can be chosen of finite type as well.*

**Proof.** Let  $X$  be a two-dimensional subshift of finite type with no aperiodic point. By Theorem 1, there is a finite set of periods  $P$  such that any configuration of  $X$  is periodic of some period  $p \in P$ . We assume that  $P$  does not contain two colinear periods, by taking their least common multiple if necessary.

For the clarity of the argument, we assume in the following that  $P$  does not contain any period colinear to  $(1, 0)$ . Since  $P$  is finite, the proof can be adapted for a different vector.

Take  $p = (p_0, p_1) \in P$ , assuming  $p_0 > 0$ , and denote  $X_p = \{x \in X : x \text{ admits } p \text{ as a period}\}$ . It is a classical argument (see for instance [9, §2.1.2] or [2, Lemma 5.2]) that  $(X_p, \sigma_{(0,1)})$  is topologically conjugate to a one-dimensional SFT, which we repeat here for completeness. Define:

$$\pi_p = \begin{cases} \Sigma^{\mathbb{Z}^2} & \rightarrow (\Sigma^{p_0-1})^{\mathbb{Z}} \\ x & \mapsto ((x_{i,j})_{0 \leq i < p_0})_{j \in \mathbb{Z}} \end{cases}$$

Denote  $Y_p = \pi(X_p)$ . It is not hard to see that  $\pi_p$  is a one-to-one function between  $X_p$  and  $Y_p$  and that  $Y_p$  is a subshift of finite type if  $X_p$  is. Furthermore,  $\pi_p \circ \sigma_{(0,1)} = \sigma \circ \pi_p$ , so it is a topological conjugacy between  $(X_p, \sigma_{(0,1)})$  and  $(Y_p, \sigma)$ .

For any  $p_1 \neq p_2 \in P$ ,  $X_{p_1} \cap X_{p_2}$  is a set of strongly periodic configurations that admit non-colinear periods  $p_1$  and  $p_2$ ; there are a finite number of such configurations, so

<sup>1</sup> Except for a finite set of points.

$|X_{p_1} \cap X_{p_2}| < +\infty$ . In other words,  $X = \bigcup_{p \in P} X_p$  and the union is disjoint except for a finite set of configurations.

Denote  $Y = \sqcup_{p \in P} Y_p$  (disjoint union).  $Y$  is a subshift on the alphabet  $\sqcup_{p \in P} \Sigma_p$ , where  $\Sigma_p$  is the alphabet of  $Y_p$ . Furthermore,  $Y$  is of finite type if every  $Y_p$  is of finite type.

Define  $\varphi : Y \rightarrow X$  by  $\varphi|_{Y_p} = \pi_p^{-1}$ . We can check that  $\varphi$  is surjective and almost everywhere bijective, and that  $\varphi \circ \sigma_1 = \sigma_{0,1} \circ \varphi$ . We have proved that  $(Y, \sigma_1)$  is almost topologically conjugate to  $(X, \sigma_{(0,1)})$ . ◀

### 3.3 Various properties of subshifts with no aperiodic points

Theorem 10 implies that the property of having no aperiodic point gives a very strong structure to a subshift. This is particularly the case for subshifts of finite type, where many problems that are undecidable in dimension 2 are completely solved in dimension 1, and these solutions carry through almost topological conjugacy.

In this section, we make use of notations that were defined in the proof of Theorem 10:  $X_p, Y_p, \pi_p$  and  $\varphi$ .

**Decision problems** have been a staple of the theory of multidimensional subshifts of finite type: the seminal paper of Wang proved that the emptiness problem (given a list of forbidden patterns  $\mathcal{F}$ , is  $X_{\mathcal{F}} = \emptyset$ ?) is decidable for two-dimensional subshifts of finite type with no aperiodic point, but Berger later proved that the problem was undecidable without this assumption [3]. We consider other classical decision problems: the extension problem, which is undecidable for multidimensional subshifts of finite type (as a consequence of the above), and the injectivity and surjectivity problems, which are undecidable even on the two-dimensional full shift [10].

► **Corollary 11.** *The following problems are decidable for two-dimensional subshifts of finite type with no aperiodic point:*

**Extension problem** given a list of forbidden patterns  $\mathcal{F}$  and a pattern  $w$ , do we have  $w \in \mathcal{L}(X_{\mathcal{F}})$ ?

**Injectivity / surjectivity problem** given a list of forbidden patterns  $\mathcal{F}$  and a cellular automata  $\Phi : \Sigma^{\mathbb{Z}^2} \rightarrow \Sigma^{\mathbb{Z}^2}$ , is  $\varphi|_{X_{\mathcal{F}}}$  injective? surjective on  $X_{\mathcal{F}}$ ?

Links between periodic points and these had already been considered in [12, 7].

**Sketch of a proof. Extension problem** Assume  $w$  has shape  $[-n, n]^2$ . By Theorem 1 have  $X = \varphi(Y)$  where  $\varphi$  is continuous on a compact space, hence uniformly continuous. In other words, for every  $n$ , there exists  $r$  such that the value of  $\varphi(y)_{[-n, n]^2}$  only depends on  $y_{[-r, r]}$ . Since the extension problem is decidable on one-dimensional subshifts of finity type, enumerate all words  $v \in \mathcal{L}(Y)$  and check whether  $w = \varphi(v)$  for some  $v$ .

**Injectivity/surjectivity problem** A configuration and its image by a cellular automaton have the same period, so any cellular automaton  $\Phi|_{X_{\mathcal{F}}}$  is injective, resp. surjective, if and only if  $\Phi|_{X_p}$  is injective, resp. surjective, for each period  $p$ . Let  $\pi_p : X_p \rightarrow Y_p$  be the continuous bijection defined in the proof of Theorem 10.  $\pi_p^{-1} \circ \Phi|_{X_{\mathcal{F}}} \circ \pi_p$  is a CA on  $Y_p$  and it shares injectivity and surjectivity with  $\Phi|_{X_{\mathcal{F}}}$ . Injectivity and surjectivity of CA is decidable for one-dimensional subshifts of finite type [7]. ◀

► **Remark.** If we did not know that  $X$  admits a finite set of periods, the first proof would still show that the extension problem is in  $\Sigma_1^0$  (RE). Since it is easy to show that it is in  $\Pi_1^0$  (co-RE), our main result is technically unnecessary here.

**Topological entropy** is a widely-used parameter in information theory (channel capacity) and dynamical systems theory (conjugacy invariant). Entropy dimension is a more refined notion for systems of entropy zero, introduced in [5] and mainly used for multidimensional subshifts [14].

► **Corollary 12.** *Any two-dimensional subshift  $X$  with no aperiodic point has zero topological entropy. Its entropy dimension is at most one.*

**Sketch of a proof.** By Theorem 1, there is a finite set of periods  $P$  such that  $\mathcal{L}(X) = \bigcup_{p \in P} \mathcal{L}(X_p)$ . Consider a pattern  $w$  of shape  $[0, n-1]^2$  in  $\mathcal{L}(X_p)$ , assuming for clarity  $p_0 \geq 0$  and  $p_1 \geq 0$ . Since  $w$  cannot contain an avoidance for  $p$ , it is entirely determined by its  $p_0$  bottommost rows and  $p_1$  leftmost columns. Therefore there are at most  $(p_0 + p_1)n$  such patterns. A similar argument applies when  $p_0 < 0$  or  $p_1 < 0$ .

It follows that there are at most  $\sum_p (|p_0| + |p_1|)n$  patterns of shape  $[0, n-1]^2$  in  $\mathcal{L}(X)$ , proving the statement. ◀

**Density of periodic points** is a typical question in dynamical systems, for example when studying chaos in the sense of Devaney. See [7] for more details, including a proof that two-dimensional subshifts of finite type do not have dense 2-periodic points in general, even under an additional irreducibility hypothesis.

► **Corollary 13.** *Any irreducible two-dimensional subshift of finite type  $X$  with no aperiodic point has dense 2-periodic points.*

**Sketch of a proof.** If  $X$  is irreducible then each  $X_p$  is irreducible, and so is each  $Y_p$  by conjugacy. One-dimensional irreducible subshifts of finite type, such as  $Y_p$ , have dense periodic points [7].  $\varphi : Y \rightarrow X$  is a continuous surjection, and the image of a periodic point in  $Y_p$  is a 2-periodic point in  $X$ , from which the statement follows. ◀

### 3.4 The full characterization of slopes of 3d SFTs

► **Definition 14.** Let  $X$  be a  $d$ -dimensional subshift,  $\theta \in (\mathbb{Q} \cup \infty)^{d-1}$  is a slope of periodicity of  $X$  if there exists a configuration  $x \in X$  such that that:

$$v\mathbb{Z} = \{v \mid \sigma_v(x) = x\}$$

A consequence of Corollary 8 is that the sets of slopes of periodicity of 3D SFTs is a  $\Sigma_2^0$  set, and together with [16] this implies the following characterization:

► **Theorem 15.**  $\Sigma_2^0$  subsets of  $S \subseteq (\mathbb{Q} \cup \{\infty\})^2$  that are  $\Sigma_2^0$  are exactly the sets realizable as sets of slopes of three dimensional subshifts of finite type.

**Proof.** We know from [16] that one can realize any such  $\Sigma_2^0$  set  $S$  as a set of slopes of a 3d subshift. Let us now show the remaining direction.

Given  $\theta$ , a slope, and  $X$  an SFT, as an input, one needs to check whether there exists a configuration in  $X$  with exactly one vector of periodicity of direction  $\theta$ . Let  $p$  be the smallest such vector, all other vectors of direction  $\theta$  are multiples of it. It thus suffices to find whether there exists some  $i \in \mathbb{N}$  such that  $ip$  is a direction of periodicity for some configuration.

Let us show that given some vector  $v \in \mathbb{Z}^3$ , checking whether there exists a point of  $X$  periodic only along  $v$  is  $\Pi_1^0$ . Using the argument as in proof of Theorem 10, notice that periodic configurations along  $v$  can be seen as configurations on one less dimension with a different alphabet. The set of configurations periodic along  $v$  thus constitutes a 2d SFT  $Y$

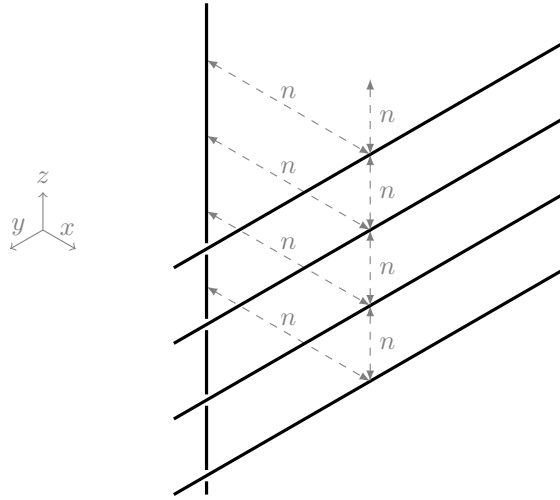
computable from  $X$  and  $v$  and the existence of a configuration of direction of periodicity  $v$  in  $X$  corresponds to the existence of an aperiodic configuration in  $Y$ . Since this is  $\Pi_1^0$  (Corollary 8), checking whether there exists a periodic configuration of direction  $ip$  for some  $i$  and thus whether there exists a configuration of slope  $\theta$  is  $\Sigma_2^0$ . ◀

**4 Counter example for dimensions  $d > 2$**

We build a counterexample to Theorem 1 in higher dimension. Take  $\Sigma = \{0, 1\}$  and define  $X$  as follows:

- All symbols 1 must form lines of direction vector  $(1, 0, 0)$  (horizontal) or  $(0, 0, 1)$  (vertical);
- There is at most one vertical line;
- All horizontal lines are repeated periodically with period  $(0, 0, n)$ , where  $n$  is the distance of the vertical line to any horizontal line.

In particular, if there is no vertical line, then there is at most one horizontal line. To sum up, a subshift configuration can be : **1.** all zeroes, **2.** one horizontal line, **3.** one vertical line, or **4.** the situation depicted in Figure 3.



■ **Figure 3** A typical configuration of  $X$ : a line of ones along  $z$  at distance  $n$  of an  $(xy)$  plane of lines along  $x$ . The only other types of configurations of  $X$  are the configurations containing only one vertical line, or no line at all.

The configuration described in Figure 3 admits  $(0, 0, n)$  as period, and no shorter period. In particular,  $X$  avoids the set of periods  $P_p$  for any  $p < (0, 0, n)$ , and thus for every  $p$   $X$  contains a configuration that avoids  $P_p$ . However,  $\Sigma$  admits no aperiodic point<sup>2</sup>. This example can easily be generalised to any  $d > 3$  by considering a  $\mathbb{Z}^d$ -subshift  $X'$  that contains a copy of  $X$  in at most one coordinate, and 0 everywhere else: that is,

$$x \in X' \Leftrightarrow \forall j \in \mathbb{Z}^{d-3}, (x_{i,j})_{i \in \mathbb{Z}^3} \in X \text{ and } (\forall j_1 \neq j_2, (x_{i,j_1})_{i \in \mathbb{Z}^3} = 0 \text{ or } (x_{i,j_2})_{i \in \mathbb{Z}^3} = 0).$$

This proves that Theorem 1 does not hold in any dimension  $d > 2$ .

<sup>2</sup> Notice that there cannot be a configuration with a single horizontal line with a single vertical line, which would be aperiodic.

## 5 Open problems

We have made clear that our main result does not hold for subshifts of dimension  $d \geq 3$ . We do not know, however, whether Theorem 10 or Corollary 12 holds in higher dimension, since the counterexample introduced in Section 4 does not contradict these results.

This counterexample is a subshift with arbitrarily large periodic but no aperiodic point. We do not know whether such a counterexamples with infinitely many directions of periodicity exist. Moreover, the structure of  $d$ -dimensional subshifts of finite type for  $d \geq 3$  remains open; the existence of this counterexample suggests that a making use of the finite type hypothesis is necessary in higher dimension.

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