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Observability of a 1D Schrödinger equation with time-varying boundaries

Duc-Trung Hoang *

Abstract

We discuss the observability of a one-dimensional Schrödinger equation on certain time dependent domain. In linear moving case, we give the exact boundary and pointwise internal observability for arbitrary time. For the general moving, we provide exact boundary observability when the curve satisfies some certain conditions. By duality theory, we establish the controllability of adjoint system.

Keywords: Observability, controllability, Schrödinger equation, moving domain, non-autonomous evolution equation.

Mathematics Subject Classification (2010): 93B07, 93C05, 35R37.

1 Introduction

Let $\tau > 0$, and $\ell(t) : [0, \tau] \rightarrow \mathbb{R}_+$ a strictly positive C^2 -function satisfying $\ell(0) = 1$ and $\frac{\ell'}{\ell} \in L_\infty$. We consider the following system as a initial boundary value problem in a time dependent domain.

$$(S_{\text{moving}}) \quad \begin{cases} i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 & x \in [0, \ell(t)] \\ u(0, t) = u(\ell(t), t) = 0 & t \geq 0 \\ u(x, 0) = u_0 & x \in [0, 1] \end{cases}$$

For Neumann boundary observations we obtain estimates like

$$c(\tau) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 dt \leq C(\tau) \|u_0\|_{H_0^1(0,1)}^2,$$

see Theorems 2.1, 2.2 and 2.3. We refer to the first estimate as observability estimate and to the second as admissibility estimate. The two first mentioned results rely on a transformation of (S_{moving}) to a non-autonomous equation on the fixed domain $[0, 1]$: the change of variables $y = \frac{x}{\ell(t)}$ and new function $w(y, t) := u(x, t)$ gives an equivalent differential equation for w , namely

$$(S_{\text{fixed}}) \quad \begin{cases} i \frac{\partial w}{\partial t} &= \frac{-1}{\ell(t)^2} \frac{\partial^2 w}{\partial y^2} + i \frac{\ell'(t)}{\ell(t)} y \frac{\partial w}{\partial y}, \\ w(0, t) &= w(1, t) = 0 \\ w_y(0, t) &= \ell(t) u_x(0, t) \quad \text{and} \\ w_y(1, t) &= \ell(t) u_x(\ell(t), t) \end{cases}$$

which can easily be obtained by the chain rule.

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To obtain Theorems 2.1 and 2.2 we apply the 'multiplier technique': This powerful method has been developped by Morawetz [26] and was later extended by Ho [11] and Lions [17]. We extend a version of Machtyngier [18] to time-dependend multipliers. The observability estimate relies then on the "uniqueness-compactness" lemma 3.5. The pitfall of this proof strategy is that it only proves existence of some positive constant, without explicit estimates. This is in contrast with Theorem 2.3 which is as specific result for the boundary curve $\ell(t) = 1 + \varepsilon t$. In this linear moving wall case, we mimic a successful approach for a one-dimensional wave-equation obtained by Haak and the author in [9] and develop the solution of (S_{moving}) into a series of eigenfunctions. This allows to use results from Fourier analysis; the obtained admissibility estimates are sharper than those obtained in the previous results, and the observation estimate is provided with explicit constants. Moreover, we obtain in this case admissibility and exact observability of internal point observations:

$$k(\tau)\|u_0\|_{L_2(0,1)}^2 \leq \int_0^\tau |u(a,t)|^2 dt \leq K(\tau)\|u_0\|_{L_2(0,1)}^2,$$

see Theorem 2.5. It is remarkable that the lower estimate cannot be true when $\varepsilon = 0$ on any rational point a ; the fact that the considered domains extend however, seem to 'middle out' this obstacle. Closely related to this observation are works of Castro and Khapalov [6, 14, 13] where on a fixed domain Ω a moving point observer is considered, with similar conclusions. We also mention results from Moyano [28, 29] where in a two-dimensional circle the radius $\ell(t)$ is used as a control parameter.

An additional result on L_p -admissibility and observability of point observations are presented as well, see Theorem 2.7.

It is well-known that exact observability for an (autonomous) wave equation implies observability for the associated Schrödinger equation, see e.g. [41, Chapter 6.7 ff.]. An inspection of the proof gives several obstacles when one passes to non-autonomous problems, and we were not able to use this approach to directly infer our results from those for the wave equation in [9]. We mention that some results on the so-called Hautus-test will be subject of an independent publication [10].

2 Main Results

Before giving precise formulations of the aforementioned results, let us start by proving that the Schrödinger equation (S_{fixed}) admits a solution: to this end, we reformulate it as an abstract non-autonomous Cauchy problem in the following way: let $X = L_2(0, 1)$ and the family of operators $\{A(t)\}$ be defined as

$$(2.1) \quad A(t)w = \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y$$

with natural domain $D(A(t)) = H^2(0, 1) \cap H_0^1(0, 1) =: D$. Moreover, by assumption, the map $t \mapsto A(t)u$ is continuously differentiable for all $u \in D$. Let $\omega > 0$. Then integration by parts gives

$$\begin{aligned} \langle (A(t) + \omega I)w, w \rangle &= \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} \bar{w} + \frac{\ell'(t)}{\ell(t)} y w_y \bar{w} + \omega |w|^2 \right) dy \\ (2.2) \quad &= \frac{-i}{\ell(t)^2} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{\ell(t)} \int_0^1 y w_y \bar{w} dy + \omega \int_0^1 |w|^2 dy \\ &= \frac{-i}{\ell(t)^2} \int_0^1 |w_y|^2 dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (|w|^2 + y w \bar{w}_y) dy + \omega \int_0^1 |w|^2 dy \end{aligned}$$

Taking real parts and observing that

$$\operatorname{Re}\left(\int_0^1 y w \overline{w_y} dy\right) = \operatorname{Re}\left(\frac{\ell'(t)}{\ell(t)} \int_0^1 y w_y \overline{w} dy\right) = -\operatorname{Re}\left(\frac{\ell'(t)}{2\ell(t)} \int_0^1 |w|^2 dy\right)$$

we obtain

$$(2.3) \quad \operatorname{Re}\left(\langle (A(t) + \omega I)w, w \rangle\right) = \left(\omega - \frac{\ell'(t)}{2\ell(t)}\right) \int_0^1 |w|^2 dy$$

For $\omega > \left\|\frac{\ell'}{2\ell}\right\|_{L^\infty}$, the left hand side of (2.3) becomes positive, and the Lumer-Philips theorem asserts that $\omega + A(t)$ generates a contraction semigroup, i.e.

$$\forall t \geq 0 \quad \left\|e^{-s A(t)}\right\| \leq e^{\omega s}$$

This ensures in particular that the family $(A(t))_{t \in [0, \tau]}$ satisfies the Kato stability condition. We apply [30, Theorem V.4.8 pp.145] to conclude that $(A(t))$ generates a unique evolution family $\{U(t, s)\}_{0 \leq s \leq t \leq \tau}$ on X satisfying $w(t) = U(t, 0)w_0$. From this we infer a solution to (S_{moving}) as well, by transforming the fixed domain back to the time-dependent domain.

Suppose that we are given observation operators $C(t) : D \rightarrow Y$ where Y is another Hilbert space. Define the output function $y(t) = C(t)w(t)$. The operator $C(t)$ is called (Y, Z) -admissible if there exist $\gamma > 0$ such that:

$$\int_0^\tau \|C(t)w(t)\|_Y^2 dt \leq \gamma \|w_0\|_Z^2.$$

We say that the system (S_{fixed}) is exactly (Y, Z) -observable in time $\tau > 0$ if there exist $\delta > 0$ such that:

$$\int_0^\tau \|C(t)w(t)\|_Y^2 dt \geq \delta \|w_0\|_Z^2.$$

If the spaces Y, Z are fixed, we simply speak of admissibility and exact observability. Exact observation in time $\tau > 0$ means that the knowledge of $y_{[0, \tau]}$ allows to recover the initial value w_0 . It is well known that exact observability is equivalent to exact controllability of the retrograde adjoint system:

$$z'(t) = -A(t)^* z(t) - C(t)^* w(t) \quad \text{with} \quad z(\tau) = 0$$

Moreover, it is easy to see that admissibility or observability of (S_{fixed}) is equivalent to those of (S_{moving}) .

Results on Neumann observations

Theorem 2.1. *Let $\tau > 0$ and $\ell : [0, \tau] \rightarrow \mathbb{R}_+^*$ be a strictly positive, twice continuously differentiable function satisfying $\frac{\ell'}{\ell} \in L^\infty$ and $\ell(0) = 1$. Then there exists a constants $C(\tau)$ such that the following admissibility inequalities hold:*

$$\int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 dt \leq C(\tau) \|u_0\|_{H_0^1(0, 1)}^2$$

An explicit estimate of constant $C(\tau)$ is given in the proof, see (3.8).

Concerning observability, we will have the following result. Let $\tau > 0$ and $\ell : [0, \tau] \rightarrow \mathbb{R}_+^*$ be a strictly positive, twice continuously differentiable function satisfying:

$$(2.4) \quad \ell'(t) > 0, \quad \ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) < \frac{1}{\pi} \quad \forall t \in (0, \tau)$$

Integrating for 0 to τ of the second condition, we have $2\tau + \pi(1 - \ell(\tau)^2) > 0$. From the condition (2.4), $\ell(t)$ is an increasing function, and then $\ell'(t) < \frac{1}{\pi}$. It follows that $\frac{\ell'(t)}{\ell(t)} < \frac{1}{\pi}$, and so the condition $\frac{\ell'}{\ell} \in L_\infty$ guaranteeing admissibility is satisfied.

Theorem 2.2. *For all τ satisfying (2.4), the following observability inequality holds:*

$$c(\tau) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau (|u_x(0,t)|^2 + |u_x(\ell(t),t)|^2) dt.$$

Here $c(\tau)$ is some positive constant depending on τ .

A direct application of theorem 2.2 can be used for periodic moving boundary $\ell(t) = 1 + \varepsilon \sin(\omega t)$ where $\varepsilon \in (0, 1)$ and $\omega \in (0, \frac{1}{\pi\varepsilon(1+\varepsilon)})$. For all $\tau \in (0, \frac{\pi}{2\omega})$, we have

$$\ell'(t) = \varepsilon\omega \cos(\omega t) > 0 \quad \text{since} \quad \omega t \in (0, \frac{\pi}{2}) \quad \forall 0 \leq t \leq \tau$$

$$\ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) = \varepsilon\omega \cos(\omega t)(1 + \varepsilon \sin(\omega t)) < \varepsilon\omega(1 + \varepsilon) < \frac{1}{\pi}$$

Hence, $\ell(t)$ satisfies the condition (2.4), so the curve is admissible. The problem of particles moving inside one dimensional square-well of oscillating width was proposed by Fermi and Ulam [19] in order to explain the mechanism of particles containing high energies. This model that plays an important role on theory of quantum chaos and it seems difficult to give an exact solution formula. Glasser [8] investigated the behavior of wave functions and energy in a given instantaneous eigenstate by assumptions on the smoothness of boundary. As far as we know, there are no results in the literature concerning observability and controllability with periodic boundary functions.

In the case that $\ell(t) = 1 + \varepsilon t$, the condition (2.4) is ensured when $\varepsilon \in (0, \frac{2}{\pi})$ and $0 < t < \frac{1}{\varepsilon}(\frac{2}{\pi} - 1)$. We have the following exact analytic solution for S_{moving} , due to Doescher and Rice [7]

$$(2.5) \quad u(x, t) = \sum_{n=1}^{+\infty} a_n \sqrt{\frac{2}{\ell(t)}} \sin\left(\frac{n\pi x}{\ell(t)}\right) e^{i\left(\frac{\varepsilon x^2}{4\ell(t)} - n^2 \pi^2 \frac{t}{\ell(t)}\right)}$$

where the coefficients (a_n) are defined by the sine-series development of the initial value u_0 . A similar exact solution in the case of two-variable moving wall can be found in [42] where the author uses the fundamental transformation to change the moving boundary problem into a solvable one side fixed boundary problem.

Based on formula (2.5) we obtain a first result on Neumann observability at the boundary $\{(x, t) : x \in \{0, \ell(t)\}\}$. Compared to Theorem 2.2 the admissibility constant is sharper. In contrast with Theorem 2.2, where we can only prove existence of some positive constant $c(\tau)$, we obtain now an explicit estimate for the observability constant. The proof is presented in section 3.

Theorem 2.3. *For every $\tau > 0$ there exist explicit constants $c(\tau, \varepsilon), C(\tau, \varepsilon)$ such that:*

$$(2.6) \quad c(\tau, \varepsilon) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau |u_x(0,t)|^2 + |u_x(\ell(t),t)|^2 dt \leq C(\tau, \varepsilon) \|u_0\|_{H_0^1(0,1)}^2$$

In particular, the Neumann observation at the boundary of the system (S_{moving}) is exact observable in any time $\tau > 0$. Moreover, the observability coefficient $c(\tau, \varepsilon)$ decays $\sim \exp\left(\frac{-2k\pi^2}{\varepsilon\tau}\right)$ where $k > \frac{3}{2}$.

Remark 2.4. *By Dirichlet condition $u(\ell(t), t) = 0$ for all t . Differentiating yields $\ell'(t)u_x(\ell(t), t) + u_t(\ell(t), t) = 0$, and so $u_x(\ell(t), t) = \frac{-1}{\varepsilon}u_t(\ell(t), t)$. As a result, observing $u_t(\ell(t), t)$ or $u_x(\ell(t), t)$ is, up to a constant, the same.*

Point observations

We now focus on point observations $u \mapsto u(a, t)$ in the case of a linearly moving wall $\ell(t) = 1 + \varepsilon t$. Observe that in the “degenerate” case that is, $\varepsilon = 0$, the (then) autonomous Schrödinger equation has the well-known solution

$$u(x, t) = \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 t} \sin(n\pi x).$$

Clearly, there is no reasonable observability possible at rational points x since infinitely many terms in the sum vanish, independently of the leading coefficient a_n . This changes when $\varepsilon > 0$: from (2.5) we obtain

$$u(a, t) = \sum_{n=1}^{+\infty} a_n \left(\frac{2}{\ell(t)}\right)^{\frac{1}{2}} \exp\left(\frac{i\varepsilon a^2}{4\ell(t)} - in^2\pi^2 \frac{t}{\ell(t)}\right) \sin\left(\frac{n\pi a}{\ell(t)}\right)$$

and so

$$(2.7) \quad \int_0^\tau |u(a, t)|^2 dt = \int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt.$$

Based on a remarkable result of Tenenbaum and Tucsnak we obtain the following result in section 3.

Theorem 2.5. *Assume $\ell(t) = 1 + \varepsilon t$. Then, for every $\tau > 0$, we have:*

$$(2.8) \quad K(\tau) \|u_0\|_{L_2(0,1)}^2 \gtrsim \int_0^\tau |u(a, t)|^2 dt \gtrsim k(\tau) \|u_0\|_{L_2(0,1)}^2$$

More precisely, $k(\tau) \approx M e^{-\frac{c}{T}}$ where $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$ and M, c are some positive constants that appear in the proof.

Corollary 2.6. *For all $a \in (0, 1)$ the point observation $C = \delta_a$ for the system (S_{moving}) is exactly observable in arbitrary short time.*

L_p -estimates of point observations

Finally we have to following L_p admissibility and observability estimates.

Theorem 2.7. *Let $\ell(t) = 1 + \varepsilon t$. We assume that $u_0 \in H_0^1(0, 1)$. For $0 < p < 2$ and $a \in (0, 1)$, we have*

$$k_p(\tau) \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p} \leq \left(\int_0^\tau |u(a, t)|^p dt \right)^{1/p} \leq K_p(\tau) \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p}$$

where $k_p(\tau)$, are constants depending on τ and p .

The upper estimate is a direct consequence of (2.8). Indeed, by the continuity of the embeddings $H_0^1 \hookrightarrow L_2 \hookrightarrow L_p$ and the boundedness of $\ell(t)$ to obtain:

$$\|u(a, t)\|_{L_p} \lesssim \|u(a, t)\|_{L_2} \stackrel{\text{from (2.8)}}{\lesssim} \|u_0\|_{L_2} \lesssim \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p}$$

Hence, it serves only to show that the lower estimate is of the right order.

3 Proof of the main results

3.1 The multiplier Lemma

We follow E. Machtyngier [18, Lemma 2.2] by using multiplier method for (S_{fixed}) : Let w be a solution to (S_{fixed}) and $q \in C^2([0, 1] \times [0, \tau])$ be a real valued function. Then, due to the differential equation (S_{fixed}) ,

$$(3.1) \quad \operatorname{Re} \left(\int_0^\tau \int_0^1 (q\overline{w}_y + \frac{1}{2}\overline{w}q_y) (iw_t + \frac{1}{\ell(t)^2}w_{yy} - i\frac{\ell'(t)}{\ell(t)}yw_y) dy dt \right) = 0$$

We separate the left hand side of (3.1) into three parts and simplify each of them.

Lemma 3.1. *The following identities hold.*

$$(3.2) \quad \left\{ \begin{array}{l} \operatorname{Re} \left(\int_0^\tau \int_0^1 (q\overline{w}_y + \frac{1}{2}\overline{w}q_y) iw_t dy dt \right) \\ = \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2}iq\overline{w}_y w \right]_{t=0}^{t=\tau} dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 iwq_t \overline{w}_y dy dt \right) \end{array} \right.$$

$$(3.3) \quad \left\{ \begin{array}{l} \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q\overline{w}_y + \frac{1}{2}\overline{w}q_y) dy dt \right) \\ = \operatorname{Re} \left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1, t)|w_y(1, t)|^2 - q(0, t)|w_y(0, t)|^2) dt \right) \\ - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} dy dt \right) \end{array} \right.$$

$$(3.4) \quad \left\{ \begin{array}{l} -\operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{iy\ell'(t)}{\ell(t)} w_y (q\overline{w}_y + \frac{1}{2}\overline{w}q_y) dy dt \right) \\ = -\operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{iy\ell'(t)}{\ell(t)} q |w_y|^2 dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} \frac{iy\ell'(t)}{\ell(t)} w_y \overline{w} q_y dy dt \right) \end{array} \right.$$

Proof. To prove (3.2), we use integration by parts. Using $\overline{w}(0, t) = \overline{w}(1, t) = 0$, we have:

$$\begin{aligned} \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \int_0^1 q_y \cdot \overline{w} w_t dy dt \right) &= \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \left(\left[\overline{w} w_t q \right]_{y=0}^{y=1} - \int_0^1 q \cdot (\overline{w}_y w_t + \overline{w} w_{ty}) dy \right) dt \right) \\ &= -\frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \int_0^1 q (\overline{w}_y w_t + \overline{w} w_{ty}) dy dt \right) \end{aligned}$$

Therefore, the left hand side of (3.2) equals

$$\begin{aligned} &\operatorname{Re} \left(\int_0^\tau \int_0^1 (q\overline{w}_y + \frac{1}{2}\overline{w}q_y) iw_t dy dt \right) = \frac{1}{2} \operatorname{Re} \left(i \int_0^1 \int_0^\tau (w_t \cdot q\overline{w}_y - q\overline{w} w_{ty}) dy dt \right) \\ &= \frac{1}{2} \operatorname{Re} \left(i \int_0^1 \left(\left[q\overline{w}_y w \right]_{t=0}^{t=\tau} - \int_0^\tau w (q_t \overline{w}_y + q \overline{w}_{yt}) dt \right) dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 q i \overline{w} w_{ty} dy dt \right) \\ &= \frac{1}{2} \operatorname{Re} \left(i \int_0^1 \left(\left[q\overline{w}_y w \right]_{t=0}^{t=\tau} \right) dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^1 \int_0^\tau iwq_t \overline{w}_y dy dt \right) \end{aligned}$$

Here, we already use the fact that

$$-\operatorname{Re}\left(\int_0^1 \int_0^\tau i q w \overline{w_{ty}}\right) = \operatorname{Re}\left(\int_0^1 \int_0^\tau i q \overline{w} w_{ty} dt dy\right).$$

To prove (3.3) we have

$$\begin{aligned} \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} q \overline{w_y} dy dt\right) &= \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{d}{dy}(|w_y|^2) \cdot \frac{1}{2\ell(t)^2} q dy dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)|w_y(1,t)|^2 - q(0,t)|w_y(0,t)|^2) dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} q_y |w_y|^2 dy dt\right) \end{aligned}$$

since we use $\operatorname{Re}(w_{yy} \overline{w_y}) = \operatorname{Re}(\overline{w_{yy}} w_y)$. Again, integration by parts shows

$$\begin{aligned} \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{2\ell(t)^2} \overline{w} q_y dy dt\right) &= \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} \overline{w} q_y d(w_y) dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \left[\frac{1}{2\ell(t)^2} \overline{w} q_y w_y\right]_{y=0}^{y=1} dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y} q_y + \overline{w} q_{yy}) w_y dy dt\right) \\ &= -\operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y} q_y + \overline{w} q_{yy}) w_y dy dt\right) \end{aligned}$$

Therefore we have:

$$\begin{aligned} &\operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q \overline{w_y} + \tfrac{1}{2} \overline{w} q_y) dy dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)w_y^2(1,t) - q(0,t)w_y^2(0,t)) dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y dy dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} dy dt\right) \end{aligned}$$

Hence, part (3.3) is proved. The last part is obvious. \square

Now summing up the three parts and using (3.1) yields

Proposition 3.2. *For any real valued function $q \in C^2([0,1] \times [0,\tau])$ and a solution w to (S_{fixed}) we have*

$$\begin{aligned} 0 &= \operatorname{Re}\left(\int_0^1 \frac{i}{2} \left[q \overline{w_y} w\right]_{t=0}^{t=\tau} dy\right) - \tfrac{1}{2} \operatorname{Re}\left(\int_0^\tau \int_0^1 i w q_t \overline{w_y} dy dt\right) \\ &\quad + \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)|w_y(1,t)|^2 - q(0,t)|w_y(0,t)|^2) dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y dy dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} dy dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} q |w_y|^2 dy dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} w_y \overline{w} q_y dy dt\right) \end{aligned}$$

3.2 Energy estimates

For a solution w to (S_{fixed}) we define the first and second energy as

$$E(t) = \tfrac{1}{2} \int_0^1 |w(y,t)|^2 dy \quad \text{and} \quad F(t) = \tfrac{1}{2} \int_0^1 |w_y(y,t)|^2 dy$$

respectively.

Lemma 3.3. *We have $\ell(\tau)E(\tau) = E(0)$.*

Proof. Taking the derivative respected to t and using S_{fixed} , we have

$$\begin{aligned}\frac{dE(t)}{dt} &= \frac{d}{dt} \frac{1}{2} \int_0^1 |w(y, t)|^2 dy = \frac{1}{2} \int_0^1 (w_t \bar{w} + w \bar{w}_t) dy \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right) \bar{w} + \overline{w \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)} \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} (w_{yy} \bar{w} - \overline{w_{yy} w}) + \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) \right) dy\end{aligned}$$

Now integration by parts gives

$$\begin{aligned}\int_0^1 \frac{i}{\ell(t)^2} (w_{yy} \bar{w} - \overline{w_{yy} w}) dy &= \int_0^1 \frac{i}{\ell(t)^2} \bar{w} d(w_y) - \int_0^1 \frac{i}{\ell(t)^2} w d(\bar{w}_y) \\ &= \frac{i}{\ell(t)^2} \left([\bar{w} w_y]_{y=0}^{y=1} - \int_0^1 |w_y|^2 \right) - \frac{i}{\ell(t)^2} \left([w \bar{w}_y]_{y=0}^{y=1} - \int_0^1 |w_y|^2 \right) \\ &= 0\end{aligned}$$

whereas

$$\begin{aligned}\int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy &= \int_0^1 \frac{\ell'(t)}{\ell(t)} y \bar{w} d(w) - \int_0^1 \frac{\ell'(t)}{\ell(t)} y w d(\bar{w}) \\ &= \left[\frac{\ell'(t)}{\ell(t)} y \bar{w} w \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (\bar{w} + y \bar{w}_y) w dy + \left[\frac{\ell'(t)}{\ell(t)} y \bar{w} w \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (w + y w_y) \bar{w} dy \\ &= -\frac{2\ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 dy - \int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy.\end{aligned}$$

Therefore,

$$\int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy = -\frac{\ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 dy,$$

so that

$$\frac{dE(t)}{dt} = -\frac{1}{2} \int_0^1 \frac{\ell'(t)}{\ell(t)} |w(y, t)|^2 dy = -\frac{\ell'(t)}{\ell(t)} E(t).$$

Using $\ell(0) = 1$, this implies easily $E(\tau) = \frac{E(0)}{\ell(\tau)}$. □

Lemma 3.4. *For all $\tau > 0$ and $\tau \in \left(0, \frac{\pi}{2\omega}\right)$, we have:*

$$\frac{\pi^2}{\ell(\tau)} E(0) \leq F(\tau) \leq \ell(\tau) F(0)$$

Proof. Concerning F we have

$$\begin{aligned}\frac{dF(t)}{dt} &= \frac{d}{dt} \frac{1}{2} \int_0^1 |w_y(y, t)|^2 dy = \frac{1}{2} \int_0^1 (w_{yt} \bar{w}_y + w_y \bar{w}_{yt}) dt \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)_y \bar{w}_y + \overline{w_y \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)_y} \\ &= \frac{i}{2\ell(t)^2} \int_0^1 (w_{yyy} \bar{w}_y - \overline{w_{yyy} w_y}) dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 ((y w_y)_y \bar{w}_y + w_y \overline{(y w_y)_y}) dy.\end{aligned}$$

The first term on the right hand side simplifies as

$$\frac{i}{2\ell(t)^2} \int_0^1 (w_{yyy} \bar{w}_y - \overline{w_{yyy} w_y}) dy$$

$$\begin{aligned}
&= \frac{i}{2\ell(t)^2} \int_0^1 \overline{w_y} d(w_{yy}) - \frac{i}{2\ell(t)^2} \int_0^1 w_y d(\overline{w_{yy}}) \\
&= \frac{i}{2\ell(t)^2} \left[\overline{w_y} w_{yy} \right]_{y=0}^{y=1} - \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 dy - \frac{i}{2\ell(t)^2} \left[\overline{w_{yy}} w_y \right]_{y=0}^{y=1} + \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 dy \\
&= \left[\frac{1}{2} \overline{w_y} (w_t - \frac{\ell'(t)}{\ell(t)} y w_y) \right]_{y=0}^{y=1} + \left[\frac{1}{2} w_y (\overline{w_t} - \frac{\ell'(t)}{\ell(t)} y \overline{w_y}) \right]_{y=0}^{y=1} = -\frac{\ell'(t)}{\ell(t)} |w_y(1, t)|^2
\end{aligned}$$

whereas the second term simplifies as follows.

$$\begin{aligned}
\frac{\ell'(t)}{2\ell(t)} \int_0^1 ((y w_y)_y \overline{w_y} + w_y \overline{(y w_y)_y}) dy &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 (w_y + y w_{yy}) \overline{w_y} + w_y \overline{(w_y + y w_{yy})} dy \\
&= \frac{\ell'(t)}{2\ell(t)} \int_0^1 2|w_y|^2 + y(w_{yy} \overline{w_y} + w_y \overline{w_{yy}}) dy \\
&= \frac{\ell'(t)}{\ell(t)} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 y d(|w_y|^2) \\
&= \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{2\ell(t)} |w_y(1, t)|^2.
\end{aligned}$$

We add both parts to obtain

$$\begin{aligned}
\frac{dF(t)}{dt} &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y(y, t)|^2 dt - \frac{1}{2} |w_y(1, t)|^2 \frac{\ell'(t)}{\ell(t)} \\
&= \frac{\ell'(t)}{\ell(t)} \left(F(t) - \frac{1}{2} |w_y(1, t)|^2 \right),
\end{aligned}$$

By Variation of constants, we get an explicit solution:

$$(3.5) \quad F(t) = \ell(t) F(0) - \ell(t) \int_0^t \frac{\ell'(s)}{2\ell(s)^2} |w_y(1, s)|^2 ds$$

One easily obtains an upper bound, namely $F(t) \leq F(0)\ell(t)$. For the lower bound, we use the Poincaré (or Wirtinger) inequality on $[0, 1]$ to obtain,

$$(3.6) \quad F(t) = \frac{1}{2} \int_0^1 |w_y(y, t)|^2 dy \geq \frac{\pi^2}{2} \int_0^1 |w(y, t)|^2 dy = \frac{\pi^2}{\ell(t)} E(0)$$

□

3.3 Admissibility of Neumann observations at the boundary

Proof of Theorem 2.2. We take the function $q(y, t) = q(y)\ell(t)$ on $(0, 1)$ satisfying $q(1) = 0$ and $q(0) = 1$. By Proposition 3.2, we have

$$\begin{aligned}
&\operatorname{Re} \left(\int_0^\tau \frac{1}{2\ell(t)^2} q(0, t) |w_y(0, t)|^2 dt \right) = \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i q \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \\
&- \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 q_y dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y w}{2\ell(t)} q_{yy} dy dt \right) \\
&- \operatorname{Re} \left(\int_0^\tau \int_0^1 i y \ell'(t) q |w_y|^2 dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} q_y dy dt \right) \\
&- \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 i w q \ell'(t) \overline{w_y} dy dt \right)
\end{aligned}$$

Therefore, we have

$$\int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt \leq A + B + C + D + E + F,$$

where we estimate all five terms separately. Concerning A , we separate the products in the real part by $ab \leq \frac{1}{2}(a^2 + b^2)$, then use Lemmata 3.3 and 3.4 to obtain

$$\begin{aligned} A &= \left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i q \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \right| \\ &\leq \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 \ell(\tau) |w(y, \tau)|^2 + |w(y, 0)|^2 + \ell(\tau) |w_y(y, \tau)|^2 + |w_y(y, 0)|^2 dy \right) \\ &= \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 2|w(y, 0)|^2 + (1 + \ell(\tau)^2) |w_y(y, 0)|^2 dy \right) \\ &\leq \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 (2\pi^2 + 1 + \ell(\tau)^2) |w_y(y, 0)|^2 dy \right). \end{aligned}$$

The second term is easily estimated by Lemma 3.3:

$$\begin{aligned} B &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 q_y dy dt \right) \right| \leq \|q_y\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 dy dt \\ &\leq \|q_y\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 |w_y(y, 0)|^2 dy dt \\ &= \|q_y\|_{L_\infty(0,1)} \tau \int_0^1 |w_y(y, 0)|^2 dy \end{aligned}$$

Part C is decoupled by Cauchy-Schwarz and then estimated using Lemma 3.4 as follows:

$$\begin{aligned} C &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y w}{2\ell(t)} q_{yy} dy dt \right) \right| \leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \frac{|w_y w|}{2\ell(t)} dy dt \right) \\ &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{1}{2\ell(t)} \left(\int_0^1 |w(y, t)|^2 dy \right)^{1/2} \left(\int_0^1 |w_y(y, t)|^2 dy \right)^{1/2} dt \right) \\ &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi}{2\ell(t)} \left(\int_0^1 |w_y(y, t)|^2 dy \right) dt \right) \\ &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi}{2} dt \right) \left(\int_0^1 |w_y(y, 0)|^2 dy \right) \\ &= \|q_{yy}\|_{L_\infty(0,1)} \frac{\pi\tau}{2} \left(\int_0^1 |w_y(y, 0)|^2 dy \right). \end{aligned}$$

For the forth part, we use Lemma 3.4 to obtain

$$\begin{aligned} D &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 i y \ell'(t) q |w_y(y, t)|^2 dy dt \right) \right| \leq \|q\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 \ell'(t) |w_y(y, t)|^2 dy dt \\ &\leq \|q\|_{L_\infty(0,1)} \left(\int_0^\tau \ell'(t) \ell(t) dt \right) \left(\int_0^1 |w_y(y, 0)|^2 dy \right) \\ &= \|q\|_{L_\infty(0,1)} \frac{\ell(\tau)^2 - 1}{2} \left(\int_0^1 |w_y(y, 0)|^2 dy \right). \end{aligned}$$

The estimate for fifth part E is similar to part C :

$$E = \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} q_y dy dt \right) \right| \leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \frac{1}{2} \ell'(t) |w_y| |\overline{w}| dy dt \right)$$

$$\begin{aligned}
&\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\ell'(t)}{2} \left(\int_0^1 |w(y,t)|^2 dy \right)^{1/2} \left(\int_0^1 |w_y(y,t)|^2 dy \right)^{1/2} dt \right) \\
&\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi \ell'(t)}{2} \left(\int_0^1 |w_y(y,t)|^2 dy \right) dt \right) \\
&\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi \ell'(t) \ell(t)}{2} dt \right) \left(\int_0^1 |w_y(y,0)|^2 dy \right) \\
&= \|q_y\|_{L_\infty(0,1)} \frac{\pi}{4} (\ell(\tau)^2 - 1) \left(\int_0^1 |w_y(y,0)|^2 dy \right).
\end{aligned}$$

Finally, the last part F is treated like part C and E :

$$\begin{aligned}
F &= \frac{1}{2} \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 i w q \ell'(t) \overline{w_y} dy dt \right) \right| \leq \frac{1}{2} \|q\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \ell'(t) |\overline{w_y}| |w| dy dt \right) \\
&\leq \|q\|_{L_\infty(0,1)} \frac{\pi}{4} (\ell(\tau)^2 - 1) \left(\int_0^1 |w_y(y,0)|^2 dy \right).
\end{aligned}$$

Summing up all five estimates, we obtain

$$(3.7) \quad \int_0^\tau \frac{1}{2\ell(t)} |w_y(0,t)|^2 dt \leq C_1(\tau) \|w_0\|_{H_0^1(0,1)}^2$$

where the constant $C_1(\tau)$ is given by

$$\begin{aligned}
(3.8) \quad C_1(\tau) &= \frac{(3 + \pi^2) \ell(\tau)^2 + \pi^2 - 1}{4} \|q\|_{L_\infty(0,1)} + \left(\tau + \frac{\pi}{4} (\ell(\tau)^2 - 1) \right) \|q_y\|_{L_\infty(0,1)} \\
&\quad + \frac{\pi \tau}{2} \|q_{yy}\|_{L_\infty(0,1)}
\end{aligned}$$

Replacing $w_y(0,t) = \ell(t)u_x(0,t)$ in (3.7) yields the admissibility inequality:

$$\int_0^\tau |u_x(0,t)|^2 dt \leq \int_0^\tau \ell(t) |u_x(0,t)|^2 dt \leq 2C_1(\tau) \|u_0\|_{H_0^1(0,1)}^2$$

The second admissibility estimate follows the same lines, using $q(y,t) = q(y)\ell(t)$ on $(0,1)$ with $q(0) = 0$ and $q(1) = 1$. \square

3.3.1 Neumann Observability at the Boundary

Recall the following lemma

Lemma 3.5. *Let E_1, E_2 and E_3 be the Hilbert spaces. We consider the continuous linear operators $T : E_1 \rightarrow E_2$, $K : E_1 \rightarrow E_3$ and $L : E_1 \rightarrow E_1$ such that K is compact, L is bounded below and:*

$$(3.9) \quad \|Lu\|_{E_1} \approx \|Tu\|_{E_2} + \|Ku\|_{E_3}$$

Then the kernel of A has finite dimension and $\|Lu\|_{E_1} \approx \|Tu\|_{E_3}$

Proof. A similar proof can be found in [38, Lemma 1 pp.1] where we just replace u by Lu . \square

Proof of Theorem 2.2. For all τ satisfying $2\tau + \pi(1 - \ell(\tau)^2) > 0$, we choose two positive constants $\eta(\tau)$ and $\delta(\tau)$ such that:

$$(3.10) \quad \eta(\tau) + \delta(\tau) < \frac{4}{1 + \ell(\tau)^3} \left(\tau - \frac{\pi}{2} (\ell(\tau)^2 - 1) \right)$$

We choose $q(y) = (1 - y)\ell(t)$ where $y \in (0, 1)$. Proposition 3.2 is then equivalent to:

$$(3.11) \quad \begin{aligned} \int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt &= \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 dy dt - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i(1 - y) \ell'(t) \overline{w_y} w dy dt \right) \\ &\quad + \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i(1 - y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) + \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} dy dt \right) \end{aligned}$$

Taking the three last formula of the right hand side to the left, then taking the absolute to get:

$$\begin{aligned} \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 dy dt &\leq \int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt + \left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i(1 - y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \right| \\ &\quad + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i(1 - y) \ell'(t) \overline{w_y} w dy dt \right) \right| \\ &\quad + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} dy dt \right) \right| \end{aligned}$$

The sum of third and fourth terms in the right hand side of above formula can be estimated as:

$$\begin{aligned} &\left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i(1 - y) \ell'(t) \overline{w_y} w dy dt \right) \right| + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} dy dt \right) \right| \\ &\leq \frac{1}{2} \int_0^\tau \int_0^1 \ell'(t) |w_y \overline{w}| dy dt + \frac{1}{2} \int_0^\tau \int_0^1 \ell'(t) |\overline{w_y} w| dy dt \\ &\leq \int_0^\tau \ell'(t) \left(\int_0^1 |w|^2 dy \right)^{1/2} \left(\int_0^1 |w_y|^2 dy \right)^{1/2} dt \\ &\leq \int_0^\tau \pi \ell'(t) \left(\int_0^1 |w_y|^2 dy \right) dt \end{aligned}$$

Due to the energy estimate in lemma 3.3 and 3.4, we have the upper bound for the second term:

$$\begin{aligned} &\left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i(1 - y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \right| \\ &\leq \frac{1}{4} \int_0^1 \left(\frac{|w(y, 0)|^2}{\eta(\tau)} + \frac{|w(y, \tau)|^2}{\eta(\tau)} + \eta(\tau) |w_y(y, 0)|^2 + \eta(\tau) \ell(\tau)^2 |w_y(y, \tau)|^2 \right) dy \\ &\leq \frac{1}{4\eta(\tau)} \left(\frac{1}{\ell(\tau)} + 1 \right) \int_0^1 |w(y, 0)|^2 dy + \frac{(1 + \ell(\tau)^3) \eta(\tau)}{4} \int_0^1 |w_y(y, 0)|^2 dy \end{aligned}$$

As a result, we combine these estimation and use (3.5) to obtain:

$$\begin{aligned} &\int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt + \frac{1}{4\eta(\tau)} \left(\frac{1}{\ell(\tau)} + 1 \right) \int_0^1 |w(y, 0)|^2 dy \\ &\geq \int_0^\tau \int_0^1 \left(\frac{1}{\ell(t)} - \pi \ell'(t) \right) |w_y(y, t)|^2 dy dt - \frac{(1 + \ell(\tau)^3) \eta(\tau)}{4} \int_0^1 |w_y(y, 0)|^2 dy \\ &= \left(\int_0^\tau (1 - \pi \ell'(t) \ell(t)) dt - \frac{(1 + \ell(\tau)^3) \eta(\tau)}{4} \right) \left(\int_0^1 |w_y(y, 0)|^2 dy \right) \\ &\quad - \int_0^\tau (1 - \pi \ell'(t) \ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 ds dt \end{aligned}$$

$$\begin{aligned}
&= \left(\tau + \frac{\pi}{2}(1 - \ell(\tau)^2) - \frac{(1 + \ell(\tau)^3)\eta(\tau)}{4} \right) \left(\int_0^1 |w_y(y, 0)|^2 dy \right) \\
&\quad - \int_0^\tau (1 - \pi \ell'(t)\ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 ds dt \\
&\geq \frac{(1 + \ell(\tau)^3)\delta(\tau)}{4} \left(\int_0^1 |w_y(y, 0)|^2 dy \right) - \int_0^\tau (1 - \pi \ell'(t)\ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 ds dt
\end{aligned}$$

where the last inequality come from (3.10). Therefore, there exist the constants A_τ and B_τ such that:

$$(3.12) \quad \int_0^1 |w_y(y, 0)|^2 dy \leq A_\tau \int_0^\tau (|w_y(0, t)|^2 + |w_y(1, t)|^2) dt + B_\tau \int_0^1 |w(y, 0)|^2 dy$$

It is sufficient to prove that there exist a constant $K > 0$ such that

$$(3.13) \quad \int_0^1 |w(y, 0)|^2 dy \leq K \left(\int_0^\tau |w_y(0, t)|^2 dt + \int_0^\tau |w_y(1, t)|^2 dt \right)$$

Let us denote the operator T from $H_0^1(0, \tau)$ to $L_2(0, \tau) \times L_2(0, \tau)$ and the operator K from $H_0^1(0, 1)$ to $L_2(0, 1)$ that maps:

$$(3.14) \quad (Tw)(t) = (w_y(0, t), w_y(1, t))$$

$$(3.15) \quad (Kw)(y) = w(y, 0)$$

From admissibility and (3.12), we have:

$$(3.16) \quad a_\tau \|Tw\|_{L_2}^2 + b_\tau \|Kw\|_{L_2}^2 \leq \|w_0\|_{H_0^1}^2 \leq A_\tau \|Tw\|_{L_2}^2 + B_\tau \|Kw\|_{L_2}^2$$

It is easy to see that K is compact operator due to Rellich's embedding lemma. In order to use the unique-compactness lemma 3.5 for $L = K$, we need to check that T is injective. Observe that $Tw = 0$ means that w satisfies (S_{fixed}) with Dirichlet conditions and zero Neumann derivative. It is well known that w vanishes in this case, see for example [39, Theorem 3] or [12, Corollary 6.1]. As a consequence,

$$c_\tau \|Tw\|_{L_2}^2 \leq \|w_0\|_{H_0^1}^2 \leq C_\tau \|Tw\|_{L_2}^2$$

for some constants $c(\tau), C(\tau) > 0$. □

3.4 Results for linear moving walls

Recall the Doescher-Rice representation formula (2.5) that yields for $t = 0$

$$(3.17) \quad u(x, 0) = \sqrt{2} \sum_{n=1}^N a_n e^{\frac{i\pi x^2}{4}} \sin(n\pi x),$$

and denote by

$$u_n(x, t) := \sqrt{\frac{2}{\ell(t)}} \sin\left(\frac{n\pi x}{\ell(t)}\right).$$

For all fixed $t > 0$, the functions $(u_n(\cdot, t))_{n \geq 1}$ form an orthonormal basis in $L_2(0, \ell(t))$, since the change of variable $y = \frac{x}{\ell(t)}$ reduces $u_n(\cdot, t)$ to the standard trigonometric system on $L_2([0, 1])$.

Lemma 3.6. *For all finitely supported sequences (a_n) we have the following relation between (a_n) and the norms of the initial data u_0 .*

$$\|u(x, 0)\|_{L_2(0,1)}^2 = \sum_{n=1}^{+\infty} |a_n|^2, \quad \|u(x, 0)\|_{H_0^1(0,1)}^2 \sim \sum_{n=1}^{+\infty} |a_n|^2 n^2$$

Proof. Observe that

$$\begin{aligned} \|e^{-\frac{i\varepsilon x^2}{4}} u_N(x)\|_{L_2(0,1)}^2 &= \|u_N(x)\|_{L_2(0,1)}^2 = 2 \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx \\ &= 2 \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx = \sum_{n=1}^{\infty} |a_n|^2. \end{aligned}$$

Since (a_n) is a finite sequence we may interchange differentiation and summation and obtain

$$\frac{d}{dx} u(x) = \sqrt{2} \sum_{n=1}^N a_n e^{\frac{i\varepsilon x^2}{4}} (ix \frac{\varepsilon}{2} \sin(n\pi x) + n\pi \cos(n\pi x))$$

so that, squaring real and imaginary parts, we find

$$\begin{aligned} \|u(x)\|_{H_0^1(0,1)}^2 &= 2 \int_0^1 \left| \sum_{n=1}^N a_n n\pi \cos(n\pi x) \right|^2 dx + 2 \int_0^1 \left| \sum_{n=1}^N a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 dx \\ &= \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + 2 \int_0^1 \left| \sum_{n=1}^N a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 dx \\ &\leq \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx \\ &= \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \sum_{n=1}^N |a_n|^2 \leq C(\varepsilon) \sum_{n=1}^N |a_n|^2 n^2 \quad \square \end{aligned}$$

Lemma 3.7. *Let $\varepsilon \in (0, \frac{\pi}{2})$ and $\tau = \frac{2}{\pi-2\varepsilon}$, then the functions $b_n(t) = \frac{\sqrt{\pi}}{\sqrt{2\ell(t)}} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}}$ for $n \geq 1$ form an orthonormal system in $L_2(0, \tau)$.*

Proof. Note that $(\frac{t}{\ell(t)})' = \frac{\ell(t) - t\ell'(t)}{\ell(t)^2} = \frac{1}{\ell(t)^2}$. Therefore, the obvious change of variable $x = \frac{t}{\ell(t)}$ reduces f_n to a standard trigonometric function on $[0, \frac{\tau}{\ell(\tau)}]$. Observe that $\frac{\tau}{\ell(\tau)} = \frac{2}{\pi-2\varepsilon} (1 + \frac{2\varepsilon}{\pi-2\varepsilon})^{-1} = \frac{2}{\pi}$. Now orthonormality easily follows. \square

Observe that the above sequence $\{b_n(t)\}_{n \geq 1}$ is not an orthonormal basis. Indeed, with $f(t) = \frac{\sqrt{\pi}}{\sqrt{2\ell(t)}} e^{3i\pi^2 \frac{t}{\ell(t)}}$, we have $\langle f(t), b_n(t) \rangle = 0$ for all $n \in \mathbb{N}$.

3.4.1 Neumann observation at the Boundary

Proof of Theorem 2.3. We start considering only the first term at $x = 0$. As in the proof of Lemma 3.6 we consider for a moment only initial data associated with finitely supported sequences (a_n) . Differentiating the representation formula (2.5) u term by term yields

$$u_x(0, t) = \sum_{n=1}^{+\infty} a_n \left(\frac{2}{\ell(t)} \right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)},$$

and therefore

$$\|u_x(0, t)\|_{L_2(0, \tau)}^2 = \int_0^\tau \frac{2\pi^2}{\ell(t)^3} \left| \sum_{n=1}^{+\infty} na_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 dt.$$

Using the monotonicity of $\ell(t)$ in $[0, \tau]$, we have $\frac{2\pi^2}{\ell(\tau)} J \leq \|u_x(0, \cdot)\|_{L_2(0, \tau)}^2 \leq 2\pi^2 J$ where

$$J = \int_0^\tau \left| \sum_{n=1}^{+\infty} na_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 \frac{dt}{\ell(t)^2}.$$

This allows to focus only on the integral J , where we abbreviate $b_n = na_n e^{-i\pi^2 n^2/\varepsilon}$ and make a change of variable $\xi = \frac{-1}{\ell(t)} + \frac{1}{2}(\frac{1}{\ell(0)} + \frac{1}{\ell(\tau)})$. Letting $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$, the above double inequality rewrites as

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-i\frac{\pi^2 n^2}{\varepsilon} \xi} \right|^2 d\xi \approx \|u_x(0, t)\|_{L_2(0, \tau)}^2$$

The sequence $\lambda_n = \frac{\pi^2 n^2}{\varepsilon}$ satisfies the hypotheses of [40, Theorem 3.1 and Corollary 3.3] so that, for all $k > \frac{3}{2}\pi^2$ and $r = \varepsilon/\pi^2$

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-i\frac{\pi^2 n^2}{\varepsilon} \xi} \right|^2 d\xi \gg e^{-\frac{2k}{r\tau}} \sum_{n=1}^{+\infty} |b_n|^2 = e^{-\frac{2k}{r\tau}} \sum_{n=1}^{+\infty} |na_n|^2.$$

On the other hand side, if $T \in [m\frac{\varepsilon}{\pi}, (m+1)\frac{\varepsilon}{\pi})$, we have by periodicity and Parseval's identity

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-in^2 \frac{\pi^2}{\varepsilon} \xi} \right|^2 d\xi \leq \int_{-(m+1)\frac{\varepsilon}{\pi}}^{(m+1)\frac{\varepsilon}{\pi}} \left| \sum_{n=1}^{+\infty} b_n e^{-in^2 \frac{\pi^2}{\varepsilon} \xi} \right|^2 d\xi = (m+1) \sum_{n=1}^{+\infty} |b_n|^2.$$

We conclude by Lemma 3.6 that

$$c(\varepsilon) \|u_0\|_{H_0^1(0,1)}^2 \leq \|u_x(0, t)\|_{L_2(0, \tau)}^2 \leq C(\varepsilon) \|u_0\|_{H_0^1(0,1)}^2.$$

This inequality being true for all u_0 leading to finitely supported sequences (a_n) , it is true for any $u_0 \in H_0^1(0, 1)$ by density.

For second term at $x = \ell(t)$, we see for finitely supported sequences (a_n) that

$$u_x(\ell(t), t) = \sum_{n=1}^{+\infty} (-1)^n a_n \left(\frac{2}{\ell(t)}\right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)} e^{i\frac{\varepsilon}{4}\ell(t)}$$

Taking the L_2 -norm, one get the equivalent between $\|u_x(\ell(t), t)\|_{L_2}$ and $\|u_x(0, t)\|_{L_2}$

$$\begin{aligned} \|u_x(\ell(t), t)\|_{L_2(0, \tau)}^2 &= \int_0^\tau \left| \sum_{n=1}^{+\infty} (-1)^n a_n \left(\frac{2}{\ell(t)}\right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)} e^{i\frac{\varepsilon}{4}\ell(t)} \right|^2 dt \\ &= \int_0^\tau \frac{2\pi^2}{\ell(t)} \left| \sum_{n=1}^{+\infty} ((-1)^n na_n) e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 \frac{dt}{\ell(t)^2}. \end{aligned}$$

Clearly, the rest proof follows the lines above. \square

3.4.2 Internal Point Observability

Proof of Theorem 2.5. Since $\ell(t) \geq 1$ for all t ,

$$\int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt \geq \int_0^\tau \frac{2}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt.$$

By definition, $\sin\left(\frac{n\pi a}{\ell(t)}\right) = \frac{1}{2i}(\exp(i\frac{n\pi a}{\ell(t)}) - \exp(-i\frac{n\pi a}{\ell(t)}))$. Therefore,

$$\begin{aligned} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) &= \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \left(e^{i\frac{n\pi a}{\ell(t)}} - e^{-i\frac{n\pi a}{\ell(t)}} \right) \\ &= \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{\varepsilon}} \left(e^{-\frac{i\pi^2 n^2}{\varepsilon \ell(t)} + i\frac{n\pi a}{\ell(t)}} - e^{-\frac{i\pi^2 n^2}{\varepsilon \ell(t)} - i\frac{n\pi a}{\ell(t)}} \right) \end{aligned}$$

For $n \in \mathbb{Z}$, we extend the series by $a_n = a_{-n}$, and $\lambda_n = \frac{\pi^2 n^2}{\varepsilon} + \text{sign}(n)n\pi a$. The sequence $\lambda_n = \frac{\pi^2 n^2}{\varepsilon}$ is regular and satisfies the hypotheses of [40, Theorem 3.1] with $r = \frac{\varepsilon}{\pi^2}$ and $C = a\pi$. We follow the lines of the proof of Theorem 2.3: changing the variable $\xi = \frac{-1}{\ell(t)}$ gives with the notation $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$,

$$\int_0^\tau \frac{1}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt = \frac{1}{\varepsilon} \int_{-T/2}^{+T/2} \left| \sum_{n \in \mathbb{Z}} e^{-\frac{i\pi^2 n^2}{\varepsilon}} a_n e^{i\lambda_n \xi} \right|^2 d\xi$$

we write $b_n = e^{-\frac{i\pi^2 n^2}{\varepsilon}} a_n$ and use [40, Corollary 3.3] with $k > \frac{3\pi^2}{2}$:

$$\frac{1}{\varepsilon} \int_{-T}^T \left| \sum_{n \in \mathbb{Z}} a_n e^{-\frac{i\pi^2 n^2}{\varepsilon}} e^{-i\lambda_n \xi} \right|^2 d\xi \gg e^{-\frac{2k}{rT}} \sum_{n \in \mathbb{Z}} \left| a_n e^{-\frac{i\pi^2 n^2}{\varepsilon}} \right|^2 \geq e^{-\frac{2k}{rT}} \sum_{n=1}^{+\infty} |a_n|^2. \quad \square$$

For the upper estimate, we use similar method as in theorem (2.3). More precisely,

$$\|u(a, t)\|_{L_2} \leq \int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 dt \lesssim (m+1) \sum_{n=1}^{+\infty} |a_n|^2$$

where m be the integer number such that $\frac{\pi\varepsilon}{T} \in [m, m+1]$ with $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$.

3.4.3 L_p -admissibility and observability

Proof of Theorem 2.7. The upper estimate yielding $K_p(\tau)$ is obtained by interpolation of the two upper estimates in Theorem 2.3. We are left with the lower estimate. Since $u \in H_0^1$, $(na_n) \in \ell_2$, and so $(a_n) \in \ell_1$ by the Cauchy-Schwarz inequality. Let $p \in (0, 2)$ and let $\theta = \frac{2}{4-p} \in (0, 1)$ which is chosen to satisfy $p\theta + 4(1-\theta) = 2$. By Hölder's inequality we then have

$$\begin{aligned} (3.18) \quad \int_0^\tau |u(a, t)|^2 dt &= \int_0^\tau |u(a, t)|^{p\theta} \cdot |u(a, t)|^{4(1-\theta)} dt \\ &\leq \left(\int_0^\tau |u(a, t)|^p dt \right)^\theta \cdot \left(\int_0^\tau |u(a, t)|^4 dt \right)^{1-\theta} \end{aligned}$$

From trivial argument on boundedness of $\sin\left(\frac{n\pi a}{\ell(t)}\right)$ and $e^{\frac{i\varepsilon a^2}{4\ell(t)} - i\pi^2 n^2 \frac{t}{\ell(t)}}$:

$$|u(a, t)|^2 = \left| \sum_{n=1}^{+\infty} a_n e^{\frac{i\varepsilon a^2}{4\ell(t)} - i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 \leq \left(\sum_{n=1}^{+\infty} |a_n| \right)^2$$

Combining with the estimate (3.18), one get:

$$(3.19) \quad \int_0^\tau |u(a, t)|^4 dt \leq \left(\sum_{n=1}^{+\infty} |a_n| \right)^2 \left(\int_0^\tau |u(a, t)|^2 dt \right)$$

From inequalities (3.18) and (3.19) and Theorem (2.5) we deduce now

$$\begin{aligned} \int_0^\tau |u(a, t)|^p dt &\geq \left(\int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left(\int_0^\tau |u(a, t)|^4 dt \right)^{\frac{\theta-1}{\theta}} \\ &\geq \left(\int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left(\sum_{n=1}^{+\infty} |a_n| \right)^{\frac{2(\theta-1)}{\theta}} \left(\int_0^\tau |u(a, t)|^2 dt \right)^{\frac{\theta-1}{\theta}} \\ &\geq k \left(\sum_{n=1}^{+\infty} |a_n|^2 \right) \left(\sum_{n=1}^{+\infty} |na_n|^2 \right)^{\frac{\theta-1}{\theta}} \geq k \|u_0\|_{L_2(0,1)}^2 \|u_0\|_{H_0^1}^{\frac{\theta-1}{\theta}}. \end{aligned}$$

Since $\frac{\theta-1}{\theta} = \frac{p-2}{2}$, the result follows. \square

4 Boundary controllability of dual problem

Since we have already stated several theorems that can be interpreted as exact observation we will briefly sketch the duality theory that allows to rephrase these assertions in terms of exact control, then the solution z to adjoint problem

$$(4.1) \quad z'(t) = -A(t)^* z(t) - C(t)^* C(t) w(t) \quad z(\tau) = 0$$

satisfies $\langle w_0, z(0) \rangle = - \int_0^\tau \frac{d}{dt} \langle w(t), z(t) \rangle dt = \int_0^\tau \|C(t)w(t)\|^2 dt$ by injection of the respective differential equations of w and z . Hence exact observability implies that the Gramian $Q : w_0 \mapsto z(0)$ satisfies $\|Qw_0\| \|w_0\| \geq \langle w_0, Qw_0 \rangle \geq \delta \|w_0\|$ to the effect that Q has closed image. Moreover, if $Q^*w_0 = 0$, taking scalar product with w_0 reveals $w_0 = 0$, so Q^* is injective and hence Q has dense range. By the open mapping theorem, Q is therefore an isomorphism on X . This means that the adjoint problem (4.1) can be steered to any state $z(0) \in X$ by an appropriate choice of the initial value w_0 . Indeed, for $u, v \in D(A(t))$ we have

$$\begin{aligned} \langle A(t)u, v \rangle_X &= \left\langle \frac{i}{\ell(t)^2} u_{yy} + \frac{\ell'(t)}{\ell(t)} y u_y, v \right\rangle_X = \int_0^1 \frac{i}{\ell(t)^2} u_{yy} \bar{v} dy + \int_0^1 \frac{\ell'(t)}{\ell(t)} y u_y \bar{v} dy \\ (\text{int. by parts}) &= -\frac{i}{\ell(t)^2} \int_0^1 u_y \bar{v}_y dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (y u \bar{v}_y + u \bar{v}) dy \\ (\text{int. by parts}) &= \frac{i}{\ell(t)^2} \int_0^1 u \bar{v}_{yy} dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (y u \bar{v}_y + u \bar{v}) dy \\ &= - \left\langle u, \frac{i}{\ell(t)^2} v_{yy} + \frac{\ell'(t)}{\ell(t)} y v_y \right\rangle - \left\langle u, \frac{\ell'(t)}{\ell(t)} v \right\rangle = \left\langle u, - \left(A(t) + \frac{\ell'(t)}{\ell(t)} \right) v \right\rangle \end{aligned}$$

It turns out that in our case $A(t)^* = -A(t) - \frac{\ell'(t)}{\ell(t)}$. So exact observation of the Schrödinger equation (S_{moving}) can be reformulated as exact control for the Schrödinger equation with zero final time. We turn back to these ideas after stating our first theorem. In the case of linear moving $\ell(t) = 1 + \varepsilon t$, let $C(t) : D(A(t)) \rightarrow \mathbb{C}$ be given by $C(t)(\varphi) := \varphi_y(b)$ where $b \in \{0, 1\}$. The (lower) estimate in theorems 2.3 and 2.2 then reformulates as exact observability of $C(t)$ for the non-autonomous Cauchy problem (2.1). Some care has to be taken since $C(t)$ is unbounded on X . Indeed, $C(t)^* : \mathbb{C} \rightarrow D(A(t))'$ is given by $C(t)^* \alpha = -\alpha \frac{d}{dy} \delta_{y=b}$, then we obtain exact controllability of (4.1) in a distributional sense:

$$z_t = \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} y z_y + \frac{\ell'(t)}{\ell(t)} z + w_y(b, t) \frac{d}{dy} \delta_{y=b} \quad \text{and} \quad z(y, \tau) = 0$$

Multiplying with a test function $\eta \in D((0,1))$, and integrating on $[0,1]$ we obtain by partial integration

$$\begin{aligned} \int_0^1 z_t \eta(y) dy &= \int_0^1 \left(\frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} (yz)_y \right) \eta(y) dy - w_y(b,t) \eta'(b) \\ &= \int_0^1 \left(\frac{i}{\ell(t)^2} z \eta''(y) - \frac{\ell'(t)}{\ell(t)} y z \eta'(y) \right) dy + \left(\frac{i}{\ell(t)^2} z(b,t) - w_y(b,t) \right) \eta'(b) \end{aligned}$$

This is possible for any test function η only if the point evaluation vanishes. The dual statement of the lower estimate in theorems 2.3 and 2.2 is thus exact controllability of a Schrödinger equation with Dirichlet control on the right boundary,

$$(4.2) \quad \begin{cases} z_t &= \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} y z_y + \frac{\ell'(t)}{\ell(t)} z & (y,t) \in (0,1) \times (0,\tau) \\ z(\bar{b},t) &= 0 & \{\bar{b}\} \cup \{b\} = \{0,1\}, t \geq 0 \\ z(b,t) &= -i\ell(t)^2 w_y(b,t) & t \geq 0 \\ z(y,\tau) &= 0 & y \in [0,1] \end{cases}$$

We reverse back to the moving boundary problem by taking $x = \ell(t)y$ and $h(x,t) = z(y,t)$. Then the problem can be written as:

$$(4.3) \quad \begin{cases} ih_t + h_{xx} - i \frac{\ell'(t)}{\ell(t)} h &= 0 & (x,t) \in (0,\ell(t)) \times (0,\tau) \\ h(\ell(t),t) &= 0 & t \geq 0 \\ h(0,t) &= -i\ell(t)^3 u_x(0,t) & t \geq 0 \\ h(x,\tau) &= 0 & x \in [0,\ell(t)] \end{cases}$$

or

$$(4.4) \quad \begin{cases} ih_t + h_{xx} - i \frac{\ell'(t)}{\ell(t)} h &= 0 & (x,t) \in (0,\ell(t)) \times (0,\tau) \\ h(0,t) &= 0 & t \geq 0 \\ h(\ell(t),t) &= -i\ell(t)^3 u_x(\ell(t),t) & t \geq 0 \\ h(x,\tau) &= 0 & x \in [0,\ell(t)] \end{cases}$$

In general situation of $\ell(t)$ satisfying condition (2.4), one take $C(t) : D(A(t)) \rightarrow \mathbb{C} \times \mathbb{C}$ be given by $C(t)(\varphi) := (\varphi_y(0), \varphi_y(1))$. Therefore, the dual operator $C(t)^* : \mathbb{C} \times \mathbb{C} \rightarrow D(A(t))'$ is given by $C(t)^*(\alpha, \beta) = -\alpha \frac{d}{dy} \delta_{y=0} - \beta \frac{d}{dy} \delta_{y=1}$. Using similar argument, we obtain exact controllability of a Schrödinger equation with Dirichlet control applied on both of boundaries

$$(4.5) \quad \begin{cases} ih_t + h_{xx} - i \frac{\ell'(t)}{\ell(t)} h &= 0 & (x,t) \in (0,\ell(t)) \times (0,\tau) \\ h(0,t) &= -i\ell(t)^3 u_x(0,t) & t \geq 0 \\ h(\ell(t),t) &= -i\ell(t)^3 u_x(\ell(t),t) & t \geq 0 \\ h(x,\tau) &= 0 & x \in [0,\ell(t)] \end{cases}$$

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