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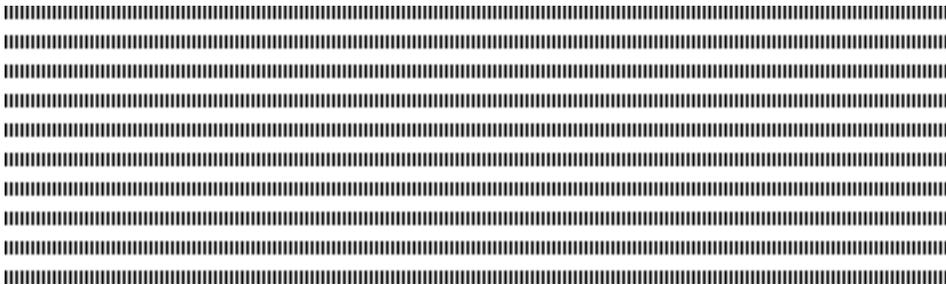
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## Optimal control of a parabolic solar collector

Nihale El Boukhari — El Hassan Zerrik

MACS Team  
Faculty of Sciences - Moulay Ismail University  
B.P. 11201 Zitoune, Meknes  
Morocco  
elboukhari.nihale@gmail.com, zerrik3@yahoo.fr

**ABSTRACT.** The aim of this paper is to study an optimal control problem for a parabolic solar collector. By means of empirical estimations, we derive a simplified bilinear distributed model, where the control models the mass flow of the heat-transfer fluid. We characterize the optimal control, that steers the fluid temperature close to the required level. Then we give an algorithm for the computation of the optimal control. The obtained results are illustrated by simulations of the simplified model, using data of Ain Beni Mathar solar plant in Morocco.

**RÉSUMÉ.** L'objet de cet article est d'étudier le problème du contrôle optimal d'un collecteur solaire parabolique. Par le biais d'estimations empiriques, on obtient un modèle simplifié bilinéaire distribué, où le contrôle modélise le débit massique du fluide caloporteur. On caractérise le contrôle optimal, qui rend la température du fluide assez proche du niveau requis. Puis on donne un algorithme pour l'implémentation numérique de ce contrôle. Les résultats obtenus sont illustrés à travers des simulations numériques, en utilisant les données de la station solaire Ain Beni Mathar au Maroc.

**KEYWORDS :** Bilinear systems, Solar collector, Parabolic trough reflector, Semigroups, Optimal control.

**MOTS-CLÉS :** Systèmes bilinéaires, Collecteur solaire, Réflecteur parabolique, Semi-groupes, Contrôle optimal.



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## 1. Introduction

Renewable energy is a clean and sustainable alternative to fossil energy. It is a promising technology to face the increasingly global demand for energy and the fluctuating prices of fossil fuels. Particularly, solar thermal energy (STE) is an abundant energy resource, which consists in collecting sunlight to generate thermal or electric energy. The efficiency of solar energy is affected by the position of the sun and by the weather. In this context, the geographic position of Morocco enables the exploitation of solar thermal energy with highly efficient power stations, which significantly reduces electricity production costs. Land availability is also an important factor that gives Morocco a tremendous potential to successfully invest in solar energy.

In this work, we are interested in the optimization of a parabolic power plant performance. Power plants usually use parabolic mirrors to concentrate sunlight onto a receiver tube, containing a heat-transfer fluid, which is often a synthetic oil. The mirrors (or reflectors) change their position along the day from east to west to collect a maximum of sunlight. The receiver tube is enclosed in a vacuum glass envelope. The vacuum significantly reduces the heat losses. Once the fluid is heated by solar irradiation, it is carried to a heat engine, where the heat energy is converted to electricity via steam turbines. For a maximal efficiency of the heat engine, the fluid temperature has to reach a required level  $T_d \simeq 673.15\text{K}$  ( $\simeq 400^\circ\text{C}$ ). The fluid temperature depends on the intensity of solar irradiation, on the optical properties of the reflectors, and on the fluid mass flow. In order to reach the required temperature, it is essential to determine the optimal fluid flow, which can be achieved via minimizing a given functional.

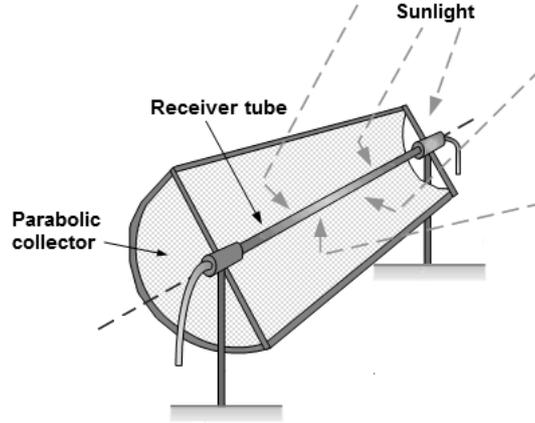
This is the aim of this paper. We consider an optimal control problem, where the control stands for the fluid mass flow. The optimal control to be searched is the minimizer of a quadratic cost functional. In order to characterize the optimal control, the original model describing the heat transfer within the collector is simplified into a distributed bilinear system. A rich literature is devoted to the optimal control of such systems (see for instance [1, 3, 5, 9, 11, 12]). However, in previous works, the control operator  $B$  is usually assumed to be relatively bounded with respect to the dynamics operator  $A$ . By contrast, the solar collector studied here is modeled by a bilinear system whose operator  $A$  is bounded, while the control operator  $B$  is unbounded and is the generator of a strongly continuous semigroup. Thereby, the optimal control of the solar collector will be studied using a new method, based on the tools of semigroup theory.

The paper is organized as follows. In section 2 the original model of the parabolic collector is simplified into a distributed bilinear system, modeling the evolution of the fluid temperature. Section 3 is devoted to the formulation of the optimal control problem, as well as the existence and the characterization of the solution. In section 4 we provide simulations of the solar collector simplified model, using data of Ain Beni Mathar solar plant in Morocco.

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## 2. The simplified model

We consider a parabolic solar collector, as depicted in figure 1. The heat-transfer fluid flows through a metallic receiver tube of length  $L$ , which is enclosed in a concentric glass tube. Both tubes are located in the focal line of a parabolic mirror, which concentrates sunlight towards the tubes.



**Figure 1.** Cylindro-parabolic solar collector [2]

The following assumptions are made on the basis of empirical estimations :

1) The fluid is incompressible, and the fluid flow is laminar, then the flow is space-independent, and depends only on time.

2) The fluid flow and temperature are uniformly distributed over the tube section, hence the receiver tube can be modeled by a one dimensional segment  $[0, L]$ .

3) There is perfect vacuum separating the receiver tube and the glass envelope, then the heat loss caused by the glass tube is negligible.

Upon the above assumptions, it has been proven (see for instance [4, 8]) that the solar collector can be modeled by a system of two partial differential equations on  $]0, L[$ , describing the temperature of the fluid  $T_f$  (K) and that of the receiver tube  $T_m$  (K), and given by

$$\begin{cases} \frac{\partial T_f}{\partial t}(x, t) = aT_f(x, t) + a_1T_m(x, t) - u(t)\frac{\partial T_f}{\partial x}(x, t) & ]0, L[ \times ]0, T[ \\ \frac{\partial T_m}{\partial t}(x, t) = b_1T_f(x, t) + b_2T_m(x, t) + b_3 & ]0, L[ \times ]0, T[ \\ (T_f(0, t), T_m(0, t)) = (T_{f,0}, T_{m,0}) & ]0, T[ \\ (T_f(x, 0), T_m(x, 0)) = (T_{f,0}, T_{m,0}) & ]0, L[ \end{cases} \quad (1)$$

where  $a, a_1, b_1, b_2$  and  $b_3$  are empirically estimated coefficients, depending on the weather and the optical properties of the collectors, and assumed to be constant in our application due to the choice of a small time interval.

$u(t) = \frac{C_f}{N_{coll}} \dot{Q}_f(t)$ , where  $\dot{Q}_f(t)$  ( $\text{kg s}^{-1}$ ) is the mass flow of the fluid,  $C_f$  ( $\text{J kg}^{-1} \text{K}^{-1}$ ) is the specific heat of the fluid, and  $N_{coll}$  is the number of collectors.  $C_f$  and  $N_{coll}$  are constant. In what follows  $u(t)$  stands for the control, and we assume that  $0 \leq u(t) \leq 1$ .

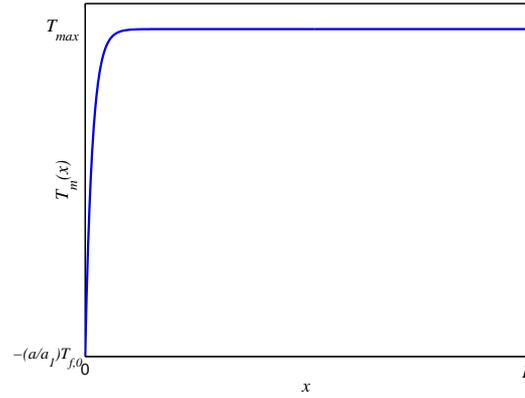
In [6], El Jai and Chalqi provide simulations of the original model (1), where it is shown that, for a relatively small fluid mass flow, the receiver tube temperature reaches rapidly a given value  $T_{max} \simeq 873 \text{ K}$  ( $\simeq 600^\circ\text{C}$ ), due to the quick metal heat absorption. They conclude that the temperature of the receiver tube  $T_m(.,.)$  can be assumed to be time-independent and close to  $T_{max}$ , while the simulated fluid temperature remains very

close to that of the original model.

Therefore the collector model (1) can be simplified into a system of one partial differential equation, given by

$$\begin{cases} \frac{\partial T_f(x, t)}{\partial t} = aT_f(x, t) + a_1T_m(x) - u(t)\frac{\partial T_f(x, t)}{\partial x} \\ T_f(0, t) = T_{f,0} \\ T_f(x, 0) = T_{f,0} \end{cases} \quad (2)$$

where the metallic tube temperature  $T_m$  is approached by a smooth function, satisfying  $T_m(0) = -\frac{a}{a_1}T_{f,0}$ , as depicted in figure 2 :



**Figure 2.** Temperature distribution in the metallic tube  $T_m$

We consider the state space  $H = L^2(0, L)$ , and we define the state  $y(t) \in H$  by

$$[y(t)](x) = T_f(x, t) - T_{f,0}$$

Denote  $B = -\frac{\partial}{\partial x}$ , and  $b = aT_{f,0} + a_1T_m \in H$ , then equation (2) takes the form of the following bilinear system

$$\begin{cases} \dot{y}(t) = u(t)By(t) + ay(t) + b \\ y(0) = 0 \end{cases} \quad (3)$$

The domain of  $B$  is  $\mathcal{D}(B) = \{z \in H^1(0, L) : z(0) = 0\}$ . Then  $y(0), b \in \mathcal{D}(B)$ . Moreover,  $B$  is the infinitesimal generator of the following  $C_0$  semigroup

$$[S(t)y](x) = \begin{cases} y(x-t) & \text{if } x-t > 0 \\ 0 & \text{if } x-t < 0 \end{cases}$$

For a maximal performance, the fluid temperature  $T_f$  has to reach a given value  $T_d = 673.15\text{K}$  ( $400^\circ\text{C}$ ) along the tube. To this end, we will search the optimal control  $u(\cdot)$  that minimizes the gap between  $y(t)$  and  $y_d = T_d - T_{f,0}$ . This is the purpose of the next section.

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### 3. Optimal control problem

We consider the following system

$$\begin{cases} \dot{y}(t) = u(t)By(t) + ay(t) + b(t) \\ y(0) = y_0 \end{cases} \quad (4)$$

where  $B$  is the infinitesimal generator of a  $C_0$  semigroup  $(S(t))_{t \geq 0}$  on a separable Hilbert space  $H$ , with a dense domain  $\mathcal{D}(B)$ .  $u \in L^2(0, T)$  is the control.  $a \in \mathbb{R}$ ,  $b \in L^2(0, T; \mathcal{D}(B))$ , and  $y_0 \in \mathcal{D}(B)$ .

Our purpose is to find a control that steers the state  $y(t)$  as close as possible to a desired state  $y_d \in H$ . Practically, this can be formulated as a minimization of the following cost functional

$$J(u) = \frac{\alpha}{2} \|y(T) - y_d\|^2 + \frac{\beta}{2} \int_0^T \|y(s) - y_d\|^2 ds + \frac{r}{2} \int_0^T u(s)^2 ds \quad (5)$$

over the set of admissible controls

$$U_{ad} = \{u \in L^2(0, T) : u_{min} \leq u(t) \leq u_{max}\}$$

where  $\alpha, \beta \geq 0$ ,  $r > 0$ , and  $0 \leq u_{min} < u_{max}$ .

Then the optimal control problem is stated as :

$$\begin{cases} \min J(u) \\ u \in U_{ad} \end{cases} \quad (6)$$

We first give the expression of the unique mild solution of system (4).

**Lemma 3.1.** *System (4) has a unique mild solution, written as*

$$y(t) = S\left(\int_0^t u(\tau) d\tau\right) y_0 + \int_0^t S\left(\int_s^t u(\tau) d\tau\right) [ay(s) + b(s)] ds \quad (7)$$

where  $(S(t))_{t \geq 0}$  is the  $C_0$  semigroup generated by  $B$ .

Moreover, since  $y_0 \in \mathcal{D}(B)$ , then  $y(t) \in \mathcal{D}(B)$ ,  $\forall t \geq 0$ .

*Proof.* For  $u \in U_{ad}$ , denote  $U(t, s) = S\left(\int_s^t u(\tau) d\tau\right)$ . Then  $(U(t, s))_{t \geq s}$  is a strongly continuous evolution family, generated by the family of operators  $B(t) = u(t)B$  (see definition 9.2, chapter VI in [7]). Moreover, there exist constants  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|U(t, s)\| \leq Me^{|\omega|u_{max}(t-s)}$ , then  $U(t, s)$  is exponentially bounded. By virtue of corollary 9.20, chapter VI in [7], it follows that the family of operators  $(u(t)B + aI)$  generates an exponentially bounded evolution family  $(\Gamma(t, s))_{t \geq s}$ , given by

$$\Gamma(t, s)y = U(t, s)y + \int_s^t U(t, \tau)a\Gamma(\tau, s)y d\tau$$

Then system (4) has a unique mild solution, written as

$$y(t) = \Gamma(t, 0)y_0 + \int_0^t \Gamma(t, s)b(s) ds$$

Replacing  $\Gamma(t, 0)$  and  $\Gamma(t, s)$  by their respective expressions, and applying Fubini's theorem yield

$$y(t) = U(t, 0)y_0 + \int_0^t U(t, s) [ay(s) + b(s)] ds$$

Hence we obtain (7). Finally, since  $y_0 \in \mathcal{D}(B)$  then, by proposition 9.3, chapter VI in [7],  $y(t) \in \mathcal{D}(B), \forall t \geq 0$ .  $\square$

In the next proposition we prove the existence of an optimal control, solution of problem (6).

**Proposition 3.2.** *There exists  $u^* \in U_{ad}$  such that*

$$J(u^*) = \inf_{u \in U_{ad}} J(u)$$

*Proof.* The functional  $J$  is nonnegative, and the set  $U_{ad}$  is nonempty, then there exists  $J^* \geq 0$  such that  $J^* = \inf\{J(u) \mid u \in U_{ad}\}$ . Let  $(u_n)_{n \in \mathbb{N}}$  be a sequence in  $U_{ad}$ , such that  $\lim_{n \rightarrow \infty} J(u_n) = J^*$ . Since  $U_{ad}$  is bounded then the sequence  $(u_n)_{n \in \mathbb{N}}$  is bounded, hence  $(u_n)_{n \in \mathbb{N}}$  has a subsequence, still denoted  $(u_n)_{n \in \mathbb{N}}$ , converging weakly to  $u^*$ . Since  $U_{ad}$  is closed and convex, then  $U_{ad}$  is weakly closed, which yields  $u^* \in U_{ad}$ .

Let  $y_n$  and  $y^*$  be the respective mild solutions of system (4), relatively to  $u_n$  and  $u^*$ . Then

$$\begin{aligned} y_n(t) - y^*(t) &= S\left(\int_0^t u_n(\tau) d\tau\right) y_0 - S\left(\int_0^t u^*(\tau) d\tau\right) y_0 \\ &\quad + \int_0^t S\left(\int_s^t u_n(\tau) d\tau\right) a[y_n(s) - y^*(s)] ds \\ &\quad + \int_0^t \left[ S\left(\int_s^t u_n(\tau) d\tau\right) y^*(s) - S\left(\int_s^t u^*(\tau) d\tau\right) y^*(s) \right] ds \\ &\quad + \int_0^t \left[ S\left(\int_s^t u_n(\tau) d\tau\right) b(s) - S\left(\int_s^t u^*(\tau) d\tau\right) b(s) \right] ds \end{aligned}$$

Denote  $\mu = \sup_{n \in \mathbb{N}} \|u_n\|$ , and let  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|S(t)\| \leq Me^{\omega t}, \forall t \geq 0$ . Let  $\widehat{M} = Me^{|\omega a| \sqrt{T} \mu}$ . Then applying Gronwall's lemma to the above equality yields

$$\begin{aligned} \|y_n(t) - y^*(t)\| &\leq \widehat{M} \left\| S\left(\int_0^t u_n(\tau) d\tau\right) y_0 - S\left(\int_0^t u^*(\tau) d\tau\right) y_0 \right\| \\ &\quad + \widehat{M} \left\| \int_0^t \left[ S\left(\int_s^t u_n(\tau) d\tau\right) y^*(s) - S\left(\int_s^t u^*(\tau) d\tau\right) y^*(s) \right] ds \right\| \\ &\quad + \widehat{M} \left\| \int_0^t \left[ S\left(\int_s^t u_n(\tau) d\tau\right) b(s) - S\left(\int_s^t u^*(\tau) d\tau\right) b(s) \right] ds \right\| \end{aligned}$$

The weak convergence  $u_n \rightharpoonup u^*$  gives  $\lim_{n \rightarrow \infty} \int_s^t u_n(\tau) d\tau = \int_s^t u^*(\tau) d\tau, \forall s, t \in [0, T]$ .

It follows that

$$\lim_{n \rightarrow \infty} \left\| S\left(\int_0^t u_n(\tau) d\tau\right) y_0 - S\left(\int_0^t u^*(\tau) d\tau\right) y_0 \right\| = 0$$

$$\lim_{n \rightarrow \infty} \left\| S \left( \int_s^t u_n(\tau) d\tau \right) y^*(s) - S \left( \int_s^t u^*(\tau) d\tau \right) y^*(s) \right\| = 0 \quad \text{a.e. on } [0, t]$$

$$\lim_{n \rightarrow \infty} \left\| S \left( \int_s^t u_n(\tau) d\tau \right) b(s) - S \left( \int_s^t u^*(\tau) d\tau \right) b(s) \right\| \quad \text{a.e. on } [0, t]$$

Then, by applying Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \left\| \int_0^t \left[ S \left( \int_s^t u_n(\tau) d\tau \right) y^*(s) - S \left( \int_s^t u^*(\tau) d\tau \right) y^*(s) \right] ds \right\| = 0$$

$$\lim_{n \rightarrow \infty} \left\| \int_0^t \left[ S \left( \int_s^t u_n(\tau) d\tau \right) b(s) - S \left( \int_s^t u^*(\tau) d\tau \right) b(s) \right] ds \right\| = 0$$

Therefore  $y_n(t) \rightarrow y^*(t)$  strongly in  $H$ , a.e. on  $[0, T]$ .

By Fatou's lemma, we obtain

$$\int_0^T \|y^*(t) - y_d\|^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^T \|y_n(t) - y_d\|^2 dt$$

and by the lower semi-continuity of norms, we have

$$\int_0^T u^*(t)^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^T u_n(t)^2 dt$$

Hence  $J(u^*) \leq \liminf_{n \rightarrow \infty} J(u_n) = J^*$ , which yields  $J(u^*) = J^*$ .  $\square$

**Proposition 3.3.** *The mapping  $u \mapsto y_u(t)$  is Gâteaux-differentiable, and its derivative at the direction  $h$  is given by*

$$\lim_{\varepsilon \rightarrow 0} \frac{y_{u+\varepsilon h}(t) - y_u(t)}{\varepsilon} = z_h(t)$$

where  $z_h$  is the mild solution of the following equation

$$\begin{cases} \dot{z}_h(t) = u(t)Bz_h(t) + az_h(t) + h(t)By_u(t) \\ z_h(0) = 0 \end{cases} \quad (8)$$

and is written as

$$z_h(t) = \int_0^t S \left( \int_s^t u(\tau) d\tau \right) [az_h(s) + h(s)By_u(s)] ds \quad (9)$$

*Proof.* Let  $u, h \in L^2(0, T)$  and  $\varepsilon \neq 0$ . Without loss of generality, we can assume that  $-1 \leq \varepsilon \|h\| \leq 1$ . Let  $y_u$  and  $y_{u+\varepsilon h}$  be the respective mild solutions of system (4), relatively to  $u$  and  $u + \varepsilon h$ , and let  $z_h$  be the mild solution of (8).

By replacing  $y_u(s)$  in (9) by its expression, we obtain

$$\begin{aligned} z_h(t) = & \int_0^t S \left( \int_s^t u(\tau) d\tau \right) az_h(s) ds + \int_0^t h(s)BS \left( \int_0^t u(\tau) d\tau \right) y_0 ds \\ & + \int_0^t \int_0^s h(s)BS \left( \int_\theta^t u(\tau) d\tau \right) [ay_u(\theta) + b(\theta)] d\theta ds \end{aligned}$$

Applying Fubini's theorem to the above expression yields

$$\begin{aligned}
z_h(t) = & \int_0^t S \left( \int_s^t u(\tau) d\tau \right) a z_h(s) ds + \left( \int_0^t h(\tau) d\tau \right) BS \left( \int_0^t u(\tau) d\tau \right) y_0 \\
& + \int_0^t \left( \int_s^t h(\theta) d\theta \right) BS \left( \int_s^t u(\tau) d\tau \right) [a y_u(s) + b(s)] ds
\end{aligned} \tag{10}$$

For  $0 \leq s \leq t \leq T$ , we denote

$$Y_0(t) = S \left( \int_0^t u(\tau) d\tau \right) y_0, \quad Y(s) = S \left( \int_s^t u(\tau) d\tau \right) a y_u(s)$$

and  $Z(s) = S \left( \int_s^t u(\tau) d\tau \right) b(s)$ . Then

$$y_u(t) = Y_0(t) + \int_0^t [Y(s) + Z(s)] ds$$

$$\begin{aligned}
y_{u+\varepsilon h}(t) = & S \left( \varepsilon \int_0^t h(\tau) d\tau \right) Y_0(t) + \int_0^t S \left( \int_s^t (u(\tau) + \varepsilon h(\tau)) d\tau \right) a y_{u+\varepsilon h}(s) ds \\
& + \int_0^t S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Z(s) ds
\end{aligned}$$

and, by expression (10), we have

$$\begin{aligned}
z_h(t) = & \left( \int_0^t h(\tau) d\tau \right) B Y_0(t) \\
& + \int_0^t \left( \int_s^t h(\tau) d\tau \right) B [Y(s) + Z(s)] ds + \int_0^t S \left( \int_s^t u(\tau) d\tau \right) a z_h(s) ds
\end{aligned}$$

which yields

$$\begin{aligned}
\frac{y_{u+\varepsilon h}(t) - y_u(t)}{\varepsilon} - z_h(t) = & \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_0^t h(\tau) d\tau \right) Y_0(t) - Y_0(t) \right] - \int_0^t h(\tau) d\tau B Y_0(t) \\
& + \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Y(s) - Y(s) \right] - \int_s^t h(\tau) d\tau B Y(s) \right] ds \\
& + \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Z(s) - Z(s) \right] - \int_s^t h(\tau) d\tau B Z(s) \right] ds \\
& + \int_0^t S \left( \int_s^t (u + \varepsilon h)(\tau) d\tau \right) a \left[ \frac{y_{u+\varepsilon h}(s) - y_u(s)}{\varepsilon} - z_h(s) \right] ds \\
& + \int_0^t \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) a z_h(s) - a z_h(s) \right] ds
\end{aligned}$$

There exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|S(t)\| \leq M e^{\omega t}$ ,  $\forall t \geq 0$ . Then applying Gronwall's lemma to the above equality yields

$$\begin{aligned}
& \left\| \frac{y_{u+\varepsilon h}(t) - y_u(t)}{\varepsilon} - z_h(t) \right\| \leq \\
& \widehat{M} \left\| \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_0^t h(\tau) d\tau \right) Y_0(t) - Y_0(t) \right] - \int_0^t h(\tau) d\tau B Y_0(t) \right\| \\
& + \widehat{M} \left\| \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Y(s) - Y(s) \right] - \int_s^t h(\tau) d\tau B Y(s) \right] ds \right\| \\
& + \widehat{M} \left\| \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Z(s) - Z(s) \right] - \int_s^t h(\tau) d\tau B Z(s) \right] ds \right\| \\
& + \widehat{M} \left\| \int_0^t \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) a z_h(s) - a z_h(s) \right] ds \right\|
\end{aligned}$$

where  $\widehat{M} = M e^{|\omega a| \sqrt{T}(\|u\|+1)}$ . By passing to the limit, we have

$$\lim_{\varepsilon \rightarrow 0} \left\| S \left( \varepsilon \int_s^t h(\tau) d\tau \right) a z_h(s) - a z_h(s) \right\| = 0 \quad \text{a.e. on } [0, t]$$

Since  $Y_0(t)$ ,  $Y(s)$  and  $Z(s)$  belong to  $\mathcal{D}(B)$ , then

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \left\| \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_0^t h(\tau) d\tau \right) Y_0(t) - Y_0(t) \right] - \int_0^t h(\tau) d\tau B Y_0(t) \right\| = 0 \\
& \lim_{\varepsilon \rightarrow 0} \left\| \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Y(s) - Y(s) \right] - \int_s^t h(\tau) d\tau B Y(s) \right\| = 0 \quad \text{a.e. on } [0, t] \\
& \lim_{\varepsilon \rightarrow 0} \left\| \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Z(s) - Z(s) \right] - \int_s^t h(\tau) d\tau B Z(s) \right\| = 0 \quad \text{a.e. on } [0, t]
\end{aligned}$$

By using appropriate bounds, and applying Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \left\| \int_0^t \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) a z_h(s) - a z_h(s) \right] ds \right\| = 0 \\
& \lim_{\varepsilon \rightarrow 0} \left\| \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Y(s) - Y(s) \right] - \int_s^t h(\tau) d\tau B Y(s) \right] ds \right\| = 0 \\
& \lim_{\varepsilon \rightarrow 0} \left\| \int_0^t \left[ \frac{1}{\varepsilon} \left[ S \left( \varepsilon \int_s^t h(\tau) d\tau \right) Z(s) - Z(s) \right] - \int_s^t h(\tau) d\tau B Z(s) \right] ds \right\| = 0
\end{aligned}$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{y_{u+\varepsilon h}(t) - y_u(t)}{\varepsilon} - z_h(t) \right\| = 0$$

□

Hereafter we characterize the derivative of the cost functional (5).

**Proposition 3.4.** *The Gâteaux-derivative of functional  $J$  at  $u \in L^2(0, T)$  is given by*

$$\begin{cases} D_u J \cdot h = \langle J'(u), h \rangle_{L^2(0, T)} \\ J'(u)(t) = \langle \varphi(t), B y(t) \rangle_H + r u(t) \end{cases} \quad (11)$$

where  $y$  is the mild solution of system (4), associated to  $u$ , and  $\varphi$  is the mild solution of the following adjoint equation

$$\begin{cases} \dot{\varphi}(t) = -u(t)B^*\varphi(t) - a\varphi(t) - \beta(y(t) - y_d) \\ \varphi(T) = \alpha(y(T) - y_d) \end{cases} \quad (12)$$

such that  $B^*$  is the adjoint operator of  $B$ .

*Proof.* By the differentiability of  $u \mapsto y_u$ , functional  $J$  is Gâteaux-differentiable over  $L^2(0, T)$ . Its derivative at the direction  $h$  is written as

$$D_u J.h = \langle \alpha(y_u(T) - y_d), z_h(T) \rangle_H + \int_0^T \langle \beta(y_u(t) - y_d), z_h(t) \rangle_H dt + \int_0^T r u(t) h(t) dt$$

For  $n \in \rho(B)$ , the resolvent set of  $B$ , let  $B_n = nB(nI - B)^{-1}$  be the Yosida approximation of  $B$ . Let  $z_n$  and  $\varphi_n$  be the respective mild solutions of the following equations

$$\begin{cases} \dot{z}_n(t) = u(t)B_n z_n(t) + a z_n(t) + h(t)B y_u(t) \\ z_h(0) = 0 \end{cases}$$

$$\begin{cases} \dot{\varphi}_n(t) = -u(t)B_n^* \varphi_n(t) - a \varphi_n(t) - \beta(y_u(t) - y_d) \\ \varphi_n(T) = \alpha(y_u(T) - y_d) \end{cases}$$

Since  $B_n$  is bounded, and  $h(t)B y_u(t) \in H$  a.e. on  $[0, T]$ , then  $\dot{z}_n, \dot{\varphi}_n \in L^2(0, T; H)$ . Therefore

$$\begin{aligned} \int_0^T \langle \beta(y_u(t) - y_d), z_n(t) \rangle_H dt &= - \int_0^T \langle \dot{\varphi}_n(t) + u(t)B_n^* \varphi_n(t) + a \varphi_n(t), z_n(t) \rangle_H dt \\ &= - \int_0^T \langle \dot{\varphi}_n(t), z_n(t) \rangle_H dt - \int_0^T \langle \varphi_n(t), u(t)B_n z_n(t) + a z_n(t) \rangle_H dt \\ &= - \int_0^T \langle \dot{\varphi}_n(t), z_n(t) \rangle_H dt - \int_0^T \langle \varphi_n(t), \dot{z}_n(t) \rangle_H dt + \int_0^T \langle \varphi_n(t), h(t)B y_u(t) \rangle_H dt \end{aligned}$$

Since  $\dot{z}_n, \dot{\varphi}_n \in L^2(0, T; H)$ , then

$$\begin{aligned} \int_0^T \langle \dot{\varphi}_n(t), z_n(t) \rangle_H dt + \int_0^T \langle \varphi_n(t), \dot{z}_n(t) \rangle_H dt &= \langle \varphi_n(T), z_n(T) \rangle_H - \langle \varphi_n(0), z_n(0) \rangle_H \\ &= \langle \alpha(y_u(T) - y_d), z_n(T) \rangle_H \end{aligned}$$

It follows that

$$\int_0^T \langle \beta(y_u(t) - y_d), z_n(t) \rangle_H dt = - \langle \alpha(y_u(T) - y_d), z_n(T) \rangle_H + \int_0^T \langle \varphi_n(t), h(t)B y_u(t) \rangle_H dt$$

By virtue of proposition 5.4, chapter 2 of [10], we have

$$\lim_{n \rightarrow +\infty} \|z_n - z_h\|_{C([0, T]; H)} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|\varphi_n - \varphi\|_{C([0, T]; H)} = 0$$

where  $z_h$  and  $\varphi$  are the respective mild solutions of (8) and (12). Then the above equality yields

$$\int_0^T \langle \beta(y_u(t) - y_d), z_h(t) \rangle_H dt + \langle \alpha(y_u(T) - y_d), z_h(T) \rangle_H = \int_0^T \langle \varphi(t), B y_u(t) \rangle_H h(t) dt$$

Therefore, the derivative of  $J$  is written as

$$\begin{aligned} D_u J.h &= \int_0^T \langle \varphi(t), By_u(t) \rangle_H h(t) dt + \int_0^T ru(t)h(t) dt \\ &= \langle J'(u), h \rangle_{L^2(0,T)} \end{aligned}$$

where  $J'(u)$  is given by (11).  $\square$

In the next proposition we characterize the optimal control, solution of problem (6).

**Proposition 3.5.** *Let  $u^*$  be an optimal control, then  $u^*$  is given by*

$$u^*(t) = \max \left( u_{min}; \min \left( u_{max}; -\frac{1}{r} \langle \varphi(t), By(t) \rangle_H \right) \right) \quad (13)$$

where  $y$  and  $\varphi$  are respectively the mild solutions of (4) and (12), associated to  $u^*$ .

*Proof.* Let  $u^*$  be an optimal control, and  $w \in U_{ad}$ . By the convexity of  $U_{ad}$  we have  $u^* + \varepsilon(w - u^*) \in U_{ad}, \forall \varepsilon \in ]0, 1[$ . Then  $J(u^* + \varepsilon(w - u^*)) \geq J(u^*), \forall \varepsilon \in ]0, 1[$ .

By passing to the limit, we obtain

$$\langle J'(u^*), w - u^* \rangle = \lim_{\varepsilon \rightarrow 0} \frac{J(u^* + \varepsilon(w - u^*)) - J(u^*)}{\varepsilon} \geq 0 \quad (14)$$

– If  $u_{min} < u^*(t) < u_{max}$  over a nonempty open set  $I \subset ]0, T[$  :

Let  $h \in \mathcal{D}(I)$  (the space of smooth functions, compactly supported). If  $\|h\|_{L^\infty(I)}$  is sufficiently small then we have  $u^* + h, u^* - h \in U_{ad}$ . It follows from inequality (14), applied to  $u^* + h$  and  $u^* - h$ , that

$$\langle J'(u^*), h \rangle = 0, \quad \forall h \in \mathcal{D}(I)$$

By the density of  $\mathcal{D}(I)$  in  $L^2(I)$ , it follows that  $J'(u^*)(t) = 0$  a.e. on  $I$ .

Therefore  $u^*(t) = -\frac{1}{r} \langle \varphi(t), By(t) \rangle_H$ .

– If  $u^*(t) = u_{min}$  over a nonempty open set  $I \subset ]0, T[$  :

Let  $h \in \mathcal{D}(I)$  such that  $h(t) \geq 0, \forall t \in I$ . Then for  $\|h\|_{L^\infty(I)}$  sufficiently small we have  $u^* + h \in U_{ad}$ . Hence inequality (14) yields  $\langle J'(u^*), h \rangle \geq 0$ .

It follows that  $J'(u^*)(t) \geq 0$  a.e. on  $I$ . Thus

$$-\frac{1}{r} \langle \varphi(t), By(t) \rangle_H \leq u^*(t) = u_{min} \quad \text{a.e. on } I$$

– Similarly to the above case, if  $u^*(t) = u_{max}$  over a nonempty open set  $I \subset ]0, T[$ , then

$$-\frac{1}{r} \langle \varphi(t), By(t) \rangle_H \geq u_{max} \quad \text{a.e. on } I$$

Therefore,  $u^* = \max \left( u_{min}; \min \left( u_{max}; -\frac{1}{r} \langle \varphi(t), By(t) \rangle_H \right) \right)$  a.e. on  $]0, T[$ .  $\square$

Finally, we give an algorithm, based on the projected gradient method, in order to implement numerically the optimal control, given by (13).

*Algorithm*

– Step 1 : Choose an initial control  $u_0 \in U_{ad}$ , a threshold accuracy  $\varepsilon > 0$ , and a step length  $\lambda$ . Initialize with  $k = 0$ .

– Step 2 : Solve the systems

$$\begin{cases} \dot{y}_k(t) = u_k(t)By_k(t) + ay_k(t) + b(t) \\ y_k(0) = y_0 \\ \dot{\varphi}_k(t) = -u_k(t)B^*\varphi_k(t) - a\varphi_k(t) - \beta(y_k(t) - y_d) \\ \varphi_k(T) = \alpha(y_k(T) - y_d) \end{cases}$$

– Step 3 : Compute

$$J'(u_k)(t) = \langle \varphi_k(t), By_k(t) \rangle + ru_k(t)$$

$$u_{k+1}(t) = \max(u_{min}; \min(u_{max}; u_k(t) - \lambda J'(u_k)(t)))$$

– Step 4 : If  $\|u_{k+1} - u_k\| > \varepsilon$ ,  $k = k + 1$ , go to step 2. Otherwise  $u^* = u_{k+1}$ .

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## 4. Simulations

We consider the captor model (2) with data of Ain Beni Mathar solar plant, in North-east Morocco (see [4]) :

$$L = 618m, a = -0.030, a_1 = 0.024, T_{f,0} = 423.15K (150^\circ C)$$

In order to determine the optimal fluid mass flow, we consider system (3) and the optimal control problem (6) with the cost functional (5), where

$$T = 3600s, \alpha = 3, \beta = 0.1, r = 5 \times 10^4$$

The required temperature is  $T_d = 673.15K (400^\circ C)$ , then the desired state is  $y_d = T_d - T_{f,0} = 250$ . The set of admissible controls is

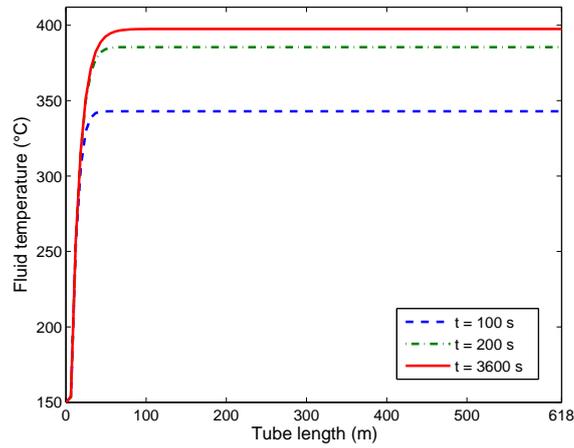
$$U_{ad} = \{u \in L^2(0, T) \mid 0 \leq u(t) \leq 1 \text{ a.e.}\}$$

According to proposition 3.5, the optimal control is given by

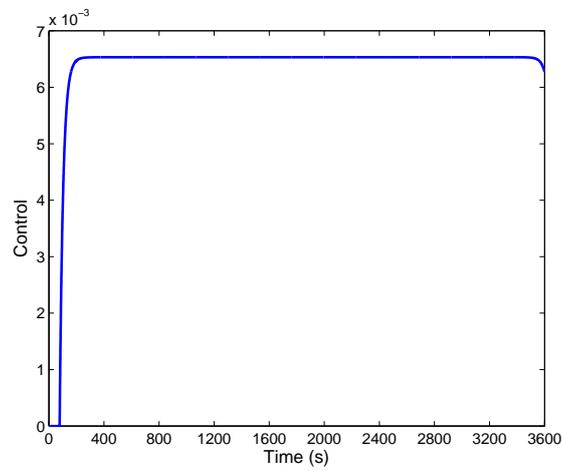
$$u^*(t) = \max \left( 0; \min \left( 1; 2 \times 10^{-5} \int_0^L \varphi(x, t) \frac{\partial}{\partial x} y(x, t) dx \right) \right)$$

where  $y$  is the solution of system (3) relatively to  $u^*$ , and  $\varphi$  is the solution of the adjoint equation (12), with  $B^* = \frac{\partial}{\partial x}$ .

Using the previous algorithm, with  $u_0 = 0$ ,  $\lambda = 10^{-6}$ , and  $\varepsilon = 10^{-5}$ , simulations lead to the following figures :



**Figure 3.** Fluid temperature at times  $t = 100s$ ,  $t = 200s$ , and  $t = 3600s$



**Figure 4.** Optimal control

Figure 3 depicts the fluid temperature ( $^{\circ}\text{C}$ ) at times  $t = 100s$ ,  $t = 200s$ , and  $t = T = 3600s$ , while the corresponding optimal control is plotted in figure 4. Figure 3 shows that the fluid temperature gets close to the desired level  $T_d = 400^{\circ}\text{C}$  within a short time interval. At the final time  $T = 3600s$ , the average fluid temperature is about  $\bar{T} \simeq 390^{\circ}\text{C}$ . As shown by the above simulations, the optimal fluid mass flow, obtained with our approach, drives the system close to the desired temperature profile. This is crucial for the performance of the solar plant.

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