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First evidence of non-locality in real band-gap metamaterials: determining parameters in the relaxed micromorphic model

Angela Madeo¹ and Gabriele Barbagallo² and Marco Valerio d'Agostino³
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Abstract

In this paper we propose the first estimate of some elastic parameters of the relaxed micromorphic model on the basis of real experiments of transmission of longitudinal plane waves across an interface separating a classical Cauchy material (steel plate) and a phononic crystal (steel plate with fluid-filled holes). A procedure is set up in order to identify the parameters of our model by superimposing the experimentally-based profile of the reflection coefficient (plotted as function of the frequency of the traveling waves) with the analogous profile obtained via simulations based upon the relaxed micromorphic model. We end up with the determination of 5 out of 6 constitutive parameters which are featured by the relaxed micromorphic model in the isotropic case, plus the determination of the micro-inertia parameter. The sixth elastic parameter, namely the Cosserat couple modulus μ_c , still remains undetermined, since experimental data concerning the transmission properties of the considered interface for transverse incident waves are not yet available. A fundamental result of the present paper is the estimate of the non-locality intrinsically associated to the underlying microstructure of the metamaterial. As a matter of fact, we appraise that the characteristic length L_c measuring the non-locality of the considered phononic crystal is of the order of $1/3$ of the diameter of the considered fluid-filled holes.

Keywords: complete band-gaps, non-local effects, relaxed micromorphic model, generalized continuum models, multi-scale modeling

AMS 2010 subject classification: 74A10 (stress), 74A30 (nonsimple materials), 74A60 (micromechanical theories), 74E15 (crystalline structure), 74M25 (micromechanics), 74Q15 (effective constitutive equations)

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1 Introduction

Mechanical band-gap metamaterials are suitably engineered microstructured materials which are able to inhibit elastic wave propagation in specific frequency ranges due to the presence of their underlying microstructure. These frequency intervals at which wave inhibition takes place are known as frequency band-gaps and their intrinsic characteristics (characteristic values of the gap frequency, extension of the band-gap, etc.) strongly depend on the metamaterial microstructure. Such unorthodox dynamical behavior can be related to two main physical phenomena occurring at the micro-level:

- local resonance phenomena (Mie resonance): the micro-structural components, excited at particular frequencies, start oscillating independently of the matrix thus capturing the energy of the propagating wave which remains confined at the level of the microstructure. Macroscopic wave propagation thus results to be inhibited,
- micro-diffusion phenomena (Bragg scattering): when the propagating wave has wavelengths which are small enough to start interacting with the microstructure of the material, reflection and transmission phenomena occur at the micro-level that globally result in an inhibited macroscopic wave propagation.

Such resonance and micro-diffusion mechanisms (usually a mix of the two) are at the basis of both electromagnetic and elastic band-gaps (see e.g. [2, 12] and they are manifestly related to the particular microstructural topologies of the considered metamaterials. In fact, it is well known (see e.g. [2, 11, 17, 25, 27]) that the characteristics of the microstructures strongly influence the macroscopic band-gap behavior.

In recent works [14, 15] we proposed a new generalized continuum model, which we called *relaxed micromorphic* which is able to account for the onset of microstructure-related frequency band-gaps [14, 15] while remaining in the macroscopic framework of continuum mechanics. Well posedness results have already been proved for this model [10, 20]. On the basis of the results obtained in our previous works, we can claim that the relaxed micromorphic model is the only macroscopic continuum model known to date which is simultaneously able to account for

- prediction of complete band-gaps in mechanical metamaterials
- non-local effects (via the introduction of higher order derivatives of the micro distortion tensor in the strain energy density).

In [16] we presented a comprehensive study of jump conditions that can be imposed at surfaces of discontinuity of the material properties in relaxed micromorphic media, so establishing a strong basis for the systematic study of reflection and transmission phenomena in real band-gap metamaterials. In this paper, we will show that the particular constraint introduced in [16] that we called “*macro internal clamp with free microstructure*” is indeed able to reproduce real situations in which a Cauchy material (e.g. steel) is connected to a phononic crystal (e.g. a steel plate with fluid-filled holes). To give a more technologically oriented taste to our investigations, we consider here the experimental investigations presented in [12] in which transmission of longitudinal plane waves at an interface between a steel plate and a phononic crystal is studied. The two main aims of the present paper can be identified as follows:

- determine, by inverse approach, the maximum possible number of constitutive elastic parameters featured by the relaxed micromorphic model in the isotropic case by direct comparison with the results proposed in [12] and based on real experiments on specific phononic crystals,
- give a first evidence of non-local effects in band-gap metamaterials, by quantifying them through the determination of the characteristic length L_c for the phononic crystal experimentally studied in [12].

The relaxed micromorphic model is, by its own nature, a “macroscopic” model, in the sense that all the constitutive parameters introduced take into account the presence of the micro-structure in an “averaged” sense. Nevertheless, it would be interesting to validate the estimate of the parameters of the relaxed micromorphic model performed in this paper against more “homogenization-oriented” methods of the type presented in [3, 26].

The present paper is organized according to the following structure:

- In section 2 we briefly recall the bulk governing equations and the associated boundary conditions that have to be used for modeling the mechanical behavior of Cauchy continua and of relaxed micromorphic media [10, 14, 15, 20]. The hypothesis of plane wave is introduced and a short discussion concerning the behavior of the dispersion relations obtained by means of our relaxed micromorphic model is performed. No redundant details about the explicit derivation of such dispersion relations are given, for this the reader is referred to [16]. It is nonetheless explicitly pointed out that the relaxed micromorphic model simultaneously allows to describe the onset of band-gaps in mechanical metamaterials together with the possibility of non-local effects.
- In section 3 we recall some results rigorously derived in [16] concerning the conservation of total energy in relaxed micromorphic media. The explicit form of the energy fluxes is presented both in the general case and using the plane wave ansatz. For completeness, the conservation of total energy is recalled also for classical Cauchy continua.
- In section 4 a particular connection between a Cauchy medium and a relaxed micromorphic medium is introduced on the basis of the results proposed in [16]. This connection has been called “*macro internal clamp with free microstructure*” and allows continuity of macroscopic displacement at the considered interface together with free motions of the microstructure on the side of the interface occupied by the relaxed micromorphic medium. Plane wave solutions are presented for the displacement in the Cauchy side and for the displacement and the micro-distortion in the relaxed micromorphic side. The unknown amplitudes of the reflected and transmitted waves are calculated by imposing the constraint of “*macro internal clamp with free microstructure*” at the Cauchy/relaxed micromorphic interface. The presented general study is particularized to the case in which only longitudinal waves travel in the considered materials.
- In section 5 reflection and transmission coefficients at a Cauchy/relaxed-micromorphic interface are defined: they measure the percentage of the energy initially carried by the incident wave which is reflected or transmitted at the interface. The particular degenerate limit case of the relaxed micromorphic model which is obtained by setting $L_c = 0$ is then introduced and some characteristic frequencies which allow to determine the bounds of the band-gaps for longitudinal waves are defined as functions of the constitutive elastic parameters of the model. Such degenerate limit case, often referred to as *internal variable model* can be used as a first rough fitting of the relaxed micromorphic model on real experimental data but it is not able to account for non-local effects. Indeed, internal variable models are able to catch the main macroscopic features of band-gap metamaterials [24, 26]. It is then argued that the fact of switching on the parameter L_c would actually allow to perform a more refined fitting on the experimental profiles of the reflection coefficient.
- In section 6 the profile of the reflection coefficient as obtained by using our relaxed micromorphic model with $L_c = 0$ is compared to the analogous profile presented in [12]. As a result of this direct comparison with experimentally-based results, four conditions on the elastic parameters are established which allow for the determination of four constitutive parameters as function of the fifth which remains free and is calibrated in order to obtain the best fitting with the experimental profile of the reflection coefficient. Finally, the characteristic length L_c is switched on and it is tuned to perform a more refined fitting until the theoretical profile of the reflection coefficient is more precisely superimposed to the experimentally-based one. The estimated value of non-local effects is found to be $L_c \simeq 0.5 \text{ mm}$, which means that such characteristic length is almost 1/3 of the diameter of the holes embedded in the considered metamaterial. Such non-local effects, which are already non-negligible, would become more and more important if considering higher contrasts of the mechanical properties at the microscopic level. This situation could be e.g. achieved by filling adjacent holes with fluids with highly contrasted properties. Only one elastic parameter, namely the Cosserat couple modulus μ_c remains undetermined at the end of the present work. In order to estimate its value (which we know to be non-vanishing for a material showing complete band-gaps [13–15]) for the phononic crystal considered here, we would need to extend our study for longitudinal waves to the case of transverse waves. A complete determination of the whole set of constitutive coefficients for the relaxed micromorphic model is left to a forthcoming contribution.

2 Dynamic formulation of the equilibrium problem

In the micromorphic model, the kinematics is enriched with respect to classical Cauchy continua by introducing an additional tensor field of **non-symmetric micro-distortions** $P : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$, beyond the classical macroscopic displacement $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then, a **non-symmetric elastic (relative) distortion** $e = \nabla u - P$ can be defined and the modeling proceeds by obtaining the constitutive relations linking elastic-distortions to stresses and by postulating a balance equation for the micro-distortion field P . All such steps can be preferably done in a variational framework such that only energy contributions need to be defined a priori. For the dynamic case, one adds in the kinetic energy density so-called micro-inertia density contributions, acting on the time derivatives $P_{,t}$ of the micro-distortion terms.

2.1 The classical Cauchy medium

In this subsection we recall that the strain energy density W and the kinetic energy T for a classical Cauchy medium in the isotropic setting take the form⁶

$$W = \mu \|\text{sym } \nabla u\|^2 + \frac{\lambda}{2} (\text{tr}(\text{sym } \nabla u))^2, \quad T = \frac{1}{2} \rho \|u_{,t}\|^2, \quad (1)$$

where λ and μ are the classical Lamé parameters and u denotes the classical macroscopic displacement field.

The associated bulk equations of motion in strong form, obtained by a classical least action principle, take the usual form:

$$\begin{aligned} \rho u_{,tt} &= \text{Div } \sigma, & \forall x \in \Omega, \\ f := \sigma \cdot n &= 0 \quad \text{or} \quad u = u_0, & \forall x \in \partial\Omega, \end{aligned} \quad (2)$$

where n is the normal to the boundary $\partial\Omega$, and σ is the symmetric elastic stress tensor defined as:

$$\sigma(\nabla u) = 2\mu \text{sym } \nabla u + \lambda \text{tr}(\nabla u) \mathbb{1}. \quad (3)$$

Considering the case of **plane waves**, we suppose that the space dependence of all introduced kinematic fields are limited to the component x_1 of x which is the direction of propagation of the wave. With this hypothesis, see [16], the equations of motion (2)₁ become

$$\underbrace{u_{1,tt} = \frac{\lambda + 2\mu}{\rho} u_{1,11}}_{\text{longitudinal}}, \quad \underbrace{u_{2,tt} = \frac{\mu}{\rho} u_{2,11}}_{\text{transverse 2}}, \quad \underbrace{u_{3,tt} = \frac{\mu}{\rho} u_{3,11}}_{\text{transverse 3}}. \quad (4)$$

We now look for solutions of the dynamic problem (4) in the form

$$u(x, t) = \alpha e^{i(kx_1 - \omega t)}, \quad \alpha \in \mathbb{R}^3. \quad (5)$$

Considering a wave traveling in an **infinite domain** no conditions on the boundary are to be imposed and, replacing the wave form expression (5) in the bulk equation (2), we can find the standard dispersion relations for Cauchy media (see also [16]) obtaining

$$\underbrace{\omega^2 = c_l^2 k^2}_{\text{longitudinal}}, \quad \underbrace{\omega^2 = c_t^2 k^2}_{\text{transverse 2}}, \quad \underbrace{\omega^2 = c_t^2 k^2}_{\text{transverse 3}}, \quad (6)$$

where we denoted by

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_t = \sqrt{\frac{\mu}{\rho}}, \quad (7)$$

⁶Here and in the sequel we denote by the subscript $,t$ the partial derivative with respect to time of the considered field.

the characteristic speeds in classical Cauchy media of longitudinal and transverse waves, respectively. The dispersion relations can be traced in the plane (ω, k) , giving rise to the standard non-dispersive behavior for a classical Cauchy continuum (see [1, 7–9, 16]). Indeed it is easily seen that for Cauchy continua the relations (6) can be inversed as:

$$\underbrace{k = \pm \frac{1}{c_l} \omega}_{\text{longitudinal}}, \quad \underbrace{k = \pm \frac{1}{c_t} \omega}_{\text{transverse 2}}, \quad \underbrace{k = \pm \frac{1}{c_t} \omega}_{\text{transverse 3}}. \quad (8)$$

2.2 The relaxed micromorphic model

Our novel relaxed micromorphic model endows Mindlin-Eringen's representation with the second order **dislocation density tensor** $\alpha = -\text{Curl}P$ instead of the full gradient ∇P .⁷ In the isotropic case the energy reads

$$\begin{aligned} W = & \underbrace{\mu_e \|\text{sym}(\nabla u - P)\|^2 + \frac{\lambda_e}{2} (\text{tr}(\nabla u - P))^2}_{\text{isotropic elastic - energy}} + \underbrace{\mu_c \|\text{skew}(\nabla u - P)\|^2}_{\text{rotational elastic coupling}} \\ & + \underbrace{\mu_{\text{micro}} \|\text{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2} (\text{tr} P)^2}_{\text{micro - self - energy}} + \underbrace{\frac{\mu_e L_c^2}{2} \|\text{Curl} P\|^2}_{\text{isotropic curvature}}, \end{aligned} \quad (9)$$

where the parameters and the elastic stress are analogous to the standard Mindlin-Eringen micromorphic model. The model is well-posed in the statical and dynamical case including when $\mu_c = 0$, see [10, 19].

In our relaxed model the complexity of the general micromorphic model has been decisively reduced featuring basically only symmetric strain-like variables and the Curl of the micro-distortion P . However, the relaxed model is still general enough to include the full micro-stretch as well as the full Cosserat micro-polar model, see [20]. Furthermore, well-posedness results for the statical and dynamical cases have been provided in [20] making decisive use of recently established new coercive inequalities, generalizing Korn's inequality to incompatible tensor fields [5, 6, 18, 21–23].

The relaxed micromorphic model counts 6 constitutive parameters in the isotropic case $(\mu_e, \lambda_e, \mu_{\text{micro}}, \lambda_{\text{micro}}, \mu_c, L_c)$. The characteristic length L_c is intrinsically related to non-local effects due to the fact that it weights a suitable combination of first order space derivatives in the strain energy density (9). For a general presentation of the features of the relaxed micromorphic model in the anisotropic setting, we refer to [4].

As for the kinetic energy, we consider that it takes the following form

$$T = \frac{1}{2} \rho \|u_t\|^2 + \frac{1}{2} \eta \|P_t\|^2, \quad (10)$$

where ρ is the value of the averaged macroscopic mass density of the considered metamaterial, while η is the micro-inertia density. In the following numerical simulations we will suppose that the macroscopic density ρ is known, while we will deduce the micro-inertia parameter η by an inverse approach. In any case, we have checked that the value of ρ does not influence the profile of the reflection coefficient, at least for the considered range of frequencies.

Defining the elastic stress $\tilde{\sigma}$, the hyper-stress tensor m , the micro-stress s as:

$$\begin{aligned} \tilde{\sigma}(\nabla u, P) &= 2\mu_e \text{sym}(\nabla u - P) + 2\mu_c \text{skew}(\nabla u - P) + \lambda_e \text{tr}(\nabla u - P) \mathbb{I}, \\ m(\text{Curl} P) &= \mu_e L_c^2 \text{Curl} P, \quad s(P) = [2\mu_{\text{micro}} \text{sym} P + \lambda_{\text{micro}} \text{tr}(P) \mathbb{I}], \end{aligned} \quad (11)$$

the associated equations of motion in strong form, obtained by a classical least action principle take the form (see [14–16, 19])

$$\begin{aligned} \rho u_{,tt} &= \text{Div} \tilde{\sigma}, & \forall x \in \Omega, \\ \eta P_{,tt} &= \tilde{\sigma} - s - \text{Curl} m, & \forall x \in \Omega, \end{aligned} \quad (12)$$

⁷The dislocation tensor is defined as $\alpha_{ij} = -(\text{Curl} P)_{ij} = -P_{ih,k} \epsilon_{jkh}$, where ϵ is the Levi-Civita tensor.

while the associated natural and kinematical boundary condition are⁸:

$$\begin{aligned} t &:= \tilde{\sigma} \cdot n = 0 & \text{or} & & u &= u_0, & \forall x \in \partial\Omega, \\ \tau &:= m \cdot \epsilon \cdot n = 0 & \text{or} & & P \cdot \nu_1 &= p_1, \quad P \cdot \nu_2 = p_2, & \forall x \in \partial\Omega, \end{aligned} \quad (13)$$

where n and ν_i ($i = 1, 2$) are the normal and tangent vectors to the boundary $\partial\Omega$, while t and τ are the resulting internal force and double force vectors.

Our approach consists in writing the micro-distortion tensor $P \in \mathbb{R}^{3 \times 3}$ by means of its Cartan-Lie decomposition as $P = \text{dev sym } (P) + \text{skew } (P) + \frac{1}{3} \text{tr } (P) \mathbf{1}$. Therefore, defining

$$\begin{aligned} P_{[ij]} &:= (\text{skew } P)_{ij} = \frac{1}{2} (P_{ij} - P_{ji}), & P^S &:= \frac{1}{3} \text{tr } (P), \\ P_{(ij)} &:= (\text{sym } P)_{ij} = \frac{1}{2} (P_{ij} + P_{ji}), & P_{(ij)}^D &:= (\text{dev sym } P)_{ij} = P_{(ij)} - P^S \delta_{ij}. \end{aligned} \quad (14)$$

we have:

$$P_{ij} = P_{(ij)}^D + P_{[ij]} + P^S \delta_{ij} = P_{(ij)} + P_{[ij]}. \quad (15)$$

Furthermore, we define

$$P^D := P_{(11)}^D = P_{(11)} - P^S, \quad P^V := P_{(22)} - P_{(33)} = P_{22} - P_{33}. \quad (16)$$

Since it will be useful in the following, let us collect some of the new variables of our problem as

$$v_1 := (u_1, P^D, P^S), \quad v_2 := (u_2, P_{(12)}, P_{[12]}), \quad v_3 := (u_3, P_{(13)}, P_{[13]}), \quad (17)$$

and, for having a homogeneous notation, let us set

$$v_4 := P_{(23)}, \quad v_5 := P_{[23]}, \quad v_6 := P^V. \quad (18)$$

with these new variables the equations of motion (12) can be written as (see also [16]):

$$\begin{aligned} \underbrace{v_{1,tt} = A_1^R \cdot v_1'' + B_1^R \cdot v_1' + C_1^R \cdot v_1}_{\text{Longitudinal}}, & \quad \underbrace{v_{\alpha,tt} = A_\alpha^R \cdot v_\alpha'' + B_\alpha^R \cdot v_\alpha' + C_\alpha^R \cdot v_\alpha}_{\text{Transverse } \alpha}, \quad \alpha = 2, 3 \\ \underbrace{v_{4,tt} = A_4^R v_4'' + C_4^R v_4}_{\text{Uncoupled}}, & \quad \underbrace{v_{5,tt} = A_5^R v_5'' + C_5^R v_5}_{\text{Uncoupled}}, & \quad \underbrace{v_{6,tt} = A_6^R v_6'' + C_6^R v_6}_{\text{Uncoupled}}, \end{aligned} \quad (19)$$

where, from now on, $(\cdot)'$ denotes the derivative of the quantity (\cdot) with respect to x_1 and we set

$$\begin{aligned} A_1^R &= \begin{pmatrix} \frac{\lambda_e + 2\mu_e}{\rho} & 0 & 0 \\ 0 & \frac{\mu_e L_c^2}{3\eta} & -\frac{2\mu_e L_c^2}{3\eta} \\ 0 & -\frac{\mu_e L_c^2}{3\eta} & \frac{2\mu_e L_c^2}{3\eta} \end{pmatrix}, & A_\alpha^R &= \begin{pmatrix} \frac{\mu_e + \mu_c}{\rho} & 0 & 0 \\ 0 & \frac{\mu_e L_c^2}{2\eta} & \frac{\mu_e L_c^2}{2\eta} \\ 0 & \frac{\mu_e L_c^2}{2\eta} & \frac{\mu_e L_c^2}{2\eta} \end{pmatrix}, & \alpha = 2, 3, \\ B_1^R &= \begin{pmatrix} 0 & -\frac{2\mu_e}{\rho} & -\frac{3\lambda_e + 2\mu_e}{\rho} \\ \frac{4}{3} \frac{\mu_e}{\eta} & 0 & 0 \\ \frac{3\lambda_e + 2\mu_e}{3\eta} & 0 & 0 \end{pmatrix}, & B_\alpha^R &= \begin{pmatrix} 0 & -\frac{2\mu_e}{\rho} & \frac{2\mu_c}{\rho} \\ \frac{\mu_e}{\eta} & 0 & 0 \\ -\frac{\mu_c}{\eta} & 0 & 0 \end{pmatrix}, & \alpha = 2, 3, \\ C_1^R &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2(\mu_e + \mu_{\text{micro}})}{\eta} & 0 \\ 0 & 0 & -\frac{(3\lambda_e + 2\mu_e) + (3\lambda_{\text{micro}} + 2\mu_{\text{micro}})}{\eta} \end{pmatrix}, & C_\alpha^R &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2(\mu_e + \mu_{\text{micro}})}{\eta} & 0 \\ 0 & 0 & -\frac{2\mu_c}{\eta} \end{pmatrix}, & \alpha = 2, 3, \\ A_4^R &= A_5^R = A_6^R = \frac{\mu_e L_c^2}{\eta}, & C_5^R &= -\frac{2\mu_c}{\eta}, & C_4^R &= C_6^R = -\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}. \end{aligned} \quad (20)$$

⁸Here and in the sequel a central dot indicates a simple contraction between tensors of suitable order. For example, $(A \cdot v)_i = A_{ij} v_j$ and $(A \cdot B)_{ij} = A_{ih} B_{hj}$. Einstein convention of sum of repeated indexes is used throughout the paper

As done for the Cauchy continuum, we consider the case of **plane waves**, by looking for solutions of the dynamic problem in the form

$$\begin{aligned} v_1 &= \beta_1 e^{i(kx_1 - \omega t)}, & v_2 &= \beta_2 e^{i(kx_1 - \omega t)}, & v_3 &= \beta_3 e^{i(kx_1 - \omega t)}, \\ v_4 &= \beta_4 e^{i(kx_1 - \omega t)}, & v_5 &= \beta_5 e^{i(kx_1 - \omega t)}, & v_6 &= \beta_6 e^{i(kx_1 - \omega t)}. \end{aligned} \quad (21)$$

where $\beta_1, \beta_2, \beta_3 \in \mathbb{R}^3$ and $\beta_4, \beta_5, \beta_6 \in \mathbb{R}$. Replacing this wave-form in equations (19) the dispersion relations for the relaxed micromorphic medium can be obtained (see [16] for details).

It is clear that the study of dispersion relations for the relaxed micromorphic continuum is intrinsically more complicated than in the case of classical Cauchy continuum due to its enriched kinematics. In the papers [14–16] it was explicitly pointed out that the wave-numbers for uncoupled waves in the relaxed micromorphic continua can be calculated as function of the frequency ω as:

$$\underbrace{k = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_s^2}}_{\text{Uncoupled } v_4}, \quad \underbrace{k = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_r^2}}_{\text{Uncoupled } v_5}, \quad \underbrace{k = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_s^2}}_{\text{Uncoupled } v_6}, \quad (22)$$

where we set

$$\omega_s = \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}}, \quad \omega_r = \sqrt{\frac{2\mu_c}{\eta}}, \quad c_m = \sqrt{\frac{\mu_e L_c^2}{\eta}}. \quad (23)$$

These relationships can be eventually easily inversed to find ω as function of k .

As far as longitudinal and transverse waves are concerned, the expressions for the wave-numbers k which allow for non-trivial solutions are by far more complicated. We refer to [16] for the complete set-up of the eigenvalue problems which must be solved to find the explicit expressions for the wave-numbers, limiting ourselves here to denote them by $\pm k_1^1, \pm k_1^2$ for longitudinal waves and $\pm k_\alpha^1, \pm k_\alpha^2$, $\alpha = 2, 3$ for transverse waves. Of course, the computed expressions for $k_1^1, k_1^2, k_\alpha^1, k_\alpha^2$ depend on the elastic constitutive parameters of the relaxed micromorphic model appearing in (9) and on the frequency ω .

We present here the **dispersion relations** for longitudinal, transverse and uncoupled waves obtained with a non vanishing Cosserat modulus $\mu_c > 0$ (Figure 1). A **complete frequency band-gap** can be recognized in the shaded intersected domain bounded from the maximum between $\omega_l = \sqrt{\frac{\mu_{\text{micro}} + 2\lambda_{\text{micro}}}{\eta}}$ and $\omega_t = \sqrt{\frac{\mu_{\text{micro}}}{\eta}}$ and the minimum between ω_r and ω_s . The existence of the complete band gap is related to $\mu_c > 0$ via the cut-off frequency $\omega_r = \sqrt{\frac{2\mu_c}{\eta}}$ of the uncoupled waves TRO and the transverse wave TO1.

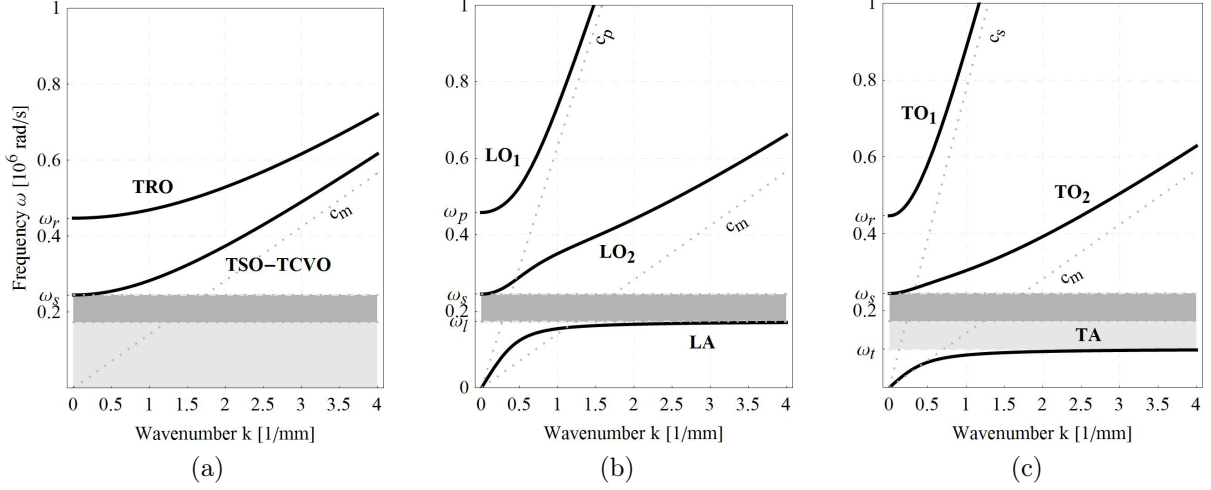


Figure 1: Dispersion relations $\omega = \omega(k)$ for the **relaxed micromorphic model** with non-vanishing Cosserat couple modulus $\mu_c > 0$. Uncoupled waves (a), longitudinal waves (b) and transverse waves (c). TRO: transverse rotational optic, TSO: transverse shear optic, TCVO: transverse constant-volume optic, LA: longitudinal acoustic, LO₁-LO₂: 1st and 2nd longitudinal optic, TA: transverse acoustic, TO₁-TO₂: 1st and 2nd transverse optic.

The relaxed micromorphic model is the only generalized continuum model (see [13, 16]) which is able to simultaneously account for:

- complete frequency band-gaps
- non-local effects involving interactions between adjacent unit cells.

Non locality is an intrinsic feature of metamaterials with heterogeneous microstructure and it is sensible that it plays a non-negligible role on the phenomena of wave propagation, reflection and transmission. We will show in the remainder of this paper, based on real experiments, that the characteristic length associated to such non-local effects may be an order of magnitude comparable with the characteristic size of the underlying microstructure, even if considering otherwise homogeneous metamaterials.

Indeed, when considering metamaterials with heterogeneous microstructures (e.g. metamaterials in which two adjacent unitary cells have strongly contrasted mechanical properties) the non-local effects may play a more and more important role on the overall behavior of the considered metamaterial.

3 Conservation of the total energy

It is known that if one considers conservative mechanical systems, like in the present paper, then conservation of total energy must be verified in the form

$$\frac{dE}{dt} + \text{div } H = 0, \quad (24)$$

where $E = T + W$ is the total energy of the considered system and H is the energy flux vector. It is clear that the explicit expressions for the total energy and for the energy flux are different depending on whether one considers a classical Cauchy model or a relaxed micromorphic one. If the expression of the total energy E is straightforward for the two mentioned cases (it suffices to look at the given expressions of T and W), the explicit expression of the energy flux H is more complicated to be obtained. The explicit expression of the energy fluxes for the Cauchy and relaxed micromorphic media have been deduced in [16], to which we refer for additional details on this subject.

3.1 The classical Cauchy medium

In classical Cauchy continua the energy flux vector H can be written as

$$H = -\sigma \cdot u_{,t}, \quad (25)$$

where the symmetric Cauchy stress tensor σ has been defined in equation (3) in terms of the displacement field. The first component of the energy flux vector given in equation (25), simplifies in the 1D case into

$$H_1 = -\dot{u}_1 [(\lambda + 2\mu) u_{1,1}] - \dot{u}_2 [\mu u_{2,1}] - \dot{u}_3 [\mu u_{3,1}]. \quad (26)$$

3.2 The relaxed micromorphic continuum

In relaxed micromorphic media, the energy flux vector H is defined as (see [16])⁹

$$H = -\tilde{\sigma}^T \cdot u_{,t} - (m^T \cdot P_{,t}) : \epsilon, \quad (27)$$

where the stress tensor $\tilde{\sigma}$ and the hyper-stress tensor m have been defined in equation (11) in terms of the basic kinematical fields and ϵ is the Levi-Civita tensor.

When considering conservation of total energy, it can be checked that the first component of the energy flux (27) can be rewritten in terms of the new variables as

$$\tilde{H}_1 = H_1^1 + H_1^2 + H_1^3 + H_1^4 + H_1^5 + H_1^6 \quad (28)$$

with

$$\begin{aligned} H_1^1 &= \dot{v}_1 \cdot \left[\begin{pmatrix} -(\lambda_e + 2\mu_e) & 0 & 0 \\ 0 & -\frac{\mu_e L_c^2}{2} & \mu_e L_c^2 \\ 0 & \mu_e L_c^2 & -2\mu_e L_c^2 \end{pmatrix} \cdot v_1' + \begin{pmatrix} 0 & 2\mu_e & (3\lambda_e + 2\mu_e) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_1 \right], \\ H_1^2 &= \dot{v}_2 \cdot \left[\begin{pmatrix} -(\mu_e + \mu_c) & 0 & 0 \\ 0 & -\mu_e L_c^2 & -\mu_e L_c^2 \\ 0 & -\mu_e L_c^2 & -\mu_e L_c^2 \end{pmatrix} \cdot v_2' + \begin{pmatrix} 0 & 2\mu_e & -2\mu_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_2 \right], \\ H_1^3 &= \dot{v}_3 \cdot \left[\begin{pmatrix} -(\mu_e + \mu_c) & 0 & 0 \\ 0 & -\mu_e L_c^2 & -\mu_e L_c^2 \\ 0 & -\mu_e L_c^2 & -\mu_e L_c^2 \end{pmatrix} \cdot v_3' + \begin{pmatrix} 0 & 2\mu_e & -2\mu_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_3 \right], \\ H_1^4 &= -2\mu_e L_c^2 (v_4)_{,1} \dot{v}_4, \quad H_1^5 = -2\mu_e L_c^2 (v_5)_{,1} \dot{v}_5, \quad H_1^6 = -\frac{\mu_e L_c^2}{2} (v_6)_{,1} \dot{v}_6. \end{aligned} \quad (29)$$

4 Interface jump conditions at a Cauchy/relaxed-micromorphic interface

In this section we present a possible choice of boundary conditions to be imposed between a Cauchy medium and a relaxed micromorphic medium. Such set of boundary conditions has been derived in [16] and allows to describe free vibrations of the microstructure at the considered interface. We will show in the remainder of this paper how this particular choice of the boundary conditions is capable to describe phenomena of wave transmission in real mechanical metamaterials. For the full presentation of the complete sets of possible connections that can be established at Cauchy/relaxed, relaxed/relaxed, Cauchy/Mindlin, Mindlin/Mindlin interfaces we refer to [16].

⁹The symbol $:$ indicates the double contraction between tensors of suitable order. For example, $(A : B) = A_{ij} B_{ji}$ and $(C : B)_i = C_{ijh} B_{hj}$.

When considering connections between a Cauchy and a relaxed micromorphic medium one can impose more kinematical boundary conditions than in the case of connections between Cauchy continua. More precisely, one can act on the displacement field u (on both sides of the interface) and also on the tangential micro-distortion P (on the side of the interface occupied by the relaxed micromorphic continuum). In what follows, we consider the “-” region occupied by the Cauchy continuum and the “+” region occupied by the micromorphic continuum, so that, accordingly, we use the following notations:

$$f = \sigma^- \cdot n^-, \quad t = \tilde{\sigma}^+ \cdot n^+, \quad \tau = \mu_e L_c^2 (\text{Curl } P^+) \cdot \epsilon \cdot n^+. \quad (30)$$

It is possible to check that considering the normal $n = (1, 0, 0)$, the normal components τ_{11}, τ_{21} and τ_{31} of the double force are identically zero. Therefore, the number of independent conditions that one can impose on the micro-distortions is 6 when considering a relaxed micromorphic model.

In this paper we focus our attention on one particular type of connection between a classical Cauchy continuum and a relaxed micromorphic one, which is sensible to reproduce the real situation in which the microstructure of the band-gap metamaterial is free to vibrate independently of the macroscopic matrix. Such particular connection guarantees continuity of the macroscopic displacement and free motion of the microstructure (which means vanishing double force) at the interface:

$$[[u]] = 0, \quad t - f = 0, \quad \tau \cdot \nu_1 = \tau \cdot \nu_2 = 0. \quad (31)$$

We explicitly remark that continuity of displacement implies continuity of the internal forces and that the conditions on the arbitrariness of micro-motions is assured by imposing that the tangent part of the double force is vanishing.

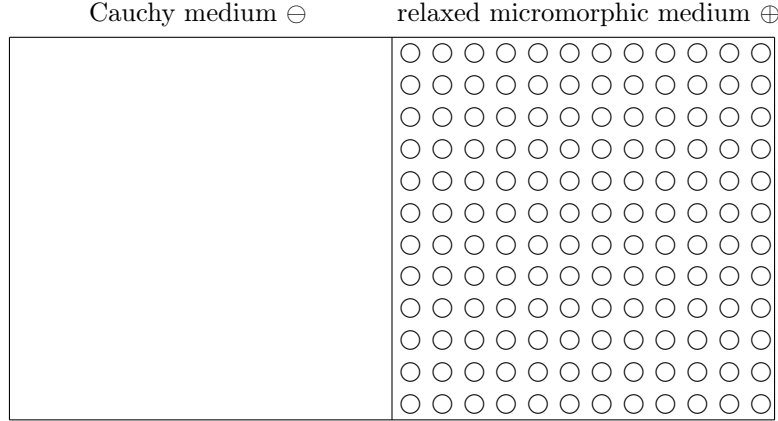


Figure 2: Schematics of a macro internal clamp with free microstructure at a Cauchy/relaxed-micromorphic interface.

Introducing the tangent vectors $\nu_1 = (0, 1, 0)$ and $\nu_2 = (0, 0, 1)$ and considering the new variables presented in (17) and (18), the boundary conditions on the jump of displacement read:

$$v_1^+ \cdot n - u_1^- = 0, \quad v_2^+ \cdot n - u_2^- = 0, \quad v_3^+ \cdot n - u_3^- = 0, \quad (32)$$

while the conditions on the internal forces become (see also [16]):

$$\begin{aligned}
\begin{pmatrix} \lambda_e + 2\mu_e \\ 0 \\ 0 \end{pmatrix} \cdot (v_1^+)' + \begin{pmatrix} 0 \\ -2\mu_e \\ -(3\lambda_e + 2\mu_e) \end{pmatrix} \cdot v_1^+ &= (\lambda + 2\mu) (u_1^-)', \\
\begin{pmatrix} \mu_e + \mu_c \\ 0 \\ 0 \end{pmatrix} \cdot (v_2^+)' + \begin{pmatrix} 0 \\ -2\mu_e \\ 2\mu_c \end{pmatrix} \cdot v_2^+ &= \mu (u_2^-)', \\
\begin{pmatrix} \mu_e + \mu_c \\ 0 \\ 0 \end{pmatrix} \cdot (v_3^+)' + \begin{pmatrix} 0 \\ -2\mu_e \\ 2\mu_c \end{pmatrix} \cdot v_3^+ &= \mu (u_3^-)'.
\end{aligned} \tag{33}$$

The conditions on the tangent part of the double force τ can be written as

$$\begin{aligned}
\tau_{22} &= \begin{pmatrix} 0 \\ -\mu_e L_c^2/2 \\ \mu_e L_c^2 \end{pmatrix} \cdot (v_1^+)' + \frac{\mu_e L_c^2}{2} (v_6^+)' = 0, & \tau_{33} &= \begin{pmatrix} 0 \\ -\mu_e L_c^2/2 \\ \mu_e L_c^2 \end{pmatrix} \cdot (v_1^+)' - \frac{\mu_e L_c^2}{2} (v_6^+)' = 0, \\
\tau_{12} &= \begin{pmatrix} 0 \\ \mu_e L_c^2 \\ \mu_e L_c^2 \end{pmatrix} \cdot (v_2^+)' = 0, & \tau_{13} &= \begin{pmatrix} 0 \\ \mu_e L_c^2 \\ \mu_e L_c^2 \end{pmatrix} \cdot (v_3^+)' = 0, \\
\tau_{23} &= \mu_e L_c^2 ((v_4^+)' + (v_5^+)') = 0, & \tau_{32} &= \mu_e L_c^2 ((v_4^+)' - (v_5^+)') = 0,
\end{aligned} \tag{34}$$

while we recall once again that the normal part of the double force is identically vanishing, i.e.:

$$\tau_{11} = 0, \quad \tau_{21} = 0, \quad \tau_{31} = 0. \tag{35}$$

4.1 Reflection and transmission of plane waves at a Cauchy/relaxed-micromorphic interface

When studying the reflection and transmission of a plane wave at a Cauchy/relaxed-micromorphic interface, we are considering that an incident wave traveling in the Cauchy medium impacts the interface. Two waves are then generated, namely one wave reflected in the Cauchy medium and one transmitted in the relaxed micromorphic medium. We explicitly remark that the reflected wave contains the longitudinal and transverse parts of the displacement field, while the transmitted wave contains a longitudinal part on the field v_1 , two transverse parts on v_α ($\alpha = 2, 3$) and the three uncoupled fields v_4 , v_5 and v_6 (see also Figure 3).

Considering the wave forms (5) and (21) for the unknown fields u^- , v_1^+ , v_α^+ , v_4^+ , v_5^+ and v_6^+ the solution of the considered problem can be written as:

$$\begin{aligned}
u_1^- &= \bar{\alpha}_1 e^{i(\omega/c_l x_1 - \omega t)} + \alpha_1 e^{i(-\omega/c_l x_1 - \omega t)}, \\
u_2^- &= \bar{\alpha}_2 e^{i(\omega/c_t x_1 - \omega t)} + \alpha_2 e^{i(-\omega/c_t x_1 - \omega t)}, \\
u_3^- &= \bar{\alpha}_3 e^{i(\omega/c_t x_1 - \omega t)} + \alpha_3 e^{i(-\omega/c_t x_1 - \omega t)}
\end{aligned} \tag{36}$$

and moreover

$$\begin{aligned}
v_1^+ &= \beta_1^1 h_1^1 e^{i(k_1^1(\omega) x_1 - \omega t)} + \beta_1^2 h_1^2 e^{i(k_1^2(\omega) x_1 - \omega t)}, & v_\alpha^+ &= \beta_\alpha^1 h_\alpha^1 e^{i(k_\alpha^1(\omega) x_1 - \omega t)} + \beta_\alpha^2 h_\alpha^2 e^{i(k_\alpha^2(\omega) x_1 - \omega t)}, & \alpha &= 2, 3, \\
v_4^+ &= \beta_4 e^{i(1/c_m \sqrt{\omega^2 - \omega_s^2} x_1 - \omega t)}, & v_5^+ &= \beta_5 e^{i(1/c_m \sqrt{\omega^2 - \omega_r^2} x_1 - \omega t)}, & v_6^+ &= \beta_6 e^{i(1/c_m \sqrt{\omega^2 - \omega_s^2} x_1 - \omega t)}.
\end{aligned} \tag{37}$$

In these formulas $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in \mathbb{R}$ are the amplitudes of the incident (longitudinal and transverse) waves traveling in the Cauchy continuum that are assumed to be known. Moreover $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ are the amplitudes of the longitudinal and transverse waves which are reflected in the Cauchy medium once the incident wave reaches the interface. Analogously, $\beta_1^1, \beta_1^2 \in \mathbb{R}$ are the amplitudes associated to the longitudinal wave transmitted in the relaxed medium, while $h_1^1, h_1^2 \in \mathbb{R}^3$ and $k_1^1, k_1^2 \in \mathbb{R}$ are the eigenvectors and eigenvalues associated to the eigenvalue problem for longitudinal waves (see [16]). In the same way $\beta_\alpha^1, \beta_\alpha^2 \in \mathbb{R}$, $h_\alpha^1, h_\alpha^2 \in \mathbb{R}^3$, $k_\alpha^1, k_\alpha^2 \in \mathbb{R}$ ($\alpha = 2, 3$) are defined for the transverse waves transmitted in the relaxed medium (see [16] for details). Finally $\beta_4, \beta_5, \beta_6 \in \mathbb{R}$ are the amplitudes of the uncoupled waves transmitted in the relaxed medium.

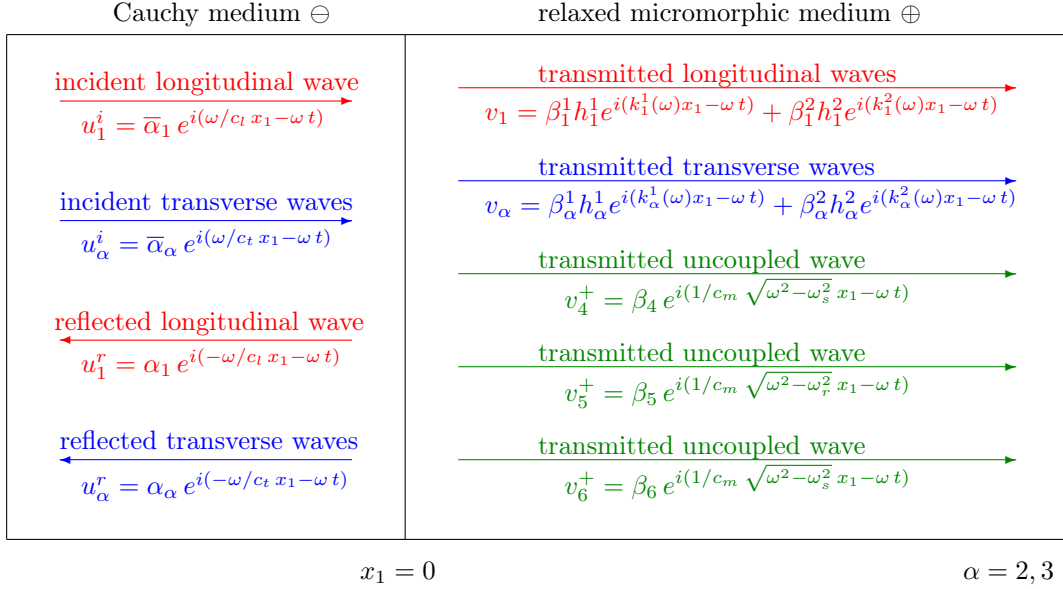


Figure 3: Incident, reflected and transmitted waves at a Cauchy/relaxed-micromorphic interface.

Assuming the amplitudes of the incident waves to be known, we can count the 12 unknown amplitudes $\alpha_1, \alpha_2, \alpha_3, \beta_1^1, \beta_1^2, \beta_\alpha^1, \beta_\alpha^2$ ($\alpha = 2, 3$), $\beta_4, \beta_5, \beta_6$ which can be calculated imposing the 12 scalar jump condition (32), (33) and (34).

4.2 Reflection and transmission of purely longitudinal waves at a Cauchy/relaxed-micromorphic interface

In the remainder of this paper, we are interested in a first calibration of the constitutive parameters of our relaxed micromorphic model on a real experiment of wave transmission in a band gap metamaterial.

To do so, we focus on the experiment proposed in [12] in which only longitudinal waves are considered. We hence consider here the solution of our relaxed problem only for what concerns the longitudinal part. In other words we are only considering the longitudinal fields v_1 and u_1 together with the field v_6 which is coupled to v_1 through the boundary conditions (34).

In summary, **the boundary value problem** for longitudinal waves can be written as (see equations (4), (19), (32), (33) and (34))

$$\begin{aligned}
 u_{1,tt}^- &= c_l^2 (u_1^-)'' , \\
 v_{1,tt}^+ &= A_1^R \cdot (v_1^+)' + B_1^R \cdot (v_1^+)' + C_1^R \cdot v_1^+ , \\
 v_{6,tt}^+ &= A_6^R (v_6^+)'' + C_6^R v_6^+ ,
 \end{aligned} \tag{38}$$

together with the boundary conditions:

$$\begin{aligned}
v_1^+ \cdot n - u_1^- &= 0, \\
\begin{pmatrix} \lambda_e + 2\mu_e \\ 0 \\ 0 \end{pmatrix} \cdot (v_1^+)' + \begin{pmatrix} 0 \\ -2\mu_e \\ -(3\lambda_e + 2\mu_e) \end{pmatrix} \cdot v_1^+ &= (\lambda + 2\mu) (u_1^-)' \\
\begin{pmatrix} 0 \\ -\mu_e L_c^2/2 \\ \mu_e L_c^2 \end{pmatrix} \cdot (v_1^+)' &= 0, \quad (v_6^+)' = 0.
\end{aligned} \tag{39}$$

The wave form solution for purely longitudinal fields is given by (see also equations (36), (37) and Figure 4):

$$\begin{aligned}
u_1^- &= \bar{\alpha}_1 e^{i(\omega/c_l x_1 - \omega t)} + \alpha_1 e^{i(-\omega/c_l x_1 - \omega t)}, \\
v_1^+ &= \beta_1^1 h_1^1 e^{i(k_1^1(\omega) x_1 - \omega t)} + \beta_1^2 h_1^2 e^{i(k_1^2(\omega) x_1 - \omega t)}, \\
v_6^+ &= \beta_6 e^{i(1/c_m \sqrt{\omega^2 - \omega_s^2} x_1 - \omega t)}.
\end{aligned} \tag{40}$$

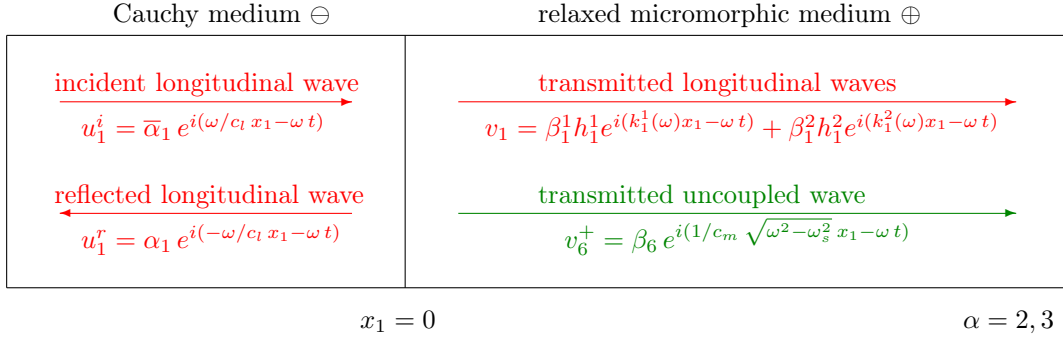


Figure 4: Incident, reflected and transmitted longitudinal waves at a Cauchy/relaxed-micromorphic interface.

Replacing the wave solution (40) in the 4 scalar jump conditions (39) and setting $x_1 = 0$ (position of the interface) we can calculate the 4 unknown amplitudes α_1 , β_1^1 , β_1^2 and β_6 .

From the condition $(v_6^+)' = 0$ it is straightforward to prove that $\beta_6 = 0$, so that finally $v_6^+ = 0 \forall x_1$ and $\forall t$. As for the other amplitudes, they have more complicated expressions that we do not explicitly show here since it does not add any fundamental information to the reasoning.

5 Reflection and transmission coefficients at a Cauchy/relaxed-micromorphic interface

We now want to define the reflection and transmission coefficients for the considered Cauchy/relaxed-micromorphic interface. To this purpose we introduce the quantities

$$J_i = \int_0^\Pi H_i(0, t) dt, \quad J_r = \int_0^\Pi H_r(0, t) dt, \quad J_t = \int_0^\Pi H_t(0, t) dt,$$

where Π is the period of the traveling plane wave and H_i , H_r and H_t are the energy fluxes of the incident, reflected and transmitted energies, respectively. The reflection and transmission coefficients can hence be defined as

$$R = \frac{J_r}{J_i}, \quad T = \frac{J_t}{J_i}. \tag{41}$$

Since the considered system is conservative, one must have $R + T = 1$.

In the particular case of reflection and transmission of longitudinal plane waves at a Cauchy/relaxed-micromorphic interface, recalling equations (26) and (28) together with the solutions (40) for the unknown fields we have:

$$H_i = \dot{u}_1^i (\lambda + 2\mu) u_{1,1}^i, \quad H_r = \dot{u}_1^r (\lambda + 2\mu) u_{1,1}^r, \quad H_t = H_1^1 + H_1^6, \quad (42)$$

where we set $u_1^i = \bar{\alpha}_1 e^{i(\omega/c_l x_1 - \omega t)}$ and $u_1^r = \alpha_1 e^{i(-\omega/c_l x_1 - \omega t)}$. We explicitly remark that the fluxes H_1^1 and H_1^6 defined in equation (29) must be calculated with the solutions v_1^+ and v_6^+ obtained for the considered constraint and given in equation (40).

Since in this particular case we have shown that v_6^+ is zero, then H_1^6 does not contribute to the evaluation of the transmitted energy. Once the solution for the energy fluxes have been calculated for the considered constraint (macro internal clamp with free microstructure), the reflection and transmission coefficients can be computed by using equations (41). We remark that R and T depend on the frequency ω of the traveling waves and that we must always have $R + T = 1$.

5.1 Transmission coefficient at a Cauchy/relaxed-micromorphic interface: the degenerate limit case $L_c = 0$ (internal variable model)

We show here that at the interface between a Cauchy continuum and a relaxed micromorphic one it is possible to model, as a degenerate limit case, the onset of two band gaps whose bounds can be identified to be (see [16]): $[\omega_l^1, \omega_s]$ and $[\omega_l^2, \omega_p]$, where:

$$\begin{aligned} \omega_l^1 &= \sqrt{\frac{a - \sqrt{a^2 - b^2}}{2\eta(\lambda_e + 2\mu_e)}}, & \omega_s &= \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}}, \\ \omega_l^2 &= \sqrt{\frac{a + \sqrt{a^2 - b^2}}{2\eta(\lambda_e + 2\mu_e)}}, & \omega_p &= \sqrt{\frac{2(\mu_e + \mu_{\text{micro}}) + 3(\lambda_e + \lambda_{\text{micro}})}{\eta}}, \end{aligned} \quad (43)$$

in which we have defined:

$$\begin{aligned} a &= 2\mu_e(3\lambda_{\text{micro}} + 2\mu_e + 4\mu_{\text{micro}}) + \lambda_e(3\lambda_{\text{micro}} + 6\mu_e + 4\mu_{\text{micro}}), \\ b^2 &= 8(\lambda_e + 2\mu_e)(\lambda_e(3\lambda_{\text{micro}}(\mu_e + \mu_{\text{micro}}) + 2\mu_{\text{micro}}(3\mu_e + \mu_{\text{micro}})) \\ &\quad + 2\mu_e(2\mu_{\text{micro}}(\mu_e + \mu_{\text{micro}}) + \lambda_{\text{micro}}(\mu_e + 3\mu_{\text{micro}}))). \end{aligned} \quad (44)$$

In Figure 5 we show a characteristic pattern of the transmission coefficient at a Cauchy/relaxed-micromorphic interface for a particular choice of the constitutive parameters and setting $L_c = 0$. The main characteristic feature of the relaxed micromorphic model with $L_c = 0$ (internal variable model) is that two separate band gaps can be determined and their bounds can be explicitly defined as functions of the constitutive parameters of the model according to equations (43).

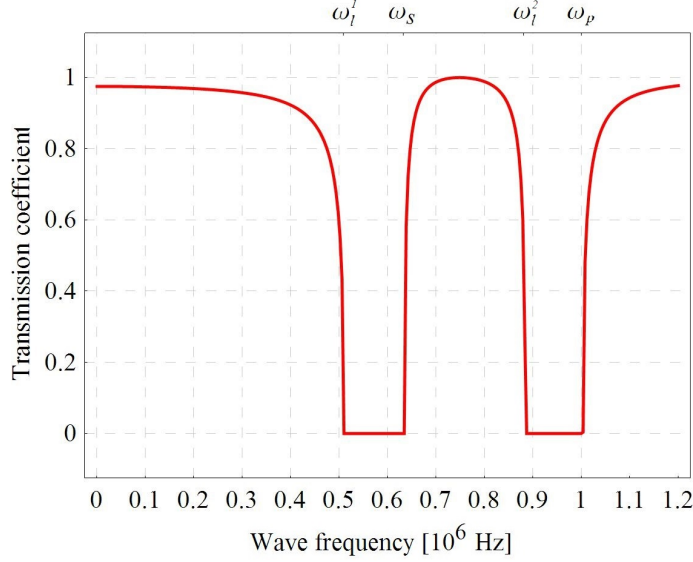


Figure 5: Transmission coefficient of the relaxed micromorphic model for $\lambda_e = \mu_e = \lambda_{\text{micro}} = \mu_{\text{micro}} = 100 \text{ GPa}$ and $\eta = 1 \text{ kg/m}$.

Switching on and slowly increasing the parameter L_c produces small changes on the reflection profile of Figure 5 basically related to the smoothening of the sharp corners that can be seen corresponding to the band-gap frequencies. We have shown in [16] that such situation in which two band gaps are precisely identified with lower bounds ω_l^1 and ω_l^2 , respectively, is a degenerate limit case of the relaxed micromorphic model obtained when setting the characteristic length L_c to be identically zero. Such model with $L_c = 0$ is also known as *internal variable model* and it has been shown to be able to catch the main features of some particular classes of band-gap metamaterials (see [24, 26]). Nevertheless, the fact of completely ignoring non-local effects in materials with heterogeneous microstructure may induce a certain amount of inaccuracy in the modeling phase which could be hard to be controlled when necessary.

In the next section we will show that we can estimate the characteristic length L_c of the metamaterial experimentally tested in [12] to be comparable to the order of magnitude of the diameters of the embedded microstructure.

We will also show that, even if the estimated value of L_c is non-negligible with respect to the characteristic size of the embedded microstructure, its effect on the amount of energy which is transmitted in the considered metamaterial is very small. This means that the error which is introduced if one uses an internal variable model instead of a relaxed micromorphic one is energetically small. On the other hand, non-locality is a fundamental feature of metamaterials with heterogeneous microstructure and as such it should always be included in their modeling. As a matter of fact, non-local effects are sensible to become more and more important when the contrast in the mechanical properties between adjacent unitary cells at the microscopic level becomes more pronounced.

As a general rule, we claim that the degenerate limit case $L_c = 0$ can be used for a first rough fitting of the elastic parameters of the relaxed micromorphic model. After that, the characteristic length L_c must be switched on in order to achieve a more accurate fitting of the experimental results. This last operation will allow for the estimate of non-local effects in real metamaterials.

The relaxed micromorphic model allows for the possibility of including non-local effects in band-gap metamaterials. In the next section we will have the twofold task of:

- fitting at best our constitutive parameters on a real metamaterial,
- estimate the order of magnitude of non-localities in such metamaterial.

6 Modeling a two dimensional phononic crystal via the relaxed micromorphic model

In this section we are interested in the modeling of the mechanical behavior of a particular metamaterial (phononic crystal) which has been seen to inhibit elastic wave propagation on an experimental basis (see [12]).

The structure presented, which is schematically shown in Figure 6(a) consists of a steel plate with liquid-filled holes in square array. The lattice constant, denoted as a , is 3.0 mm, the thickness t of the plate is 15 mm, the diameter d of the hole is 1.8 mm and the width of the cavity, w , is 1.5 mm.

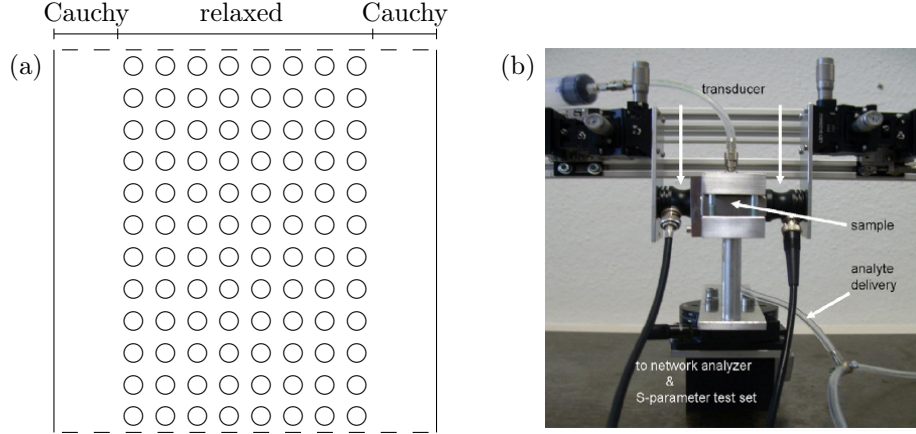


Figure 6: Schematics of the sample structure (a) and the experimental setup (b) ([12, Figure 1(b)]).

6.1 Experiments of wave transmission at a Cauchy/phononic-crystal interface

We show in Figure 7 the obtained experimental transmission spectrum of the considered phononic crystal, i.e. with 8 rows of liquid-filled holes (see Figure 6(a)) as a function of the frequency of the traveling wave. Given the geometry of the specimen shown in Figure 6, a longitudinal wave is sent in the Cauchy medium left side and the transmission coefficient is evaluated when the wave leaves the metamaterial on the opposite side. With liquid filled holes the band gap edge crosses the -3 dB-level at $\omega_1 = 586 \text{ kHz}$. Transmission of acoustic waves is suppressed until the upper edge at $\omega_3 = 918 \text{ kHz}$ but a single peak arises $\omega_2 = 793 \text{ kHz}$, which can be attributed to the resonance of the liquid-filled holes. The periodic variation of transmission at lower frequencies is caused by Bragg resonances. The second transmission band extends to about $\omega_4 \simeq 1 \text{ MHz}$.

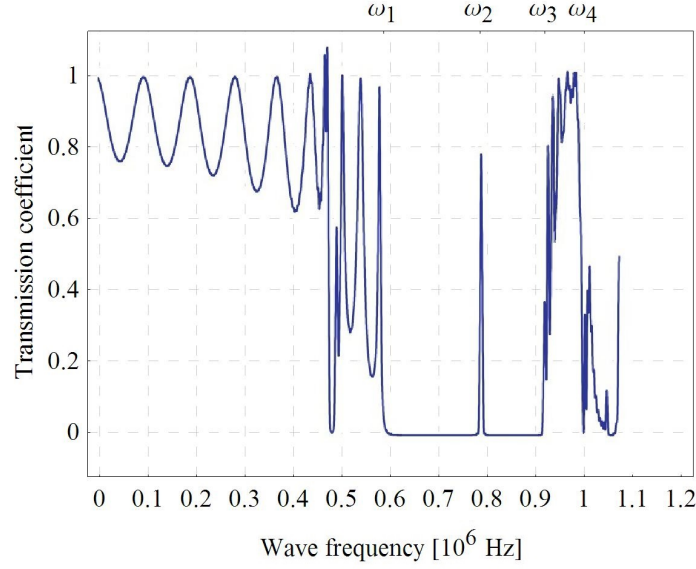


Figure 7: Transmission spectrum of the phononic crystal presented in Figure 6a, with liquid-filled holes [12, Figure 2b].

6.2 Identification of the parameters

In this subsection we present the procedure that we used in order to fit in the best possible way the maximum possible number of parameters of our relaxed micromorphic model on the available data based on a real phononic crystal. To start with, we fix the macroscopic mass density to be known as the averaged density of steel with fluid-filled holes. In particular we choose $\rho = 5000 \text{ kg/m}^3$. Nevertheless, we verify a posteriori that the value of ρ indeed does not sensibly affect the profile of the reflection coefficient for frequencies between 0 and 1 MHz. This fact is sensible if, with reference to [15,16] and to Figure 1, we notice that the parameter ρ only intervenes in the definition of the oblique asymptote $c_p = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}}$ for longitudinal waves. Such asymptote governs the slope of the optic wave LO1 which starts playing a significant role for frequencies higher than ω_p . In the considered example ω_p will be set to be equal to ω_3 which is experimentally seen to be close to 1 MHz. For frequencies higher than 1 MHz variations of ρ could eventually produce more tangible changes in the profile of the reflection coefficient.

To perform the fitting of the remaining parameters, we started by imposing the following identities:

$$\begin{aligned} \omega_l^1(\mu_e, \mu_{\text{micro}}, \lambda_e, \lambda_{\text{micro}}, \eta) &= \omega_1, & \omega_p(\mu_e, \mu_{\text{micro}}, \lambda_e, \lambda_{\text{micro}}, \eta) &= \omega_3, \\ \omega_l^2(\mu_e, \mu_{\text{micro}}, \lambda_e, \lambda_{\text{micro}}, \eta) &= \omega_2, & \omega_s(\mu_e, \mu_{\text{micro}}, \eta) &= \omega_2 - 8 \text{ kHz}. \end{aligned} \quad (45)$$

where we recall that the explicit expression of ω_l^1 , ω_s , ω_l^2 , ω_p as function of the elastic parameters of the relaxed micromorphic model is given in equations (43). We hence have 4 conditions to determine the 5 elastic parameters $\mu_e, \mu_{\text{micro}}, \lambda_e, \lambda_{\text{micro}}, \eta$ setting in a first instance $L_c = 0$. If analogous experiments as the one proposed in [12] for longitudinal waves would be reproduced on the same metamaterial but for transverse waves, extra conditions on the parameters of the relaxed micromorphic model would be available that would permit a more accurate fitting.

We start by numerically solving the system of four equations (45) with respect to the parameters

$\lambda_e, \mu_{\text{micro}}, \lambda_{\text{micro}}$ and η leaving free the parameter μ_e . The obtained solution is¹⁰

$$\lambda_e = 4.58 \mu_e, \quad \mu_{\text{micro}} = 7.21 \mu_e, \quad \lambda_{\text{micro}} = -2.57 \mu_e, \quad \eta = 2.66 \cdot 10^{-11} \mu_e, \quad (46)$$

The free parameter μ_e is then varied in order to evaluate its influence on the reflection coefficient. A parametric study on the free coefficient μ_e is performed giving rise to the profiles of the transmission coefficients shown in Figure 8.

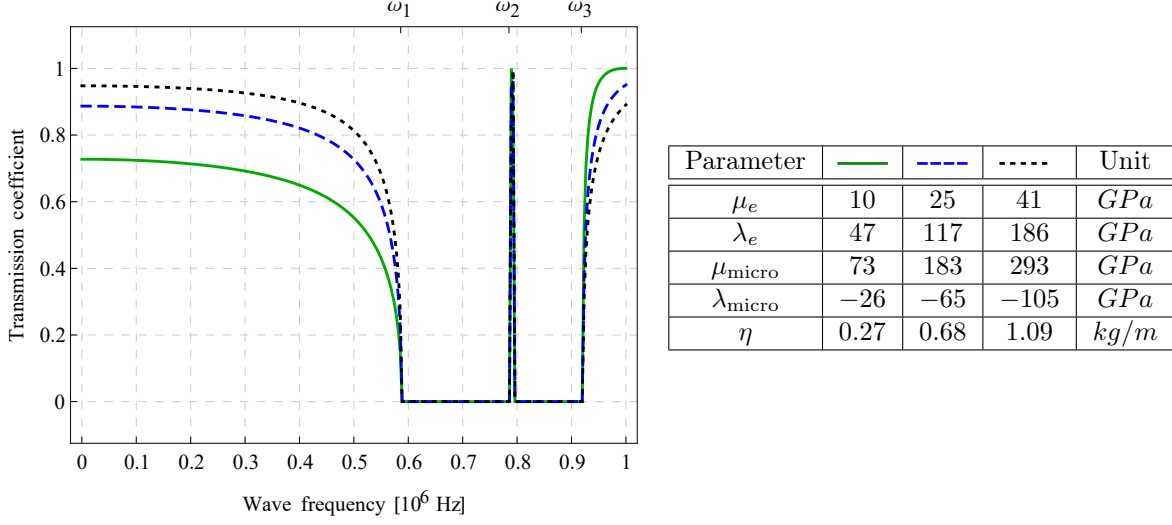


Figure 8: Profiles of the transmission coefficients obtained for different arbitrary values of the parameter μ_e . The other parameters corresponding to any given value of μ_e are obtained according to the solution (46) and are shown in the table given on the right.

At this point, we are able to choose the value of the parameter μ_e which respects the conditions (46) and which fits at best the profile of Figure 7. We conclude that, based on the described fitting procedure, the values of the parameters that best fits the profile associated to the real phononic crystal are those presented in Table 1b.

(a)	Condition	Value	Unit
	$\omega_l^1 = \omega_1$	586	<i>kHz</i>
	$\omega_s = \omega_2 - 8 \text{ kHz}$	785	<i>kHz</i>
	$\omega_l^2 = \omega_2$	793	<i>kHz</i>
	$\omega_p = \omega_3$	918	<i>kHz</i>

(b)	Parameter	Value	Unit
	μ_e	25	<i>GPa</i>
	λ_e	117	<i>GPa</i>
	μ_{micro}	183	<i>GPa</i>
	λ_{micro}	-65	<i>GPa</i>
	η	0.68	<i>kg/m</i>

Table 1: Conditions used for the parameters identification (a) and corresponding values of the obtained elastic parameters of the relaxed micromorphic model (b).

The corresponding profile of the transmission coefficients as compared to the one presented in [12] is given in Figure 9.

¹⁰We explicitly mention that, additionally to the solution (46) we obtain a second solution which, nevertheless must be excluded since it violates the positive definiteness of the strain energy density W . Solution (46) is then the only possible solution which can be used to fit the profile of the transmission coefficient. We checked that it is possible to leave free any other parameter rather than μ_e to perform the desired fitting of the transmission coefficient and that it yields comparable results for the obtained values of the constitutive parameters.

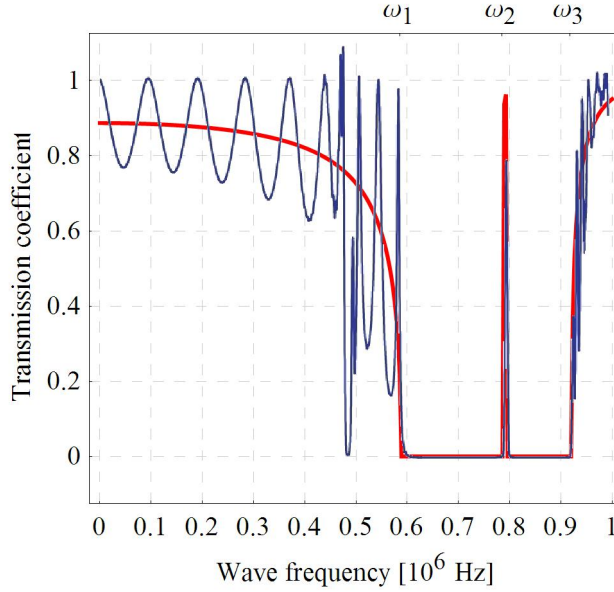


Figure 9: Comparison of the profile obtained in [12] based on a real metamaterial and the one obtained with the relaxed micromorphic model with the values given in Table 1 and $L_c = 0$.

Figure 9 shows the comparison between the profile of the transmission coefficient obtained in [12] for a real phononic crystal and the one obtained with our relaxed micromorphic model when setting $L_c = 0$.

It can be seen that a very good fitting can be obtained up to frequencies of the order of 1 MHz.

In particular, the oscillatory behavior observed for lower frequencies and which, according to the authors of [12], is due to Bragg scattering phenomena is caught by our model in an “averaged” sense.

The fitting for higher frequencies is almost perfect up to reaching 1 MHz, while for frequencies higher than 1 MHz the relaxed micromorphic model loses its predictivity due to the fact that the corresponding wavelengths are so small that the continuum hypothesis is sensible to become inaccurate.

We need to explicitly remark that the peak of reflection, which is obtained around the frequency ω_2 and that is experimentally related to a resonant behavior of the fluid inside the walls is slightly overestimated by the simulation via the relaxed micromorphic model with respect to the one observed in [12]. This peak magnification can be related to the fact that no dissipation is accounted for in our model, while the fluid viscosity may perhaps play here a non-negligible role.

We now come back to the point where we set $L_c = 0$ in order to start fitting our constitutive parameters (see subsection 5.1). This fact allowed us to obtain here the values of the elastic parameters of our model by a first fitting with the profile of the transmission coefficient (see Table 1).

On the other hand, as expected, switching on the characteristic length L_c allows an even better fitting as shown in Figure 10.

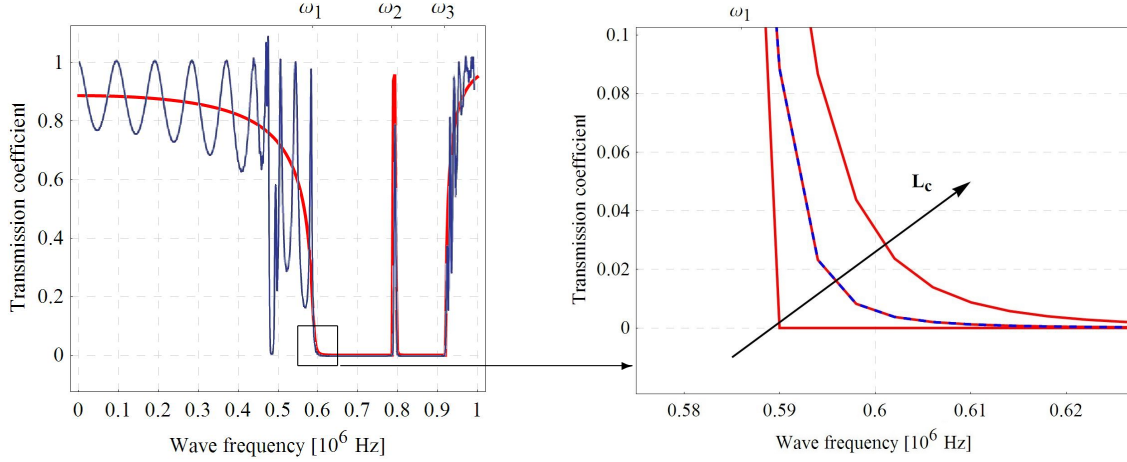


Figure 10: Fitting of the parameter L_c on the experimental profile of the transmission coefficients.

Indeed, we can notice from Figure 10 that the degenerate limit case $L_c = 0$ lets the calculated transmission coefficient slightly deviate from the experimental one (sharp corners). Small variations of the numerical profile can be perceived as far as $L_c \in [0, 0.5 \text{ mm})$. On the other hand, as far as $L_c = 0.5 \text{ mm}$ an almost perfect fitting is achieved (dashed line in Figure 10 on the right)). This means that we have been able to estimate the non-locality of the considered metamaterial to be of the order of 0.5 mm , i.e. $\sim 1/3$ of the diameters of the holes.

We need to explicitly say that the value of the macroscopic density ρ might slightly affect the variation of the transmission coefficient as function of L_c . Nevertheless, we need to consider a density of 1 order of magnitude higher ($50\,000 \text{ kg/m}^3$) in order to appreciate a sensible deviation of the profiles shown in Figure 10. We leave to a subsequent work the aim of determining also the macroscopic mass density ρ by using extra conditions provided by the fact of considering also measurements on transverse waves.

The determination of the parameter L_c completes the fitting of the elastic parameters of our relaxed micromorphic model on the band-gap metamaterial experimentally tested in [12] (see also Table 1). The Cosserat couple modulus parameter μ_c cannot be measured as far as only longitudinal waves are considered here and it thus remains to be determined. We have to explicitly remark that if an analogous fitting procedure would have been possible for transverse waves, we would have had more conditions than parameters to be determined. The extra conditions could have been used as validation of the fitted parameters.

It is out of the scope of this paper to envisage a general procedure for the optimal fitting of the whole set of parameters of the relaxed micromorphic model. We leave this fundamental objective to a forthcoming paper where such general procedure will be deeply discussed based on experiments of real interest.

The main scope of the present paper that we think to have successfully achieved is threefold:

- we give the very first estimation of the maximum possible number of constitutive parameters of the relaxed micromorphic model based on a simple measurement of transmission of longitudinal waves at a Cauchy/band-gap-metamaterial interface
- we give the very first evidence of the non-locality in band-gap metamaterials based upon real experiments
- we elucidate the physical meaning of the constraint which has been introduced in [16] and that we called “*internal clamp with free microstructure*”: such constraint allows for the description of continuity of displacement in the solid phase at the Cauchy/metamaterial interface, while the fluid in the embedded microstructure is free to vibrate. It is exactly the freedom which is left to the micro-motions that allows for the description of the local resonant peak around the frequency ω_2 which is indeed not possible for other types of constraints (see [16]).

7 Conclusions

In the present paper we give the very first estimate of the elastic coefficients of the relaxed micromorphic model based upon the experimentally-based results presented in [12] which concern the measurement of the transmitted energy as a function of the frequency of the traveling wave for a particular band-gap metamaterial. More particularly, restricting our attention to the problem of studying reflection and transmission of longitudinal waves at a Cauchy/relaxed-micromorphic interface, we are able to reproduce the main characteristic features which are observed in [12] for a phononic crystal obtained by means of an aluminum plate with small fluid-filled holes (diameter $\sim 1.8\text{ mm}$).

Suitably choosing the values of the parameters of our relaxed micromorphic model, we are able to fit the profile of the transmission coefficient proposed in [12] as function of the frequency of the traveling waves.

Two band-gaps which almost collapse to form a unique band-gap can be observed both in [12] and as a result of the simulations based upon our relaxed micromorphic model.

The continuity of such extended band-gap is broken due to the presence of a resonant peak of transmitted energy that is seen to be related to the internal resonance of the fluid embedded in the microstructure.

We present a detailed procedure that we use to fit almost all the parameters of our relaxed-micromorphic model except the Cosserat couple modulus μ_c which remains undetermined. This indeterminacy is due to the fact that experiments concerning reflection and transmission of transverse waves in the considered metamaterial have not been performed yet.

We leave to a forthcoming contribution the problem of determining the whole set of parameters of the relaxed micromorphic model for real band-gap metamaterials.

The results presented in this paper allow to give the first physical interpretation of the boundary conditions that can be imposed at a Cauchy/relaxed-micromorphic interface based upon a real experiment.

We conclude our paper with the finding that we believe to be the most important to be pointed out. Indeed, we showed by direct comparison of our relaxed micromorphic model with available evidences that non-local effects are an intrinsic feature of band-gap metamaterials.

A characteristic length $L_c = 0.5\text{ mm}$ has been estimated for the real phononic crystal studied in [12] which is almost 1/3 of the diameter of the holes in the embedded microstructure.

Even if the energetic contribution associated to the underlying non-locality is very small (only small changes in the transmission coefficient can be appreciated when increasing L_c from 0 to 0.5 mm), such non-locality is intrinsically present in any microstructured material and as such it should be always accounted for when modeling their mechanical behavior.

The macroscopic effects of non-localities are sensible to become more and more energetically significant when considering stronger contrasts in the mechanical properties at the microscopic level (e.g. unitary cells with very different stiffnesses). The relaxed micromorphic model should be always used when one wants to model band-gap metamaterials in order to account for such non-localities.

In a subsequent work we will provide a more complete determination of the constitutive parameters of the relaxed micromorphic model for real band-gap metamaterials and we will discuss further the importance of non-local effects in such microstructured materials.

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