Introduction to homogenisation. Application to viscoelastic fluid flow in porous media, semi-solid alloys and klinkenberg flow in porous media

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ABSTRACT

Modelling finely heterogeneous physical systems turns out to be a difficult task. Nonetheless, an heterogeneous system may be modelled by an equivalent macroscopic continuous system if the condition of separation of scales is verified, [2, 5]

$$\varepsilon = \frac{l}{L} \ll 1, \qquad (1)$$

where l and L are the characteristic lengths of the heterogeneities and of the macroscopic sample or excitation, respectively. The macroscopic equivalent model is obtained from the description at the heterogeneity scale by [1]: i) assuming the medium to be periodic, without loss of generality; ii) writing the local description in a dimensionless form; iii) evaluating the dimensionless numbers with respect to the scale ratio ε; iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε; v) solving the successive boundary value problems that are obtained after introducing these expansions in the local dimensionless description. The macroscopic equivalent model is obtained from compatibility conditions which are the necessary conditions for the existence of solutions to the boundary value problems. The main advantages of the method rely upon the possibility of: a) avoiding prerequisites at the macroscopic scale; b) modelling finite size macroscopic samples; c) modelling macroscopically non-homogeneous media or phenomena; d) modelling problems with several separations of scales; e) modelling several simultaneous phenomena; f) determining whether the system "medium+phenomena" is homogenisable or not; g) providing the domains of validity of the macroscopic models. Below, we review three applications of the method.

1 Viscoelastic flow in porous media

In this section we address the problem of the transient linear viscoelastic fluid flow in porous media (see [4] for details). The fluid is assumed to be incompressible. The local description under consideration is the following

$$\rho \frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \qquad (2)$$

$$\tilde{\tau} + \sum_{p=1}^{n} (\lambda_p)^p \frac{\partial^p \tilde{\tau}}{\partial t^p} = 2\mu (\tilde{D} + \sum_{k=1}^{n} (\theta_k)^k \frac{\partial^k \tilde{D}}{\partial t^k}), \qquad (3)$$

which supposes that the relaxation times λ_p and the retardation times θ_k are small compared to the transit time l/v; therefore, the local Deborah number is small, $De_l = \mathrm{Max}(\lambda_p v/l, \theta_k v/l) \ll 1$. The filtration law is investigated for a transient excitation of characteristic time T which is of same order as both the relaxation and the retardation times, $T/\lambda_p = O(1), T/\theta_k = O(1)$. On the other hand, viscous local term is balanced by the macroscopic pressure gradient, which yields $Q_l^F = \dfrac{p/l}{\mu v/l^2} = O(\varepsilon^{-1})$. Under these conditions the homogenisation process yields macroscopic flow behaviour which is described by a Darcy-like law in Fourier space

$$< v_i > = -K_{ij} \frac{\partial p}{\partial x_j}, \qquad (4)$$

where the generalized permeability tensor $\mathbf{K}$ is complex valued and frequency dependent. The macroscopic behaviour appears to be strongly dependent on the relative values of the λ_p's and the θ_k's.

2 Semi-solid alloys

Semi-solid alloys can be modelled at the heterogeneity scale by a porous matrix saturated by an incompressible newtonian fluid [3]. The porous matrix material itself is an incompressible Odqvist medium, whose deviatoric parts of the rate of deformation and stress are related by the following power-law

$$\overline{\mathbf{D}}^S = A(\overline{\sigma}^S : \overline{\sigma}^S)^{\frac{n-1}{2}} \overline{\sigma}^S \qquad (5)$$

The pressures in the matrix and in the fluid and the deviatoric stress in the matrix are assumed to be of similar order of magnitude. In this study, the key dimensionless number is the ratio $Q_l^F = O(\frac{p^F/l}{\mu^F v^F/l^2})$ of the pressure term to the viscous term in the fluid. Three different behaviours are obtained with respect to the value of Q_l^F

- $Q_l^F = O(\varepsilon^{-1})$: The macroscopic model is a two-phase model, of Biot-type, but with an Odqvist power-law of exponent n for the porous matrix. Then, segregation exists. The fluid flow follows a classical Darcy's law.
- $Q_l^F = O(\varepsilon^0)$: The macroscopic model is a single-phase model and the equivalent medium is incompressible, with an Odqvist viscous power-law of exponent n. There is no segregation. The fluid interacts by its pressure, only.
- $Q_l^F = O(\varepsilon^1)$: The macroscopic model is a single-phase model and the equivalent medium is viscous non-linear and incompressible. There is a strong influence of the liquid viscosity. However, unlike the previous case, the behaviour is not given by a power-law. There is no segregation.

3 Klinkenberg flow in porous media

At low pressure, the molecular mean free path λ of a gas increases. The local Knudsen number of gas flow in a porous medium, $Kn_l = \lambda/l$, becomes larger and the adherence condition on the pore surface is no longer valid. A wall-slip velocity exists, which is proportional to the shear rate of deformation on the pore surface. This effect is measured by the ratio of the wall-slip velocity to the gas pore velocity, which ratio is also given by the Knudsen number.

The study [6] is conducted for a very small Reynolds number Re and a quite small Knudsen number

$$Re \ll \varepsilon \ll Kn \ll 1, \qquad (6)$$

where ε is the small scale ratio. We assume that Navier-Stokes equations remain valid in the pores. Therefore, as in the first part, the flow is caused by a macroscopic pressure gradient which is balanced by the local viscous forces, $Q_l^F = \frac{p/l}{\mu v/l^2} = O(\varepsilon^{-1})$. Under these conditions, a tensorial Klinkenberg law is obtained by homogenisation

$$< v >_i = -\frac{K_{ij}}{\mu(p)}(I_{jk} + \frac{\beta}{p}H_{jk})\frac{\partial p}{\partial x_k}, \qquad (7)$$

where $\mathbf{K}$ is the classical positive and symmetric permeability tensor, $\beta > 0$ and $\mathbf{H}$ is a positive tensor. The symmetry of $\mathbf{H}$ is not demonstrated.

References

[1] J.-L. Auriault, Heterogeneous medium: Is an equivalent description possible?, *Int. J. Eng. Sc.*, vol. 29, 1991, pp. 785–795.

[2] A. Bensoussan, J.-L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[3] C. Geindreau and J.-L. Auriault, Investigation of the viscoplastic behavior of alloys in the semi-solid state by homogenisation, *Mech. Mat.*, vol. 31, 1999, pp. 535–551.

[4] B. Khuzhayorov, J.-L. Auriault and P. Royer, Derivation of macroscopic filtration law for transient linear viscoelastic fluid flow in porous media, *Int. J. Eng. Sc.*, vol. 38, 2000, pp. 497–504.

[5] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, vol. 127, Springer, Berlin, 1980.

[6] E. Skjetne and J.-L. Auriault, Homogenisation of wall-slip gas flow through porous media, *Transp. in Porous Media*, vol. 36, 1999, pp. 293–306.