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Homogenization of Gas Flow through Dual-Porosity Media

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Part I

Introduction

A dual-porosity medium consists of two interacting porous systems of distinctly different fluid transport properties as it is the case in a fractured porous medium: one porous structure is associated with the fractures and the second one with the porous matrix. Modelling such an heterogeneous system turns out to be a difficult task. Nonetheless, the internal disorder repetition allows a large-scale continuous description. In other words, physical processes can be described by means of equations with transfer coefficients that are independent of the macroscopic boundary conditions. Two kinds of approaches for deriving such continuous mathematical models may be distinguished: i) Phenomenological approaches; ii) Upscaling methods.

The first investigations were on the basis of phenomenological approaches, i.e. directly macroscopic approaches (Barenblatt et al., 1960), (Barenblatt, 1963), (Warren and Root, 1963). The first model (Barenblatt et al., 1960) shows an important feature of dual-porosity systems: the interporosity flow, i.e. the fluid exchange between both media. In these phenomenological models, a pseudo-steady state flow is described. They are based upon the conjecture that the interporosity flow occurs in response to the fracture-porous matrix difference in pressure.

On the other hand, homogenization techniques allow the determination of an equivalent macroscopic behaviour by upscaling the local description. By definition, this macroscopically equivalent medium behaves "in average" like the initial heterogeneous medium under a given excitation. The condition required for applying these methods is the separation of scales. Homogenization techniques have proved to be efficient for modelling fluid flow in porous media. In particular, the homogenization method for periodic structures leads to precise descriptions since no macroscopic prerequisite is required. When looking for a macroscopic equivalent description of fluid flow in a fractured porous medium, three separated scales may be under consideration: the pore scale, the fracture scale and the macroscopic scale. An innovative three-scale homogenization method for modelling fluid flow in fractured porous media was established in (Auriault and Boutin, 1992), (Auriault and Boutin, 1993), (Auriault and Boutin, 1994), (Royer and Auriault, 1994), (Royer, 1994).

The goals of this lecture are two-fold: i) To give a general introduction to homogenization theory; ii) To present the derivation of the models for gas flow in a fractured porous medium via homogenization. In Part II, general features of homogenization theory, and the detailed procedure of the method for periodic structures are presented. Part III is

devoted to the derivation of Darcy's law via homogenization. The cases of an incompressible and then of a highly compressible fluid are successively considered. Finally, Part IV specifically relates to modelling gas flow in dual-porosity media by the three-scale homogenization approach.

Part II

Introduction to Homogenization Theory

1 Fundamental Concepts

Homogenization methods aim at representing the evolution of a given physical process in a given heterogeneous medium by an equivalent macroscopic continuous behaviour. As a continuous description, the derived macroscopic behaviour is intrinsic to the medium and to the solicitation and is independent of the macroscopic boundary conditions.
The macroscopic behaviour is derived from the description that describes the physical process over a Representative Elementary Volume (REV). The existence of such a volume is required for any continuous macroscopic representation of the physical system, and, as a consequence, is required for applying any homogenization technique. By definition the REV is
i) sufficiently large for representing the heterogeneity scale,
ii) small compared to the macroscopic volume.
As a consequence, a condition of separation of scales is required. This fundamental condition can be expressed as follows:

$$\frac{l}{L} << 1, \tag{1}$$

where l and L are the characteristic lengths at the REV scale and at the macroscopic scale, respectively.
This definition intuitively conjures up a geometrical separation of scales, whereas this fundamental condition must also be verified regarding the excitation (i.e. the physical process). For instance, consider the propagation of a wave in an heterogeneous medium. The wavelength actually constitutes a third characteristic length. Intuitively, we see that a continuous approach for describing this physical process will be possible only if the heterogeneity scale is small compared to the wavelength; a wavelength of the order of the heterogeneity length-scale would lead to wave-trapping effects, which could not be described by constitutive equations at the macroscopic scale. For fluid flow in porous media, as the medium is excited by a pressure gradient, the characteristic length of the excitation is thus related to the pressure gradient.
Therefore, the fundamental condition of separation of scales is expressed as $\frac{l}{L} << 1$, where L is the macroscopic characteristic length and is either geometrical or related to

the excitation.
Thus, the existence of the REV and, as a consequence, the condition of separation of scales, are not only constrained to geometrical considerations but also related to the excitation (i.e. the physical process). The analysis focuses on what we call the physical system, that consists of both the medium and the excitation.

2 A Definition of Homogenization

2.1 How to Model an Heterogeneous Medium?

The choice of the approach for describing physical processes in heterogeneous media is conditioned by the existence of a Representative Elementary Volume (REV). If there is no REV, then, the modelling approach must be on the basis of an exact description of all heterogeneities over the whole macroscopic volume. These approaches are called discrete approaches.
On the other hand, the existence of the REV allows a continuous approach. Thus, the derived description is intrinsic to the material and to the excitation and is independent of the macroscopic boundary conditions. Continuous approaches lead to the definition of average parameters (effective parameters) that characterise the physical system (i.e. the medium and the excitation) at the macroscopic scale. There are two ways of deriving this macroscopic description.
The first one is a directly macroscopic approach, which is often associated with experiments and is called phenomenological approach. Many physical laws have been first derived by this kind of approaches. It is the case of Darcy's law (Darcy, 1856).
The second kind of continuous approaches allows to derive the macroscopic behaviour from the local description. This is an upscaling technique, or an homogenization method. The equivalent description is called the homogenized description. Both the constitutive equations and the effective parameters are determined. The knowledge of the physical parameters and of the geometry is required over the REV only. Therefore, homogenization techniques are continuous approaches that allow the derivation of an equivalent macroscopic continuous description from the description at the REV scale.

2.2 The Fundamental Condition

Firstly, it is important to notice that homogenization cannot be applied in all cases. As already mentioned, the fundamental condition is that the scales must be separated. The heterogeneity scale is called the microscopic scale, in contrast with the macroscopic scale at which the equivalent medium is defined. If l is a characteristic dimension of the

heterogeneities and L is a characteristic dimension of the volume of the material (or of the phenomenon) under consideration, then the separation of scales is expressed as follows:

$$\frac{l}{L} = \varepsilon << 1$$

ε is a measure of the separation of scales.
This is the fundamental condition which is common to all homogenization techniques, even if it is only implicit for some methods. We will see that this condition is clearly stated in the case of the method for periodic structures, as the method is based on the small parameter ε. The idea behind this condition is that the macroscopic behaviour is fully governed by the local behaviour. Thus, stating that the scales are separated means that it is possible to deduce the macroscopic description from the microscopic description. It is therefore obvious that the medium must contain a large number of heterogeneities.

3 Homogenization for Periodic Structures

3.1 Introduction

We now assume that the fundamental condition of separation of scales is verified. All homogenization methods can derive the macroscopic description using macroscopic quantities which are derived by averaging. The choice of the method and therefore of the average depends on the structure of the medium. Thus, we may distinguish materials with a random structure and materials with a periodic structure.
Materials with periodic structures are investigated by means of the method of homogenization for periodic structures, which is also called the method of asymptotic expansions. This method has been introduced by (Bensoussan et al., 1978) and (Sanchez-Palencia, 1980). Recently, a more physical methodology based on dimensionless analysis has been introduced by (Auriault, 1991). This approach highlights the conditions under which homogenization can be applied. The presentation of this methodology is the purpose of the present section.

3.2 Fundamental Hypotheses

Consider an heterogeneous medium. l and L are the characteristic lengths at the microscopic and the macroscopic scales, respectively. We assume the scales to be separated:

$$\varepsilon = \frac{l}{L} << 1$$

The method can be applied only if ε is very small. The development of the method is based on the small parameter ε. The medium is also assumed to be periodic, and l is the

characteristic size of the period.
Note that the condition of periodicity is actually not a restriction; it allows derivation of the macroscopic description with no preliminary hypothesis on the form of the macroscopic constitutive equations.

3.3 Representative Elementary Volume

If the scales are separated in the medium under consideration, then, it means that the medium contains a large number of heterogeneities whose dimension is very small compared to the macroscopic size. As a result, there exists a volume which is very small compared to the macroscopic volume, and which is also large enough to contain a sufficient number of heterogeneities to represent the problem at the local scale. In other words, there exists a Representative Elementary Volume. In the case of a periodic medium, the REV is simply the period. For random media, the definition of the REV is more difficult.

3.4 Separation of Space Variables

Let $\vec{X}$ be the physical space variable of the medium. From the two characteristic lengths, two dimensionless space variables can be defined:

$$\begin{aligned} \vec{y} &= \frac{\vec{X}}{l} \quad \text{dimensionless microscopic space variable} \\ \vec{x} &= \frac{\vec{X}}{L} \quad \text{dimensionless macroscopic space variable} \end{aligned}$$

As a consequence of the separation of scales, $\vec{y}$ and $\vec{x}$ are two separated variables. Any quantity ϕ required to describe the physical process is, a priori, a function of these two separated variables:

$$\phi = \phi(\vec{y}, \vec{x}).$$

3.5 Interpretation of Periodicity and Separation of Scales

With both assumptions, periodicity and separation of scales (figure 1), we will be able to derive the macroscopic behaviour from the complete description over the REV, i.e. over the period.
The existence of the REV, and therefore the separation of scales, imply that the properties of the physical system (e.g. porosity, saturation, ...) vary very smoothly over the whole macroscopic volume. If the medium is periodic, then these variations are periodic. Thus, the assumption of separation of scales entails the existence of an REV and the separation of the space variables, while the assumption of periodicity implies that the REV is the

Figure 1: *Periodicity and separation of scales*

period and leads to the property of local periodicity.
As a consequence of both the separation and scales and the periodicity, all physical quantities can be looked for in the form of asymptotic expansions in power of ε (Bensoussan et al., 1978):

$$\phi(\vec{y}, \vec{x}) = \phi^0(\vec{y}, \vec{x}) + \varepsilon^1 \phi^1(\vec{y}, \vec{x}) + \varepsilon^2 \phi^2(\vec{y}, \vec{x}) + ... \tag{2}$$

where the ϕ^i's are $\vec{y}$-periodic.
Note that expression (2) represents both assumptions of separation of scales and periodicity: the condition of separation of scales is expressed via the dependence of ϕ upon the two separated variables $\vec{x}$ and $\vec{y}$, and the periodicity of the medium is shown through the condition of periodicity assigned to the ϕ^i's.

3.6 Methodology

The methodology presented below is that introduced by (Auriault, 1991).

3.6.1 Normalisation

The first step, called normalisation aims at writing all governing equations in a dimensionless form. For this purpose, all dimensionless numbers introduced by the governing equations must be estimated with respect to ε. A number Q is said to be $O(\varepsilon^p)$ if:

$$\varepsilon^{p+1} << Q << \varepsilon^{p-1}$$

To estimate dimensionless numbers, we may choose a characteristic length (l or L) as the reference length. When Q is estimated with the microscopic length l, the resulting estimation is designated by Q_l and this choice is referred as the "microscopic point of view". Both points of view are absolutely equivalent. We just need to be consistent. For example, the determination of the dimensionless partial derivative with respect to the physical space variable, $\vec{X}$, requires the use of the reference length. In effect, according to the definition of $\vec{y}$ and $\vec{x}$, the partial derivative with respect to $\vec{X}$ is the following:

$$\frac{\partial}{\partial X_i} = \frac{\partial}{\partial y_i}\frac{\partial y_i}{\partial X_i} + \frac{\partial}{\partial x_i}\frac{\partial x_i}{\partial X_i} = \frac{1}{l}\frac{\partial}{\partial y_i} + \frac{1}{L}\frac{\partial}{\partial x_i} \tag{3}$$

Thus, if we have chosen the microscopic point of view, the dimensionless partial derivative with respect to $\vec{X}$ is written as follows:

$$l\frac{\partial}{\partial X_i} = \frac{\partial}{\partial y_i} + \varepsilon\frac{\partial}{\partial x_i} \tag{4}$$

As a consequence, the dimensionless gradient is expressed as:

$$l\vec{\nabla}_X = \vec{\nabla}_y + \varepsilon\vec{\nabla}_x \tag{5}$$

Once all dimensionless numbers have been estimated, the governing equations can then be written in a dimensionless form. As a result, they will all be expressed as follows:

$$\sum_i \varepsilon_i^p \; A_i = 0 \qquad \text{where } \; A_i = O(1) \quad (\text{i.e. } \; \varepsilon << A_i << \varepsilon^{-1})$$

The A_i's are some dimensionless operators and, as a consequence, are $O(1)$ with respect to ε. The influence of the value of the dimensionless numbers is shown through the terms ε_i^p.

3.6.2 Derivation of the Macroscopic Behaviour

The procedure is the following:

• Introduce the asymptotic expansions in the set of dimensionless equations,

• Deduce the relevant boundary-value problems arising at the successive orders of ε.

At a given order of ε, the local balance equation will give an equation of the following type:

$$\underbrace{\vec{\nabla}_y \, . \, \phi^{i+1}}_{microscopic} + \underbrace{\vec{\nabla}_x \, . \, \phi^i}_{macroscopic} = 0$$

This latter equation will lead to the macroscopic description. This equation actually expresses the balance over the period of the quantity ϕ^{i+1}, in presence of the source term $-\vec{\nabla}_x \, . \, \phi^i$. Therefore, we can write it as follows:

$$div \; \vec{q} - source = 0$$

Integrating this over the period gives what we call the "compatibility condition". This compatibility condition must be checked, otherwise the original equation is not valid. It is written as:

$$< div \; \vec{q} >_\Omega - < source >_\Omega = 0 \tag{6}$$

where $< . >_\Omega$ denotes the average over the period and is defined by:

$$< . >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_y} . \; d\Omega$$

Now, using the divergence theorem, the first term of equation (6) becomes:

$$< div \; \vec{q} >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_y} div \; \vec{q} \; d\Omega = \frac{1}{\mid \Omega \mid} \int_{\delta\Omega} \vec{q} \, . \, \vec{n} \; dS$$

Thus, the compatibility condition (6) becomes:

$$\frac{1}{\mid \Omega \mid} \int_{\delta\Omega} \vec{q} \, . \, \vec{n} \; dS - < source >_\Omega = 0 \tag{7}$$

Equation (7) gives the macroscopic behaviour. The term $\int_{\delta\Omega} \vec{q} \, . \, \vec{n} \; dS$ can be simplified using the boundary conditions and the property of periodicity. In many simple cases, the macroscopic behaviour is simply $< source >_\Omega = 0$.

Part III

Derivation of Darcy's Law via Homogenization

4 Incompressible Fluid

4.1 Introduction

In this section we are going to derive Darcy's law by homogenization. The first theoretical derivation of Darcy's law by homogenization technique is due to (Ene and Sanchez-Palencia, 1975). The presentation below is taken from (Auriault, 1991).
The problem is to derive, when possible, the macroscopic description of the flow of an incompressibility Newtonian fluid at constant temperature in a rigid porous medium.

We consider a rigid porous matrix, which is periodic. The period Ω is of the order of l. The characteristic size of the macroscopic sample is L. We assume the scales to be separated:

$$\frac{l}{L} = \varepsilon << 1$$

The fluid and the solid occupy the domains Ω_f and Ω_s, respectively. Γ is the surface of the solid (figure 2).
Let $\vec{X}$ be the physical space variable. We define the two following dimensionless space variables:

$$\begin{cases} \vec{y} &= \dfrac{\vec{X}}{l}, \\ \vec{x} &= \dfrac{\vec{X}}{L}. \end{cases}$$

$\vec{y}$ and $\vec{x}$ are the microscopic and the macroscopic dimensionless space variables, respectively. As a result of the separation of scales, they are independent variables.

4.2 Local Description

The local description consists of the set of equations that describe fluid flow over the period. Therefore, it must express the balance of momentum and mass and appropriate boundary conditions on the solid/fluid interface.
For a steady-state and slow flow, the momentum balance is described by Stokes equation:

$$\mu \Delta \vec{v} - \vec{\nabla} P = \vec{0} \quad \text{in } \Omega_f \tag{8}$$

Figure 2: *The periodic cell*

For an incompressible fluid, the mass balance is given by:

$$\vec{\nabla} \, . \, \vec{v} = 0 \quad \text{in } \Omega_f \tag{9}$$

The no-slip condition on the boundary is expressed as:

$$\vec{v}/_{\Gamma} = \vec{0} \tag{10}$$

Equations (8), (9) and (10) constitute the local description.

4.3 Normalisation

Let us take l as the reference length (microscopic point of view). Equation (8) introduces the following dimensionless number:

$$Q = \frac{\mid \vec{\nabla} P \mid}{\mid \mu \Delta \vec{v} \mid}$$

When using l as the reference length, the estimate of Q is:

$$Q_l = \frac{\frac{\delta P}{l}}{\frac{\mu V}{l^2}}, \tag{11}$$

where δP and V are characteristic values of the pressure drop and the velocity, respectively.

Now, the flow is forced by the macroscopic pressure gradient $O(\frac{\delta P}{L})$. Equation (8) expresses the fact that, as a result of this driving force, a fluid flow occurs within the pore space, i.e. at the local scale. This local viscous term is $O(\frac{\mu V}{l^2})$. Hence, Stokes equation expresses the balance between the macroscopic pressure gradient and the local viscous forces. Therefore, we have:

$$\frac{\delta P}{L} = O(\frac{\mu V}{l^2}) \tag{12}$$

Incorporating (12) into (11) gives:

$$Q_l = O(\varepsilon^{-1}) \tag{13}$$

Note that estimate (13) is imposed by the physics. The fact that $\vec{\nabla} P$ is said to be of the order of $\frac{\delta P}{L}$ is related to what has previously been mentioned as the separation of scales regarding the excitation.
Equations (9) and (10) are already normalised. We are now able to write the equations in a dimensionless form. For the sake of simplicity, we keep the same notations, but all quantities are now dimensionless.

$$\varepsilon \Delta \vec{v} - \vec{\nabla} P = \vec{0} \quad \text{in } \Omega_f \tag{14}$$

$$\vec{\nabla} \,.\, \vec{v} = 0 \quad \text{in } \Omega_f \tag{15}$$

$$\vec{v}/_{\Gamma} = \vec{0} \tag{16}$$

In these equations, the operators $\vec{\nabla}$ and Δ are also dimensionless. Since we have chosen the microscopic point of view, we have:

$$\begin{cases} \vec{\nabla} &= \vec{\nabla}_y + \varepsilon \vec{\nabla}_x \\ \Delta &= \Delta_y + 2\varepsilon \vec{\nabla}_y \,.\, \vec{\nabla}_x + \varepsilon^2 \Delta_x \end{cases}$$

4.4 Homogenization

The unknowns are the pressure and the velocity fields. They are looked for in the form of asymptotic expansions in terms of ε:

$$\begin{cases} P(\vec{x},\vec{y}) &= P^0(\vec{x},\vec{y}) &+ \varepsilon P^1(\vec{x},\vec{y}) &+ \varepsilon P^2(\vec{x},\vec{y}) &+ \ldots \\ \vec{v}(\vec{x},\vec{y}) &= \vec{v}^0(\vec{x},\vec{y}) &+ \varepsilon \vec{v}^1(\vec{x},\vec{y}) &+ \varepsilon \vec{v}^2(\vec{x},\vec{y}) &+ \ldots \end{cases}$$

where the P^i's and the $\vec{v}^i$'s are $\vec{y}$-periodic and dimensionless.
The upscaling process consists in introducing these expansions in the set of dimensionless

equations (14-16), then in deducing the relevant boundary-value problems at the successive orders and solving them.
Let us expand the equations at the two first orders.
Momentum-balance equation (14):

$$\vec{\nabla}_y P^0 = \vec{0} \tag{17}$$

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{18}$$

Mass-balance equation (15):

$$\vec{\nabla}_y \,.\, \vec{v}^0 = 0 \tag{19}$$

$$\vec{\nabla}_y \,.\, \vec{v}^1 + \vec{\nabla}_x \,.\, \vec{v}^0 = 0 \tag{20}$$

No-slip condition on the boundary (16):

$$\vec{v}^0/_\Gamma = \vec{0} \tag{21}$$

$$\vec{v}^1/_\Gamma = \vec{0} \tag{22}$$

We may now identify the appropriate boundary-value problems at the successive orders and solve them.
<u>First problem</u>:
The first problem gives the pressure at the first order. Consider equation (17):

$$\vec{\nabla}_y P^0 = \vec{0} \tag{23}$$

Thus, at the first order, the pressure field is locally constant:

$$P^0 = P^0(\vec{x}) \tag{24}$$

<u>Second problem</u>:
This problem describes the local flow at the first order and consists of equations (18), (19) and (21):

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{25}$$

$$\vec{\nabla}_y \,.\, \vec{v}^0 = 0 \tag{26}$$

$$\vec{v}^0/_\Gamma = \vec{0} \tag{27}$$

where $\vec{v}^0$ and P^1 are Ω-periodic.
This is a linear problem whose unknowns are $\vec{v}^0$ and P^1. The solution of this set of equations is:

$$\vec{v}^0 = -\tilde{k}(\vec{y})\vec{\nabla}_x P^0 \tag{28}$$

$$P^1 = -\vec{\tau}(\vec{y}) \ . \ \vec{\nabla}_x P^0 + \tilde{P}^1(\vec{x}) \tag{29}$$

The detailed derivation of this solution is presented in the next paragraph.

Third problem:

Consider the mass balance equation at the second order (20):

$$\vec{\nabla}_y \ . \ \vec{v}^1 + \vec{\nabla}_x \ . \ \vec{v}^0 = 0 \tag{30}$$

Let us integrate this equation over the period:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_y \ . \ \vec{v}^1 \ d\Omega + \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_x \ . \ \vec{v}^0 \ d\Omega = 0 \tag{31}$$

From the divergence theorem, we get:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_y \ . \ \vec{v}^1 \ d\Omega = \int_{\partial\Omega_f} \vec{v}^1 \ . \ \vec{n} \ dS \tag{32}$$

Since $\vec{v}^1/_\Gamma = \vec{0}$ and $\vec{v}^1$ is $\vec{y}$-periodic, we have:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_y \ . \ \vec{v}^1 \ d\Omega = \int_{\partial\Omega_f} \vec{v}^1 \ . \ \vec{n} \ dS = 0 \tag{33}$$

Thus, equation (31) becomes:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_x \ . \ \vec{v}^0 \ d\Omega = 0 \tag{34}$$

which can also be written as follows:

$$\vec{\nabla}_x \ . \ < \vec{v}^0 >_\Omega = 0 \tag{35}$$

where

$$< \ . \ >_{\Omega_f} = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \ . \ d\Omega \tag{36}$$

Using the expression for $\vec{v}^0$ (28), we get:

$$< \vec{v}^0 >_\Omega = -\tilde{K} \vec{\nabla}_x P^0 \tag{37}$$

where

$$\tilde{K} = < k >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \tilde{k} \ d\Omega \tag{38}$$

Equation (37) is Darcy' law.

Incorporating (37) into (35) leads to the macroscopic behaviour:

$$\vec{\nabla}_x \ . \ (\tilde{K} \vec{\nabla}_x P^0) = 0 \tag{39}$$

We recognize here the classical Laplace's equation.

4.5 Solving the Second-Order Problem

We have to solve the following system:

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{40}$$

$$\vec{\nabla}_y \ . \ \vec{v}^0 = 0 \tag{41}$$

$$\vec{v}^0/_\Gamma = \vec{0} \tag{42}$$

where $\vec{v}^0$ and P^1 are $\vec{y}$-periodic.
These three equations form a linear problem, of variable $\vec{y}$ and of unknowns $\vec{v}^0$ and P^1. This problem has a unique solution. To solve it, we look for an equivalent variational formulation of the system of equations.
In order to find out a weak formulation for the system (40-42), let us multiply equation (40) by a vector $\vec{\alpha}(\vec{y})$, which is periodic, of zero divergence and that is zero on Γ, and integrate the resulting equation over Ω_f. We get:

$$\int_{\Omega_f} \vec{\alpha} \cdot \Delta_y \vec{v}^0 \ d\Omega - \int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_y P^1 \ d\Omega - \int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_x P^0 \ d\Omega = 0 \tag{43}$$

We may now simplify expression (43).
The first term in equation (43) can be transformed as follows:

$$\begin{aligned}
\int_{\Omega_f} \vec{\alpha} \cdot \Delta_y \vec{v}^0 \ d\Omega &= \int_{\Omega_f} \alpha_i \frac{\partial^2 v_i^0}{\partial y_j \partial y_j} \ d\Omega \\
&= \int_{\Omega_f} \frac{\partial}{\partial y_j}(\alpha_i \frac{\partial v_i^0}{\partial y_j}) \ d\Omega - \int_{\Omega_f} \frac{\partial \alpha_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} d\Omega \\
&= \int_{\delta\Omega_f} \alpha_i \frac{\partial v_i^0}{\partial y_j} n_j \ dS - \int_{\Omega_f} \frac{\partial \alpha_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \ d\Omega
\end{aligned}$$

Since $\vec{\alpha}/_\Gamma = \vec{0}$ and $\vec{\alpha}$ is $\vec{y}$-periodic, we have:

$$\int_{\delta\Omega_f} \alpha_i \frac{\partial v_i^0}{\partial y_j} n_j \ dS = 0$$

Therefore,

$$\int_{\Omega_f} \vec{\alpha} \cdot \Delta_y \vec{v}^0 \ d\Omega = - \int_{\Omega_f} \frac{\partial \alpha_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \ d\Omega \tag{44}$$

The second term in (43) becomes:

$$\begin{aligned}
\int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_y P^1 \ d\Omega &= \int_{\Omega_f} \alpha_i \frac{\partial P^1}{\partial y_i} \ d\Omega \\
&= \int_{\Omega_f} \frac{\partial(\alpha_i P^1)}{\partial y_i} \ d\Omega - \int_{\Omega_f} P^1 \frac{\partial \alpha_i}{\partial y_i} \ d\Omega \\
&= \int_{\delta\Omega_f} \alpha_i P^1 n_i \ dS - \int_{\Omega_f} P^1 \frac{\partial \alpha_i}{\partial y_i} \ d\Omega
\end{aligned}$$

Since $\vec{\alpha}/_{\Gamma} = \vec{0}$ and $\vec{\alpha}$ is $\vec{y}$-periodic, we get:

$$\int_{\delta\Omega_f} \alpha_i P^1 n_i \; dS = 0$$

Moreover, since $\dfrac{\partial \alpha_i}{\partial y_i} = 0$, then

$$\int_{\Omega_f} P^1 \frac{\partial \alpha_i}{\partial y_i} \; d\Omega = 0$$

Therefore,

$$\int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_y P^1 \; d\Omega = 0 \tag{45}$$

Using (44) and (45), equation (43) reduces to:

$$\int_{\Omega_f} \frac{\partial \alpha_i}{\partial y_j} \frac{\partial vi^0}{\partial y_j} \; d\Omega + \int_{\Omega_f} \alpha_i \frac{\partial P^0}{\partial x_i} \; d\Omega = 0 \tag{46}$$

or

$$\int_{\Omega_f} \vec{\nabla}_y \vec{\alpha} \cdot \vec{\nabla}_y \vec{v}^0 \; d\Omega + \int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_x P^0 \; d\Omega = 0 \tag{47}$$

Now, let us define the following operator:

$$\Phi(\vec{\alpha}, \vec{\beta}) = \int_{\Omega_f} \vec{\nabla}_y \vec{\alpha} \cdot \vec{\nabla}_y \vec{\beta} \; d\Omega, \tag{48}$$

The operator $\Phi(\vec{\alpha}, \vec{\beta})$ has the following properties:

- $(1): \; \Phi(\vec{\alpha}, \vec{\beta}) = \Phi(\vec{\beta}, \vec{\alpha})$
- $(2): \; \Phi(\vec{\alpha}, \vec{\beta} + \vec{\gamma}) = \Phi(\vec{\alpha}, \vec{\beta}) + \Phi(\vec{\alpha}, \vec{\gamma})$
- $(3): \; \Phi(\lambda\vec{\alpha}, \vec{\beta}) = \lambda\Phi(\vec{\alpha}, \vec{\beta})$
- $(4): \; \Phi(\vec{\alpha}, \vec{\alpha}) \geq 0$
- $(5): \; \Phi(\vec{\alpha}, \vec{\alpha}) = 0 \Rightarrow \vec{\alpha} = \vec{0}$

This means that $\Phi(\vec{\alpha}, \vec{\beta})$ is an inner product (properties 1 to 3) with an associated norm (properties 4 and 5).
Now consider the space $\mathcal{W}$ of periodic vectors $\vec{\alpha}$ such that: $\vec{\nabla}_y \, . \, \vec{\alpha} = 0$ and $\vec{\alpha}/_{\Gamma} = \vec{0}$. $\mathcal{W}$ is equipped with the operator $\Phi(\vec{\alpha}, \vec{\beta})$. Note that the properties of $\mathcal{W}$ match equations (40), (41) and (42). By definition, since $\Phi(\vec{\alpha}, \vec{\beta})$ is an inner product, with an associated norm, hence $\mathcal{W}$ is an Hilbert space. As an inner product with an associated norm, $\Phi(\vec{\alpha}, \vec{\beta})$ is now denoted as follows:

$$\Phi(\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}, \vec{\beta})$$

The advantage of having defined an Hilbert space is that we will be able to use useful theorems which ensure the existence of the solution.

The equivalent variational formulation or the weak formulation of the system (40-42) is deduced from (47) and is expressed as follows:

$$\forall \vec{\alpha} \in \mathcal{W} : (\vec{\alpha}, \vec{v}^0) = L(\vec{\alpha}) \qquad \text{where } L(\vec{\alpha}) = -\int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_x P^0 \, d\Omega \tag{49}$$

$L(\vec{\alpha})$ is a linear form, i.e. it is a linear application defined over $\mathcal{W}$ which values are in $\Re$. We can apply the following lemma, which is called "Lax-Milgram's Lemma":
The problem: $\forall \vec{\alpha} \in \mathcal{W} : (\vec{\alpha}, \vec{\beta}) = L(\vec{\alpha})$, where

- $\bullet \mathcal{W}$ is an Hilbert space which is equipped with the inner product $(\vec{\alpha}, \vec{\beta})$
- $\bullet L(\vec{\alpha})$ is a linear form

has one unique solution for $\vec{\beta}$.

Therefore, the problem (49), which is the weak formulation of our initial system, has one unique solution for $\vec{v}^0$. Let us now determine this solution:

$$(\vec{\alpha}, \vec{v}^0) = -\int_{\Omega_f} \vec{\alpha} \cdot \vec{\nabla}_x P^0 \, d\Omega$$

Let $\vec{k}_j$ be the particular solution in the case where:

- $\bullet \dfrac{\partial P^0}{\partial x_i} = 1$ if $i = j$
- $\bullet \dfrac{\partial P^0}{\partial x_i} = 0$ if $i \neq j$

Thus, the general solution is:

$$v_i^0 = -\tilde{k}_{ij} \frac{\partial P^0}{\partial x_j} \qquad \text{or} \qquad \vec{v}^0 = -\tilde{k}(\vec{y}) \vec{\nabla}_x P^0 \tag{50}$$

Now, to determine the expression for P^1, let us combine equation (40) together with the solution for $\vec{v}^0$ (50):

$$\vec{\nabla}_y P^1 = -\mu(\Delta_y \tilde{k} + \tilde{\delta}) \vec{\nabla}_x P^0 \tag{51}$$

where $\tilde{\delta}$ is the identity tensor.
We get the following general solution for P^1:

$$P^1 = -\vec{\tau}(\vec{y}) \cdot \vec{\nabla}_x P^0 + \tilde{P}^1(\vec{x}) \tag{52}$$

where $\tilde{P}^1(\vec{x})$ is an arbitrary function of $\vec{x}$. If we consider two couples of solutions $(\vec{v}^0, P^1)$ and $(\vec{v}^{0^*}, P^{1^*})$, then we get:

- $\bullet \vec{v}^0 = \vec{v}^{0^*}$
- $\bullet P^1 - P^{1^*} = constant$

Therefore, the unique solution of the system:

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0}$$
$$\vec{\nabla}_y \,.\, \vec{v}^0 = 0$$
$$\vec{v}^0/_\Gamma = \vec{0}$$

where $\vec{v}^0$ and P^1 are Ω-periodic is:

$$\vec{v}^0 = -\tilde{k}(\vec{y})\vec{\nabla}_x P^0$$
$$P^1 = -\vec{\tau}(\vec{y}) \cdot \vec{\nabla}_x P^0 + \tilde{P}^1(\vec{x})$$

Note that P^1 is determined modulo a constant.

4.6 Properties of the Effective Permeability Tensor

We have derived the following macroscopic behaviour:

$$\vec{\nabla}_x \cdot (\tilde{K}\vec{\nabla}_x P^0) = 0 \tag{53}$$

The effective permeability tensor $\tilde{K}$ is defined by:

$$\tilde{K} =< \tilde{k} >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \tilde{k} \; d\Omega, \tag{54}$$

where the local tensor $\tilde{k}$ is such that:

$$\vec{v}^0 = -\tilde{k}(\vec{y})\vec{\nabla}_x P^0 \tag{55}$$

which is the solution of the following problem:

$$\forall \vec{\alpha} \in \mathcal{W} : \int_{\Omega_f} \frac{\partial \alpha_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \; d\Omega = -\int_{\Omega_f} \alpha_i \frac{\partial P^0}{\partial x_i} \; d\Omega \tag{56}$$

$\vec{k}_j$ is the solution of the particular case where:

- $\dfrac{\partial P^0}{\partial x_i} = 1$ if $i = j$
- $\dfrac{\partial P^0}{\partial x_i} = 0$ if $i \neq j$

4.6.1 Symmetry

Let us consider $\vec{\alpha} = \vec{k}^q$ and $\vec{v}^0 = \vec{k}^p$. Then, the weak formulation (56) gives:

$$\int_{\Omega_f} \frac{\partial k_i^q}{\partial y_j} \frac{\partial k_i^p}{\partial y_j} \, d\Omega = -\int_{\Omega_f} k_p^q \, d\Omega = -|\Omega| \; K_{pq}$$

Repeating this procedure while considering $\vec{\alpha} = \vec{k}^p$ and $\vec{v}^0 = \vec{k}^q$ yields:

$$\int_{\Omega_f} \frac{\partial k_i^p}{\partial y_j} \frac{\partial k_i^q}{\partial y_j} \, d\Omega = -\int_{\Omega_f} k_q^p \, d\Omega = -|\Omega| \; K_{qp}$$

As

$$\int_{\Omega_f} \frac{\partial k_i^q}{\partial y_j} \frac{\partial k_i^p}{\partial y_j} \, d\Omega = \int_{\Omega_f} \frac{\partial k_i^p}{\partial y_j} \frac{\partial k_i^q}{\partial y_j} \, d\Omega,$$

we deduce that:

$$K_{pq} = K_{qp} \tag{57}$$

Therefore, **the effective permeability tensor is symmetric**. This property is a consequence of the symmetry of the inner product and is deduced from the microscopic balance equations.

4.6.2 Positiveness

Consider the weak formulation (56) with $\vec{\alpha} = \vec{v}^0$. We get:

$$\int_{\Omega_f} \frac{\partial v_i^0}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \, d\Omega = -\int_{\Omega_f} v_i^0 \frac{dP^0}{dx_i} \, d\Omega$$
$$= \int_{\Omega_f} k_{ij} \frac{dP^0}{dx_i} \frac{dP^0}{dx_i} \, d\Omega = K_{ij} \frac{dP^0}{dx_i} \frac{dP^0}{dx_i}$$

The equation:

$$\int_{\Omega_f} \frac{\partial v_i^0}{\partial y_j} \frac{\partial v_i^0}{\partial y_j} \, d\Omega = K_{ij} \frac{dP^0}{dx_i} \frac{dP^0}{dx_i} \tag{58}$$

expresses that the average of the local density of dissipation is equal to the macroscopic density of dissipation. Since the density of dissipation is a positive quantity, we deduce that:

$$K_{ij} \frac{dP^0}{dx_i} \frac{dP^0}{dx_i} > 0$$

Thus, **the effective permeability tensor is positive**.

4.7 Physical Meaning of the Macroscopic Quantities

The macroscopic description can be written as follows:

$$\vec{\nabla}_x \cdot < \vec{v}^0 >_\Omega = 0 \tag{59}$$

where

$$< \vec{v}^0 >_\Omega = -\tilde{K}\vec{\nabla}_x P^0 \tag{60}$$

The definition of the macroscopic pressure P^0 is not posing any problem since it is the same at both scales. In contrast, the physical meaning of the macroscopic velocity $< \vec{v}^0 >_\Omega$ is debatable. In effect, $< \vec{v}^0 >_\Omega$ is defined as a volume average, whereas a velocity, as a flux, is a surface average. Therefore, for the sake of physical consistency of the macroscopic description, we must prove that $< \vec{v}^0 >_\Omega$ reduces to a surface average. Consider the following identity:

$$\frac{\partial}{\partial y_j}(v_j^0 y_i) \equiv \frac{\partial v_j^0}{\partial y_j} y_i + v_j^0 \delta_{ij} \tag{61}$$

Now, let us integrate identity (61) over Ω_f:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \frac{\partial}{\partial y_j}(v_j^0 y_i)\ d\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \frac{\partial v_j^0}{\partial y_j} y_i\ d\Omega + \frac{1}{\mid \Omega \mid} \int_{\Omega_f} v_j^0 \delta_{ij}\ d\Omega \tag{62}$$

For the left hand side, we get:

$$\int_{\Omega_f} \frac{\partial}{\partial y_j}(v_j^0 y_i)\ d\Omega = \int_{\Gamma \cup (\delta\Omega_f \cap \delta\Omega)} v_j^0 y_i n_j\ dS = \int_{\delta\Omega_f \cap \delta\Omega} v_j^0 y_i n_j\ dS \tag{63}$$

The right hand side becomes:

$$\int_{\Omega_f} \frac{\partial v_j^0}{\partial y_j} y_i\ d\Omega + \int_{\Omega_f} v_j^0 \delta_{ij}\ d\Omega = \int_{\Omega_f} v_j^0 \delta_{ij}\ d\Omega = \int_{\Omega_f} v_i^0\ d\Omega \tag{64}$$

Using (63) and (64), equation (62) yields:

$$< v_i^0 >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\delta\Omega_f \cap \delta\Omega} v_j^0 y_i n_j\ dS \tag{65}$$

Let l_i be the length of the period along the y_i axis and Σ_i be the cross section of the period at $y_i = l_i$. The quantity $v_j^0 y_i$ is Ω-periodic in the y_k direction, where $k \neq i$, and is equal to zero at $y_i = 0$. Thus, (65) can be written as:

$$< v_i^0 >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\delta\Omega_f \cap \Sigma_i} v_i^0 l_i\ dS = \frac{1}{\mid \Sigma_i \mid} \int_{\delta\Omega_f \cap \Sigma_i} v_i^0\ dS \tag{66}$$

This latter equation shows that the volume average that defines the macroscopic velocity is equal to a surface average. Therefore, $< v_i^0 >_\Omega$ is well defined as a flux.

5 Compressible Fluid

In this section, we consider the flow of a highly compressible fluid in a rigid porous medium. This study was performed by (Auriault et al., 1990).

5.1 Local Description

5.2 Estimations

From equations (**??**) and (**??**) the following dimensionless numbers can be defined:

$$Q = \frac{\mid \vec{\nabla} P \mid}{\mid \mu \Delta \vec{v} \mid} \tag{67}$$

$$Re = \frac{\mid \rho (\vec{v} \vec{\nabla}) \vec{v} \mid}{\mid \mu \Delta \vec{v} \mid} \qquad \text{Reynolds number} \tag{68}$$

$$Rt = \frac{\mid \rho \dfrac{\partial \vec{v}}{\partial t} \mid}{\mid \mu \Delta \vec{v} \mid} \qquad \text{Transient Reynolds number} \tag{69}$$

$$S = \frac{\mid \dfrac{\partial \rho}{\partial t} \mid}{\mid \vec{\nabla} \cdot (\rho \vec{v}) \mid} = \frac{Rt}{Re} \qquad \text{Strouhal number} \tag{70}$$

Note that Q has the same definition as in section 4. Therefore, the reasoning performed in (4.3) is still valid:

$$Q_l = O(\varepsilon^{-1}) \;\; \Rightarrow Q_L = O(\varepsilon^{-2}) \tag{71}$$

The following restrictions on the orders of magnitude of Re, Rt and S are required for homogenization to be possible:

$$Re_l \leq O(1) \Rightarrow Re_L \leq O(\varepsilon^{-1}) \tag{72}$$

$$Rt_l < O(1) \Rightarrow Rt_L < O(\varepsilon^{-2}) \tag{73}$$

$$S_l \leq O(\varepsilon) \Rightarrow S_L \leq O(1) \tag{74}$$

These restrictions will be clarified in paragraph 5.5. They are homogenizability conditions. Physical situations that do not respect these restrictions may exist. In these cases, there is no equivalent macroscopic description. Several behaviours of interest may be distinguished:

- Steady-state flows: $Rt = 0$, $S = 0$.

• If $Re_l \leq O(\varepsilon)$ $(\Rightarrow Re_L \leq O(1))$, then the flow is macroscopically governed by classical linear Darcy's law.

• If $Re_l = O(1)$ $(\Rightarrow Re_L = O(\varepsilon^{-1}))$, Darcy's law is no longer valid and the drag law becomes non-linear.

- Transient flows: $S_L = O(1)$.

• If $Rt_l = O(\varepsilon^2)$ $(\Rightarrow Rt_L = O(1))$ and $Re_l = O(\varepsilon)$ $(\Rightarrow Re_L = O(1))$, then the transient term in the mass-balance equation is accounted for, whereas the transient term in the momentum-balance equation is neglected. The flow will macroscopically be governed by Darcy's law and a transient and non-linear mass-balance equation.

• If $Rt_l = O(\varepsilon)$ $(\Rightarrow Rt_L = O(\varepsilon^{-1}))$ and $Re_l = O(1)$ $(\Rightarrow Re_L = O(\varepsilon^{-1}))$, Darcy's law is no longer valid and the drag law becomes non-linear.

In this section we will successively consider a steady-state flow with $Re_l = O(\varepsilon)$ and then a transient flow with $S_L = O(1)$, $Rt_l = O(\varepsilon^2)$ and $Re_l = O(\varepsilon)$.

Note that in section 4, for the flow of an incompressible fluid, we have directly considered Stokes equation:

$$\mu \Delta \vec{v} - \vec{\nabla} P = \vec{0} \tag{75}$$

Thus, a steady-sate flow has been assumed, i.e. $Rt = 0$. For the steady-sate flow of an incompressible fluid, Navier-Stokes equation is written as:

$$\mu \Delta \vec{v} - \vec{\nabla} P = \rho (\vec{v} \vec{\nabla}) \vec{v} \tag{76}$$

Therefore, considering equation (75), also means that Re has implicitly be assumed to be such that $Re_l = O(\varepsilon)$.

5.3 Steady-State Flow

5.3.1 Normalisation

As mentioned above, a steady-state flow is defined by the following orders of magnitude:

$$Rt = 0, \;\; S = 0.$$

The corresponding (dimensional) local description is therefore written as:

$$\mu \Delta \vec{v} + (\lambda + \mu) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} P = \rho (\vec{v} \vec{\nabla}) \vec{v} \qquad \text{in } \Omega_f \tag{77}$$

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad \text{in } \Omega_f \tag{78}$$

$$f(p, \rho) = 0 \qquad \text{in } \Omega_f \tag{79}$$

$$\vec{v}/_{\Gamma} = \vec{0} \qquad \text{on } \Gamma \tag{80}$$

For the reasons mentioned above, let consider the case where $Re_l = O(\varepsilon)$, i.e. $Re_L = O(1)$. Let us choose L as the reference characteristic length. We are therefore interested in the following orders of magnitude:

$$Q_L = O(\varepsilon^{-2}), \;\; Re_L = O(1) \tag{81}$$

Hence, the dimensionless local description is written as:

$$\varepsilon^2 \Delta \vec{v} + \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} P = \varepsilon^2 \rho (\vec{v} \vec{\nabla}) \vec{v} \qquad \text{in } \Omega_f \tag{82}$$

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad \text{in } \Omega_f \tag{83}$$

$$f(p, \rho) = 0 \qquad \text{in } \Omega_f \tag{84}$$

$$\vec{v}/_{\Gamma} = \vec{0} \qquad \text{on } \Gamma \tag{85}$$

in which all quantities are now dimensionless quantities. In these equations, the operators $\vec{\nabla}$ and Δ are also dimensionless. Since we have chosen the macroscopic point of view (L is the reference characteristic length), we have (see section 3.6):

$$\left\{ \begin{array}{lcl} \vec{\nabla} & = & \varepsilon^{-1} \vec{\nabla}_y + \vec{\nabla}_x \\ \Delta & = & \varepsilon^{-2} \Delta_y + 2\varepsilon^{-1} \vec{\nabla}_y \; . \; \vec{\nabla}_x + \Delta_x \end{array} \right.$$

We now introduce the multiple scale variables

$$\left\{ \begin{array}{lcl} \vec{y} & = & \dfrac{\vec{X}}{l} \\ \vec{x} & = & \dfrac{\vec{X}}{L} \end{array} \right.$$

and the perturbation expansions for the pressure, velocity and density fields:

$$\left\{ \begin{array}{lclclclcl} P(\vec{x}, \vec{y}) & = & P^0(\vec{x}, \vec{y}) & + & \varepsilon P^1(\vec{x}, \vec{y}) & + & \varepsilon P^2(\vec{x}, \vec{y}) & + & ... \\ \vec{v}(\vec{x}, \vec{y}) & = & \vec{v}^0(\vec{x}, \vec{y}) & + & \varepsilon \vec{v}^1(\vec{x}, \vec{y}) & + & \varepsilon \vec{v}^2(\vec{x}, \vec{y}) & + & ... \\ \rho(\vec{x}, \vec{y}) & = & \rho^0(\vec{x}, \vec{y}) & + & \varepsilon \rho^1(\vec{x}, \vec{y}) & + & \varepsilon \rho^2(\vec{x}, \vec{y}) & + & ... \end{array} \right.$$

where the $\vec{v}^i$'s, P^i's and ρ^i's are $\vec{y}$-periodic.

5.3.2 Homogenization

At the first two orders equations (82) through (80) give:
Momentum-balance equation (82):

$$\vec{\nabla}_y P^0 = \vec{0} \tag{86}$$

$$\Delta_y \vec{v}^0 + \vec{\nabla}_y(\vec{\nabla}_y \cdot \vec{v}^0) - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{87}$$

Mass-balance equation (83):

$$\vec{\nabla}_y \ \cdot (\rho^0 \vec{v}^0) = 0 \tag{88}$$

$$\vec{\nabla}_y \ \cdot (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) + \vec{\nabla}_x \ \cdot (\rho^0 \vec{v}^0) = 0 \tag{89}$$

Fluid state equation (84):

$$f(P^0, \rho^0) = 0 \tag{90}$$

$$P^1 \left(\frac{\partial f}{\partial P}\right)^0 + \rho^1 \left(\frac{\partial f}{\partial \rho}\right)^0 = 0 \tag{91}$$

No-slip condition on the boundary (80):

$$\vec{v}^0/_\Gamma = \vec{0} \tag{92}$$

$$\vec{v}^1/_\Gamma = \vec{0} \tag{93}$$

From (86) and (90) we get

$$P^0 = P^0(\vec{x}), \tag{94}$$

$$\rho^0 = \rho^0(\vec{x}). \tag{95}$$

As a result, (88) reduces to:

$$\vec{\nabla}_y \ \cdot \vec{v}^0 = 0 \tag{96}$$

Thus, the boundary-value problem for $\vec{v}^0$ and P^1 is given by the following equations:

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{97}$$

$$\vec{\nabla}_y \ . \ \vec{v}^0 = 0 \tag{98}$$

$$\vec{v}^0/_\Gamma = \vec{0} \tag{99}$$

where $\vec{v}^0$ and P^1 are $\vec{y}$-periodic.
This boundary-value problem is identical that obtained for an incompressible fluid (equations (25) through (27) and whose solution is:

$$\vec{v}^0 = -\tilde{k}(\vec{y})\vec{\nabla}_x P^0 \tag{100}$$

$$P^1 = -\vec{\tau}(\vec{y}) \ . \ \vec{\nabla}_x P^0 + \tilde{P}^1(\vec{x}) \tag{101}$$

where $\tilde{k}$ is a second-rank tensor and where $\tilde{P}^1(\vec{x})$ is an arbitrary function of $\vec{x}$.
Now, let consider equation (89):

$$\vec{\nabla}_y \ \cdot (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) = -\vec{\nabla}_x \ \cdot (\rho^0 \vec{v}^0) \tag{102}$$

This equation expresses the balance of the periodic quantity $\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1$ in presence of the source term $-\vec{\nabla}_x \ \cdot (\rho^0 \vec{v}^0)$. Integrating this equation over the period gives:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_y \ \cdot (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \ d\Omega = -\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_x \ \cdot (\rho^0 \vec{v}^0) \tag{103}$$

Using the divergence theorem, the left hand side of (103) becomes:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_y \ \cdot (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \ d\Omega = \frac{1}{\mid \Omega \mid} \int_{\delta\Omega_f} (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \cdot \vec{n} \ dS \tag{104}$$

As a consequence of the periodicity on $\delta\Omega_f \cap \delta\Omega$ and of the no-slip condition on Γ, we get

$$\begin{aligned} &\frac{1}{\mid \Omega \mid} \int_{\delta\Omega_f} (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \cdot \vec{n} \ dS = \\ &\frac{1}{\mid \Omega \mid} \left(\int_{\delta\Omega_f \cap \delta\Omega} (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \cdot \vec{n} \ dS + \int_{\Gamma} (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) \cdot \vec{n} \ dS \right) = 0 \end{aligned} \tag{105}$$

Therefore, the compatibility condition (see paragraph 3.6.2) arising from the integration of equation (89) over the period is:

$$\frac{1}{\mid \Omega \mid} \int_{\Omega_f} \vec{\nabla}_x \ \cdot (\rho^0 \vec{v}^0) = 0, \tag{106}$$

which can also be written as:

$$\vec{\nabla}_x \ \cdot (\rho^0 < \vec{v}^0 >_\Omega) = 0 \tag{107}$$

where

$$< . >_{\Omega_f} = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \ . \ d\Omega \tag{108}$$

Using the expression for $\vec{v}^0$ (100), we get:

$$< \vec{v}^0 >_\Omega = -\tilde{K} \vec{\nabla}_x P^0 \tag{109}$$

where

$$\tilde{K} = < k >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_f} \tilde{k} \ d\Omega \tag{110}$$

Therefore, the macroscopic behaviour is given by:

$$\vec{\nabla}_x \ \cdot (\rho^0 < \vec{v}^0 >_\Omega) = 0, \tag{111}$$

$$< \vec{v}^0 >_\Omega = -\tilde{K}\vec{\nabla}_x P^0, \quad \tilde{K} =< \tilde{k} >_\Omega, \tag{112}$$

$$f(P^0, \rho^0) = 0. \tag{113}$$

Therefore, the flow is governed by Darcy's law (112) and non-linearities due to the compressibility of the fluid appear in the mass-balance equation (111).
For a linear fluid state equation, i.e. when (79) is replaced by $\rho = AP$, A is constant, the macroscopic behaviour becomes:

$$\vec{\nabla}_x \cdot (P^0 \tilde{K} \vec{\nabla}_x P^0) = 0 \tag{114}$$

5.4 Transient Flow

5.4.1 Normalisation

As mentioned above the situation that leads to a transient mass-balance with a linear drag law is defined by the following orders of magnitude for Re, Rt, and S:

$$S_L = O(1), \;\; Rt_l = O(\varepsilon^2), \;\; Re_l = O(\varepsilon). \tag{115}$$

Let take the macroscopic point of view: L is taken as the reference characteristic length. We are therefore interested in the following orders of magnitude:

$$Q_L = O(\varepsilon^{-2}), \;\; Re_L = O(1), \;\; Rt_L = O(1), \;\; S_L = O(1). \tag{116}$$

Thus, the corresponding dimensionless local description is the following:

$$\varepsilon^2 \Delta \vec{v} + \varepsilon^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} P = \varepsilon^2 \rho (\frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v}) \qquad \text{in } \Omega_f \tag{117}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad \text{in } \Omega_f \tag{118}$$

$$f(p, \rho) = 0 \qquad \text{in } \Omega_f \tag{119}$$

$$\vec{v}/_\Gamma = \vec{0} \qquad \text{on } \Gamma \tag{120}$$

5.4.2 Homogenization

Introducing the asymptotic expansions for $\vec{v}$, P and ρ leads, at the first two orders, to the following developments for equations (117) trough (120):
Momentum-balance equation (117):

$$\vec{\nabla}_y P^0 = \vec{0} \tag{121}$$

$$\Delta_y \vec{v}^0 + \vec{\nabla}_y (\vec{\nabla}_y \cdot \vec{v}^0) - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \vec{0} \tag{122}$$

Mass-balance equation (118):

$$\vec{\nabla}_y \cdot (\rho^0 \vec{v}^0) = 0 \tag{123}$$

$$\frac{\partial \rho^0}{\partial t} + \vec{\nabla}_y \cdot (\rho^1 \vec{v}^0 + \rho^0 \vec{v}^1) + \vec{\nabla}_x \cdot (\rho^0 \vec{v}^0) = 0 \tag{124}$$

Fluid state equation (119):

$$f(P^0, \rho^0) = 0 \tag{125}$$

$$P^1 \left(\frac{\partial f}{\partial P}\right)^0 + \rho^1 \left(\frac{\partial f}{\partial \rho}\right)^0 = 0 \tag{126}$$

No-slip condition on the boundary (120):

$$\vec{v}^0/_\Gamma = \vec{0} \tag{127}$$

$$\vec{v}^1/_\Gamma = \vec{0} \tag{128}$$

From the steady-state case, only equation (124) is modified by an additional source term, from the steady-state case. Thus, as in paragraph 5.3.2 we get (with an additional time dependence):

$$P^0 = P^0(\vec{x}, t), \tag{129}$$

$$\rho^0 = \rho^0(\vec{x}, t), \tag{130}$$

and equation (123) reduces to:

$$\vec{\nabla}_y \cdot \vec{v}^0 = 0 \tag{131}$$

The boundary-value problem for $\vec{v}^0$ and P^1 is that of an incompressible fluid (97 - 99) and leads to:

$$\vec{v}^0 = -\tilde{k} \vec{\nabla}_x P^0 \tag{132}$$

Due to the additional term, $\frac{\partial \rho^0}{\partial t}$, in equation (124), the compatibility condition becomes:

$$n \frac{\partial \rho^0}{\partial t} + \vec{\nabla}_x \cdot (\rho^0 \vec{v}^0) = 0, \tag{133}$$

where n is the porosity

$$n = \frac{\mid \Omega_f \mid}{\mid \Omega \mid} \tag{134}$$

The macroscopic behaviour is therefore given by:

$$\vec{\nabla}_x \cdot (\rho^0 < \vec{v}^0 >_\Omega) + n \frac{\partial \rho^0}{\partial t} = 0 \tag{135}$$

$$< \vec{v}^0 >_\Omega = -\tilde{K}\vec{\nabla}_x P^0, \quad \tilde{K} =< \tilde{k} >_\Omega, \tag{136}$$

$$f(P^0, \rho^0) = 0. \tag{137}$$

For a linear fluid state equation, ($\rho = AP$, A is constant), the macroscopic behaviour becomes:

$$n\frac{\partial \rho^0}{\partial t} + \vec{\nabla}_x \cdot (P^0 \tilde{K}\vec{\nabla}_x P^0) = 0 \tag{138}$$

5.5 Other Orders of Magnitude for Re, Rt and S

Let us now clarify the restrictions introduced in 5.2.

5.5.1 Restrictions on Re

•$Re_L = O(\varepsilon^{-1})$
In this case, the normalised form of equation (**??**) is:

$$\varepsilon^2 \Delta \vec{v} + \varepsilon^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} P = \rho\varepsilon(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v}) \tag{139}$$

Thus, the boundary-value problem that defines $\vec{v}^0$ and P^1 is the following (compare with (97-99):

$$\Delta_y \vec{v}^0 - \vec{\nabla}_y P^1 - \vec{\nabla}_x P^0 = \rho^0(\vec{v}^0 \vec{\nabla}_y)\vec{v}^0 \tag{140}$$

$$\vec{\nabla}_y \,.\, \vec{v}^0 = 0 \tag{141}$$

$$\vec{v}^0/_\Gamma = \vec{0} \tag{142}$$

where $\vec{v}^0$ and P^1 are Ω-periodic.
It turns out that $\vec{v}^0$ is a non-linear vectorial function of $\vec{\nabla}_x P^0$ and ρ^0. In other words, the drag law becomes non-linear.

Therefore, if $Re_L = O(\varepsilon^{-1})$ or $Re_l = O(1)$, Darcy's law is no longer valid and the drag law becomes non-linear.

•$Re_L = O(\varepsilon^{-2})$
The normalised form of equation (**??**) becomes:

$$\varepsilon^2 \Delta \vec{v} + \varepsilon^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \vec{\nabla} P = \rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v}) \tag{143}$$

The development of equation (143) at the order of ε^{-1} is:

$$-\vec{\nabla}_y P^0 = \rho^0(\vec{v}^0 \vec{\nabla}_y)\vec{v}^0, \tag{144}$$

which gives the following boundary-value problem for $\vec{v}^0$ and P^0:

$$\vec{\nabla}_y \cdot (\rho^0 \vec{v}^0) = 0, \tag{145}$$

$$\vec{v}^0/_\Gamma = \vec{0} \tag{146}$$

If we assume that the solution is unique, then it gives:

$$\vec{v}^0 = \vec{0}, \quad \rho^0 = \rho^0(\vec{x}) \tag{147}$$

Now, $\vec{v}^0 = \vec{0} \Rightarrow \mid \vec{v} \mid = O(\varepsilon)$, which is incompatible with the order of magnitude $Re_L = O(\varepsilon^{-2})$ which assumed $\mid \vec{v} \mid = O(1)$.
Therefore, a flow with $Re_L = O(\varepsilon^{-2})$, i.e. $Re_l = O(\varepsilon^{-1})$ is not homogenizable.
Thus, the situation is homogenizable for $Re_l \leq O(1)$, i.e. for $Re_L \leq O(\varepsilon^{-1})$. The drag law is linear for $Re_l \leq O(\varepsilon)$, i.e. for $Re_L \leq O(1)$ and it is non-linear when $Re_l = O(1)$, i.e. when $Re_L = O(\varepsilon^{-1})$ (see figure 3). Note that the appropriate order of magnitude for the analysis of Re is the local Reynolds number, Re_l.

Figure 3: *Restrictions on Re*

5.5.2 Restrictions on S

$\bullet S_L = O(\varepsilon^{-1})$
In this case the normalised mass-balance equation is rewritten as:

$$\varepsilon^{-1} \frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0 \tag{148}$$

Therefore, from the mass-balance equation at the first order, we get (instead of (123)):

$$\frac{\partial \rho^0}{\partial t} + \vec{\nabla}(\rho^0 \vec{v}^0) = 0 \tag{149}$$

In contrast with (123), this equation introduces a compatibility condition. In effect, integrating (149) over the period gives:

$$< \frac{\partial \rho^0}{\partial t} >_\Omega = 0, \tag{150}$$

from which we get

$$\frac{\partial \rho^0}{\partial t} = 0. \tag{151}$$

The order of magnitude $S = O(\varepsilon^{-1})$ is modified by relationship (151), which actually gives $S = O(1)$.
Therefore, flows with $S_L = O(\varepsilon^{-1})$ or $S_l = O(1)$ are not homogenizable.

Thus, the situation is homogenizable for $S_L \leq O(1)$, i.e. for $S_l \leq O(\varepsilon)$ (see figure 4). We get a steady-state flow for $S_L \leq O(\varepsilon)$, and a transient flow for $S_L = O(1)$. The appropriate order of magnitude for analysing S is the macroscopic Strouhal number, S_L.

Figure 4: *Restrictions on S*

5.5.3 Restrictions on Rt

Since $Rt = S \times Re$, we deduce the restrictions on Rt from those on Re and S. It turns out that the situation is homogenizable for $Rt_l \leq O(\varepsilon)$ (see figure 5).
Thus, a local Reynolds number of the order of 1 is impossible. Hence, transient effects in Navier-Stokes equations cannot be taken into account. Such flows can occur but they are not homogenizable. Transient terms of Navier-Stokes equations can only be accounted for in the specific case of acoustics of a saturated porous medium under small amplitude waves.

Figure 5: *Restrictions on Rt*

Part IV

Gas Flow through Dual-Porosity Media

6 Problem Description

6.1 Introduction

A dual-porosity medium is constituted of two porous systems of distinctly different porosities and permeabilities. This concept was introduced in (Barenblatt et al., 1960) for the investigation of fissured media porous media: one porous structure is associated with the fracture network and the second one is the porous matrix. The behaviour of such media when subject to fluid flow is of interest in many engineering fields such as petroleum engineering, mining engineering or environmental engineering. It has consequently been the subject of intensive research. The first investigations of dual-porosity media dealt with fissured reservoirs by means of phenomenological approaches (Barenblatt et al., 1960), (Barenblatt, 1963), (Warren and Root, 1963). In these studies the fluid is assumed to be slightly compressible and the derived models are therefore linear. The Barenblatt-Zheltov model (Barenblatt et al., 1960) shows an important feature of dual-porosity systems, the interporosity flow, i.e. the fluid exchange between both porous structures which will be referred as coupling effects below. In the model of Warren and Root (Warren and Root, 1963), the flow through the porous matrix is neglected. This simplification is a subject of debate as mentioned in (Chen, 1989) and (Chen, 1990) where it is claimed that the complete Barenblatt-Zheltov model, that accounts for the flow in the porous matrix, should be solved. The Barenblatt-Zheltov model and the model of Warren and Root have widely been applied in reservoir engineering. States of the art in the knowledge of dual-porosity behaviours derived by this kind of approaches can be found in (Gringarten, 1984) and (Chen, 1989).

Homogenization techniques may also be used for modelling fluid flow in dual-porosity media. (Arbogast et al., 1990) derived a dual-porosity model for slightly compressible fluid flow via homogenization and by considering Darcy's law at the microscopic levels. It is preferable to consider Stokes equations within the pores and the fractures so as to highlight local effects influence. This was performed in (Levy, 1990), in which the filtration of an incompressible fluid in a fissured microporous rigid rock is investigated in order to show that the case of interest is when both the pores and the fractures are connected.

In a dual-porosity medium, three separated scales may be considered: the pore scale, the fracture scale and the macroscopic scale (figure 6). A three-scale homogenization

Figure 6: *The three scales of a fractured porous medium.*

procedure was established In (Auriault and Boutin, 1992), (Auriault and Boutin, 1993) and (Auriault and Boutin, 1994) for investigating the flow of an incompressible fluid in deformable dual-porosity media. An important result from these works is that the macroscopic behaviour strongly depends upon the scale ratios. This three-scale homogenization procedure is used here for investigating the flow of a highly compressible fluid in a fractured porous medium. The derivation of the mathematical models which is presented below in a quite detailed way is taken from (Royer and Auriault, 1994) and (Royer, 1994).

6.2 Medium under Consideration

To fit the homogenization method for periodic structures to three-scale problems, the medium is assumed to be doubly periodic (figure 7). No specific internal geometry is at issue, the work is aimed towards deriving general macroscopic models. At the pore scale, consider the medium to be Ω-periodic and its characteristic length to be l. The solid and

Figure 7: *Double periodicity*

the pores occupy the domains Ω_s and Ω_p, respectively, and their common boundary is Γ. A second periodic structure exists at the fracture level, whose period is Ω' and whose characteristic length l' is such that $l' >> l$. The porous matrix and the fractures occupy domains Ω'_{sp} and Ω'_f, respectively and their common boundary is Γ'. In a given medium l and l' are defined but the macroscopic length, l'', must be chosen such that $l'' >> l'$. Therefore, the dual-porosity medium exhibits two separations of scales instead of one in the single-porosity case:

- $l'/l'' << 1$ between the fracture scale and the macroscopic level,
- $l/l' << 1$ between the pore scale and the fracture scale.

If the first condition is not checked the homogenization cannot be applied because the macroscopic scale and the fracture scale are not separated. If the second condition is not checked, it means that the medium is a two-scale medium (single-porosity medium).

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \alpha\beta = \frac{l}{l''}$$

Thus, let assume both conditions $\alpha << 1$ and $\beta << 1$ to be checked, so that the medium is a three-scale medium and homogenization theory can be applied.

We will see that the macroscopic behaviour of gas flow in such a medium depends upon the relative order of magnitude between both scale ratios α and β. In other words, the macroscopic behaviour will depend upon the value of n, where n is such that:

$$\beta = \alpha^n \tag{152}$$

As in (Auriault and Boutin, 1992) and (Auriault and Boutin, 1992) the three following cases will be investigated:

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \frac{l}{l''} = \alpha\beta.$$

$$\beta = O\left(\alpha^n\right)$$

$$\begin{array}{lll} \underline{\text{Case I:}} & n = 2, & \beta = O(\alpha^2) = O(\varepsilon^2) \\ \underline{\text{Case II:}} & n = 1 & \beta = O(\alpha) = O(\varepsilon) \\ \underline{\text{Case III:}} & n = 1/2 & \alpha = O(\beta^2) = O(\varepsilon^2) \end{array} \tag{153}$$

6.3 Local Description

In the pores (Ω_p) and in the fractures (Ω'_f), gas flow is controlled by Navier-Stokes equations. For slow flows, inertial and transient terms of Navier-Stokes equations can be neglected. For the sake of simplicity, we consider a linear equation of state for the fluid. Thus, gas flow is governed by the following equations with $k = p$ in the pores and $k = f$ in the fractures.

Momentum-balance in Ω_p and Ω'_f

$$\mu\Delta\vec{v}_k + (\lambda + \mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_k) - \vec{\nabla}p_k = \vec{0} \quad (k = p, f) \tag{154}$$

Mass-balance in Ω_p and Ω'_f

$$\frac{\partial\rho_k}{\partial t} + \vec{\nabla}\cdot(\rho_k\vec{v}_k) = 0 \quad (k = p, f) \tag{155}$$

Equation of state in Ω_p and Ω'_f

$$\rho_k = Ap_k \quad A = \frac{\rho_0}{p_0} \text{ is constant} \quad (k = p, f) \tag{156}$$

No-slip condition on Γ

$$\vec{v}_p|_\Gamma = \vec{0} \tag{157}$$

Continuity of velocities on Γ'

$$\vec{v}_f|_{\Gamma'} =< \vec{v}_p >_\Omega |_{\Gamma'} = \left(\frac{1}{\mid \Omega \mid}\int_{\Omega_p}\vec{v}_p d\Omega\right)_{\Gamma'} \tag{158}$$

Continuity of pressures on Γ'

$$p_f|_{\Gamma'} = p_p|_{\Gamma'} \tag{159}$$

6.4 Order of Magnitude of the Velocity Ratio

An important feature of fluid flow in a fractured porous medium is that the fluid velocity in the pores is very low compared to the fluid velocity in the fractures. This can be figured out as follows:

$$\frac{V_p}{V_f} = O(\varepsilon^h) \quad h \text{ is a positive integer} \tag{160}$$

in which V_p and V_f are characteristic values of the velocity in the pores and in the fractures, respectively.

The purpose of this paragraph is to determine h for the three cases defined in (153).

Gas flow through the fractured porous medium is forced by a time-dependent pressure gradient. Stokes equation in both the pores and the fractures (154) expresses the balance between the pressure gradient and the viscous term:

$$\frac{\delta'' P}{l''} = O\left(\mu \frac{V_f^P}{l'^2}\right), \tag{161}$$

$$\frac{\delta' P}{l'} = O\left(\mu \frac{V_p^P}{l^2}\right), \tag{162}$$

in which $\delta'' P$ and $\delta' P$ are characteristic macroscopic pressure drops at the fracture scale and the pore scale, respectively and where V_f^P and V_p^P are the characteristic velocities due to the pressure gradient, in the fractures and in the pores, respectively.

It follows that:

$$V_f^P = O\left(\frac{\delta'' P \; l'^2}{\mu \; l''}\right), \tag{163}$$

$$V_p^P = O\left(\frac{\delta' P \; l^2}{\mu \; l'}\right). \tag{164}$$

Since $\rho_k = \frac{\rho_0}{p_0} p_k$, the mass-balance equation can be rewritten as:

$$\frac{\rho_0}{p_0} \frac{\partial p_k}{\partial t} + \frac{\rho_0}{p_0} \vec{\nabla} \cdot (\rho_k \vec{v}_k) = 0 \tag{165}$$

As a result of the compressibility of the fluid, the pressure field p_k can be expressed as follows:

$$p_k = p_0 + \bar{p}_k, \tag{166}$$

in which $\bar{p}_k$ is the pressure increment.

Thus, the mass-balance equation (155) expresses the fact that the fluid is subject to a temporal variation of pressure and we get:

$$\frac{\rho_0}{p_0} \omega \bar{p}_f = O\left(\rho_0 \frac{V_f^t}{l''}\right), \tag{167}$$

$$\frac{\rho_0}{p_0}\omega\bar{p}_p = O\left(\rho_0\frac{V_p^t}{l'}\right), \tag{168}$$

in which $\frac{\partial p_k}{\partial t}$ has been estimated as:

$$\frac{\partial p_k}{\partial t} = O(\omega\bar{p}_k),$$

and V_k^t is the velocity due to the temporal variation of pressure. It follows that:

$$V_f^t = O\left(\frac{\bar{p}_f}{p_o}\ \omega l''\right), \tag{169}$$

$$V_p^t = O\left(\frac{\bar{p}_p}{p_o}\ \omega l'\right). \tag{170}$$

We are interested in the physical situations for which the fluid compressibility does act at the macroscopic scale that are characterised by:

$$V_f = V_f^P = V_f^t. \tag{171}$$

Thus, according to the orders of magnitude obtained for V_f^P and V_f^t, respectively, we get:

$$\frac{\delta'' P\ l'^2}{\mu\ l''} = O\left(\frac{\bar{p}_f}{p_o}\ \omega l''\right), \tag{172}$$

from which we deduce:

$$\frac{p_0}{\mu\omega} = O\left(\frac{\bar{p}_f}{\delta'' P}\right) \times O\left(\frac{l''^2}{l'^2}\right) = O\left(\frac{\bar{p}_f}{\delta'' P}\right) \times O\left(\frac{l}{l'}\right)^{2n}. \tag{173}$$

Now let estimate the ratio of V_p^P to V_p^t:

$$\frac{V_p^P}{V_p^t} = O\left(\frac{\delta' P\ l^2}{\mu\ l'}\right) \times \left(\frac{p_o}{\bar{p}_p\omega l'}\right) = O\left(\frac{\delta' P}{\bar{P}_p}\right) \times O\left(\frac{p_0}{\mu\omega}\right) \times O\left(\frac{l^2}{l'^2}\right) \tag{174}$$

$\frac{p_0}{\mu\omega}$ is given by (173). Thus, expression (174) becomes:

$$\frac{V_p^P}{V_p^t} = O\left(\frac{\delta' P}{\bar{P}_p}\right) \times O\left(\frac{\bar{P}_f}{\delta'' P}\right) \times O\left(\frac{l}{l'}\right)^{-2n+2} \tag{175}$$

Now, according to (166), it turns out that:

$$\frac{\delta' P}{\bar{P}_p} = O\left(\frac{\delta'' P}{\bar{P}_f}\right) \tag{176}$$

It follows that:

$$\frac{V_p^P}{V_p^t} = O\left(\frac{l}{l'}\right)^{-2n+2} \tag{177}$$

Therefore,

$$\bullet\ if\ n<1: \quad \frac{V_p^P}{V_p^t} < O(1) \quad \Rightarrow V_p = V_p^t$$
$$\bullet\ if\ n=1: \quad \frac{V_p^P}{V_p^t} = O(1) \quad \Rightarrow V_p = V_p^t = V_p^P$$
$$\bullet\ if\ n>1: \quad \frac{V_p^P}{V_p^t} > O(1) \quad \Rightarrow V_p = V_p^P$$

We are now able to precise the order of magnitude $\frac{V_p}{V_f} = O(\varepsilon^h)$ for the three cases under consideration (153):

$$\begin{aligned}
&\underline{\text{Case I:}} \quad n=2 \\
&\qquad \frac{V_p}{V_f} = O\left(\frac{V_p^t}{V_f^t}\right) = O\left(\frac{l'}{l''}\right) = O(\varepsilon^2) \\
&\underline{\text{Case II:}} \quad n=1 \\
&\qquad \frac{V_p}{V_f} = O\left(\frac{V_p^t}{V_f^t}\right) = O\left(\frac{l'}{l''}\right) = O(\varepsilon) \\
&\underline{\text{Case I:}} \quad n=1/2 \\
&\qquad \frac{V_p}{V_f} = O\left(\frac{V_p^P}{V_f^t}\right) = O\left(\frac{V_p^t}{V_f^t}\right) \times O\left(\frac{l}{l'}\right)^{-2n+2} = O\left(\frac{l'}{l''}\right) \times O\left(\frac{l}{l'}\right)^{-2n+2} = O(\varepsilon^3)
\end{aligned} \tag{178}$$

In summary, we obtain:

$$\frac{V_p}{V_f} = O(\varepsilon^h) \text{ with: } \begin{cases} h=2 & \text{in case I} \\ h=1 & \text{in case II} \\ h=3 & \text{in case III} \end{cases} \tag{179}$$

6.5 Space Variables

Let $\vec{X}$ be the physical space variable. From the three characteristic lengths, l, l' and l'', three dimensionless space variables may be defined:

$$\vec{x} = \frac{\vec{X}}{l} \quad \text{(pore scale dimensionless space variable)}, \tag{180}$$

$$\vec{x}' = \frac{\vec{X}}{l'} \quad \text{(fracture scale dimensionless space variable)}, \tag{181}$$

$$\vec{x}'' = \frac{\vec{X}}{l''} \quad \text{(macroscopic scale dimensionless space variable)}, \tag{182}$$

Thus we have:

$$\vec{x}'' = \beta\vec{x}' = \gamma\vec{x} \tag{183}$$

Hence, as a result of both separations of scales ($\beta << 1$, $\gamma << 1$), these three space variables may be considered as independent space variables. Thus, in Ω_p, any physical variables is a priori a function of $\vec{x}$, $\vec{x}'$ and $\vec{x}''$ and in Ω'_f, any physical variable is a priori a function of $\vec{x}'$ and $\vec{x}''$.

6.6 Normalisation

From the momentum balance (154) and the mass balance (155), the following dimensionless numbers may be defined:

$$Q_k = \frac{\mid \vec{\nabla} p_k \mid}{\mid \mu \Delta \vec{v}_k \mid} \quad (k = p, f), \tag{184}$$

$$S_k = \frac{\mid \frac{\partial \rho_k}{\partial t} \mid}{\mid \vec{\nabla} \cdot (\rho_k \vec{v}_k) \mid} \quad (k = p, f) \quad \text{(Strouhal number)} \tag{185}$$

The estimation of these dimensionless numbers actually requires the choice of both a characteristic length and a characteristic time. In effect, a dual-porosity behaviour is due to the strong difference between the characteristic times of the flow in the pores and in the fractures, which is a linked to the strong difference between fluid velocities. Thus, instead of a characteristic length and a characteristic time, a characteristic length and a characteristic velocity will be equivalently considered here, namely l'' and V_f. With respect to these characteristic quantities, Q_k and S_k are estimated as follows:

$$Q_f = Q_p = \frac{\frac{\delta'' P}{l''}}{\mu \frac{V_f}{l''^2}} \tag{186}$$

Since (see 4.3)

$$\frac{\delta'' P}{l''} = O\left(\mu \frac{V_f}{l'^2}\right), \tag{187}$$

we therefore deduce

$$Q = Q_f = Q_p = O\left(\frac{l''^2}{l'^2}\right) = O(\beta^{-2}) \tag{188}$$

$$S_f = S_p = O\left(\frac{\omega}{V_f l''}\right) \tag{189}$$

According to the analysis carried out in 5.2, and since we are interested here in a transient flow, the following orders of magnitude will be considered:

$$S = S_f = S_p = O(1) \tag{190}$$

6.7 Dimensionless Local Description

According to the above estimations, the normalised local description is therefore the following:

$$\beta^2 \Delta \vec{v}_k + \beta^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}_k) - \vec{\nabla} p_k = \vec{0} \quad \text{in } \Omega_p \text{ and } \Omega'_f \tag{191}$$

$$\frac{\partial \rho_k}{\partial t} + \vec{\nabla} \cdot (\rho_k \vec{v}_k) = 0 \quad \text{in } \Omega_p \text{ and } \Omega'_f \tag{192}$$

$$\rho_k = A p_k \quad A = \frac{\rho_0}{p_0} \text{ is constant} \quad \text{in } \Omega_p \text{ and } \Omega'_f \tag{193}$$

$$\vec{v}_p = \vec{0} \quad \text{on } \Gamma \tag{194}$$

$$\vec{v_f} =< \vec{v}_p >_\Omega \quad \text{on } \Gamma' \tag{195}$$

$$p_f = p_p \quad \text{on } \Gamma' \tag{196}$$

in which all quantities are now dimensionless quantities and where the dimensionless gradient operator is written as:

$$\vec{\nabla}_{x''} + \beta^{-1} \vec{\nabla}_{x'} + \gamma^{-1} \vec{\nabla}_x \tag{197}$$

The method consists now in writing the physical unknowns in the form of asymptotic expansions as follows:

$$\begin{array}{llllll}
\phi_p = & \phi_p^0(\vec{x}, \vec{x}', \vec{x}'', t) & +\varepsilon\ \phi_p^1(\vec{x}, \vec{x}', \vec{x}'', t) & +... & & \phi = p, \rho \\
\vec{v}_p = & \varepsilon^h\ \vec{v}_p^h(\vec{x}, \vec{x}', \vec{x}'', t) & +\varepsilon^{h+1}\ \vec{v}_p^{h+1}(\vec{x}, \vec{x}', \vec{x}'', t) & ... & & \\
\phi_f = & \phi_f^0(\vec{x}', \vec{x}'', t) & +\varepsilon^r\ \phi_f^r(\vec{x}', \vec{x}'', t) & +... & \beta = \varepsilon^r & \phi = \vec{v}, p, \rho
\end{array}$$

The physical quantities in the pores are asymptotic expansions in power of ε. As a result of the order of magnitude of the velocity ratio, the first order of the expansion of $\vec{v}_p$ is in ε^h. The separation between the fracture scale and the macroscopic scale is measured by the small parameter β. Therefore, the expansions of the quantities related to the fracture scale are expansions in power of β.

The physical variables in the pores are Ω and Ω'- periodic whereas the physical variables related to the fractures are Ω'- periodic.

The method consists now in incorporating these expansions, together with the expression obtained for the dimensionless gradient operator, in the set of dimensionless equations. The derivation of the macroscopic models is the purpose of the next section.

7 Homogenization

7.1 Homogenization in Case I

7.1.1 Characteristics of Case I

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \frac{l}{l''} = \alpha\beta.$$

$$\beta = O\left(\alpha^n\right)$$

Case I has been defined by the following relationships between the scale ratios:

$$n = 2, \;\; \beta = O(\alpha^2) = O(\varepsilon^2) \tag{198}$$

The characteristics of case I are the following:

Velocity ratio:	$\frac{V_p}{V_f} = O(\varepsilon^2)$
Space variables:	$\vec{x} = O(\varepsilon^{-3})\vec{x}'', \;\; \vec{x}' = O(\varepsilon^{-2})\vec{x}'', \;\; \vec{x}''.$
Gradient:	$\vec{\nabla}_{x''} + \varepsilon^{-2}\vec{\nabla}_{x'} + \varepsilon^{-3}\vec{\nabla}_{x'}$
Dimensionless numbers:	$Q = O(\varepsilon^{-4}), \;\; S = O(1).$

7.1.2 Derivation of the Macroscopic Model

In the pores

Pressure and density fields at the first order:
Equation (191) at the order ε^{-3} and equation (193) at the order ε^0.

$$\vec{\nabla}_x p_p^0 = \vec{0}, \tag{199}$$

$$\rho_p^0 = A\; p_p^0. \tag{200}$$

Therefore, the first order pressure and density fields are independent of the local space variable $\vec{x}$:

$$p_p^0 = p_p^0(\vec{x}', \vec{x}'', t) \tag{201}$$

$$\rho_p^0 = \rho_p^0(\vec{x}', \vec{x}'', t) \tag{202}$$

Pressure and density fields at the first and second orders:
Equation (191) at the order ε^{-2} and equation (193) at the order ε^1.

$$\vec{\nabla}_x p_p^1 + \vec{\nabla}_{x'} p_p^0 = \vec{0}, \tag{203}$$

$$\rho_p^1 = A\ p_p^1. \tag{204}$$

From (203) and by virtue of periodicity, it follows that:

$$\vec{\nabla}_x p_p^1 = \vec{0}, \tag{205}$$

$$\vec{\nabla}_{x'} p_p^0 = \vec{0}, \tag{206}$$

Thus,

$$p_p^0 = p_p^0(\vec{x}'', t), \tag{207}$$

$$p_p^1 = p_p^1(\vec{x}', \vec{x}'', t). \tag{208}$$

According to (200) and (204) we also get

$$\rho_p^0 = \rho_p^0(\vec{x}'', t), \tag{209}$$

$$\rho_p^1 = \rho_p^1(\vec{x}', \vec{x}'', t). \tag{210}$$

Pressure and density fields at the second and third orders:
Equation (191) at the order ε^{-1} and equation (193) at the order ε^2.

$$\vec{\nabla}_x p_p^2 + \vec{\nabla}_{x'} p_p^1 = \vec{0}, \tag{211}$$

$$\rho_p^2 = A\ p_p^2. \tag{212}$$

This problem is identical to the previous one. We therefore deduce:

$$p_p^1 = p_p^1(\vec{x}'', t), \tag{213}$$

$$p_p^2 = p_p^2(\vec{x}', \vec{x}'', t), \tag{214}$$

$$\rho_p^1 = \rho_p^1(\vec{x}'', t), \tag{215}$$

$$\rho_p^2 = \rho_p^2(\vec{x}', \vec{x}'', t). \tag{216}$$

First-order filtration in the pores:
Equation (191) at the order ε^0, equation (192) at the order ε^{-1} and equation (194) at the order ε^2.

$$\Delta_x \vec{v}_p^2 + \vec{\nabla}_x(\vec{\nabla}_x \cdot \vec{v}_p^2) - \vec{\nabla}_x p_p^3 - \vec{\nabla}_{x'} p_p^2 - \vec{\nabla}_{x''} p_p^0 = \vec{0}, \tag{217}$$

$$\vec{\nabla}_x \cdot \vec{v}_p^2 = 0, \tag{218}$$

$$\vec{v}_p^2 = \vec{0}, \quad \text{on } \Gamma \tag{219}$$

in which $\vec{v}_p^2$ and p_p^3 are Ω-periodic.
Equations (217) through (219) constitute a linear boundary-value problem of variables $\vec{x}$ and whose unknowns are $\vec{v}_p^2$ and p_p^3. The solution takes the form (see 4.5):

$$\vec{v}_p^2 = -\tilde{k}_p(\vec{x})(\vec{\nabla}_{x''} p_p^0 + \vec{\nabla}_{x'} p_p^2), \tag{220}$$

$$p_p^3 = -\vec{\tau}_p(\vec{x}) \cdot (\vec{\nabla}_{x''} p_p^0 + \vec{\nabla}_{x'} p_p^2) + \bar{p}_p^3(\vec{x}', \vec{x}'', t). \tag{221}$$

Mass-balance in the porous matrix:
Equation (192) at the order ε^0.

$$\frac{\partial \rho_p^0}{\partial t} + \vec{\nabla}_{x'} \cdot (\rho_p^0 \vec{v}_p^2) + \vec{\nabla}_x \cdot (\rho^0 \vec{v}_p^3 + \rho_p^1 \vec{v}_p^2) = 0. \tag{222}$$

Integration of this equation over the period Ω leads, by virtue of periodicity and the no-slip condition on Γ, to:

$$< \frac{\partial \rho_p^0}{\partial t} >_\Omega + \vec{\nabla}_{x'} \cdot (\rho_p^0 < \vec{v}_p^2 >_\Omega) = 0, \tag{223}$$

where the average, over the periodic cell Ω is defined by:

$$< \cdot >_\Omega = \frac{1}{\mid \Omega \mid} \int_{\Omega_p} \cdot \, d\Omega \tag{224}$$

Since ρ_p^0 is independent of the local space variable $\vec{x}$ and $\vec{x}'$, equation (223) is accordingly rewritten as:

$$\phi \frac{\partial \rho_p^0}{\partial t} + \rho_p^0 \, \vec{\nabla}_{x'} \cdot (< \vec{v}_p^2 >_\Omega) = 0, \tag{225}$$

where

$$\phi = \frac{\mid \Omega_p \mid}{\mid \Omega \mid} \tag{226}$$

is the porosity of the microporous medium.
According to (220) and (202), equation (225) becomes:

$$\phi \frac{\partial \rho_p^0}{\partial t} + \rho_p^0 \, \vec{\nabla}_{x'} \cdot (\tilde{K}_p(\vec{\nabla}_{x''} p_p^0 + \vec{\nabla}_{x'} p_p^2)) = 0, \tag{227}$$

where

$$\tilde{K}_p = < \tilde{k}_p >_\Omega . \tag{228}$$

Equation (227) describes the flow at the porous matrix scale, i.e. in Ω'_{sp}.

<u>In the fractures</u>

Pressure and density fields at the first order:

Equation (191) at the order ε^{-2} and equation (193) at the order ε^0.

$$\vec{\nabla}_{x'} p_f^0 = \vec{0}, \tag{229}$$

$$\rho_f^0 = A\ p_f^0. \tag{230}$$

Thus,

$$p_f^0 = p_f^0(\vec{x}'', t), \tag{231}$$

$$\rho_f^0 = \rho_f^0(\vec{x}'', t), \tag{232}$$

Hence, the first-order pressure field in both the pores and the fractures verifies:

$$p_p^0 = p_p^0(\vec{x}'', t), \tag{233}$$

$$p_f^0 = p_f^0(\vec{x}'', t), \tag{234}$$

$$p_p^0 = p_f^0 \quad \text{on } \Gamma' \tag{235}$$

Therefore, we have:

$$p_p^0 = p_f^0 = p^0(\vec{x}'', t), \tag{236}$$

and consequently

$$\rho_p^0 = \rho_f^0 = \rho^0(\vec{x}'', t), \tag{237}$$

First-order filtration in the fractures:

Equation (191) at the order ε^{-1}, equation (192) at the order ε^{-2} and equation (195) at the order ε^0.

$$\Delta_{x'} \vec{v}_f^0 + \vec{\nabla}_{x'}(\vec{\nabla}_{x'} \cdot \vec{v}_f^0) - \vec{\nabla}_{x''} p^0 - \vec{\nabla}_{x'} p_f^2 = \vec{0}, \tag{238}$$

$$\vec{\nabla}_{x'} \cdot \vec{v}_f^0 = 0, \tag{239}$$

$$\vec{v}_f^0 =< \vec{v}_p^0 >_\Omega = \vec{0} \quad \text{on } \Gamma', \tag{240}$$

$\vec{v}_f^0$ and p_f^2 are Ω'-periodic.

The solution of this linear boundary-value problem is:

$$\vec{v}_f^0 = -\tilde{k}_f(\vec{x}')\vec{\nabla}_{x''} p^0, \tag{241}$$

$$p_f^2 = -\vec{\tau}_f(\vec{x}') \cdot \vec{\nabla}_{x''} p^0 + \bar{p}_f^2(\vec{x}'', t). \tag{242}$$

Macroscopic mass-balance:
Equation (192) at the order ε^0.

$$\frac{\partial \rho^0}{\partial t} + \vec{\nabla}_{x''} \cdot (\rho^0 \vec{v}_f^0) + \vec{\nabla}_{x'} \cdot (\rho^0 \vec{v}_f^2 + \rho_f^2 \vec{v}_f^0) = 0, \tag{243}$$

which can also be written as:

$$\frac{\partial p^0}{\partial t} + \vec{\nabla}_{x''} \cdot (p^0 \vec{v}_f^0) + \vec{\nabla}_{x'} \cdot (p^0 \vec{v}_f^2 + p_f^2 \vec{v}_f^0) = 0, \tag{244}$$

Integration of this equation over Ω' leads to:

$$< \frac{\partial p^0}{\partial t} >_{\Omega'} + \vec{\nabla}_{x''} \cdot < p^0 \vec{v}_f^0 >_{\Omega'} + < \vec{\nabla}_{x'} \cdot (p^0 \vec{v}_f^2 + p_f^2 \vec{v}_f^0) >_{\Omega'} = 0, \tag{245}$$

which

$$< \cdot >_{\Omega'} = \frac{1}{\mid \Omega' \mid} \int_{\Omega_f} \cdot \, d\Omega \tag{246}$$

Now, by the divergence theorem we get:

$$\begin{aligned} < \vec{\nabla}_{x'} \cdot (p_f^2 \vec{v}_f^0) >_{\Omega'} &= \frac{1}{\mid \Omega' \mid} \int_{\Gamma'} p_f^2 \vec{v}_f^0 \cdot \vec{n}' \, dS \\ &= \frac{1}{\mid \Omega' \mid} \int_{\Gamma'} p_f^2 < \vec{v}_p^0 >_{\Omega} \cdot \vec{n}' \, dS \\ &= \vec{0}. \end{aligned} \tag{247}$$

Now, the term $< \vec{\nabla}_{x'} \cdot (p_f^0 \vec{v}_f^2) >_{\Omega'}$ can be expressed as:

$$< \vec{\nabla}_{x'} \cdot (p_f^0 \vec{v}_f^2) >_{\Omega'} = p^0 \frac{1}{\mid \Omega' \mid} \int_{\Gamma'} < \vec{v}_p^2 >_{\Omega} \cdot \vec{n}' \, dS \tag{248}$$

Integration over Ω'_{sp} of the mass-balance equation in the porous matrix (225) gives:

$$\phi(1-\phi') \frac{\partial p^0}{\partial t} - p^0 \frac{1}{\mid \Omega' \mid} \int_{\Gamma'} < \vec{v}_p^2 >_{\Omega} \cdot \vec{n}' \, dS = 0, \tag{249}$$

from which we deduce:

$$< \vec{\nabla}_{x'} \cdot (p_f^0 \vec{v}_f^2) >_{\Omega'} = \phi(1-\phi') \frac{\partial p^0}{\partial t}. \tag{250}$$

Therefore, according to (247) and (250), equation (245) reduces to:

$$[\phi' + (1-\phi')\phi] \frac{\partial p^0}{\partial t} + \vec{\nabla}_{x''} \cdot (p^0 < \vec{v}_f^0 >_{\Omega'}) = 0, \tag{251}$$

or, according to the expression obtained for $\vec{v}_f^0$:

$$[\phi' + (1-\phi')\phi] \frac{\partial p^0}{\partial t} - \vec{\nabla}_{x''} \cdot (p^0 \tilde{K}_f \vec{\nabla}_{x''} p^0) = 0, \tag{252}$$

in which

$$\tilde{K}_f =< \tilde{k}_f >_{\Omega'} . \tag{253}$$

Therefore, the flow in the fractured porous medium is, at the first order of approximation, macroscopically described by:

$$[\phi' + (1-\phi')\phi]\frac{\partial p^0}{\partial t} - \vec{\nabla}_{x''} \cdot (p^0 \tilde{K}_f \vec{\nabla}_{x''} p^0) = 0$$

where

- $\tilde{K}_f$ is the fracture permeability,
- $p^0 = p^0_p = p^0_f$ is independent of $\vec{x}'$ Ω'_{sp} and Ω'_f, (254)
- $\phi = \dfrac{\mid \Omega_p \mid}{\mid \Omega \mid}$ is the porous matrix porosity,
- $\phi' = \dfrac{\mid \Omega'_f \mid}{\mid \Omega' \mid}$ is the fracture porosity.

This macroscopic behaviour is a non-linear mass balance equation with classical Darcy's law in the fractures. The flow in the pores in negligible regarding the flow in the fractures and is ignored at the first order of approximation. The main feature of this model relies upon the source term $[\phi' + (1-\phi')\phi]\dfrac{\partial p^0}{\partial t}$ which highlights the role of fluid reservoir played by the porous matrix.

7.2 Homogenization in Case II

7.2.1 Characteristics of Case II

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \frac{l}{l''} = \alpha\beta.$$

$$\beta = O\left(\alpha^n\right)$$

Case II has been defined by the following relationships between the scale ratios:

$$n = 1, \;\; \beta = O(\alpha) = O(\varepsilon) \tag{255}$$

The characteristics of case II are the following:

<u>Velocity ratio:</u> $\dfrac{V_p}{V_f} = O(\varepsilon)$

<u>Space variables:</u> $\vec{x} = O(\varepsilon^{-2})\vec{x}'', \;\; \vec{x}' = O(\varepsilon^{-1})\vec{x}'', \;\; \vec{x}''.$

<u>Gradient:</u> $\vec{\nabla}_{x''} + \varepsilon^{-1}\vec{\nabla}_{x'} + \varepsilon^{-2}\vec{\nabla}_{x'}$

<u>Dimensionless numbers:</u> $Q = O(\varepsilon^{-2}), \;\; S = O(1).$

7.2.2 Derivation of the Macroscopic Model

In the pores

Pressure and density fields at the first order:

Equation (191) at the order ε^{-2} and equation (193) at the order ε^0.

$$\vec{\nabla}_x p_p^0 = \vec{0}, \tag{256}$$

$$\rho_p^0 = A\ p_p^0. \tag{257}$$

Thus,

$$p_p^0 = p_p^0(\vec{x}', \vec{x}'', t) \tag{258}$$

$$\rho_p^0 = \rho_p^0(\vec{x}', \vec{x}'', t) \tag{259}$$

First-order filtration in the pores:

Equation (191) at the order ε^{-1}, equation (192) at the order ε^{-1} and equation (194) at the order ε^1.

$$\Delta_x \vec{v}_p^1 + \vec{\nabla}_x(\vec{\nabla}_x \cdot \vec{v}_p^1) - \vec{\nabla}_x p_p^1 - \vec{\nabla}_{x'} p_p^0 = \vec{0}, \tag{260}$$

$$\vec{\nabla}_x \cdot \vec{v}_p^2 = 0, \tag{261}$$

$$\vec{v}_p^2 = \vec{0}, \quad \text{on } \Gamma \tag{262}$$

in which $\vec{v}_p^2$ and p_p^3 are Ω-periodic.

$$\vec{v}_p^1 = -\tilde{k}_p(\vec{x})\vec{\nabla}_{x'} p_p^0 \tag{263}$$

$$p_p^1 = -\vec{\tau}_p(\vec{x}) \cdot \vec{\nabla}_{x'} p_p^0 + \bar{p}_p^1(\vec{x}', \vec{x}'', t) \tag{264}$$

Mass-balance in the porous matrix:

Equation (192) at the order ε^0.

$$\frac{\partial \rho_p^0}{\partial t} + \vec{\nabla}_{x'} \cdot (\rho_p^0 \vec{v}_p^1) + \vec{\nabla}_x \cdot (\rho_p^0 \vec{v}_p^2 + \rho_p^1 \vec{v}_p^1) = 0 \tag{265}$$

Integration of this equation over the period Ω yields:

$$< \frac{\partial \rho_p^0}{\partial t} + \vec{\nabla}_{x'} \cdot (\rho_p^0 \vec{v}_p^1) >_\Omega = 0, \tag{266}$$

from which we deduce the behaviour in Ω'_{sp}:

$$\phi \frac{\partial \rho_p^0}{\partial t} - \vec{\nabla}_{x'} \cdot (\rho_p^0 \tilde{K}_p \vec{\nabla}_{x'} p_p^0) = 0, \tag{267}$$

which can also be rewritten as follows:

$$\phi \frac{\partial p_p^0}{\partial t} - \vec{\nabla}_{x'} \cdot (p_p^0 \tilde{K}_p \vec{\nabla}_{x'} p_p^0) = 0, \tag{268}$$

where ϕ is the porosity and $\tilde{K}_p$ the effective permeability of the porous matrix.

<u>In the fractures</u>

Pressure and density fields at the first order:

Equation (191) at the order ε^{-1} and equation (193) at the order ε^0.

$$\vec{\nabla}_{x'} p_f^0 = \vec{0}, \tag{269}$$

$$\rho_f^0 = A\ p_f^0. \tag{270}$$

Thus,

$$p_f^0 = p_f^0(\vec{x}'', t), \tag{271}$$

$$\rho_f^0 = \rho_f^0(\vec{x}'', t). \tag{272}$$

First-order filtration in the fractures:

Equation (191) at the order ε^0, equation (192) at the order ε^{-1} and equation (195) at the order ε^0.

$$\Delta_{x'} \vec{v}_f^0 + \vec{\nabla}_{x'}(\vec{\nabla}_{x'} \cdot \vec{v}_f^0) - \vec{\nabla}_{x''} p^0 - \vec{\nabla}_{x'} p_f^1 = \vec{0}, \tag{273}$$

$$\vec{\nabla}_{x'} \cdot \vec{v}_f^0 = 0, \tag{274}$$

$$\vec{v}_f^0 =< \vec{v}_p^0 >_\Omega = \vec{0} \quad \text{on } \Gamma', \tag{275}$$

$\vec{v}_f^0$ and p_f^1 are Ω'-periodic.

The solutions of this boundary-value problem are:

$$\vec{v}_f^0 = -\tilde{k}_f(\vec{x}') \vec{\nabla}_{x''} p_f^0, \tag{276}$$

$$p_f^1 = -\vec{\tau}_f(\vec{x}') \cdot \vec{\nabla}_{x''} p_f^0 + \bar{p}_f^1(\vec{x}'', t). \tag{277}$$

Macroscopic mass-balance:

Equation (192) at the order ε^0.

$$\frac{\partial \rho_f^0}{\partial t} + \vec{\nabla}_{x''} \cdot (\rho_f^0 \vec{v}_f^0) + \vec{\nabla}_{x'} \cdot (\rho_f^0 \vec{v}_f^1 + \rho_f^1 \vec{v}_f^0) = 0. \tag{278}$$

Integration of this equation leads to:

$$\phi' \frac{\partial \rho_f^0}{\partial t} + \vec{\nabla}_{x''} \cdot (\rho_f^0 < \vec{v}_f^0 >_{\Omega'}) + < \vec{\nabla}_{x'} \cdot (\rho_f^0 \vec{v}_f^1) >_{\Omega'} = 0, \tag{279}$$

where ϕ' is the fracture porosity.
Now, by the divergence theorem and the appropriate boundary condition on Γ' we get:

$$< \vec{\nabla}_{x'} \cdot (\rho_f^0 \vec{v}_f^1) >_{\Omega'} = \frac{\rho_f^0}{\mid \Omega' \mid} \int_{\Gamma'} < \vec{v}_p^1 >_{\Omega} \cdot \vec{n}' \; dS, \tag{280}$$

and integration of the mass-balance equation in the porous matrix (266) together with the condition

$$p_f^0 = p_p^0 \quad \text{on } \Gamma', \tag{281}$$

yields

$$\int_{\Gamma'} < \vec{v}_p^1 >_{\Omega} \cdot \vec{n}' \; dS = \frac{1}{\rho_f^0} \phi \frac{\partial}{\partial t} \left(\int_{\Omega'_{sp}} \rho_p^0 \; d\Omega \right). \tag{282}$$

Finally,

$$< \vec{\nabla}_{x'} \cdot (\rho_f^0 \vec{v}_f^1) >_{\Omega'} = \phi \frac{\partial < \rho_p^0 >_{eff}}{\partial t}, \tag{283}$$

where

$$< \rho_p^0 >_{eff} = \frac{1}{\mid \Omega' \mid} \int_{\Omega'_{sp}} \rho_p^0 \; d\Omega. \tag{284}$$

Thus, equation (278) becomes:

$$\phi' \frac{\partial \rho_f^0}{\partial t} + \phi \frac{\partial < \rho_p^0 >_{eff}}{\partial t} - \vec{\nabla}_{x''} \cdot (\rho_f^0 \tilde{K}_f(\vec{x}'') \vec{\nabla}_{x''} p_f^0) = 0, \tag{285}$$

which can also be written as:

$$\phi' \frac{\partial p_f^0}{\partial t} + \phi \frac{\partial < p_p^0 >_{eff}}{\partial t} - \vec{\nabla}_{x''} \cdot (p_f^0 \tilde{K}_f(\vec{x}'') \vec{\nabla}_{x''} p_f^0) = 0. \tag{286}$$

Therefore, in case II, the flow is described by:

$$\phi' \frac{\partial p_f^0}{\partial t} + \phi \frac{\partial < p_p^0 >_{eff}}{\partial t} - \vec{\nabla}_{x''} \cdot (p_f^0 \tilde{K}_f(\vec{x}'') \vec{\nabla}_{x''} p_f^0) = 0$$

where

- $\tilde{K}_f$ is the fracture porosity,
- $< p_p^0 >_{eff} = \frac{1}{\mid \Omega' \mid} \int_{\Omega'_{sp}} p_p^0 \, d\Omega$,
- p_p^0 is defined by the following boundary-value problem on Ω'_{sp}: (287)

$$\phi \frac{\partial p_p^0}{\partial t} - \vec{\nabla}_{x'} \cdot (p_p^0 \tilde{K}_p \vec{\nabla}_{x'} p_p^0) = 0,$$

$$p_p^0 = p_f^0 \quad \text{on } \Gamma',$$

$$p_p^0 : \Omega'\text{-periodic.}$$

- $p_p^0 = \mathcal{F}(P_f^0)$

$\mathcal{F}$ is a non-linear, time-dependent fonctional with memory effects.

7.3 Homogenization in Case III

7.3.1 Characteristics of Case III

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \frac{l}{l''} = \alpha\beta.$$

$$\beta = O\left(\alpha^n\right)$$

Case III has been defined by the following relationships between the scale ratios:

$$n = 1/2, \;\; \alpha = O(\beta^2) = O(\varepsilon^2) \tag{288}$$

The characteristics of case III are the following:

<u>Velocity ratio:</u> $\frac{V_p}{V_f} = O(\varepsilon^3)$

<u>Space variables:</u> $\vec{x} = O(\varepsilon^{-3})\vec{x}'', \;\; \vec{x}' = O(\varepsilon^{-1})\vec{x}'', \;\; \vec{x}''.$

<u>Gradient:</u> $\vec{\nabla}_{x''} + \varepsilon^{-1}\vec{\nabla}_{x'} + \varepsilon^{-3}\vec{\nabla}_{x'}$

<u>Dimensionless numbers:</u> $Q = O(\varepsilon^{-2}), \;\; S = O(1).$

7.3.2 Derivation of the Macroscopic Model

<u>In the fractures</u>

Pressure and density fields at the first order:

Equation (191) at the order ε^{-1} and equation (193) at the order ε^0.

$$\vec{\nabla}_{x'} p_f^0 = \vec{0}, \tag{289}$$

$$\rho_f^0 = A\ p_f^0. \tag{290}$$

Hence,

$$p_f^0 = p_f^0(\vec{x}'', t), \tag{291}$$

$$\rho_f^0 = \rho_f^0(\vec{x}'', t). \tag{292}$$

First-order filtration in the fractures:
Equation (191) at the order ε^0, equation (192) at the order ε^{-1} and equation (195) at the order ε^0.

$$\Delta_{x'} \vec{v}_f^0 + \vec{\nabla}_{x'}(\vec{\nabla}_{x'} \cdot \vec{v}_f^0) - \vec{\nabla}_{x''} p^0 - \vec{\nabla}_{x'} p_f^1 = \vec{0}, \tag{293}$$

$$\vec{\nabla}_{x'} \cdot \vec{v}_f^0 = 0, \tag{294}$$

$$\vec{v}_f^0 =< \vec{v}_p^0 >_\Omega = \vec{0} \quad \text{on } \Gamma', \tag{295}$$

$\vec{v}_f^0$ and p_f^1 are Ω'-periodic.
This boundary-value problem leads to:

$$\vec{v}_f^0 = -\tilde{k}_f(\vec{x}')\vec{\nabla}_{x''} p_f^0, \tag{296}$$

$$p_f^1 = -\vec{\tau}_f(\vec{x}') \cdot \vec{\nabla}_{x''} p_f^0 + \bar{p}_f^1(\vec{x}'', t). \tag{297}$$

Macroscopic mass-balance:
Equation (192) at the order ε^0.

$$\frac{\partial \rho_f^0}{\partial t} + \vec{\nabla}_{x''} \cdot (\rho_f^0 \vec{v}_f^0) + \vec{\nabla}_{x'} \cdot (\rho_f^0 \vec{v}_f^1 + \rho_f^1 \vec{v}_f^0) = 0. \tag{298}$$

Integrating this equation over the period Ω' leads to:

$$< \frac{\partial \rho_f^0}{\partial t} + \vec{\nabla}_{x''} \cdot (\rho_f^0 \vec{v}_f^0) >_{\Omega'} = 0, \tag{299}$$

which gives the macroscopic behaviour:

$$\phi' \frac{\partial p_f^0}{\partial t} - \vec{\nabla}_{x''} \cdot (p_f^0 \tilde{K}_f(\vec{x}'')\vec{\nabla}_{x''} p_f^0) = 0. \tag{300}$$

Therefore, the behaviour at the pore scale is not required for deriving the macroscopic description of the flow. Thus, flows at both scales are not coupled at the first order of approximation in this case. The derived macroscopic behaviour describes the flow in a single-porosity medium. The flow in the pores is ignored at the first order.

8 Conclusions

This study shows that the macroscopic behaviour of a compressible fluid flow in a fractured porous medium is strongly dependent upon the scale ratios. In the three cases of interest, very different macroscopic behaviours are obtained:

$$\alpha = \frac{l}{l'}, \;\; \beta = \frac{l'}{l''}, \;\; \gamma = \frac{l}{l''} = \alpha\beta.$$

$$\beta = O\left(\alpha^n\right)$$

<u>Case I:</u> $n = 2, \; \beta = O(\alpha^2) = O(\varepsilon^2)$

$$[\phi' + (1-\phi')\phi]\frac{\partial p^0}{\partial t} - \vec{\nabla}_{x''} \cdot (p^0 \tilde{K}_f \vec{\nabla}_{x''} p^0) = 0$$

The porous matrix plays the role of a fluid reservoir.

<u>Case II:</u> $n = 1, \; \beta = O(\alpha) = O(\varepsilon)$

$$\phi' \frac{\partial p_f^0}{\partial t} + \phi \frac{\partial < p_p^0 >_{eff}}{\partial t} - \vec{\nabla}_{x''} \cdot (p_f^0 \tilde{K}_f(\vec{x}'') \vec{\nabla}_{x''} p_f^0) = 0$$

Non-linear behaviour with memory effects.

<u>Case III:</u> $n = 1/2, \; \alpha = O(\beta^2) = O(\varepsilon^2)$

$$\phi' \frac{\partial p_f^0}{\partial t} - \vec{\nabla}_{x''} \cdot (p_f^0 \tilde{K}_f(\vec{x}'') \vec{\nabla}_{x''} p_f^0) = 0$$

Single-porosity behaviour.

In case I, the macroscopic behaviour is a non-linear mass-balance equation with Darcy's law for the fracture network. The behaviour is quite similar to a single-porosity behaviour. The peculiarity of this model is its source term $[\phi' + (1-\phi')\phi]\frac{\partial p^0}{\partial t}$, in which the porous matrix influence does appear. At the first order of approximation, the pore flow is accounted for via this term, only. The porous matrix plays the role of a compressible fluid reservoir.

In case II, the flow through the pores strongly influences the macroscopic behaviour: it imposes memory effects and enhanced non-linearities at the macroscopic scale. This is the case of greatest interest, which shows how local effects may strongly affect the macroscopic behaviour.

Finally in case III, the pores are entirely ignored. It is the result that would be obtained by a straight upscaling from the fracture scale up to the macroscopic level. In this case the medium can be considered as a single-porosity medium at the first order of approximation.

The non-linearities in the three models are due to the high compressibility of the fluid and would disappear at lower fluid compressibility.
This strong dependence upon the scale ratios was already shown in (Auriault and Boutin, 1992), (Auriault and Boutin, 1993) and (Auriault and Boutin, 1994). As in those papers, the largest coupling effects are obtained in Case II, i.e. for equal scale ratios.
In order to interpret these results let us define the characteristic times at which the fluid occurs by:

$$\tau_p = \frac{l'}{V_p} \quad \text{in the pores,} \tag{301}$$

$$\tau_f = \frac{l''}{V_f} \quad \text{in the fractures.} \tag{302}$$

It turns out that

$$\frac{\tau_p}{\tau_f} = O(\beta\varepsilon^{-h}), \tag{303}$$

which gives the following results for the three cases under consideration:

$$\begin{array}{lll} \text{Case I:} & \frac{V_p}{V_f} = O(\varepsilon^2) & \frac{\tau_p}{\tau_f} = O(1) \\ \text{Case II:} & \frac{V_p}{V_f} = O(\varepsilon) & \frac{\tau_p}{\tau_f} = O(1) \\ \text{Case III:} & \frac{V_p}{V_f} = O(\varepsilon^3) & \frac{\tau_p}{\tau_f} = O(\varepsilon^{-2}) \end{array}$$

These orders of magnitude show that in cases I and II, both flows are initiated at the same time, but the velocity ratio is lower in case I than in case II which explains the fact that case II leads to the strongest coupling effects. In case III, the flows don't occur at the same time: for short times $O(\tau_f)$, the flow occurs in the fractures, only. As a result, the pore flow has no effect at the macroscopic scale at the first order of approximation.

Via classical approaches, transport phenomena in dual-porosity media are directly modelled at the macroscopic scale. Thus, the influence of local heterogeneities upon the macroscopic behaviour cannot be perfectly disclosed. Furthermore, phenomena such as flow in the pores or storage capacity in the fractures are simply discarded. The present work shows that the homogenization method for periodic structures allows the derivation of the macroscopic behaviour from the complete microscopic description. Since there is no preliminary assumption on the form of the macroscopic model, the influence of the local effects are conveyed to the macroscopic level. The above models constitute the first mathematical models of gas flow through a dual-porosity medium that accounts for the high compressibility of the fluid.

References

Arbogast, T., Douglas, J. and Hornung, U. (1990). Derivation of Double Porosity of Single-Phase Flow via Homogenization Theory. *SIAM J. Math. Anal.*, 21 (4), 823-836.

Auriault, J.-L. (1991). Heterogeneous Medium. Is an Equivalent Macroscopic Decsription Possible? *International Journal of Engineering Science*, 29(7):785 – 795.

Auriault, J.-L, Boutin C. (1992). Deformable Porous Media with Double Porosity. Quasi-Statics I: Coupling Effects. *Transport in Porous Media*, 7, 63–82.

Auriault, J.-L, Boutin C. (1993). Deformable Porous Media with Double Porosity. Quasi-Statics II: Memory Effects. *Transport in Porous Media*, 10, 153–169.

Auriault, J.-L, Boutin C. (1994). Deformable Porous Media with Double Porosity. III: Acoustics. *Transport in Porous Media*, 14, 143–162.

Auriault, J.-L., Strzelecki, T., Bauer, J., and He, S. (1990). Porous Deformable Media Saturated by a Very Compressible Fluid: Quasi-Statics. *European Journal of Mechanics, A/Solids*, 9(4):373 – 392.

Barenblatt, G.I. (1963). On Certain Boundary Value Problems for the Equations of Seepage of Liquid in Fissured Rocks. *PMM*, 27 (2), 348-350.

Barenblatt, G.I., Zheltov, Iu, and Kochina, I.N (1960). Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks. *PMM*, 24 (5), 852-864.

Bensoussan, A., Lions, J.-L., and Papanicolaou, G. (1978). *Asymptotic Analysis for Periodic Structures.* North-Holland Publishing Company, Amsterdam.

Chen, Z.X. (1989). Transient Flow of Slightly Compressible Fluids through Double-Porosity, Double-Permeability Systems - A State of the Art Review. *Transport in Porous Media*, 4, 147-184.

Chen, Z.X. (1990). Analytical Solutions for the Double-Porosity, Double-Permeability and Layered Systems. *J. Petrol. Sci. eng.*, 5, 1-24.

Darcy, H. (1856). *Les fontaines publiques de la ville de Dijon.* V. Dalmont, Paris.

Gringarten, A.C. (1984). Interpretation of Tests in Fissured and Multilayered Reservoirs with Double Porosity Behavior: Theory and Practice. *J. Petrol. Technol, Distinguished Authors Series*, 4, 549-464.

Levy, T. (1990). Filtration in a Porous Fissured Rock. Influence of the Fissure Connexity. *Eur. J. Mech. B/Fluids*, 9, 309-327.

Royer, P. (1994). *Contribution de l'homogénéisation à l'étude de la filtration d'un gaz en milieu déformable à double porosité - Application à l'étude du système gaz-charbon.* Thèse de l'Université Joseph Fourier, Grenoble, France.

Royer, P. and Auriault, J.-L. (1994). Transient Quasi-Static Gas Flow through a Rigid Porous Medium with Double Porosity. *Transport in Porous Media*, 17, 33-57.

Sanchez-Palencia, E. (1980). Non-homogeneous media and vibration theory. In *Lecture Notes in Physics*, volume 127. Springer-Verlag, Berlin.

Warren, J.O., and Root, P.J. (1963). The Behavior of Naturally Fractured Reservoirs. Soc. Petrol. Eng. J.,Sept, 254-255.