HAL
open science

Signature morphisms from the Cremona group over a non-closed field

Stéphane Lamy, Susanna Zimmermann

To cite this version:

Stéphane Lamy, Susanna Zimmermann. Signature morphisms from the Cremona group over a non-closed field. Journal of the European Mathematical Society, 2020, 10.4171/JEMS/983 . hal-01719067

HAL Id: hal-01719067

https://hal.science/hal-01719067v1

Submitted on 27 Feb 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SIGNATURE MORPHISMS FROM THE CREMONA GROUP OVER A NON-CLOSED FIELD

STÉPHANE LAMY AND SUSANNA ZIMMERMANN

Abstract. We prove that the plane Cremona group over a perfect field with at least one Galois extension of degree 8 is a non-trivial amalgam, and that it admits a surjective morphism to a free product of groups of order two.

Introduction

The Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is the group of birational symmetries of the projective plane defined over a field $\mathbf{k}$. Its elements are of the form

$$[x : y : z] \dashrightarrow [f_0(x, y, z) : f_1(x, y, z) : f_2(x, y, z)]$$

where $f_0, f_1, f_2 \in \mathbf{k}[x, y, z]$ are homogeneous polynomials of equal degree with no common factor, and such that there exists an inverse of the same form. Equivalently, working in an affine chart one can define the Cremona group as the group of birational selfmaps of the affine plane, which is also (anti-)isomorphic to the group $\mathrm{Aut}_{\mathbf{k}}\,\mathbf{k}(x, y)$ of $\mathbf{k}$-automorphisms of the fraction field $\mathbf{k}(x, y)$. The Cremona group contains the group of polynomial automorphisms of the affine plane over the field $\mathbf{k}$. In particular it is a rather huge group. It is neither finitely generated (see [Can17, Proposition 3.6]), nor finite dimensional, even when working over a finite base field. It was recently shown that $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is not a simple group, over any base field $\mathbf{k}$ [CL13, Lon16]. Then it is natural to ask for nice quotients of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, for instance abelian ones. Over an algebraically closed field, it is known (see also §4.3) that the automorphism group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{k})$, or the Jonquières group $\mathrm{PGL}_2(\mathbf{k}(T)) \rtimes \mathrm{PGL}_2(\mathbf{k})$, both embed in any quotient of the Cremona group. In particular in this situation any morphism from $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ to an abelian group, or to a finite group, is trivial. On the other hand it was shown by the second author in [Zim15] that the situation is drastically different over the field $\mathbf{R}$ of real numbers. The real Cremona group admits an uncountable collection of morphisms to $\mathbf{Z}/2\mathbf{Z}$, and precisely we have the following result about the abelianization of $\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)$:

$$\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)/[\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)] \simeq \bigoplus_{(0,1]} \mathbf{Z}/2\mathbf{Z}. \qquad (\dagger)$$

In this paper we explore a similar question, over any perfect base field $\mathbf{k}$ that admits at least one Galois extension of degree 8. Observe that this condition corresponds to a large collection of fields, which includes the case of all number fields and finite fields.

The special role of degree 8 extensions is explained by their relation to Bertini involutions. Indeed, given a point of degree 8 on $\mathbb{P}^2$, that is, an orbit of cardinal 8 under the natural action of the absolute Galois group of the base field $\mathbf{k}$, we can consider the surface S obtained by blowing-up this orbit. If the point is sufficiently general, the surface S is del Pezzo and admits another birational morphism to $\mathbb{P}^2$, and the induced birational selfmap of $\mathbb{P}^2$ is an example of a Bertini involution. Let $\mathcal{B} \subset \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ be a set of representatives of such Bertini involutions with a base point of degree 8, up to conjugacy by automorphisms. We prove that as soon as $\mathbf{k}$ admits at least one Galois extension of degree 8, then the set $\mathcal{B}$ is quite large. Namely $\mathcal{B}$ has at least the same

Date: December 1, 2017.
2010 *Mathematics Subject Classification.* 14E07, 14E30.
Key words and phrases. Cremona group; Sarkisov program; abelianization; non-closed fields.

cardinality than $\mathbf{k}$, and in the case of a finite field $\mathbf{F}_q$ one can be more precise and give a lower bound for the cardinal of $\mathcal{B}$ which is polynomial in q (see §4.2).

These Bertini involutions are part of a system of elementary generators $\mathcal{E}$ for the group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ which was found by Iskovskikh [Isk91]. The set $\mathcal{E}$ is always a huge set, because for instance it contains all Jonquières maps (up to left-right composition by automorphisms). Before stating our results we introduce a bit more notation. For each Bertini involution $b \in \mathcal{B}$, we set $G_b = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), b\rangle$. Moreover, we denote $G_e = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), \mathcal{E} \smallsetminus \mathcal{B}\rangle$ the subgroup generated by automorphisms and all non-Bertini elementary generators. Our first result gives an amalgamated product structure for $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, in terms of these subgroups.

Theorem A. *Let $\mathbf{k}$ be a perfect field admitting at least one Galois extension of degree 8. Consider the subgroups G_i as defined above, for $i \in \mathcal{B} \cup \{e\}$. Then for all $i \neq j$ we have $G_i \cap G_j = \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, and we have an amalgamated product structure*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \simeq \mathop{\ast}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_i.$$

Moreover, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ acts faithfully on the corresponding Bass-Serre tree.

It was shown by Cornulier (appendix of [CL13]) that the Cremona group over an algebraically closed field is not a non-trivial amalgam of two groups. In contrast, we deduce from the above theorem the following structure result, where we denote $G_{\mathcal{B}} = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), \mathcal{B}\rangle$ the subgroup generated by all G_b:

Corollary B. *Let $\mathbf{k}$ be a perfect field admitting at least one Galois extension of degree 8. Then $G_e \cap G_{\mathcal{B}} = \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, and*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \simeq G_{\mathcal{B}} *_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_e,$$

and it acts faithfully on its Bass-Serre tree.

It turns out that each subgroup G_b admits a structure of free product, and this allows to obtain a lot of morphisms from the Cremona group to $\mathbf{Z}/2\mathbf{Z}$:

Theorem C. *Let $\mathbf{k}$ be a perfect field with at least one Galois extension of degree 8. Then:*

(1) *For each $b \in \mathcal{B}$ we have $G_b \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) * \mathbf{Z}/2\mathbf{Z}$.*
(2) *There is a surjective morphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \to \mathop{\ast}_{\mathcal{B}} \mathbf{Z}/2\mathbf{Z}$$

whose kernel is the smallest normal subgroup containing G_e, and which sends each $b \in \mathcal{B}$ to the corresponding generator on the right-hand side.
(3) *In particular, the abelianization of the Cremona group over $\mathbf{k}$ contains a subgroup isomorphic to $\bigoplus_{\mathcal{B}} \mathbf{Z}/2\mathbf{Z}$.*

We see that even if there was no Bertini involutions involved in the paper [Zim15], we obtain a similar looking (even if less precise) result. In particular, the $\mathbf{Z}/2\mathbf{Z}$ in the target group in (†) have nothing to do with the fact that the absolute Galois group of $\mathbf{R}$ has order 2, but rather with the fact that we are able to produce a natural set of generators for the Cremona group that contains involutions. In this sense, we like to think of the above morphisms $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \to \mathbf{Z}/2\mathbf{Z}$ as some analogues of the classical signature morphism on the symmetric group. The huge collection of such morphisms corresponds to the existence of a system of generators with a lot of non-conjugate involutions. In this paper, we focus on Bertini involutions associated to a base point of degree 8 because they seem to be the easiest to handle technically. When the base field is $\mathbf{R}$, a similar role was played by the so-called "standard quintic involutions". It seems quite plausible that other "signature morphisms" exist on the Cremona group, associated to other type of involutions, such as the Geiser involution associated to a

base point of degree 7. Also, we mention that we see no obvious obstruction why such morphisms could not exist in higher dimension, even over the field of complex numbers.

The strategy to prove the above results is to use the Sarkisov Program. The Sarkisov Program is a way to factorize a given birational map between Mori fiber spaces into elementary links. We recall that even if the starting map is a birational selfmap of a given variety X (for instance $X = \mathbb{P}^2$), the elementary links are not in general elements of the group $\mathrm{Bir}(X)$. In other words, even if one is primarily interested in the group $\mathrm{Bir}(X)$, the Sarkisov Program naturally produces generators for the pseudo-group of birational maps $Y \dashrightarrow Z$, where Y, Z can be any Mori fiber spaces birational to X. Nevertheless the Sarkisov Program turns out to be an efficient tool to produce some systems of generators for $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, and also to describe relations between them [Isk91, IKT93, Isk96]. The Sarkisov Program was revisited recently in light of the progresses in the theory of the Minimal Model Program, and is now established in any dimension (over $\mathbf{C}$) [HM13]. Moreover the relations between Sarkisov links were described by Kaloghiros [Kal13].

In Section 2 we encode Sarkisov links and relations between them in a square complex $\mathcal{X}$ on which the group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ acts naturally. Then in Section 3 we give an account of the proof of the Sarkisov Program in the simpler case of surfaces, but working over an arbitrary perfect field. This allows to prove that the square complex $\mathcal{X}$ is connected and simply connected.

In Section 4 we recall the notion of elementary generators for the group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, following the work of Iskovskikh. Among the elementary generators we discuss in particular the Bertini involutions and prove their existence (and in fact, their abundance). We also discuss the Jonquières maps, and in any dimension we recall the following basic dichotomy: given a morphism φ from the Cremona group to another group H, either the subgroup generated by the Jonquières maps lies in the kernel, or φ induces an embedding of this subgroup into H.

Finally in Section 5 we use Bass-Serre theory to prove our results. The general idea is that the Bass-Serre trees of the various amalgams appearing in Theorem A, Corollary B and Theorem C are realized either as a quotient or as a subcomplex of the square complex $\mathcal{X}$.

When one encounters a cube complex in geometric group theory, a natural question is whether this complex has non-positive curvature. It turns out that this is not the case for our square complex $\mathcal{X}$, however we should mention that $\mathcal{X}$ is essentially a subcomplex of an infinite dimensional CAT(0) cube complex associated with the Cremona group that was constructed by Lonjou in her PhD thesis. We thank Anne Lonjou for many useful discussions at an early stage of this project, Anne-Sophie Kaloghiros for clarifying to us some fine points in her work [Kal13], and Jérémy Blanc and Andrea Fanelli for discussions on issues with birational maps over non-perfect fields.

1. Birational maps between surfaces over an arbitrary field

In this section we review some results about the birational geometry of surfaces, with a focus on the case of an arbitrary perfect base field.

1.1. **Factorization into blow-ups.** Let $\mathbf{k}$ be a perfect field, and $\mathbf{k}^a$ an algebraic closure. All field extensions of $\mathbf{k}$ that we shall consider will be supposed to lie in $\mathbf{k}^a$. By a *surface* (over $\mathbf{k}$) we shall mean a smooth projective surface defined over $\mathbf{k}$. We denote by $S(\mathbf{k})$ the set of $\mathbf{k}$-rational points on S. The Galois group $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ acts on $S \times_{\mathrm{Spec}\,\mathbf{k}} \mathrm{Spec}\,\mathbf{k}^a$ through the second factor. In particular, $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ acts on the set $S(\mathbf{k}^a)$ of $\mathbf{k}^a$-rational points. By a *point of degree d* on S we mean an orbit $p = \{p_1, \dots, p_d\} \subset S(\mathbf{k}^a)$ of cardinal d under the action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$. Observe that the points in $S(\mathbf{k})$ are exactly the fixed points for the action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ on $S(\mathbf{k}^a)$, or in other words the points of degree 1. Let $\mathbf{L}/\mathbf{k}$ be a field extension such that the p_i are $\mathbf{L}$-rational points. We call blow-up of p the blow-up of these d points, which is a

morphism $\pi\colon S' \to S$ defined over $\mathbf{k}$, with exceptional divisor $E = C_1 + \cdots + C_d$, where the C_i are disjoint (-1)-curves defined over $\mathbf{L}$, and $E^2 = -d$. We shall refer to this situation by saying that E is an *exceptional divisor of degree d*.

We recall the following classical factorization results (see e.g. [Liu02, Theorems 9.2.2 and 9.2.7]).

Proposition 1.1. *Let $\pi\colon S' \to S$ be a birational morphism between smooth projective surfaces defined over $\mathbf{k}$. Then $\pi = \pi_1 \circ \cdots \circ \pi_n$, where each $\pi_i\colon S_i \to S_{i-1}$ is the blow-up of a point of degree $d_i \geqslant 1$ on S_{i-1}, with exceptional divisor E_i on S_i satisfying $E_i^2 = -d_i$ (in particular $S = S_0$, and $S' = S_n$).*

Proposition 1.2. *Let $\varphi\colon S \dashrightarrow S'$ be a birational map between smooth surfaces over $\mathbf{k}$. Then there exists Z a smooth surface over $\mathbf{k}$, and sequence of blow-ups $\pi\colon Z \to S$, $\pi'\colon Z' \to S'$ of orbits of points under $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$, such that $\pi' = \varphi \circ \pi$.*

We should mention that even if the Cremona group was explicitly defined in the introduction in terms of homogeneous polynomials, in practice we almost always think of an element of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ as given by two sequences of blow-ups defined over $\mathbf{k}$, as provided by Proposition 1.2.

Remark 1.3. Over a non-perfect field $\mathbf{k}$, there is no reason why the base points of a birational map should be defined over a separable closure of $\mathbf{k}$, and so we can no longer identify closed points with Galois orbits as we did in the statements of Propositions 1.1 and 1.2. As a simple example of this phenomenon, consider $\mathbf{k} = \mathbf{F}_2(t)$, and denote by $t^{\frac{1}{2}}$ the unique square root of t in $\mathbf{k}^a$. Then $\mathbf{k}(t^{\frac{1}{2}})/\mathbf{k}$ is a non separable extension. Now consider the birational involution $f \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ given by

$$f\colon [x_0 : x_1 : x_2] \dashrightarrow [x_0x_2 : x_1x_2 : x_0^2 + tx_1^2]$$

As a quadratic birational map, f admits 3 base points defined over $\mathbf{k}^a$, which are

$$p_1 = [0 : 0 : 1], \qquad\qquad p_2 = [t^{\frac{1}{2}} : 1 : 0],$$

and p_3 a point infinitely near to p_2. In particular p_2 is not defined over a separable extension of $\mathbf{k}$.

1.2. **Negative maps, minimal and ample models, scaling.** Let S be a surface defined over $\mathbf{k}$, and S^a the same surface over $\mathbf{k}^a$. We define the Néron-Severi space $N^1(S^a)$ as the space of numerical classes of $\mathbf{R}$-divisors:

$$N^1(S^a) := \mathrm{Div}(S^a) \otimes \mathbf{R}/\equiv .$$

The action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ on $N^1(S^a)$ factors through a finite group, and we denote by $N^1(S)$ the subspace of invariant classes. Since we only consider surfaces with $S(\mathbf{k}) \neq \varnothing$ and $\mathbf{k}^a[S^a]^* = (\mathbf{k}^a)^*$, $N^1(S)$ is also the space of classes of divisors defined over $\mathbf{k}$ (see [San81, Lemma 6.3(iii)]). The dimension of this finite dimensional $\mathbf{R}$-vector space is called the Picard number of S over $\mathbf{k}$, and denoted by $\rho(S)$.

Remark 1.4. When working on a surface S, we can identify the space $N^1(S)$ of divisors and the space $N_1(S)$ of 1-cycles, and similarly the subspaces $\mathrm{Eff}(S)$ or $\mathrm{NE}(S)$ of effective divisors or 1-cycles. In the sequel we shall use the notation which seem most natural in view of the extension of the results in higher dimension. For instance the Cone Theorem 1.7 is about 1-cycles, so there we use the notation $\mathrm{NE}(S)$.

Let $\pi\colon S' \to S$ be a birational morphism between smooth surfaces over $\mathbf{k}$, and D' a $\mathbf{Q}$-divisor on S' with push-forward $D = \pi_*(D')$. By Proposition 1.1, we can write $\pi = \pi_1 \circ \ldots \pi_n$, where $\pi_i\colon S_i \to S_{i-1}$ is the blow-up of a point of degree d_i, with $S = S_0$ and $S' = S_n$. For any i, we denote by E_i the exceptional divisor of π_i, and by D_i the push-forward of D' on S_i. We say that π is *D'-negative* if $D_i \cdot E_i < 0$ for all i. Observe also that on S' we can write

$$D' = \pi^* D + \sum a_i E_i$$

for some $a_i \in \mathbf{Q}$, where here the E_i denote strict transforms on S'.

Lemma 1.5. *With the above notation, the morphism π is D'-negative if and only if $a_i > 0$ for all i.*

Proof. On S_i, we have $D_i = \pi_i^* D_{i-1} + a_i E_i$, so that

$$0 = \pi^* D_{i-1} \cdot E_i = D_i \cdot E_i - a_i E_i^2.$$

Since $E_i^2 = -d_i$, where $d_i \geqslant 1$ is the degree of the point blown-up by π_i, we get $a_i = -\frac{D_i \cdot E_i}{d_i}$, so that a_i and $D_i \cdot E_i$ have opposite signs as expected. □

If $\pi\colon S' \to S$ is a D'-negative birational morphism, and $D = \pi_*(D')$ is nef, we call S a *D'-minimal model* of S'. Such a model, if it exists, is unique:

Lemma 1.6 ([Mat02, p. 94])**.** *Let S_1, S_2 be two D'-minimal models of S', then the induced map $S_1 \dashrightarrow S_2$ is an isomorphism.*

We shall use the above setting for divisors D' of the form $D' = K_{S'} + A$, with A an ample $\mathbf{Q}$-divisor (or $A = 0$). Observe that a $(K_{S'} + A)$-negative birational morphism is also a $K_{S'}$-negative morphism, hence simply a sequence of inverse of blow-ups. There exist surfaces with infinitely many (-1)-curves: a classical example is given by $\mathbb{P}^2$ blown-up at (sufficiently general) 9 points. This gives a countable collection of curves C_i with $K_{S'} \cdot C_i < 0$ and $C_i^2 < 0$. However, after perturbing $K_{S'}$ by adding any $\mathbf{Q}$-ample divisor, we get a finite collection:

Theorem 1.7 ([Rei97, Cone Theorem D.3.2])**.** *Let S be a smooth surface defined over $\mathbf{k}^a$. Then if $\rho(S) \geqslant 3$, all K_S-negative extremal rays of the cone $NE(S)$ are of the form $\mathbf{R}_{>0}C$ with C a (-1)-curve. Moreover, for any ample $\mathbf{Q}$-divisor A on S, there are only finitely many (-1)-curves C_i such that $(K_S + A) \cdot C_i < 0$.*

Comments on the proof. In the Cone Theorem, the main delicate point to check when working in arbitrary characteristic is vanishing. If A is ample, by Serre duality we have $H^2(S, K_S + A) = H^0(S, -A)^* = 0$. So by Riemann-Roch on a surface we have, for the divisor $D = K + A$:

$$h^0(S, D) \geqslant h^0(S, D) - h^1(S, D) = \frac{1}{2} D(D - K_S) + \chi(\mathcal{O}_S). \tag{1}$$

In fact when S is rational, we even have $H^1(S, K_S + A) = 0$ (Kodaira vanishing in positive characteristic can only fail for surfaces of Kodaira dimension $\geqslant 1$, see [Ter99, Theorem 1.6]). But this extra information is not necessary in the argument given by Reid, as Inequality (1) is enough. □

Remark 1.8. (1) Over an arbitrary perfect field $\mathbf{k}$, we have an equivariant version of the Cone Theorem with respect to the action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$, see [KM98, p. 48]. Essentially we only have to change "(-1)-curve" by "orbit of pairwise disjoint (-1)-curves under the action of $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$". By Castelnuovo Contraction Theorem, we can contract such an orbit and obtain a new smooth projective surface. Thus by running the Minimal Model Program with respect to the canonical divisor K, or more generally with respect to $K + A$ with A ample, we stay in the category of smooth surfaces, and at posteriori this justifies that we restrict ourselves to this setting.

(2) In the case where the Picard number $\rho(S)$ is equal to 2, $\mathrm{NE}(S)$ is a convex cone in a real 2-dimensional vector space, thus we have at most two extremal rays, which correspond either to the contraction of an exceptional divisor or to a Mori fibration. This case is particularly interesting in a relative setting, and is then often referred to as a *two rays game*. Precisely, we start with a morphism $\pi\colon S \to Y$ from a surface S, with relative Picard number equal to 2. We assume that any curve C contracted by π satisfies $K_S \cdot C < 0$. Then there exist exactly two morphisms of relative Picard number

1, $\pi_i\colon S \to Y_i$, $i = 1, 2$, such that π factors through each of the π_i:

$$\begin{tikzcd} & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\pi"] & \\ Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

Theorem 1.9 (Base Point Free Theorem, see [Rei97, D.4.1])**.** *Let S be a smooth projective surface, and let D be a nef divisor on S such that $D - \varepsilon K_S$ is ample for some $\varepsilon > 0$. Then the linear system $|mD|$ is base point free for all sufficiently large m.*

If D is nef and of the form $D = K + A$ with A ample, we denote by φ_D the morphism from S associated to the linear system $|mD|$ for $m \gg 0$. This morphism has connected fibers, and it contracts precisely the curves C such that $(K + A) \cdot C = 0$. Now we extend the definition of φ_D to any pseudo-effective divisor $D = K + A$, by using the notion of scaling.

Let S be a smooth surface, and A, Δ ample $\mathbf{Q}$-divisors on S (we also admit the case $A = 0$). The $(K + A)$ *minimal model program with scaling* Δ is a sequence of birational morphisms $\pi_i\colon S_{i-1} \to S_i$ defined iteratively as follows. We set $S_0 = S$. If S_i is constructed, we denote A_i and Δ_i the direct images of A and Δ on S_i, and we consider t_i such that $D_i = K_{S_i} + A_i + t_i\Delta_i$ is nef but not ample on S_i. Then the morphism π_{i+1} from S_i is obtained by applying Theorem 1.9 to D_i. If $\pi_{i+1}(S_i) = S_{i+1}$ is a surface, we repeat the construction. At some point π_{i+1} is a fibration to a curve or a point, in which case we reached a Mori fiber space and the program stops. In particular this process gives a finite sequence of rational numbers

$$t_0 > t_1 > \cdots > t_n$$

such that $K_{S_i} + A_i + t\Delta_i$ is ample on S_i for any $t > t_i$. We shall say that a birational morphism $\pi\colon S \to S'$ is $(K + A)$-negative with scaling Δ if S' is one of the S_i in the above process.

Now assume that D is a pseudo-effective $\mathbf{Q}$-divisor on S of the form $D = K + \Delta$, with Δ ample. We run the K minimal model program with scaling Δ, and we look where the coefficient $t = 1$ corresponding to D fits into the sequence $t_0 > t_1 > \cdots > t_n$. Precisely, we set $j = \max\{i \mid t_i \geqslant 1\}$ and we denote $\varphi_D = \pi_j \circ \cdots \circ \pi_1$. Observe that the morphism φ_D is birational if and only if $j \leqslant n - 1$, and in any case $\varphi_D(D)$ is ample. We say that φ_D is the *ample model* of D.

Remark 1.10. (1) The above construction only depends on the numerical class of D, which would not be the case for more general D (that is, not of the form $K + \Delta$ with Δ ample), see [KKL14, Example 4.8].

(2) The morphism φ_D coincides with the morphism from S to $\mathrm{Proj}(\bigoplus H^0(Z, mD))$, see [KKL14, Remark 2.4]. In particular, if we write $D = K + \Delta_1 + \Delta_2$ with Δ_1, Δ_2 ample, and run the $K + \Delta_1$ minimal model program with scaling Δ_2, we will get the same morphism φ_D, but possibly by another sequence of contractions.

2. A square complex associated to the Cremona group

In this section we construct a square complex that encodes Sarkisov links and relations between them. First we introduce the key notion of rank r fibration.

2.1. **Rank r fibrations.** If not stated otherwise, all varieties and morphisms are defined over $\mathbf{k}$. Let S be a smooth surface, and $r \geqslant 1$ an integer. We say that S is a *rank r fibration* if there exists a morphism $\pi\colon S \to B$, where B is a point or a smooth curve, with relative Picard number equal to r, and such that the anticanonical divisor $-K_S$ is π-ample. The last condition means that for any curve C contracted to a point by π, we have $K_S \cdot C < 0$. Observe that the condition on the Picard number is that $\rho(S) = r$ if B is a point, and $\rho(S) = r + 1$ if B is isomorphic to $\mathbb{P}^1$. If S is a rank r

fibration, we will write S/B if we want to emphasize the basis of the fibration, and S^r when we want to emphasize the rank. An isomorphism between two fibrations S/B and S'/B' (necessarily of the same rank r) is an isomorphism $S \xrightarrow{\simeq} S'$ such that there exists an isomorphism on the bases (necessarily uniquely defined) that makes the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\simeq"] \arrow[d, "\pi"'] & S' \arrow[d, "\pi'"] \\ B \arrow[r, "\simeq"] & B' \end{tikzcd}$$

As the following examples make it clear, there are sometimes several choices for a structure of rank r fibration on a given surface, that may even correspond to distinct ranks.

Example 2.1. (1) $\mathbb{P}^2$ with the morphism $\mathbb{P}^2 \to \mathrm{pt}$, or the Hirzebruch surface $\mathbb{F}_n$ with the morphism $\mathbb{F}_n \to \mathbb{P}^1$, are rank 1 fibrations.

(2) $\mathbb{F}_1$ with the morphism $\mathbb{F}_1 \to \mathrm{pt}$ is a rank 2 fibration. Idem for $\mathbb{F}_0 \to \mathrm{pt}$. The blow-up $S^2 \to \mathbb{F}_n \to \mathbb{P}^1$ of a Hirzebruch surface along a point of degree d, such that each point of the orbit is in a distinct fiber, is a rank 2 fibration over $\mathbb{P}^1$.

(3) The blow-up of two distinct points on $\mathbb{P}^2$, or of two points of $\mathbb{F}_n$ not lying on the same fiber, gives examples of rank 3 fibrations, with morphisms to the point or to $\mathbb{P}^1$ respectively.

Remark 2.2. Observe that the definition of a rank r fibration puts together several well-known notions. If B is a point, then S is just a del Pezzo surface of Picard rank r (over the base field $\mathbf{k}$). If B is a curve, then S is just a conic bundle of relative Picard rank r: a general fiber is isomorphic to $\mathbb{P}^1$, and (over $\mathbf{k}^a$) any singular fiber is the union of two (-1)-curves secant at one point. Remark also that rank 1 fibrations are exactly the usual 2-dimensional Mori fiber spaces.

We will be interested only in rational surfaces, and we call *marking* on a rank r fibration S/B a choice of a birational map $\varphi\colon S \dashrightarrow \mathbb{P}^2$. Observe that if S is rational and B is a curve, then B is isomorphic to $\mathbb{P}^1$. We say that two marked fibrations $\varphi\colon S/B \dashrightarrow \mathbb{P}^2$ and $\varphi'\colon S'/B' \dashrightarrow \mathbb{P}^2$ are *equivalent* if $\varphi'^{-1} \circ \varphi\colon S/B \to S'/B'$ is an isomorphism of fibrations. We denote by $(S/B, \varphi)$ an equivalence class under this relation. The Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ acts on the set of equivalence classes of marked fibrations by post-composition:

$$f \cdot (S/B, \varphi) := (S/B, f \circ \varphi).$$

If S'/B' and S/B are marked fibrations of respective rank $r' > r \geqslant 1$, we say that S'/B' *factorizes* through S/B if the birational map $S' \to S$ induced by the markings is a morphism, and moreover there exists a (uniquely defined) morphism $B \to B'$ such that the following diagram commutes:

$$\begin{tikzcd} S' \arrow[rr, "\pi'"] \arrow[dr] & & B' \\ & S \arrow[r, "\pi"] & B \arrow[u] \end{tikzcd} \tag{2}$$

In fact if $B' = \mathrm{pt}$ the last condition is empty, and if $B' \simeq \mathbb{P}^1$ it means that $S' \to S$ is a morphism of fibration over a common basis $\mathbb{P}^1$.

2.2. **Square complex.** We define a 2-dimensional complex $\mathcal{X}$ as follows. Vertices are equivalence classes of marked rank r fibrations, with $3 \geqslant r \geqslant 1$. We put an oriented edge from $(S'/B', \varphi')$ to $(S/B, \varphi)$ if S'/B' factorizes through S/B. If $r' > r$ are the respective ranks of S'/B' and S/B, we say that the edge has type r', r. For each triplets of pairwise linked vertices $(S''^3/B'', \varphi'')$, $(S'^2/B', \varphi')$, $(S^1/B, \varphi)$, we glue a triangle. In this way we obtain a 2-dimensional simplicial complex on which the Cremona group acts.

Lemma 2.3. *For each edge of type* $3,1$ *from* S''/B'' *to* S/B*, there exist exactly two triangles that admit this edge as a side.*

Proof. In short, the proof is a two rays game (see Remark 1.8). By assumption S''/B'' factorizes through S/B, so by setting $Y = S$ (if $B \simeq B'$) or $Y = B$ (if $B \simeq \mathbb{P}^1$ and $B' = \mathrm{pt}$), we obtain via Diagram (2) a morphism $\pi\colon S'' \to Y$ with relative Picard number $\rho(S''/Y)$ equal to 2. We have exactly two extremal rays in the cone $\mathrm{NE}(S''/Y)$, and since $\rho(S'') = 3$ or 4, both correspond to divisorial contractions. Denote by $S'' \to S'$ and $S'' \to \tilde{S}'$ these two contractions. Then the two expected triangles are S''/B'', S'/B', S/B and S''/B'', $\tilde{S}'/B'$, S/B. □

In view of the lemma, by gluing all the pairs of triangles along edges of type 3,1, and keeping only edges of types 3,2 and 2,1, we obtain a square complex that we still denote $\mathcal{X}$. We call *vertices of type* $\mathbb{P}^2$ the vertices in the orbit of the vertex $(\mathbb{P}^2/\mathrm{pt}, \mathrm{id})$ under the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. When drawing subcomplexes of $\mathcal{X}$ we will often drop part of the information which is clear by context, about the markings, the equivalence classes and/or the fibration. For instance S/B must be understood as $(S/B, \varphi)$ for an implicit marking φ, and $(\mathbb{P}^2, \varphi)$ as $(\mathbb{P}^2/\mathrm{pt}, \varphi)$.

Example 2.4. Let S be the surface obtained by blowing-up $\mathbb{P}^2$ in two distinct points a and b of degree 1. Denote by $\mathbb{F}_{1,a}/\mathbb{P}^1_a$, $\mathbb{F}_{1,b}/\mathbb{P}^1_b$ the two intermediate Hirzebruch surfaces with their fibrations to $\mathbb{P}^1$. Finally, denote by $\mathbb{F}_0$ the surface obtained by contracting the strict transform on S of the line through a and b. All these surfaces fit into the subcomplex of $\mathcal{X}$ pictured on Figure 1, where the dotted arrows are the edges of type 3,1 that we need to remove from the simplicial complex in order to get a square complex.

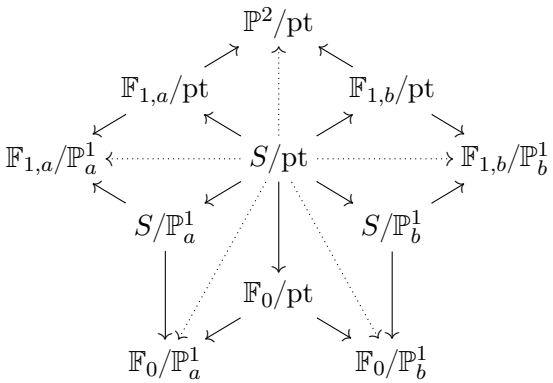

FIGURE 1

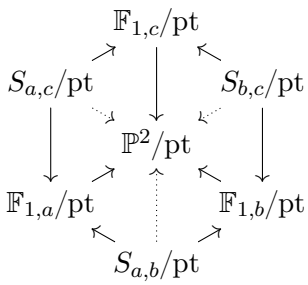

FIGURE 2

Example 2.5. Consider the blow-ups of 3 points a, b, c on $\mathbb{P}^2$. These gives three squares around the corresponding vertex of type $\mathbb{P}^2$ (see Figure 2). In particular the square complex $\mathcal{X}$ is not CAT(0), as mentioned in the introduction.

2.3. **Sarkisov links and elementary relations.** In this section we show that the complex $\mathcal{X}$ encodes the notion of Sarkisov links, and of elementary relation between them.

First we rephrase the usual notion of Sarkisov links between 2-dimensional Mori fiber spaces. Let $(S/B, \varphi)$, $(S'/B', \varphi')$ be two marked rank 1 fibrations. We say that the induced birational map $S \dashrightarrow S'$ is a *Sarkisov link* if there exists a marked rank 2 fibration S''/B'' that factorizes through both S/B and S'/B'. Equivalently, the vertices corresponding to S/B and S'/B' are at distance 2 in the complex $\mathcal{X}$, with middle vertex S''/B'':

$$\begin{tikzcd} & S''/B'' \arrow[dl] \arrow[dr] & \\ S/B & & S'/B' \end{tikzcd}$$

This definition is in fact equivalent to the usual definition of a link of type I, II, III or IV from S/B to S'/B' (see [Kal13, Definition 2.14] for the definition in arbitrary dimension). Below we recall these definitions in the context of surfaces, in terms of commutative diagrams where each morphism has relative Picard number 1 (such a diagram corresponds to a "two rays game"), and we give some examples. Remark that these diagrams are not part of the complex $\mathcal{X}$: in each case, the corresponding subcomplex of $\mathcal{X}$ is just a path of two edges, as described above.

• Type I: B is a point, $B' \simeq \mathbb{P}^1$, and $S' \to S$ is the blow-up of a point of degree $d \geqslant 1$ such that we have a diagram

$$\begin{tikzcd} & S' \arrow[dl] \arrow[dr] & \\ S \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

Then we take $S''/B'' := S'/\mathrm{pt}$.

Examples are given by the blow-up of a point of degree 1 or 4 on $S = \mathbb{P}^2$. The fibration $S'/\mathbb{P}^1$ corresponds respectively to the lines through the point of degree 1, or to the conics through the point of degree 4.

• Type II: $B = B'$, and there exist two blow-ups $S'' \to S$ and $S'' \to S'$ that fit into a diagram of the form:

$$\begin{tikzcd} & S'' \arrow[dl] \arrow[dr] & \\ S \arrow[dr] & & S' \arrow[dl] \\ & B & \end{tikzcd}$$

Then we take $S''/B'' := S''/B$.

An example is given by blowing-up a point of degree 2 on $S = \mathbb{P}^2$, and then by contracting the transform of the unique line through this point. The resulting surface S' is a del Pezzo surface of degree 8, which has rank 1 over $\mathbf{k}$, but has rank 2 over $\mathbf{k}^a$ (being isomorphic to $\mathbb{P}^1_{\mathbf{k}^a} \times \mathbb{P}^1_{\mathbf{k}^a}$). Other examples, important for this paper, are provided by blowing-up a point of degree 8 on $\mathbb{P}^2$: see §4.2.

• Type III: symmetric situation of a link of type I.

• Type IV: (S, φ) and (S', φ') are equal as marked surfaces, but the fibrations to B and B' are distinct. In this situation B and B' must be isomorphic to $\mathbb{P}^1$, and we have a diagram

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & B' \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

Then we take $S''/B'' := S/\mathrm{pt}$.

For rational surfaces, a type IV link always corresponds to the two rulings on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, that is, $S/B = \mathbb{F}_0/\mathbb{P}^1$ is one of the rulings, $S'/B' = \mathbb{F}_0/\mathbb{P}^1$ the other one, and $S''/B'' = \mathbb{F}_0/\mathrm{pt}$. See [Isk96, Theorem 2.6 (iv)] for other examples in the context of non-rational surfaces.

A *path of Sarkisov links* is a finite sequence of marked rank 1 fibrations

$$(S_0/B_0, \varphi_0), \dots, (S_n/B_n, \varphi_n),$$

such that for all $0 \leqslant i \leqslant n-1$, the induced map $g_i \colon S_i/B_i \dashrightarrow S_{i+1}/B_{i+1}$ is a Sarkisov link.

Proposition 2.6. *Let $(S'/B, \varphi)$ be a marked rank 3 fibration. Then there exist finitely many squares in $\mathcal{X}$ with S' as a corner, and the union of these squares is a subcomplex of $\mathcal{X}$ homeomorphic to a disk with center corresponding to S'.*

Proof. Since $\rho(S') = 3$ or 4, we can factorize the fibration S'/B into

$$S' \to S \to Y \to B,$$

where $S' \to S$ is a divisorial contraction, and Y is either a surface of a rational curve. By playing the two rays game on S'/Y (see Remark 1.8), we obtain another surface $\tilde{S}$ and a divisorial contraction $S' \to \tilde{S}$ that fits into a commutative diagram:

$$\begin{tikzcd}
 & S \arrow[dr] & & \\
S' \arrow[ur] \arrow[dr] & & Y \arrow[r] & B \\
 & \tilde{S} \arrow[ur] & &
\end{tikzcd}$$

If Y is a surface, we obtain the following square in $\mathcal{X}$:

$$\begin{tikzcd}
 & S'/B \arrow[dl] \arrow[dr] & \\
S/B \arrow[dr] & & \tilde{S}/B \arrow[dl] \\
 & Y/B &
\end{tikzcd}$$

On the other hand if $Y \simeq \mathbb{P}^1$ is a curve (and so B is a point), then we obtain the following two squares in $\mathcal{X}$:

$$\begin{tikzcd}
 & S'/\mathrm{pt} \arrow[dl] \arrow[d] \arrow[dr] & \\
S/\mathrm{pt} \arrow[d] & S'/\mathbb{P}^1 \arrow[dl] \arrow[dr] & \tilde{S}/\mathrm{pt} \arrow[d] \\
S/\mathbb{P}^1 & & \tilde{S}/\mathbb{P}^1
\end{tikzcd}$$

Now in both cases we consider the two rays game on $\tilde{S}/B$: this produces $\tilde{Y}$, which is either a surface or a curve, and which fits into a diagram:

$$\begin{tikzcd}
 & S \arrow[dr] & & \\
S' \arrow[ur] \arrow[dr] & & Y \arrow[dr] & \\
 & \tilde{S} \arrow[ur] \arrow[dr] & & B \\
 & & \tilde{Y} \arrow[ur] &
\end{tikzcd}$$

Then by considering the two rays game on $S'/\tilde{Y}$, we produce one or two new squares in $\mathcal{X}$ that are adjacent to the previous ones. After finitely many such steps, the process must stop and produce the expected disk, because by Theorem 1.7 there are only finitely many divisorial contractions that we can use to factor S'/B. $\square$

Remark 2.7. If $B \simeq \mathbb{P}^1$, then at each step Y is a surface, and in this case the disk produced by the proof consists of exactly four squares in $\mathcal{X}$. On the other hand if B is a point the number of squares might vary: for instance in Example 2.4 we saw a situation with 5 squares.

In the situation of Proposition 2.6, by going around the boundary of the disk we obtain a path of Sarkisov links whose composition is the identity (or strictly speaking, an automorphism). We say that this path is an *elementary relation* between Sarkisov links, coming from S'^3/B. More generally, any composition of Sarkisov links that corresponds to a loop in the complex $\mathcal{X}$ is called a *relation* between Sarkisov links.

3. Relations in the Sarkisov program in dimension 2

In this section we prove that the complex $\mathcal{X}$ is connected and simply connected, which will be the key in proving Theorem A. These connectedness results will follow from the Sarkisov program, and more precisely from the study of relations in the Sarkisov program, that we can state as follows:

Theorem 3.1 (Sarkisov Program)**.**

(1) *Any birational map $f\colon S \dashrightarrow S'$ between rank 1 fibrations is a composition of Sarkisov links.*
(2) *Any relation between Sarkisov links is a composition of elementary relations.*

In arbitrary dimension over $\mathbf{C}$, these results correspond to [HM13, Theorem 1.1] and [Kal13, Theorem 1.3]. We give an account of the proof of these results in the simpler case of surfaces, but working over an arbitrary perfect base field.

3.1. **Polyhedral decomposition.** Let $\{(S_1/B_1, \varphi_1), \dots, (S_r/B_r, \varphi_r)\}$ be a finite collection of marked rank 1 fibrations. In the context of Theorem 3.1, we will take $r = 2$, $S_1 = S$, $S_2 = S'$ when proving (1), or the entire collection of rank 1 fibrations visited by a relation of Sarkisov links when proving (2).

By repetitively applying Proposition 1.2, we produce a marked surface Z dominating all the S_i, that is, such that all induced birational maps $f_i\colon Z \to S_i$ are morphisms. We pick a sufficiently small ample $\mathbf{Q}$-divisor A on Z such that for all i, f_i is $(K + A)$-negative, and $-K_{S_i} - f_{i*}(A)$ is relatively ample over B_i. The point of choosing such an ample divisor A is to ensure that there exist only finitely many $(K + A)$-negative birational morphisms from Z (up to post-composition by an isomorphism). Indeed this follows from Theorem 1.7, which says that at each step there are only finitely possible divisorial contractions. If there are only finitely many birational morphisms from Z (for instance if Z is a del Pezzo surface), we also admit the choice $A = 0$.

For each $i = 1, \dots, r$, applying the following two steps we construct an (effective) ample divisor Δ_i on Z such that $f_i\colon Z \to S_i$ is $(K + A)$-negative with scaling Δ_i, and more precisely f_i will be a $(K + A + \Delta_i)$-ample model of Z:

(1) If $B_i \simeq \mathbb{P}^1$, pick G_i a large multiple of the fiber of S_i/B_i such that $-K_{S_i} - f_{i*}(A) + G_i$ is ample on S_i. If $B_i = \{pt\}$, then $-K_{S_i} - f_{i*}(A)$ is already ample, so we just set $G_i = 0$. In both cases G_i is a nef divisor on S_i.

(2) Now pick P_i an effective $\mathbf{Q}$-divisor on S_i, equivalent to the ample divisor $-K_{S_i} - f_{i*}(A) + G_i$, and set $\Delta_i = A + f_i^*(P_i)$, which is ample as the sum of ample and nef divisors. One checks that

$$\begin{aligned} f_{i*}(K + A + \Delta_i) &= K_{S_i} + f_{i*}(A) + f_{i*}(A) - K_{S_i} - f_{i*}(A) + G_i \\ &= f_{i*}(A) + G_i, \end{aligned}$$

which is an ample divisor on S_i as expected.

We can assume that the Δ_i generate $N^1(Z)$ (throw in more ample divisors if necessary). We choose some rationals $r_i > 0$ such that $K + A + r_i\Delta_i$ is ample. We say that

a $\mathbf{Q}$-divisor Δ is a *subconvex combination* of the $r_i\Delta_i$ if

$$\Delta = \sum t_i r_i \Delta_i \text{ with } t_i \geqslant 0, \sum t_i \leqslant 1.$$

Let $\{g_j\}_{j=1}^s$ be the finite collection of $(K+A+\Delta)$-negative birational morphisms (up to isomorphism on the range), for all choices of a subconvex combination Δ as above. Observe that this collection is indeed finite because of Theorem 1.7. Remark also that the initial f_i are part of this collection by construction, and also the identity $\mathrm{id}\colon Z \to Z$ (corresponding for instance to taking one of the t_i equal to 1, and all the other equal to 0, because then $D = K + A + r_i\Delta_i$ is ample by assumption, and the corresponding embedding φ_D is equivalent to the identity morphism).

Now in the vector space $N^1(Z)$ we consider the cone over the convex hull

$$\mathrm{Conv}(K+A, K+A+r_1\Delta_1, \dots, K+A+r_s\Delta_s),$$

that we intersect with the pseudo-effective cone $\overline{\mathrm{Eff}}(Z)$ to get a cone $\mathcal{C}^\circ$. Explicitly:

$$\mathcal{C}^\circ = \left\{ D = \lambda \left(K + A + \sum t_i r_i \Delta_i \right);\ \lambda, t_i \geqslant 0, \sum t_i \leqslant 1, D \text{ pseudo-effective} \right\} \subseteq N^1(Z).$$

We denote by $\mathcal{C}$ the intersection of $\mathcal{C}^\circ$ with an affine hyperplane defined by $K + A$:

$$\mathcal{C} := \{D \in \mathcal{C}^\circ;\ (K+A)\cdot D = -1\}.$$

Recall from Proposition 1.1 that each $g_j\colon Z \to S_j$ is a finite sequence of contractions of exceptional divisors, that is, orbits under $\mathrm{Gal}(\mathbf{k}^a/\mathbf{k})$ of pairwise disjoint (-1)-curves. We denote $\{C_i\}_{i\in I}$ the finite collection of classes in $N^1(Z)$ obtained as pull-back of such exceptional divisors contracted by g_j, for all $j = 1, \dots, s$. We introduce a notation for the hyperplane and half-spaces defined by C_i in $N^1(Z)$:

$$C_i^\perp = \{D \in N^1(Z); D\cdot C_i = 0\},$$
$$C_i^\geqslant = \{D \in N^1(Z); D\cdot C_i \geqslant 0\},$$

and similarly for $C_i^>$, $C_i^\leqslant$, $C_i^<$.

Let $D \in \mathcal{C}$ be a big divisor. Then we get a partition $I = I^+ \cup I^-$ such that

$$D \in \bigcap_{i\in I^+} C_i^> \cap \bigcap_{i\in I^-} C_i^\leqslant.$$

There exists j such that the morphism φ_D associated with D (see discussion after Theorem 1.9) coincides with $g_j : Z \to S_j$. The classes C_i, $i \in I^-$, correspond to the curves contracted by $\varphi_D = g_j$.

Lemma 3.2. *The cone $\mathcal{C}^\circ \subset N^1(Z)$ is rational polyhedral, hence also the affine section $\mathcal{C}$.*

Proof. First we prove that

$$\mathrm{nef}(Z) \cap \mathcal{C}^\circ = \bigcap_{i\in I} C_i^\geqslant \cap \mathrm{Conv}(K+A, K+A+r_1\Delta_1, \dots, K+A+r_s\Delta_s),$$

from which it follows that $\mathrm{nef}(Z) \cap \mathcal{C}^\circ$ is a rational polyhedral cone.

If $D = K + A + \Delta \in \mathcal{C}^\circ$ is not nef, then by construction of φ_D there exists $i \in I$ such that the exceptional divisor C_i is contracted by φ_D, and $D\cdot C_i < 0$. On the other hand if $D \in \mathcal{C}^\circ$ is nef, then for any irreducible component C in the support of one of the C_i we have $D\cdot C \geqslant 0$, hence $D\cdot C_i \geqslant 0$ for all i.

Now we show that $\mathcal{C}^\circ$ is a rational polyhedral cone. Let $D \in \mathcal{C}^\circ$ be a big divisor, and $\varphi_D\colon Z \to S$ the associated birational morphism. Up to reordering the C_i, we can assume that $C_1, \dots, C_r$ are the classes contracted by φ_D, where r is the relative Picard number of Z over S. Then we can write

$$D = {\varphi_D}^*(\varphi_{D*}(D)) + \sum_{i=1}^r a_i C_i,$$

where ${\varphi_D}^*(\varphi_{D*}(D)) \in \mathrm{nef}(Z) \cap \mathcal{C}^\circ$, and the a_i are positive by Lemma 1.5. Together with $K + A + r_i\Delta_i \in \mathrm{nef}(Z) \cap \mathcal{C}^\circ$ this gives

$$\mathcal{C}^\circ \subseteq \mathrm{Conv}(\mathrm{nef}(Z) \cap \mathcal{C}^\circ, K + A) \bigcap \mathrm{Conv}(\mathrm{nef}(Z) \cap \mathcal{C}^\circ, C_i \mid i \in I)$$

This inclusion is in fact an equality because the first cone on the right-hand side is just $\mathrm{Conv}(K + A, K + A + r_1\Delta_1, \dots, K + A + r_s\Delta_s)$ and the second cone consists of pseudo-effective divisors, so the right hand side is contained in $\mathcal{C}^\circ$. It follows that $\mathcal{C}^\circ$ is rational polyhedral, as expected. □

We set

$$\mathcal{A}_j := \bigcap_{i \in I^+} C_i^{>} \cap \bigcap_{i \in I^-} C_i^{\leqslant} \cap \mathcal{C}.$$

In particular, $\mathcal{A}_j$ is a rational polyhedral subset of $\mathcal{C}$, and is equal to the set of divisors $D \in \mathcal{C}$ such that $\varphi_D = g_j$. Observe that the chamber $\mathcal{A}$ of ample divisors in $\mathcal{C}$ is one of the $\mathcal{A}_j$, associated to $g_j = \mathrm{id}$, and to the partition $I^+ = I$, $I^- = \varnothing$. Clearly the $\mathcal{A}_j$ form a partition of the interior of $\mathcal{C}$.

We just reproved [KKL14, Theorem 4.2] (which is stated in arbitrary dimension, but over $\mathbf{C}$):

Theorem 3.3. *The interior of the cone $\mathcal{C}$ admits a finite partition in polyhedral chambers* $\mathrm{Int}(\mathcal{C}) = \bigcup_{j \in J} \mathcal{A}_j$.

For further reference we sum-up the above discussion:

Set-Up 3.4.

- We start with a finite collection $\{(S_1/B_1, \varphi_1), \dots, (S_r/B_r, \varphi_r)\}$ of marked rank 1 fibrations.
- We pick Z a common resolution with Picard number $\rho(Z) \geqslant 4$.
- We choose A an ample $\mathbf{Q}$-divisor on Z such that each map $Z \to S_i$ is $(K + A)$-negative with scaling by an ample divisor Δ_i. If there are finitely many birational morphisms from Z, we allow $A = 0$.
- We construct a convex cone $\mathcal{C}^\circ$ in $N^1(Z)$, by considering the union of all segments $[\Delta, K + A] \cap \overline{\mathrm{Eff}}(Z)$, for all convex combinations Δ of the ample divisors $K + A + r_i\Delta_i$, and by taking the cone over these. In practice, we work with $\mathcal{C}$, the section of $\mathcal{C}^\circ$ by the affine hyperplane corresponding to classes D such that $(K + A) \cdot D = -1$.
- Each class D in the interior of $\mathcal{C}$ corresponds to a $(K + A)$-birational morphism $\varphi_D \colon Z \to S_D$.
- Conversely, given $g_j \colon Z \to S_j$ a $(K + A)$-negative birational morphism, the set $\mathcal{A}_j$ of divisors D in $\mathcal{C}$ such that $g_j = \varphi_D$ form a polyhedral chamber $\mathcal{A}_j$ with non-empty interior.

Remark 3.5. In higher dimension several complications arise that we avoided in the above discussion. In particular in dimension $\geqslant 3$ it is not true anymore that each $\mathcal{A}_j$ spans $N^1(Z)$, because of the appearance of small contractions.

We should also mention that in dimension 2, the decomposition of Theorem 3.3 can be phrased in terms of Zariski decompositions (see [BKS04]). Namely, in each chamber $\mathcal{A}_j$ the support of the negative part of the Zariski decomposition is constant, and corresponds to the support of the classes C_i with $i \in I^-$.

Finally, as already noticed in Remark 1.10, since we only consider adjoint divisors of the form $K + \Delta$ with Δ ample, we can work directly in the Néron-Severi space $N^1(Z)$ instead of choosing a subspace of the space of Weil divisors, as in [HM13, Kal13].

Example 3.6. In [Kal13, Figures 1 and 6] the above construction is illustrated by the case of $\mathbb{P}^2$ blown up at 2 or 3 distinct points. Here we consider the case of $\mathbb{P}^2$ blown up at two points (of degree 1), with one infinitely near the other. Precisely, let Z be the surface obtained from $\mathbb{P}^2$ by blowing-up a point $p = L \cap L'$ intersection of two lines,

producing an exceptional divisor E, and then $p' = E \cap L'$, producing an exceptional divisor E' (see Figure 3, where the numbers in bracket denote self-intersection, and where we use the same notation for a curve and its strict transforms).

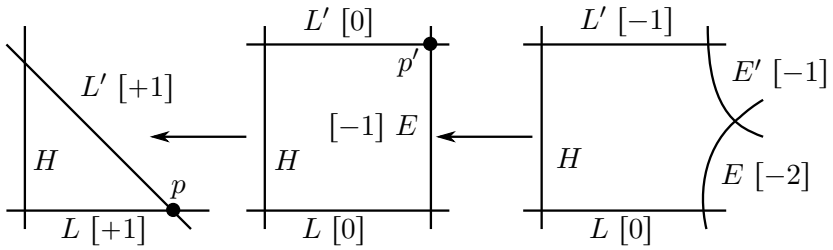

FIGURE 3. $\mathbb{P}^2$ blown up at p and p'.

On Z, the curves E, E', L' are the only irreducible divisors with negative self-intersection, and they generate the pseudo-effective cone $\overline{\mathrm{Eff}}(S)$. We also denote by H the class of a generic line from $\mathbb{P}^2$, and as usual K is the canonical divisor. On Z we have

$$H = L' + E + 2E', \qquad -K = 3L' + 2E + 4E'.$$

The classes C_i of contracted curves are

$$C_1 = L', \qquad C_2 = E', \qquad C_3 = E + E'.$$

In this simple example, it turns out that any class D in the pseudo-effective cone $\overline{\mathrm{Eff}}(Z)$ corresponds to a birational morphism φ_D. There are 6 possibilities for this morphism φ_D, and we have the following chamber decomposition for the whole $\overline{\mathrm{Eff}}(Z)$:

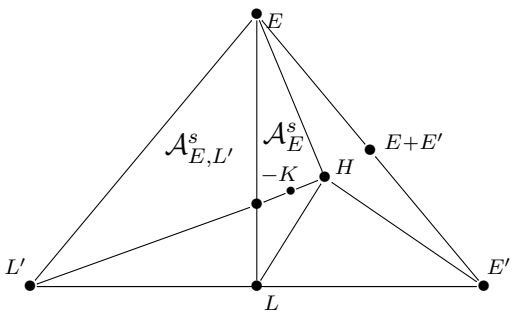

FIGURE 4

However, Set-Up 3.4 only guaranties the chamber decomposition for classes in $\mathcal{C}$, which is pictured on Figure 5 (where we work with the choice $A = 0$). Observe in particular that chambers $\mathcal{A}^s_E$ and $\mathcal{A}^s_{E,L'}$ in Figure 4 corresponds to singular surfaces, namely the blow-down of E, or the blow-down of E, L'. Equivalently, a wall between chambers in Figure 4 does not always correspond to the contraction of a (-1)-curve. In contrast, chambers $\mathcal{A}_1, \ldots, \mathcal{A}_4$ in Figure 5 correspond to smooth surfaces, and the chambers are delimited by the hyperplanes $C_i^\perp$.

The following two propositions correspond to [HM13, Theorem 3.3], in the case of surfaces.

Proposition 3.7. *Assume Set-Up 3.4, let $\mathcal{A}_j$ be one of the polyhedral chambers given by Theorem 3.3, and $g_j \colon Z \to S_j$ the associated birational morphism. If $D \in \bar{\mathcal{A}}_j \smallsetminus \mathcal{A}_j$ with associated morphisms $\varphi_D \colon Z \to Y$, then there exists a unique morphism $f \colon S_j \to Y$ such that $\varphi_D = f \circ g_j$.*

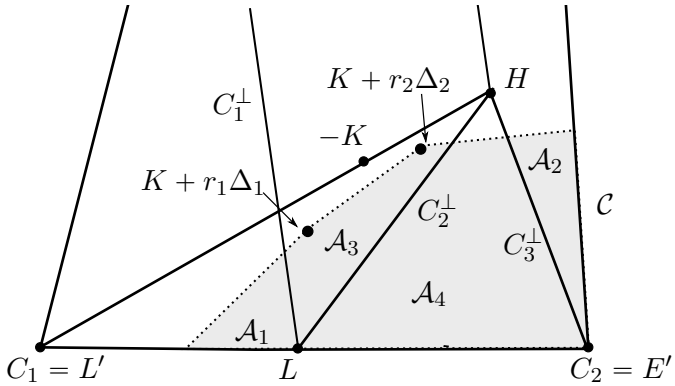

FIGURE 5. The partition $\mathrm{Int}(\mathcal{C}) = \bigcup_{i=1}^4 \mathcal{A}_i$ (in gray).

Proof. Set $D_0 = D \in \bar{\mathcal{A}}_j \smallsetminus \mathcal{A}_j$, and pick $D_1 \in \mathcal{A}_j$. Then for all $t \in]0,1]$, we have $D_t := tD_1 + (1-t)D_0 \in \mathcal{A}_j$. The curves contracted by $g_j = \varphi_{D_t}$ are the curves C such that $D_t \cdot C \leqslant 0$ for $t \in]0,1]$, and so also for $t = 0$. Hence $\mathrm{NE}(\varphi_D) \subset \mathrm{NE}(g_j)$, and the Rigidity Lemma below gives the expected morphism $f\colon S_j \to Y$. □

Lemma 3.8 (Rigidity Lemma, see [Deb01, Proposition 1.14])**.** *Let X, Y, Y' be projective varieties and let $\pi\colon X \to Y$, $\pi'\colon X \to Y'$ be morphisms with connected fibers. If $NE(\pi) \subset NE(\pi')$, there is a unique morphism $f\colon Y \to Y'$ such that $\pi' = f \circ \pi$.*

Proposition 3.9. *Assume Set-Up 3.4. Let j, k be two indexes such that $\bar{\mathcal{A}}_j \cap \mathcal{A}_k \neq \varnothing$, and let $g_{j,k}\colon S_j \to S_k$ be the morphism (given by Proposition 3.7) such that $g_k = g_{j,k} \circ g_j$. Then the relative Picard number of $g_{j,k}\colon S_j \to S_k$ is equal to the codimension of $\bar{\mathcal{A}}_j \cap \bar{\mathcal{A}}_k$ in $\bar{\mathcal{A}}_j$.*

Proof. By definition there exist two partitions of the set of indexes $I = I_j^+ \cup I_j^- = I_k^+ \cup I_k^-$ such that

$$\mathcal{A}_j = \bigcap_{i \in I_j^+} C_i^{>} \cap \bigcap_{i \in I_j^-} C_i^{\leqslant} \cap \mathcal{C},$$
$$\mathcal{A}_k = \bigcap_{i \in I_k^+} C_i^{>} \cap \bigcap_{i \in I_k^-} C_i^{\leqslant} \cap \mathcal{C}.$$

The condition $\bar{\mathcal{A}}_j \cap \mathcal{A}_k \neq \varnothing$ means that $I_j^- \subsetneq I_k^-$. Let $\{i_1, \dots, i_t\} = I_j^+ \cap I_k^-$. Then

$$\bar{\mathcal{A}}_j \cap \bar{\mathcal{A}}_k = \bigcap_{i \in I_k^+} C_i^{\geqslant} \cap \bigcap_{i \in \{i_1, \dots, i_t\}} C_i^{\perp} \cap \bigcap_{i \in I_j^-} C_i^{\leqslant} \cap \mathcal{C}.$$

The morphism $g_{j,k}$ corresponds to the contraction of the classes $C_{i_1}, \dots, C_{i_t}$, and t is by construction the codimension of $\bar{\mathcal{A}}_j \cap \bar{\mathcal{A}}_k$ in $\bar{\mathcal{A}}_j$. □

3.2. **Boundary of $\mathcal{C}$.** Assuming Set-up 3.4, we define $\partial^+\mathcal{C}$ as the intersection of $\mathcal{C}$ with the boundary of the pseudo-effective cone in $N^1(Z)$, or equivalently, as the classes in $\mathcal{C}$ that are not big. If $\rho = \rho(Z)$ is the Picard number of Z, then by assumption $\mathcal{C}^\circ$ is a cone of full dimension ρ, hence the affine section $\mathcal{C}$ is homeomorphic to a ball $B^{\rho-1}$ and $\partial^+\mathcal{C}$ is homeomorphic either to a ball $B^{\rho-2}$, or to a sphere $S^{\rho-2}$, depending whether the pseudo-effective cone is contained in $\mathcal{C}$ or not. For instance the sphere situation arises if Z is del Pezzo, $A = 0$, and the initial collection of rank 1 fibrations S_i/B_i is the full (finite) collection of such fibrations dominated by Z.

Now we put a structure of polyhedral complex on $\partial^+\mathcal{C}$. We consider the set of chambers $\mathcal{A}_j$ such that $\bar{\mathcal{A}}_j \cap \partial^+\mathcal{C}$ contains a codimension 1 face $\mathcal{W}$ of the closed polytope $\bar{\mathcal{A}}_j$. We call such a face $\mathcal{W}$ a *window* of $\mathcal{A}_j$. Then the collection of such $\mathcal{W}$ uniquely defines a structure of polyhedral complex on $\partial^+\mathcal{C}$, such that the $\mathcal{W}$ are the codimension

0 faces of the complex. More generally we shall denote $\mathcal{F}^k$ a (closed) codimension k face of $\partial^+\mathcal{C}$. We say that $\mathcal{F}^k$ is an *inner* face if it intersects the relative interior of $\partial^+\mathcal{C}$. Equivalently, $\mathcal{F}^k$ is inner if it can be written as the intersection of $k+1$ windows.

For a window $\mathcal{W}$ of the chamber $\mathcal{A}_j$, D in the relative interior of $\mathcal{W}$, and sufficiently small $\varepsilon > 0$, the divisor $D' := D - \varepsilon(K+A)$ is in $\mathcal{A}_j$, and the images $S = \varphi_{D'}(Z)$ and $B = \varphi_D(Z)$ correspond to a Mori fibration S/B that depends only on $\mathcal{W}$ (and not on the particular choice of D or ε).

We see that the codimension 0 faces of $\partial^+\mathcal{C}$ are in bijection with the rank 1 fibrations S^1/B dominated by a $(K+A)$-negative map from Z, and such that S^1/B is $(-K-A)$-ample. More generally, we have:

Proposition 3.10. (1) *Let $\mathcal{F}^k$ be an inner codimension k face of the polyhedral complex $\partial^+\mathcal{C}$. Then there exists a uniquely defined rank $k+1$ fibration S^{k+1}/B such that:*

- *S^{k+1}/B is $(-K-A)$-ample;*
- *The induced map $Z \to S^{k+1}$ is equal to g_j for some $g \in \{1, \cdots, s\}$, in particular this is a $(K+A)$-negative birational morphism;*
- *The chamber $\mathcal{A}_j$ associated with g_j satisfies $\bar{\mathcal{A}}_j \cap \partial^+\mathcal{C} = \mathcal{F}^k$.*

(2) *If moreover $\mathcal{F}^k \subset \mathcal{F}^j$ is contained in a codimension j face $\mathcal{F}^j$, then the rank $j+1$ fibration associated to $\mathcal{F}^j$ factorizes through the rank $k+1$ fibration associated to $\mathcal{F}_k$.*

Proof. (1) Let D be a class in the relative interior of $\mathcal{F}^k$. By definition, there exists an ample class $\Delta \in \mathcal{C}^\circ$ such that D lies in the segment $[\Delta, K+A]$. Moreover for sufficiently small ε, $D' = D - \varepsilon(K+A)$ lies in a chamber $\mathcal{A}_S$ (in fact in the interior of $\mathcal{A}_S$, since $K+A$ is negative against the exceptional curves C_i), where S does not depend on D nor ε. Let $Z \to B$ be the morphism associated to D, and $Z \to S$ the morphism associated to $D - \varepsilon(K+A)$. The induced morphism (Proposition 3.7) $S \to B$ admits a factorization (obtained by running a relative MMP over B)

$$S^{r+1} \to S^r \to \cdots \to S^1 \to B,$$

where each $S^{i+1} \to S^i$ is the contraction of one exceptional curve. By Proposition 3.9, $\mathcal{A}_{S^{r+1}}$ and $\mathcal{A}_{S^1}$ share a codimension r face, and the intersection of this face with $\partial^+\mathcal{C}$ has codimension r in $\partial^+\mathcal{C}$.

(2) With the same notation as above, there exists a small ball $B(\Delta, \delta)$ inside the ample chambers, such that if $D' = (1-t')\Delta + t'(K+A)$, then all divisors of the form $(1-t')B(\Delta,\delta) + t'(K+A) \subset \mathcal{A}_S$. Now there exists $1 \geqslant t \geqslant t'$ such that $D = (1-t)\Delta + t(K+A)$. The open set $(1-t)B(\Delta,\delta) + t(K+A)$ meets $\mathcal{F}^j$ by assumption, hence also the associated chamber $\mathcal{A}_j$. So we can pick $\Delta' \in B(\Delta,\delta)$ such that the segment $[\Delta', K+A]$ meets successively $\mathcal{A}_S$ and $\mathcal{A}_j$. □

Corollary 3.11. *If the intersection $\mathcal{W}_i \cap \mathcal{W}_j$ of two windows has codimension 1 in $\mathcal{W}_i$ (hence also in $\mathcal{W}_j$), then there is a Sarkisov link between the corresponding Mori fiber spaces.*

Proof. By Proposition 3.10, there exists a rank 2 fibration corresponding to the codimension 1 face $\mathcal{F}^1 = \mathcal{W}_i \cap \mathcal{W}_j$, that factorizes through the rank 1 fibrations associated respectively to $\mathcal{W}_i$ and $\mathcal{W}_j$. This is exactly our definition of a Sarkisov link. □

Remark 3.12. The above corollary corresponds to [Kal13, Lemma 3.17], but the situation for surfaces is simpler (see Figure 6). Let $\mathcal{A}_i$, $\mathcal{A}_j$ be the two chambers with window $\mathcal{W}_i$, $\mathcal{W}_j$, and S_i/B_i, S_j/B_j the corresponding rank 1 fibrations. We write $\mathcal{F}^1 = \mathcal{W}_i \cap \mathcal{W}_j$. We distinguish 3 cases in term of the codimension of the intersection $\overline{\mathcal{A}_i} \cap \overline{\mathcal{A}_j}$.

(a) If $\overline{\mathcal{A}_i} = \overline{\mathcal{A}_j}$, that is, $\mathcal{W}_i$ and $\mathcal{W}_j$ are two windows of a same chamber, then we have a link of type IV. In the case of a rational surface, the only possibility is the

change of ruling on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The fibrations S_i/B_i and S_j/B_j correspond to the two rulings on $\mathbb{F}_0$, and the codimension 1 cell $\mathcal{F}^1$ to the rank 2 fibration $\mathbb{F}_0/\mathrm{pt}$.

(b) If $\overline{\mathcal{A}_i}$ and $\overline{\mathcal{A}_j}$ share a codimension 1 face, then we have a link of type I or III. Up to reordering we can assume that we have a blow-up map $S_j \to S_i$. Then $B_j = \mathbb{P}^1$, $B_i = \mathrm{pt}$, and $\mathcal{F}^1$ corresponds to the rank 2 fibration S_j/pt.

(c) Otherwise, let S^2/B be the rank 2 fibration associated to $\mathcal{F}^1$, as given by Proposition 3.10. The chamber $\mathcal{A}_{S^2}$ is distinct from $\mathcal{A}_i$ and $\mathcal{A}_j$, otherwise we would be in one of the two previous cases. Then $\mathcal{A}_{S^2}$ shares a codimension 1 face with $\overline{\mathcal{A}_i}$ and with $\overline{\mathcal{A}_j}$, and we have a link of type II.

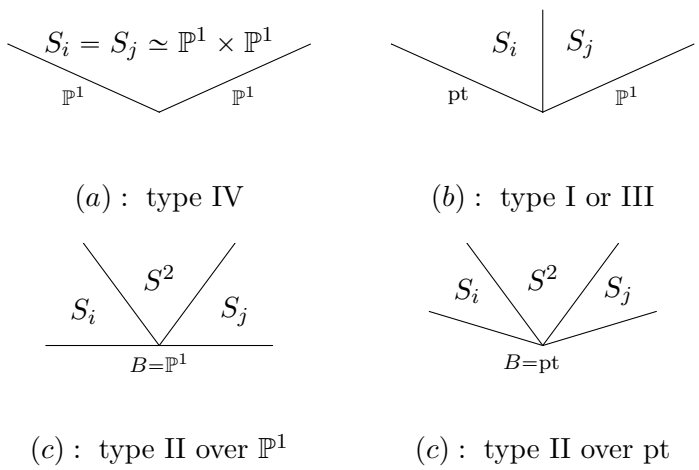

FIGURE 6. Adjacent windows and Sarkisov links

Corollary 3.13. *Let $\mathcal{F}^2$ be an inner codimension 2 cell in $\partial^+\mathcal{C}$. Let S^3/B be the associated rank 3 fibration, as given by Proposition 3.10. Then the elementary relation associated to S^3/B correspond to the finite collection of windows $\mathcal{W}_1, \dots, \mathcal{W}_k$ containing $\mathcal{F}^2$ in their closure, and ordered such that $\mathcal{W}_j$ and $\mathcal{W}_{j+1}$ share a codimension 1 face for all j (where indexes are in $\mathbf{Z}/k\mathbf{Z}$).*

The following two propositions correspond to the two assertions in Theorem 3.1.

Proposition 3.14. *Any birational map $f\colon S \dashrightarrow S'$ between rank 1 fibrations is a composition of Sarkisov links, and in particular the complex $\mathcal{X}$ is connected.*

Proof. We want to prove that two vertices in $\mathcal{X}$ corresponding to two rank 1 fibrations are connected by a path. Let S_1/B_1 and S_2/B_2 be these two fibrations, and apply Set-Up 3.4 to this collection of two fibrations. Let Δ_1, Δ_2 be ample divisors in $\mathcal{C}$ such that for $i = 1, 2$, the fibration S_i/B_i corresponds to a $(K + A)$-negative map with scaling Δ_i. Up to a small perturbation of Δ_1 and Δ_2, we can assume that the 2 dimensional affine plane containing Δ_1, Δ_2 and $K + A$ intersects transversally the faces of $\partial^+\mathcal{C}$. This means that the windows $\mathcal{W}_1$ and $\mathcal{W}_2$ corresponding to S_1/B_1 and S_2/B_2 are connected by a finite sequence of windows, where two successive windows share a codimension 1 face. By Corollary 3.11, this corresponds to a sequence of Sarkisov links, hence the expected path in the complex $\mathcal{X}$. □

Proposition 3.15. *Any relation between Sarkisov links is a composition of elementary relations, and in particular the complex $\mathcal{X}$ is simply connected.*

Proof. Let γ be a loop in $\mathcal{X}$. Without loss in generality, we can assume that the image of γ lies in the 1-skeleton of $\mathcal{X}$. We can also assume that the loop visits only vertices of rank 1 or 2. Indeed, using elementary relations, that is, moving around the boundary of disks as provided by Proposition 2.6, we can avoid all vertices of rank 3. Moreover, we can

assume that the base point is a vertex of rank 1, S_1/B_1. Now the sequence of vertices of rank 1 visited by the loop corresponds to a sequence $S_1/B_1, S_2/B_2, \dots, S_r/B_r = S_1/B_1$, where for each i, the map from S_i/B_i to S_{i+1}/B_{i+1} is a Sarkisov link. Consider the surface Z and the complex $\mathcal{C}$ associated to this collection, as in Set-Up 3.4, and denote as above $\partial^+\mathcal{C}$ the non-big boundary with its induced structure of polyhedral complex.

Now we consider the barycentric subdivision $\mathcal{B}\partial^+\mathcal{C}$ of the polyhedral complex $\partial^+\mathcal{C}$, which is defined as follows (see e.g. [Mun84, p. 83]). For each face $\mathcal{F}^k$ of codimension k ($\rho - 2 \geqslant k \geqslant 0$), we denote by $p(\mathcal{F}^k)$ the barycenter of the vertices of $\mathcal{F}^k$. The d-simplexes of $\mathcal{B}\partial^+\mathcal{C}$ are defined as the convex hulls of the $p(\mathcal{F}^k)$, for each sequence of nested faces $\mathcal{F}^{i_d} \subset \mathcal{F}^{i_{d-1}} \subset \cdots \subset \mathcal{F}^{i_1} \subset \mathcal{F}^{i_0}$, where $i_d > \cdots > i_1 > i_0$. If $\mathcal{F}^{i_d}$ is an inner face (hence also all other faces of the sequence), we say that the corresponding simplex is an inner simplex of $\mathcal{B}\partial^+\mathcal{C}$. The inner simplexes form a subcomplex $\mathcal{I}\partial^+\mathcal{C} \subset \mathcal{B}\partial^+\mathcal{C}$, and $\mathcal{I}\partial^+\mathcal{C}$ is a deformation retract of the interior of $\mathcal{B}\partial^+\mathcal{C}$: this follows from Lemma 3.16 below, with $X = \mathcal{B}\partial^+\mathcal{C}$, $A = \mathcal{I}\partial^+\mathcal{C}$ and C the boundary of $\mathcal{B}\partial^+\mathcal{C}$. Since $\mathcal{B}\partial^+\mathcal{C}$ is homeomorphic to a ball or a sphere of dimension $\rho(Z) - 2 \geqslant 2$, it is simply connected, so that $\mathcal{I}\partial^+\mathcal{C}$ also is simply connected. By Corollaries 3.11 and 3.13, the 2-skeleton of the complex $\mathcal{I}\partial^+\mathcal{C}$ is exactly the (simplicial version) of the subcomplex of $\mathcal{X}$ generated by the S_i/B_i (see §2.2). Being the 2-skeleton of a simply connected complex, this 2-complex also is simply connected (see [Hat02, Corollary 4.12]). Moreover, by Proposition 2.6, each face $\mathcal{F}^2$ of codimension 2 in $\mathcal{I}\partial^+\mathcal{C}$ is the center of a disk whose boundary corresponds to an elementary relation. By construction, the interiors of these disks are pairwise disjoint, and any triangle of $\mathcal{I}\partial^+\mathcal{C}$ belongs to one of the disks. In conclusion, we can perform the required homotopy of our initial loop to a constant loop inside this subcomplex of $\mathcal{X}$ by using elementary relations. □

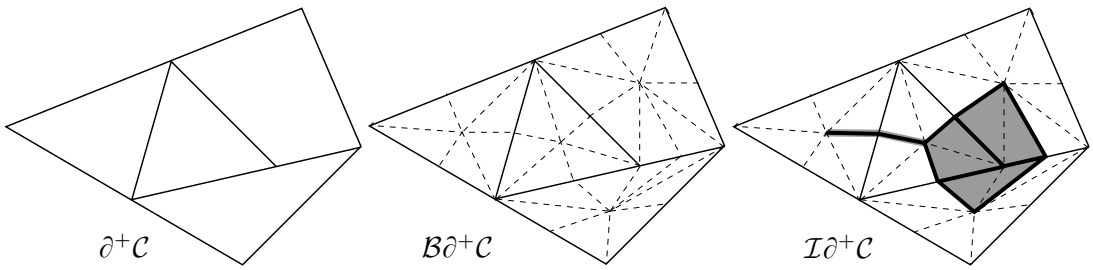

FIGURE 7. The complexes $\partial^+\mathcal{C}$, $\mathcal{B}\partial^+\mathcal{C}$ and $\mathcal{I}\partial^+\mathcal{C}$.

Lemma 3.16 ([Mun84, Lemma 70.1])**.** *Let A be a full subcomplex of a finite simplicial complex X. Let C consist of all simplices of X that are disjoint from A. Then A is a deformation retract of $X \smallsetminus C$.*

4. Elementary generators

4.1. **Definition.** As in the previous section we consider a path of Sarkisov links

$$(S_0/B_0, \varphi_0) \dots, (S_n/B_n, \varphi_n),$$

that is, for each $0 \leqslant i \leqslant n-1$, there is a Sarkisov link $g_i \colon S_i/B_i \dashrightarrow S_{i+1}/B_{i+1}$. If in such a path S_0, S_n are both isomorphic to $\mathbb{P}^2$, but no intermediate vertex S_i is isomorphic to $\mathbb{P}^2$, we say that $\varphi_n g_{n-1} \cdots g_1 g_0 \varphi_0^{-1} \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is an *elementary generator*.

We denote by $\mathcal{E}$ a choice of representatives of elementary generators in $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, up to right and left composition by elements of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$. From the Sarkisov program in dimension 2 it directly follows:

Proposition 4.1. *Any $f \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is a composition of elementary generators, up to an automorphism. In particular, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), \mathcal{E} \rangle$.*

Proof. By Theorem 3.1(1), there exists a sequence of Sarkisov links decomposing f. Then we can cut this sequence at each intermediate surface isomorphic to $\mathbb{P}^2$. □

Remark 4.2. The set $\mathcal{E}$ contains the list of generators given in [IKT93], which contains all Jonquières transformations. The set $\mathcal{E}$ is really huge: over an algebraically closed field, in fact every element of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is in $\mathcal{E}$, which does not seem a very clever choice of generators. However, when working over a non-closed field, and since we never try to work with an explicit presentation of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ by generators and relations, the immensity of $\mathcal{E}$ is not a draw-back.

In the following sections we study two particular examples of elementary generators.

4.2. **Bertini involutions.** Let $p = \{p_1, \dots, p_8\} \in \mathbb{P}^2$ a point of degree 8. We say that p is *general*, or equivalently that the p_i are in *general position*, if the blow-up of p is a del Pezzo surface of degree 1, that is, if no line (defined over $\mathbf{k}^a$) contains 3 of the p_i, no conic contains six of them, and no cubic is singular at one of them and contains all the others.

Let S be the surface obtained by blowing-up such a general point p of degree 8. Then S is a rank 2 fibration with exactly two outgoing arrows, and there exists another contraction $S \to \mathbb{P}^2$ that fits into a diagram

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "b"'] & & \mathbb{P}^2 \end{tikzcd} \tag{3}$$

where b is a Bertini involution (this link is noted χ_8 in [IKT93]). Recall that geometrically, b is defined as follows. Since p is general, the linear system Γ of cubics through p is a pencil whose general member is smooth. The base locus of the pencil is equal to $p \cup q$, where q is a point of degree 1. For $x \in \mathbb{P}^2$ a general point, the unique member of Γ through x is a smooth cubic, that we can see as an elliptic curve with neutral element q. Then $b(x) = -x$, where $-x$ means the opposite of x with respect to the group law on the elliptic curve.

In particular, such a link b is an elementary generator as defined above. Now the crucial but easy observation is:

Lemma 4.3. *In diagram (3), each contraction $S \to \mathbb{P}^2$ corresponds to an edge in $\mathcal{X}$ which is not contained in any square.*

Proof. If $S/\mathrm{pt} \to \mathbb{P}^2/pt$ corresponds to an edge of a square, we would have a rank 3 fibration S'/pt that factorizes through S/pt. But such a surface S' would be a del Pezzo surface, and would be a blow-up of S which is already a del Pezzo surface of degree 1; contradiction. □

Up to changing the initial choice of representatives $\mathcal{E}$, we can assume that a representive of a Bertini involution is an involution. We denote by $\mathcal{B} \subseteq \mathcal{E}$ the subset of representatives of Bertini involutions.

Example 4.4. Since the above construction relies on the existence of a Galois extension of degree 8, we recall a few examples of such extensions:

(1) $\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbf{Q}$ is Galois with Galois group isomorphic to $(\mathbf{Z}/2\mathbf{Z})^3$. This is the splitting field of $(X^2-2)(X^2-3)(X^2-5)$.

(2) The cyclotomic extension $\mathbf{Q}(e^{2i\pi/15})/\mathbf{Q}$ has degree $\varphi(15) = 8$ and Galois group isomorphic to $(\mathbf{Z}/15\mathbf{Z})^* \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. This is the splitting field of the 15th cyclotomic polynomial $\Phi_{15}(X) \in \mathbf{Z}[X]$ ([Mor96, Corollary 7.8]).

(3) $\mathbf{Q}(\sqrt[4]{2}, i)/\mathbf{Q}$ is Galois with Galois group isomorphic to the dihedral group D_8. This is the splitting field of $(X^4-2)(X^2+1)$. Generators for the Galois group are r, s where

$$r(\sqrt[4]{2}) = i\sqrt[4]{2},\ r(i) = i \text{ and } s(\sqrt[4]{2}) = \sqrt[4]{2},\ s(i) = -i.$$

(4) Let $\alpha = e^{i\pi/4}$, and set $\mathbf{k} = \mathbf{Q}(\alpha)$. Then pick $\beta \in \mathbf{k}$ which is not a square in $\mathbf{k}$: for instance $\beta = 3$ is a possible choice, but $\beta = 2$ is not because $\alpha + \alpha^7 = \sqrt{2}$. Then $\mathbf{k}(\sqrt[8]{\beta})/\mathbf{k}$ is a cyclic extension of degree 8, that is, Galois with Galois group isomorphic to $\mathbf{Z}/8\mathbf{Z}$ ([Mor96, Corollary 9.6]).

(5) $\mathbf{Q}(\theta)/\mathbf{Q}$ with $\theta^2 = (2+\sqrt{2})(2+\sqrt{3})(3+\sqrt{6})$ is Galois with Galois group isomorphic to the quaternionic group. This is the splitting field of $X^8 - 72X^6 + 180X^4 - 144X^2 + 36$ ([Dea81]).

(6) Let $\mathbf{F}_q$ be a finite field, and $\mathbf{F}_{q^n}/\mathbf{F}_q$ the (essentially unique) extension of degree n (in particular one can take $n = 8$). Then this extension is Galois, with Galois group isomorphic to $\mathbf{Z}/n\mathbf{Z}$, generated by the Frobenius automorphism $x \mapsto x^q$ ([Mor96, Corollary 6.7]).

Now we turn to the problem of proving that the set $\mathcal{B}$ is large, that is, there exist many Bertini involutions, even modulo the action of $\mathrm{PGL}_3(\mathbf{k})$. We shall produce points of degree 8 in general position by using nodal cubics. The following set-up about the group structure on the smooth locus of a nodal cubic is classical, see for instance [Sil09, §III.2, Proposition 2.5], and also Remark 4.8. However since we want to work over an arbitrary field (typically non-algebraically closed), we find convenient to make a slightly different choice of normal form.

Set-Up 4.5. Consider the plane nodal cubic curve C_P defined over $\mathbf{k}$, given by the equation

$$xyz = P(x,z),$$

where $P(x,z) = c_0x^3 - c_0z^3$, $c_0 \in \mathbf{k}^*$. In the affine chart $z = 1$, the equation becomes

$$y = \frac{P(x,1)}{x} \left(= \frac{c_0x^3 - c_0}{x} \right).$$

Observe also that the singular point $[0:1:0]$ is the only intersection point between C_P and the line at infinity $z = 0$.

We shall be interested in nodal cubics up to the action of $\mathrm{PGL}_3(\mathbf{k})$, and we shall use the above set-up as a normal form. Observe that we cannot assume $c_0 = 1$ even after applying a diagonal element of $\mathrm{PGL}_3(\mathbf{k})$, because c_0 might not be a cube in $\mathbf{k}$.

Lemma 4.6. *Let $C \subset \mathbb{P}^2$ be an irreducible nodal cubic with singular point at $[0:1:0]$, and tangent cone at this point given by $xz = 0$. Then C admits an equation of the form*

$$xyz = c_0x^3 + c_1x^2z + c_2xz^2 + c_3z^3, \qquad c_0, c_3 \in \mathbf{k}^*, c_1, c_2 \in \mathbf{k},$$

and C is equivalent under the action of $\mathrm{PGL}_3(\mathbf{k})$ to a cubic from Set-Up 4.5 if and only if $-\frac{c_0}{c_3}$ is a cube in $\mathbf{k}$.

Proof. The assumption on the singular point implies that C admits an equation of the form $xyz = P(x,z)$ with P a homogeneous polynomial of degree 3. Moreover $P(0,1) \neq 0$ and $P(1,0) \neq 0$, otherwise C would be reducible. So $P(x,z) = c_0x^3 + c_1x^2z + c_2xz^2 + c_3z^3$, with $c_0, c_3 \in \mathbf{k}^*$, and no condition on c_1, c_2.

By applying $(x,y,z) \mapsto (x, y + c_1x + c_2z, z)$ we can assume $c_1 = c_2 = 0$. Then, if $-\frac{c_0}{c_3} = a^3$ for some $a \in \mathbf{k}^*$, by applying $(x,y,z) \mapsto (x, a^{-1}y, az)$ we get $c_0 = -c_3$. □

Lemma 4.7. *Assume Set-up 4.5, and consider a collection of three or six pairwise distinct $a_i \in \mathbf{k}^*$. Then:*

(1) *The points $p_i = \left(a_i, \frac{P(a_i,1)}{a_i}\right) \in C_P$, $i = 1,2,3$, are on a same line if and only if $a_1a_2a_3 = 1$.*
(2) *The points $p_i = \left(a_i, \frac{P(a_i,1)}{a_i}\right) \in C_P$, $i = 1,\dots,6$, are on a same conic if and only if $a_1 \dots a_6 = 1$.*

Proof. (1) The general equation of a line that does not intersect the cubic C_P at infinity is $y + Ax + B = 0$. If the p_i are on a same line, then replacing $y = \frac{P(x,1)}{x}$ in the equation of the line we get

$$P(x,1) + Ax^2 + Bx = c_0(x - a_1)(x - a_2)(x - a_3),$$

hence, by comparing the constant terms and dividing by $-c_0$, we get $1 = a_1a_2a_3$ as expected. Conversely, if $a_1a_2a_3 = 1$, then the above relation allows to define A, B in function of the a_i such that the line $y + Ax + B = 0$ contains the three points p_i.

(2) The proof is similar, working with the general equation of a conic that does not intersect the cubic at infinity:

$$y^2 + Ayx + By + Cx^2 + Dx + E = 0.$$

Replacing $y = \frac{P(x,1)}{x}$ we get

$$P(x,1)^2 + Ax^2P(x,1) + BxP(x,1) + Cx^4 + Dx^3 + Ex^2$$
$$= c_0^2x^6 + \cdots + c_0^2 = c_0^2\prod_{i=1}^{6}(x - a_i),$$

and we conclude as in the previous case. □

Remark 4.8. On the smooth locus C^{sm} of a nodal cubic C, with a choice of e a flex point, recall that there exists a group law defined similarly as in the case of an elliptic curve. Given $p, q \in C^{sm}$, define $p \circ q$ as the third point of intersection of C with the line through p and q (or the tangent by p if $p = q$). Then by setting $p \cdot q := (p \circ q) \circ e$, we get a group law $\cdot$ with neutral element e. The previous lemma shows that in the case of C_P given by Set-Up 4.5, where one can check that $e = (1, 0)$ is a flex point, the map $x \mapsto (x, \frac{P(x,1)}{x})$ is a group morphism from $\mathbf{k}^*$ to C_P^{sm}.

We shall need the following result about singular fibers of a pencil of cubic curves:

Lemma 4.9. *Let Γ be a pencil of plane cubic curves. Then Γ contains at most 12 nodal cubics.*

Proof. Consider the surface S obtained by blowing-up $\mathbb{P}^2$ at the nine base points of the pencil. Then S admits an elliptic fibration and has Euler number $c_2(S) = 12$, which by Ogg's formula is equal to the sum over the singular fibers of the valuation $v(\Delta)$ of the minimal discriminant. Independently of the characteristic of the base field, each nodal cubic contributes by 1 to this sum, hence the result (see e.g. [Lie13, §5.3]). □

Now we apply this set-up to the case of a field which admits a Galois extension of degree 8.

Proposition 4.10. *Assume Set-Up 4.5. Let $\mathbf{L}/\mathbf{k}$ be a Galois field extension of degree 8, $b_1, \ldots, b_8 \in \mathbf{L}$ be an orbit under $\mathrm{Gal}(\mathbf{L}/\mathbf{k})$, and $\lambda \in \mathbf{k}^*$. Set $a_i = \lambda b_i$, so that $a_1, \ldots, a_8 \in \mathbf{L}$ also is a Galois orbit. Then, except for at most 6 values of λ, the points $p_i = (a_i, \frac{P(a_i,1)}{a_i}) \subset \mathbb{A}^2 \subset \mathbb{P}^2$ are in general position.*

Before giving the proof we establish a corollary.

Corollary 4.11. *Let $\mathbf{k}$ be an infinite field that admits at least one Galois extension of degree 8. Then the set $\mathcal{B}$ of representatives of Bertini involutions up to conjugacy by $\mathrm{PGL}_3(\mathbf{k})$ has at least the same cardinality than the field $\mathbf{k}$.*

Proof. Assume that $a_1, \ldots, a_8$ is a Galois orbit such that the points $p_i = \left(a_i, \frac{P(a_i,1)}{a_i}\right)$ are in general position. It is sufficient to prove that for any $\lambda \in \mathbf{k}^*$ except finitely many, the points $q_i = \left(\lambda a_i, \frac{P(\lambda a_i,1)}{\lambda a_i}\right)$ also are in general position and are not equivalent to the p_i under the action of $\mathrm{PGL}_3(\mathbf{k})$. First by Proposition 4.10, by avoiding 6 values of λ we can assume that the q_i are in general position. Assume that $g \in \mathrm{PGL}_3(\mathbf{k})$ sends

the p_i on the q_i. By assumption the nodal cubic C_P from Set-Up 4.5 contains both the p_i and the q_i. Then $g^{-1}(C_P)$ is a nodal cubic through the p_i, and by Lemma 4.9 we know that there are at most 12 of them. Moreover a given nodal cubic is stabilized by finitely many elements of $\mathrm{PGL}_3(\mathbf{k})$, hence the result. □

We separate the proof of Proposition 4.10 into the next three lemmas, where we always assume Set-Up 4.5.

Lemma 4.12. *There is no singular irreducible cubic passing through all the p_i, with one of them the double point.*

Proof. If C is such a cubic, and $C' = \sigma(C)$ is the image of C under a non trivial element $\sigma \in \mathrm{Gal}(\mathbf{L}/\mathbf{k})$, then $C \cdot C' = 2 + 2 + \underbrace{1 + \cdots + 1}_{6} = 10$, a contradiction. □

Lemma 4.13. *Any three points among the $p_i = \left(a_i, \frac{P(a_i,1)}{a_i}\right)$ are not on a line.*

Proof. Assume the contrary, and denote by L a line containing three of the p_i. Since the p_i lie on a nodal cubic, the line L contains exactly three points among the p_i. Then taking the Galois orbit of L we obtain a configuration of 8 lines, each line containing 3 of the p_i, and with 3 lines through each p_i. This is the classical Möbius-Kantor configuration 8_3, see figure 8, where 7 of the lines are represented as lines of the plane, and the label of a vertex indicates the first coordinate of the corresponding point in $\mathbb{A}^2$. Precisely, a, b, c are among the a_i, and the other labels are expressed in terms of a, b, c using Lemma 4.7.

Multiplying the labels of the bottom line we find $c^3 = 1$. But a cubic root of the unity either lives in $\mathbf{k}$ or in a quadratic extension of $\mathbf{k}$, in contradiction with the assumption that c, being any one of the a_i, should satisfy $\mathbf{k}(c) = \mathbf{L}$. □

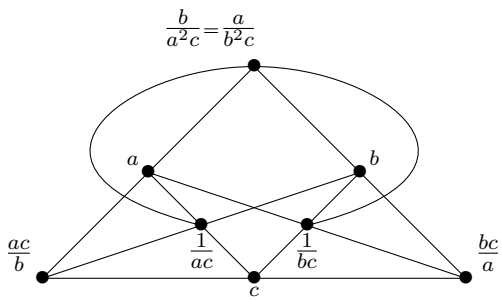

FIGURE 8. The Möbius-Kantor configuration 8_3

Lemma 4.14. *Except for at most 6 values of λ, any 6 points among the points p_i associated to the orbit λb_i are not on a conic.*

Proof. Assume C is a conic through 6 of the p_i. Since the p_i lie on the cubic C_P, Bézout Theorem implies that the 2 remaining points p_i are not on the conic C. Consider σ in the Galois group $\mathrm{Gal}(\mathbf{L}/\mathbf{k})$. Then $\sigma(C)$ is either equal to C, or shares exactly 4 intersection points with C. The only possibility is that the Galois orbit of C consists of 4 conics, and we can group the p_i into 4 pairs, such that each conic passes through three of the four pairs of points. Then the combinatorics is a configuration 4_3: by each pair pass 3 conics, and each conic contains three pairs of points. Denote by $\pi_1, \pi_2, \pi_3, \pi_4$ the product of the first coordinates of each pair of points. By Lemma 4.7 we have

$$\pi_1\pi_2\pi_3 = 1, \qquad \pi_1\pi_2\pi_4 = 1,$$
$$\pi_1\pi_3\pi_4 = 1, \qquad \pi_2\pi_3\pi_4 = 1.$$

It follows that $\pi_1 = \pi_2 = \pi_3 = \pi_4$, and then $\pi_i^3 = 1$ for all i. Finally pick an element $\lambda \in \mathbf{k}^*$, such that $\lambda^6 \neq 1$. Now if we replace a_i by λa_i, and the corresponding points on C_P are still in non-general position, then the same argument gives

$$\lambda^6 = \lambda^6 \pi_i^3 = (\lambda a_i \tau(\lambda a_i))^3 = 1,$$

a contradiction. We conclude that if there is a bad point of degree 8 associated to an orbit a_i, then for all $\lambda \in \mathbf{k}^*$ except maybe the 6th roots of unity, the point of degree 8 associated with the orbit λa_i is in general position with respect to conics. □

In the case of finite fields, for small cardinal the previous statements are empty, and anyways for arbitrary cardinal one would like an estimate of the cardinality of $\mathcal{B}$. We adapt the previous argument as follows.

Lemma 4.15. *Assume Set-Up 4.5 for a finite field $\mathbf{k} = \mathbf{F}_q$. Let $a_1, \dots, a_8 \in \mathbf{F}_{q^8}$ be an orbit under $\mathrm{Gal}(\mathbf{F}_{q^8}/\mathbf{F}_q)$, and assume that the points $p_i = (a_i, \frac{P(a_i,1)}{a_i}) \subset \mathbb{A}^2 \subset \mathbb{P}^2$ are in non-general position. For $\beta \in \mathbf{F}_{q^4}^*$, set $b_i = \beta^{q^{i-1}} a_i$, so that $b_1, \dots, b_8 \in \mathbf{F}_{q^8}$ is also a Galois orbit.*

If $\beta \notin \{x \in \mathbf{F}_{q^4}^*;\ x^6 = 1, x^2 \in \mathbf{F}_q\}$, then the points $q_i = (b_i, \frac{P(b_i,1)}{b_i})$ are in general position.

Proof. Lemmas 4.12 and 4.13 are valid for any β, so the delicate point is only the general position with respect to conics. With the same notation as in the proof of Lemma 4.14, since 6 among the p_i lie on a conic C then we obtain that the products π_i are all equal to a same 3rd root of unity. Now we use the fact that over a finite field, $\tau : x \mapsto x^{q^4}$ is the only element of order 2 in $\mathrm{Gal}(\mathbf{F}_{q^8}/\mathbf{F}_q)$. In particular, τ is the element of order 2 that fixes C. It also fixes all conjugates of C and hence interchanges the two elements in each pair of p_i. Then the product π_i is invariant under the Galois group $\mathrm{Gal}(\mathbf{F}_{q^8}/\mathbf{F}_q)$, so we have $\pi_i \in \mathbf{F}_q$. Pick $\beta \in \mathbf{F}_{q^4} \smallsetminus \{x \in \mathbf{F}_{q^4}^*;\ x^6 = 1, x^2 \in \mathbf{F}_q\}$, and replace the orbit of a_1 by the orbit of $b_1 = \beta a_1$, which still has cardinal 8. Then $\pi_1 \in \mathbf{F}_q$ is replaced by $\beta^2 \pi_1$, which either is not a 3rd root of the unity anymore, or is not an element of $\mathbf{F}_q$. Thus for each such choice of β the points $q_i = (b_i, \frac{P(b_i,1)}{b_i})$ are in general position with respect to conics. □

Lemma 4.16. *Let x, x' be two conjugate elements in $\mathbf{F}_{q^8} \smallsetminus \mathbf{F}_{q^4}$, that are also in the same orbit under the action of $\mathbf{F}_{q^4}^*$ by left multiplication. Assume that one of the following conditions holds:*

(i) x is a generator of $\mathbf{F}_{q^8}$;
(ii) $q = 2$.

Then $x = x'$.

Proof. Assume $x \neq x'$. We have $x' = x^{q^i}$, for some $1 \leqslant i \leqslant 7$, and $x'/x = x^{q^i - 1} \in \mathbf{F}_{q^4}^*$. In particular $x^{(q^i-1)(q^4-1)} = 1$, so that $(q^i - 1)(q^4 - 1)$ is a non-zero multiple of $\mathrm{order}(x)$.

If x is a generator of $\mathbf{F}_{q^8}$, we get $(q^i - 1)(q^4 - 1) = d(q^8 - 1)$ for some $d > 0$. This implies $5 \leqslant i \leqslant 7$, and reducing modulo q^4 we find $d \equiv -1 \mod q^4$, hence $d \geqslant q^4 - 1$ which gives a contradiction.

If $q = 2$, the group $\mathbf{F}_{2^8}^*$ is cyclic of order $255 = 3 \cdot 5 \cdot 17$. Observe that an element $x \in \mathbf{F}_{2^8}^*$ is in $\mathbf{F}_{2^8} \smallsetminus \mathbf{F}_{2^4}$ if and only if $\mathrm{order}(x)$ is a multiple of 17 (namely, the possibilities are $17, 51, 85$ and 255). Then one checks that for $1 \leqslant j \leqslant 7$, $2^j - 1$ is not a multiple of 17, which gives the expected contradiction. □

Proposition 4.17. *Let $\mathbf{F}_q$ be a finite field. Then the number of Bertini involutions with a base point of degree 8, up to conjugacy by $\mathrm{PGL}_3(\mathbf{F}_q)$, is at least M_q, where $M_2 = 2$, $M_3 = 12$, and for $q \geqslant 4$,*

$$M_q = \frac{1}{640}(q^6 - 1).$$

In particular, $M_q \geqslant q$ for all $q \geqslant 2$.

Proof. First we count the number of nodal cubics equivalent to the ones from Set-Up 4.5.

We need to choose a point in $\mathbb{P}^2(\mathbf{F}_q)$, and then two distinct lines defined over $\mathbf{F}_q$ through this point. The number of such choices is:

$$N_1 = (q^2+q+1)\frac{q(q+1)}{2}.$$

Using the action of $\mathrm{PGL}_3(\mathbf{k})$, we can assume that the two lines are $x=0$ and $z=0$, and by Lemma 4.6 we need to count nodal cubics with equation of the form

$$xyz = c_0x^3 + c_1x^2z + c_2xz^2 + c_3z^3, \qquad c_0, c_3 \in \mathbf{k}^*, c_1, c_2 \in \mathbf{k},$$

and $-c_0/c_3$ a cube in $\mathbf{F}_q$. The number of choices is at least

$$N_2 = \frac{(q-1)^2q^2}{3}.$$

In fact, if 3 does not divides $q-1$, then any element of $\mathbf{F}_q$ is a cube, and we do not need to divide by 3 in the above formula (we shall use this remark below, for $q=2$ or 3).

Consider a Galois orbit $a_1,\dots,a_8$, and the associated points $p_i = \left(a_i, \frac{P(a_i,1)}{a_i}\right)$. If the p_i are in non-general position, then Lemma 4.15 says that by multiplying by $\beta \in \mathbf{F}_{q^4}^* \smallsetminus \{x;\ x^6=1, x^2 \in \mathbf{F}_q\}$ we can produce Galois orbits in general position. Moreover if a_i is a generator of $\mathbf{F}_{q^8}^*$, by Lemma 4.16 these orbits are pairwise disjoint. The number of such orbits in general position is at least $(q^4-7)/(q^4-1)$, which is greater than $9/10$ for $q \geqslant 3$. In fact, for $q=2$ the only third root of unity with square in $\mathbf{F}_2$ is 1, so again we get the ratio $(2^4-2)/(2^4-1) > 9/10$.

The number of generators for $\mathbf{F}_{q^8}^*$ is equal to $\varphi(q^8-1)$, where φ is the Euler function. We have the following lower bound for the Euler function [RS62, Theorem 15]:

$$\varphi(n) \geqslant \frac{n}{e^\gamma \log(\log n) + \frac{3}{\log(\log n)}},$$

where γ is the Euler constant. On can check that for $q \geqslant 3$, this implies $\varphi(q^8-1) \geqslant q^6-1$. So we get at least

$$N_3 = \frac{9}{10}\frac{q^6-1}{8}$$

Galois orbits of cardinal 8 in general position on a given nodal cubic. Finally by Lemma 4.9, a given orbit belongs to at most 12 nodal cubics, and we also have to mod out by the action of $\mathrm{PGL}_3(\mathbf{F}_q)$ whose cardinal is:

$$|\mathrm{PGL}_3(\mathbf{F}_q)| = \frac{1}{q-1}(q^3-1)(q^3-q)(q^3-q^2) = q^3(q^3-1)(q^2-1).$$

Finally:

$$\begin{aligned} M_q &= \frac{N_1 \cdot N_2 \cdot N_3}{12|\mathrm{PGL}_3(\mathbf{F}_q)|} \\ &= \frac{9}{2\cdot 3\cdot 80\cdot 12}\frac{(q^2+q+1)q(q+1)\cdot(q-1)^2q^2\cdot(q^6-1)}{q^3(q^3-1)(q^2-1)} \\ &= \frac{1}{640}(q^6-1). \end{aligned}$$

For $q=3$, we do not need the 3 in the denominator of N_2, and we can replace the coarse estimate 3^6-1 in the formula of N_3 by the exact number $\varphi(3^8-1)=2560$, so that we get the better bound:

$$\begin{aligned} M_3 &= \frac{9\cdot 2560}{2\cdot 80\cdot 12}\frac{(3^2+3+1)3(3+1)\cdot(3-1)^2 3^2}{3^3(3^3-1)(3^2-1)} \\ &= 12 \end{aligned}$$

Finally for $q = 2$, by Lemma 4.16 we can use all 240 elements of $\mathbf{F}_{2^8} \smallsetminus \mathbf{F}_{2^4}$ in the estimate for N_3, and not only the generators of $\mathbf{F}_{2^8}^*$. Moreover as above we can discard the 3 in the denominator of N_2, and we find:

$$N_1 = 21,\ N_2 = 4,\ N_3 = 27,\ |\mathrm{PGL}_3(\mathbf{F}_2)| = 168,$$

and

$$M_2 = \frac{21 \cdot 4 \cdot 27}{12 \cdot 168} = \frac{9}{8} > 1,$$

which we can round-up to $M_2 = 2$. □

4.3. **Jonquières maps.** Let $\mathbf{k}$ be any field, $n \geqslant 2$ any dimension. We define the *Jonquières subgroup* $J \subset \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ as the subgroup isomorphic to $\mathrm{PGL}_2(\mathbf{k}(x_2, \dots, x_n))$, via the choice of an affine chart $\mathbb{A}^n \subset \mathbb{P}^n$ and the formula

$$(x_1, \dots, x_n) \dashrightarrow \left(\frac{A(x_2, \dots, x_n)x_1 + B(x_2, \dots, x_n)}{C(x_2, \dots, x_n)x_1 + D(x_2, \dots, x_n)}, x_2, \dots, x_n \right).$$

Recall that a group G is called *perfect* if it is equal to its commutator subgroup $G^{(1)}$. We define the special Jonquières subgroup as the commutator subgroup of J; this is a group isomorphic to $\mathrm{PSL}_2(\mathbf{k}(x_2, \dots, x_n))$. Let G be the subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ generated by the special Jonquières subgroup $J^{(1)}$ and $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)^{(1)} = \mathrm{PSL}_{n+1}(\mathbf{k})$.

Proposition 4.18. (1) *If $g \neq \mathrm{id}$ is an element in the special Jonquières subgroup or in $\mathrm{PSL}_n(\mathbf{k})$, then the normal subgroup $\ll g \gg$ generated by g in G is equal to G;*
(2) *G is a perfect group.*

Proof. (1) The groups $\mathrm{PSL}_2(\mathbf{k}(x_2, \dots, x_n))$ and $\mathrm{PSL}_{n+1}(\mathbf{k})$ are simple (recall that we assume $n \geqslant 2$, so we avoid the non simple groups $\mathrm{PSL}_2(\mathbf{F}_2)$ and $\mathrm{PSL}_2(\mathbf{F}_3)$), and they have a non trivial intersection, as they both contain for instance the translation $(x_1 + 1, x_2, \dots, x_n)$. From these facts it follows that $\ll g \gg$ contains both $\mathrm{PSL}_2(\mathbf{k}(x_2, \dots, x_n))$ and $\mathrm{PSL}_{n+1}(\mathbf{k})$, hence G.

(2) Follows from the fact that $\mathrm{PSL}_2(\mathbf{k}(x_2, \dots, x_n))$ and $\mathrm{PSL}_{n+1}(\mathbf{k})$ are both perfect groups (because simple and non-abelian). □

Corollary 4.19. (1) *If $\varphi\colon \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \to A$ is a morphism to an abelian group A, then $G = \langle \mathrm{PSL}_2(\mathbf{k}(x_2, \dots, x_n)), \mathrm{PSL}_{n+1}(\mathbf{k}) \rangle \subset \ker \varphi$.*
(2) *If a morphism $\varphi\colon \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \to H$ sends a non-trivial element $g \in G$ onto 1_H, then $G \subset \ker \varphi$.*

Now we come back to the case of dimension 2. An equivalent definition of Jonquières map is that $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is Jonquières if it admits a base point q of degree 1 and f preserves a general member of the pencil of lines through q. For instance a quadratic map with one base point of degree 1 and one base point of degree 2 is Jonquières, but a quadratic map with a unique base point of degree 3 is not Jonquières. With the identification $(x, y) \in \mathbb{A}^2 \mapsto [x : y : 1] \in \mathbb{P}^2$, a Jonquières map is written $(x, y) \mapsto \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right)$ and admits the degree 1 point $q = [1 : 0 : 0]$ as a base point. Over a perfect field $\mathbf{k}$ one can factorize such a map into Sarkisov links by first blowing-up q to get a surface $\mathbb{F}_1$, then performing a sequence of type II links over $\mathbb{P}^1$ between Hirzebruch surfaces, and a last contraction to come back to $\mathbb{P}^2$. In particular, any Jonquières map f is an elementary generator, so that $\mathcal{E}$ contains a representative equivalent to f.

Recall that by Noether-Castelnuovo Theorem, over an algebraically closed field $\mathbf{k}$ the Jonquières group J and the automorphism group $\mathrm{PGL}_3(\mathbf{k})$ generate the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. In this context Corollary 4.19 implies that J and $\mathrm{PGL}_3(\mathbf{k})$ embed into any non-trivial quotient of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, in particular such a quotient cannot be finite, nor abelian. Observe also that there exist some non-algebraically closed field $\mathbf{k}$ such that

$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) = \langle J, \mathrm{PGL}_3(\mathbf{k})\rangle$, so that the same remark applies: by [Isk91], it is sufficient that $\mathbf{k}$ does not admit any extension of degree $\leqslant 8$. For instance, one can take $\mathbf{k}$ to be the composite (which in this case is also the union) of all towers of extensions of $\mathbf{Q}$ of degree at most 8.

Finally, remark that a Jonquières map can have base points of arbitrary degree, and in particular of degree 8. For instance, over a suitable field $\mathbf{k}$ let $p, q \in \mathbb{P}^2$ be points of respective degree 8 and 1. Then blowing-up the point q, and performing an elementary link from $\mathbb{F}_1$ to $\mathbb{F}_1$ by blowing up p and contracting the orbit of 8 fibers through p, we get a Jonquières map $j \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, that we can choose to be an involution (up to composing by an automorphism of $\mathbb{P}^2$). On the other hand, by blowing up only p, we construct a Bertini involution which has nothing to do with j. In fact it follows from Theorem A that b and j generate an infinite dihedral group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

5. Amalgamated structure and morphism to a free product

5.1. **Bass-Serre tree of an amalgam.** Our reference for this section is [Ser03]. Let G be a group, A a subgroup, and $(G_i)_{i\in I}$ a collection of (proper) subgroups generating G and such that $A \subset G_i$ for all i.

One constructs a graph $\mathcal{G}$ on which G acts as follows. The vertices are left cosets gA and gG_i in G/A and G/G_i. Then for each $g \in G$ and each $i \in I$, we put an edge between the vertices gA and gG_i. The group G acts on the resulting graph $\mathcal{G}$ by

$$f \cdot gA := (fg)A, \qquad\qquad f \cdot gG_i := (fg)G_i.$$

One says that G is the *amalgamated product* of the G_i over A, denoted $G = \ast_A G_i$, if it satisfies the following universal property: for any group H, and any collection of morphisms $\varphi_i \colon G_i \to H$ that coincide on A, there exists a (necessarily unique) morphism $\varphi \colon G \to H$ that extends all the φ_i. In this case, one can show that $A = G_i \cap G_j$ for all $i \neq j$ (see [Ser03, Remark after Theorem 1 p.3]).

Recall that a *star graph* is a tree of diameter 2. We call *central vertex* the unique center of a star graph, and *peripheral vertices* the other vertices. When the group G is the amalgamated product of the G_i over A then the graph $\mathcal{G}$ is a tree, and a fundamental domain with respect to the action of G is the star graph pictured on Figure 9, where we label each vertex by its stabilizer. Conversely we have the following basic result from Bass-Serre Theory:

Theorem 5.1 ([Ser03, §4, Theorem 10])**.** *With the notation above, suppose that G acts on a tree $\mathcal{T}$ with fundamental domain as in Figure 9, such that*

- *A is the stabilizer of the central vertex in the fundamental domain*
- *the G_i are the stabilizers of the peripheral vertices in the fundamental domain.*

Then $G = \ast_A G_i$ is the amalgamated product of the G_i along A, and the graph $\mathcal{G}$ constructed above is isometric to $\mathcal{T}$.

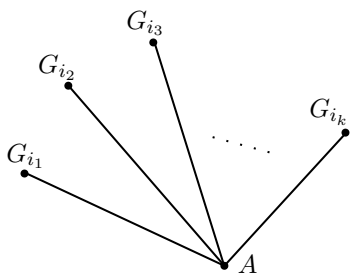

Figure 9. Fundamental domain for the Bass-Serre tree of the amalgam $\ast_A G_i$.

Remark 5.2. In the case where there are only two subgroups G_1 and G_2 with intersection $A = G_1 \cap G_2$, it is customary to remove the vertices of valence 2 corresponding to the left cosets modulo A, and to consider that the fundamental domain is a single edge with the following stabilizers:

$$\overset{G_1}{\bullet}\!\!\overset{A}{\text{———}}\!\!\overset{G_2}{\bullet}$$

5.2. **Subcomplexes.** First we define two subcomplexes of the complex $\mathcal{X}$ constructed in §2.2. Recall that we say that a vertex in $\mathcal{X}$ has type $\mathbb{P}^2$ is it is of the form $(\mathbb{P}^2, \varphi)$ for some $\varphi \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.

Let $\mathcal{X}_{\mathcal{B}} \subset \mathcal{X}$ be the subgraph whose edges correspond to blow-ups of degree 8 points in $\mathbb{P}^2$. If $b \in \mathcal{B}$, we denote $\eta_b \colon S_b \to \mathbb{P}^2$ the blow-up of the base point of degree 8 of b. In particular, for any $\varphi \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ and any $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, we obtain the following two edges in $\mathcal{X}_{\mathcal{B}}$:

$$\begin{tikzcd} & (S_b, \varphi\alpha\eta_b) \arrow[dl, "\eta_b"'] \arrow[dr] & \\ (\mathbb{P}^2, \varphi) = (\mathbb{P}^2, \varphi\alpha) & & (\mathbb{P}^2, \varphi\alpha b) = (\mathbb{P}^2, \varphi\alpha b\alpha^{-1}) \end{tikzcd}$$

Conversely, any edge in $\mathcal{X}_{\mathcal{B}}$ has this form, and any two vertices of type $\mathbb{P}^2$ at distance 2 in $\mathcal{X}_{\mathcal{B}}$ differ by a Bertini involution $\alpha b \alpha^{-1}$, for some $b \in \mathcal{B}$ and $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$.

We define another subcomplex $\mathcal{X}_e \subset \mathcal{X}$, by taking the closure of the complement of $\mathcal{X}_{\mathcal{B}}$ in $\mathcal{X}$.

Lemma 5.3. *The intersection of the two subcomplexes $\mathcal{X}_{\mathcal{B}}$ and $\mathcal{X}_e$ is exactly the set of vertices of type $\mathbb{P}^2$:*

$$\mathcal{X}_{\mathcal{B}} \cap \mathcal{X}_e = \{(\mathbb{P}^2, \varphi) \mid \varphi \in \mathrm{Bir}(\mathbb{P}^2)\}$$

Proof. Lemma 4.3 states that an edge in $\mathcal{X}$ corresponding to the blow-up of a point in $\mathbb{P}^2$ of degree 8 is not contained in any square. Therefore, $\mathcal{X}_{\mathcal{B}} \cap \mathcal{X}_e$ contains only vertices. Now as observed before, there are two types of vertices in $\mathcal{X}_{\mathcal{B}}$. A vertex of the form $(S_b, \varphi\alpha\eta_b)$ belongs to exactly two edges of $\mathcal{X}$, which by definition are edges of the graph $\mathcal{X}_{\mathcal{B}}$, therefore such a vertex does not belong to $\mathcal{X}_e$. On the other hand, any vertex of type $\mathbb{P}^2$ belongs to edges from both $\mathcal{X}_{\mathcal{B}}$ and $\mathcal{X}_e$, associated to blow-up of points of respective degree 8 or distinct from 8. □

We denote respectively $\mathcal{X}_{\mathcal{B}}^{\circ} \subset \mathcal{X}_{\mathcal{B}}$ and $\mathcal{X}_e^{\circ} \subset \mathcal{X}_e$ the connected components containing $(\mathbb{P}^2, \mathrm{id})$.

Lemma 5.4. (1) *Both $\mathcal{X}_e$ and $\mathcal{X}_{\mathcal{B}}$ are invariant under the action of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.
(2) *The graph $\mathcal{X}_{\mathcal{B}}^{\circ}$ is a tree.*

Proof. (1) An edge in $\mathcal{X}_{\mathcal{B}}$ has the form $(S, \varphi\eta) \to (\mathbb{P}^2, \varphi)$ for some $\varphi \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ and η a blow-up of a point of degree 8. Now $g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ sends this edge to $(S, g\varphi\eta) \to (\mathbb{P}^2, g\varphi)$, which is again an edge of the same form, hence in $\mathcal{X}_{\mathcal{B}}$. This gives that $\mathcal{X}_{\mathcal{B}}$ is invariant under the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, thus the same is true for the closure of its complement.

(2) Assume that the graph $\mathcal{X}_{\mathcal{B}}^{\circ}$ is not a tree. Then there exists a sequence of edges $e_1, \ldots, e_r$ in $\mathcal{X}_{\mathcal{B}}^{\circ}$ that form an embedded loop. Recall that by Proposition 3.15 the complex $\mathcal{X}$ is simply connected. By collapsing in $\mathcal{X}$ all edges of this loop except e_1, we obtain a space which is still simply connected, and which is the connected sum of a circle (corresponding to e_1) and another space. By van Kampen theorem such a space should have fundamental group of the form $\mathbf{Z} * G$ for some G, contradiction. □

We recall the following definitions of subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ that were given in the introduction:

$$G_e := \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), \mathcal{E} \smallsetminus \mathcal{B} \rangle \qquad G_b := \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), b \rangle,\ b \in \mathcal{B}, \qquad G_{\mathcal{B}} := \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), \mathcal{B} \rangle.$$

Observe that $G_{\mathcal{B}} = \langle G_b \mid b \in \mathcal{B} \rangle$.

Lemma 5.5. *Let $g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. Then*

(1) $g(\mathcal{X}_e^\circ) = \mathcal{X}_e^\circ$ *if and only if* $g \in G_e$.
(2) $g(\mathcal{X}_{\mathcal{B}}^\circ) = \mathcal{X}_{\mathcal{B}}^\circ$ *if and only if* $g \in G_{\mathcal{B}}$.

Proof. (1) Let $g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ such that $g(\mathcal{X}_e^\circ) = \mathcal{X}_e^\circ$, and in particular $g \cdot (\mathbb{P}^2, \mathrm{id}) = (\mathbb{P}^2, g) \in \mathcal{X}_e^\circ$. Pick a path γ of edges connecting $(\mathbb{P}^2, g)$ and $(\mathbb{P}^2, \mathrm{id})$ inside $\mathcal{X}_e^\circ$. By Lemma 5.3, the path γ does not involve any edge from $\mathcal{X}_{\mathcal{B}}$. By cutting at each intermediate vertex of type $\mathbb{P}^2$, we write γ as a composition of paths γ_i, where each γ_i links two vertices of type $\mathbb{P}^2$ whose markings differ by an element of $\mathcal{E} \smallsetminus \mathcal{B}$. It follows that $g \in G_e$.

Conversely if $g \in G_e$ we write $g = g_1 \cdots g_n$ for some $g_1, \dots, g_n \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) \cup \mathcal{E} \smallsetminus \mathcal{B}$. Then for each i, there exists a path from $(\mathbb{P}^2, g_1 \cdots g_i)$ to $(\mathbb{P}^2, g_1 \cdots g_{i+1})$ inside $\mathcal{X}_e^\circ$. By joining them we obtain a path in $\mathcal{X}_e^\circ$ starting at $(\mathbb{P}^2, \mathrm{id})$ and ending at $(\mathbb{P}^2, g) = g \cdot (\mathbb{P}^2, \mathrm{id})$, so that the connected component $g(\mathcal{X}_e^\circ)$ coincides with $\mathcal{X}_e^\circ$.

(2) The proof is entirely similar, and left to the reader. □

For each $b \in \mathcal{B}$, we define $\mathcal{T}_b$ to be the subgraph of $\mathcal{X}_{\mathcal{B}}^\circ$ obtained as the orbit of the edge between $(\mathbb{P}^2, \mathrm{id})$ and (S_b, η_b), under the action of $G_b = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), b \rangle$. Since b stabilizes (S_b, η_b) and $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ stabilizes $(\mathbb{P}^2, \mathrm{id})$, we obtain that $\mathcal{T}_b$ is connected, that is, $\mathcal{T}_b$ is a subtree of $\mathcal{X}_{\mathcal{B}}^\circ$.

Lemma 5.6. *Let b, b' be two elements in $\mathcal{B}$, and $g, g' \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. If $g(\mathcal{T}_b)$ and $g'(\mathcal{T}_{b'})$ are distinct, then the intersection $g(\mathcal{T}_b) \cap g'(\mathcal{T}_{b'})$ is either empty or equal to a single vertex of type $\mathbb{P}^2$.*

Proof. By using the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, we can assume that $g' = \mathrm{id}$. If $g \notin G_{\mathcal{B}}$, then by Lemma 5.5(2) $g(\mathcal{T}_b)$ and $\mathcal{T}_{b'}$ are in distinct connected components of $\mathcal{X}_{\mathcal{B}}$, and so are disjoint. Now we assume $g \in G_{\mathcal{B}}$, so that $g(\mathcal{T}_b)$ and $\mathcal{T}_{b'}$ are two subtrees of $\mathcal{X}_{\mathcal{B}}^\circ$.

• First we consider the case $b \neq b'$.

Suppose that $g(\mathcal{T}_b)$ and $\mathcal{T}_{b'}$ contain a common edge. Then in particular they contain a vertex of the form $(S_b, \varphi\eta_b)$, and the two edges from this vertex. But then the markings of the two corresponding vertices of type $\mathbb{P}^2$ should differ by composition by an element of the form $\alpha b \alpha^{-1} = \alpha' b' \alpha'^{-1}$ for some $\alpha, \alpha' \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, and this contradicts our assumption that b, b' are two distinct representatives of Bertini involutions. Now, since $g(\mathcal{T}_b)$, $\mathcal{T}_{b'}$ are two subtrees of $\mathcal{X}_{\mathcal{B}}^\circ$ without a common edge, they cannot have more than one common vertex, which has to be of type $\mathbb{P}^2$.

• Now we assume $b = b'$.

By definition of $\mathcal{T}_b$, we have $\mathcal{T}_b = g(\mathcal{T}_b)$ if and only if $g \in G_b$. So we can assume that $g \in G_{\mathcal{B}} \smallsetminus G_b$. Now if $g(\mathcal{T}_b) \cap \mathcal{T}_b$ contains a vertex of the form $(S_b, \varphi\eta_b)$, then it also contains the two neighbor vertices of type $\mathbb{P}^2$. But if $(\mathbb{P}^2, \varphi) \in g(\mathcal{T}_b) \cap \mathcal{T}_b$, then we should have $\varphi = gf_1 = f_2$ with $f_1, f_2 \in G_b$, in contradiction with $g \notin G_b$. So we conclude that $g(\mathcal{T}_b) \cap \mathcal{T}_b$ is empty. □

5.3. **Quotients.** Let $\mathcal{Y}$ be any connected subcomplex of $\mathcal{X}$. We define a star graph $\mathrm{star}(\mathcal{Y})$ associated with $\mathcal{Y}$, by requiring that the peripheral vertices of $\mathrm{star}(\mathcal{Y})$ are in one-to-one correspondence with the vertices of $\mathcal{Y}$ of type $\mathbb{P}^2$. Then we have a uniquely defined simplicial map from $\mathcal{Y}$ to $\mathrm{star}(\mathcal{Y})$, which is a bijection in restriction to the vertices of type $\mathbb{P}^2$, and which sends any other vertex to the central vertex of $\mathrm{star}(\mathcal{Y})$. We call $\mathrm{star}(\mathcal{Y})$ together with the map $\mathcal{Y} \to \mathrm{star}(\mathcal{Y})$ the *star quotient* of $\mathcal{Y}$.

Now assume that $(\mathcal{Y}_i)$ is a collection of connected subcomplexes of $\mathcal{X}$, such that $\mathcal{X} = \bigcup_i \mathcal{Y}_i$, and for any i, j either $\mathcal{Y}_i = \mathcal{Y}_j$ or $\mathcal{Y}_i \cap \mathcal{Y}_j$ contains only vertices of type $\mathbb{P}^2$. Then we can put together all star quotients $\mathcal{Y}_i \to \mathrm{star}(\mathcal{Y}_i)$ in order to get a map from $\mathcal{X}$ to a well defined connected graph.

Now we check that we can apply this construction to the family of subcomplexes

$$(\mathcal{Y}_i)_i = \{g(\mathcal{X}_e^\circ);\ g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)\} \cup \{g(\mathcal{T}_b);\ g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2), b \in \mathcal{B}\}.$$

First, the $g(\mathcal{T}_b)$, $g \in G_{\mathcal{B}}$, $b \in \mathcal{B}$, form a cover of $\mathcal{X}_{\mathcal{B}}^{\circ}$: any vertex in $\mathcal{X}_{\mathcal{B}}^{\circ}$ at distance 2 from $(\mathbb{P}^2, \mathrm{id})$ has the form $(\mathbb{P}^2, \alpha b)$ for some $b \in \mathcal{B}$ and $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, and any edge of $\mathcal{X}_{\mathcal{B}}^{\circ}$ can be mapped to an edge issued from $(\mathbb{P}^2, \mathrm{id})$ by applying an element of $G_{\mathcal{B}}$. This implies that the family of complexes $(\mathcal{Y}_i)_i$ is a cover of $\mathcal{X}$. Observe that the $g(\mathcal{X}_e^{\circ})$ are exactly the connected components of $\mathcal{X}_e$, and in particular they are pairwise equal or disjoint. Moreover, by Lemma 5.3, any intersection $g(\mathcal{X}_e^{\circ}) \cap g'(\mathcal{T}_b)$ contains only vertices of type $\mathbb{P}^2$. Finally, by Lemma 5.6, any intersection $g(\mathcal{T}_b) \cap g'(\mathcal{T}_{b'})$ between distinct subcomplexes is either empty or is equal to a single vertex of type $\mathbb{P}^2$.

We denote by $\mathcal{T}_Q$ the resulting quotient graph, and $\sigma\colon \mathcal{X} \to \mathcal{T}_Q$ the associated simplicial map. By a slight abuse of notation we shall use the same notation $(\mathbb{P}^2, \varphi)$ either for a vertex of type $\mathbb{P}^2$ in $\mathcal{X}$, or for the corresponding vertex in $\mathcal{T}_Q$.

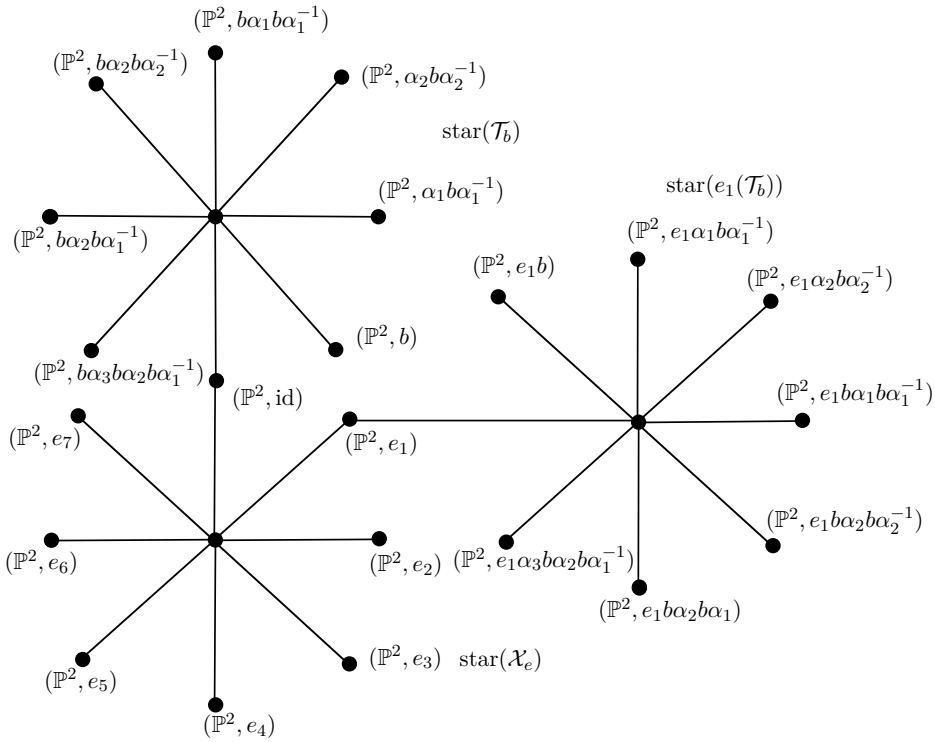

FIGURE 10. A few vertices of the tree $\mathcal{T}_Q$, where $e_i \in G_e$, $b \in \mathcal{B}$ and $\alpha_i \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$.

Lemma 5.7. *The connected graph $\mathcal{T}_Q$ is a tree.*

Proof. Let γ be a loop in $\mathcal{T}_Q$. We can assume that γ is parametrized by arc length, with base point a vertex of the form $(\mathbb{P}^2, \varphi)$. For each even i, $[\gamma(i), \gamma(i+2)]$ is a segment connecting two peripheral vertices in $\mathrm{star}(\mathcal{Y})$ for some connected subcomplex $\mathcal{Y} \subset \mathcal{X}$. In particular, we can lift this segment as a path in $\mathcal{Y}$. Thus we obtain a lift $\tilde{\gamma}$ of the entire path γ, and this lift is also a closed loop because the map π is a bijection in restriction to vertices of the form $(\mathbb{P}^2, \varphi)$. Now by Proposition 3.15 the loop $\tilde{\gamma}$ is trivial in $\pi_1(\mathcal{X})$, hence the push-forward $\sigma_*(\tilde{\gamma}) = \gamma$ is trivial in $\pi_1(\mathcal{T}_Q)$. □

Lemma 5.8. *The tree $\mathcal{T}_Q$ inherits the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ on $\mathcal{X}$, and*

(1) *the group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ is the stabilizer of $(\mathbb{P}^2, \mathrm{id})$,*
(2) *the group G_e is the stabilizer of the central vertex of $\mathrm{star}(\mathcal{X}_e^{\circ})$,*
(3) *for each $b \in \mathcal{B}$, the group G_b is the stabilizer of the central vertex of $\mathrm{star}(\mathcal{T}_b)$.*

Proof. By construction, the family

$$(\mathcal{Y}_i)_i = \{g(\mathcal{X}_e^{\circ});\ g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)\} \cup \{g(\mathcal{T}_b);\ g \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2), b \in \mathcal{B}\}$$

of subcomplexes involved in the construction of $\mathcal{T}_Q$ is invariant under the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, hence the action descends to $\mathcal{T}_Q$. Since the quotient map $\sigma\colon \mathcal{X} \to \mathcal{T}_Q$ is a

bijection in restriction to vertices of type $\mathbb{P}^2$, the stabilizers of these vertices remain the same in $\mathcal{T}_Q$: this gives (1). The stabilizer of the central vertex of $\mathrm{star}(\mathcal{Y}_i)$ corresponds to the stabilizer of the subcomplex $\mathcal{Y}_i$ for the initial action on $\mathcal{X}$. So the two remaining assertions follow from Lemma 5.5(1) and the definition of $\mathcal{T}_b$. □

Observe that each vertex of $\mathcal{T}_Q$ is either of the form $(\mathbb{P}^2, \varphi)$, or is the central vertex of $\mathrm{star}(\mathcal{Y})$ for some subcomplex $\mathcal{Y}$ in the family. These two types of vertices are preserved by the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, so that $\mathcal{T}_Q$ has a natural structure of a bicolored tree.

5.4. **Structure of the Cremona group.** In this last section we prove the results stated in the introduction.

Proof of Theorem A. Since $\mathcal{T}_Q$ is a bicolored tree, and the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ on vertices of type $\mathbb{P}^2$ is transitive, we can look for a fundamental domain of the action inside the ball of center $(\mathbb{P}^2, \mathrm{id})$ and radius 1. In fact the whole ball is a fundamental domain, indeed $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ is the stabilizer of $(\mathbb{P}^2, \mathrm{id})$ (see Lemma 5.8), and we now check that $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ also fixes each neighbor vertex. First, for each $b \in \mathcal{B}$ the central vertex of $\mathrm{star}(\mathcal{T}_b)$ is a neighbor of $(\mathbb{P}^2, \mathrm{id})$, and by Lemma 5.8 the corresponding stabilizer is G_b. Then the last remaining neighbor vertex is the central vertex of $\mathrm{star}(\mathcal{X}_e^\circ)$, whose stabilizer is G_e, by the same lemma.

By applying Theorem 5.1 to the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ on the tree $\mathcal{T}_Q$, we get that $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is isomorphic to $\ast_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_i$ where $I = \mathcal{B} \cup \{e\}$.

Now we prove that the action of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ on $\mathcal{T}_Q$ is faithful, by proving that the intersection of the stabilizers of $(\mathbb{P}^2, \mathrm{id})$ and $(\mathbb{P}^2, b)$ is trivial, for any $b \in \mathcal{B}$. If $g \in \mathrm{Stab}(\mathbb{P}^2, \mathrm{id}) \cap \mathrm{Stab}(\mathbb{P}^2, b)$, we have $g \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and $bgb = g' \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, so that $bg = g'b$. But these two maps cannot have the same base points unless $g = \mathrm{id}$, because any automorphism of $\mathbb{P}^2$ preserving 8 points in general position is the identity. □

Remark 5.9. If the field $\mathbf{k}$ does not have any Galois extension of degree 8, i.e. if $\mathcal{B}$ is empty, we have $\mathcal{X}_{\mathcal{B}} = \varnothing$, $\mathcal{X}_e^\circ = \mathcal{X}_e = \mathcal{X}$ and $\mathcal{T}_Q = \mathrm{star}(\mathcal{X}_e^\circ)$. This reflects the fact that in this case $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) = G_e$. In fact, trivially we have $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) *_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) \cap G_e} G_e = G_e$, and $\mathcal{T}_Q$ is its Bass-Serre tree, whose fundamental domain is the edge between $(\mathbb{P}^2, \mathrm{id})$ and the central vertex of $\mathrm{star}(\mathcal{X}_e^\circ)$.

Proof of Corollary B. Let $I = \mathcal{B} \cup \{e\}$. Then Theorem A gives

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) = \mathop{\ast}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_i = \left(\mathop{\ast}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_b \right) *_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_e.$$

Now we have

$$\mathop{\ast}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_b = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), G_b \mid b \in \mathcal{B} \rangle = G_{\mathcal{B}},$$

from which the claim follows. The reason why $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ acts faithfully on the Bass-Serre tree of $G_{\mathcal{B}} *_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_e$ is the same as in the proof of Theorem A. □

Proof of Theorem C. (1) Let $b \in \mathcal{B}$ be a Bertini involution, and consider the edge in the tree $\mathcal{X}_{\mathcal{B}}^\circ$ between the vertices $(\mathbb{P}^2, \mathrm{id})$ and (S_b, η_b), where $\eta_b \colon S_b \to \mathbb{P}^2$ is the blow-up of the base point of degree 8 of b. Recall from Lemma 5.5(2) that the group $G_{\mathcal{B}}$ acts on $\mathcal{X}_{\mathcal{B}}^\circ$. The involution b fixes the vertex (S_b, η_b) and exchanges the two edges attached to it. In particular b does not fix the vertex $(\mathbb{P}^2, \mathrm{id})$. On the other hand, the subgroup $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ stabilizes $(\mathbb{P}^2, \mathrm{id})$, and any $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ that also stabilizes (S_b, η_b) must be the identity, because any automorphism of $\mathbb{P}^2$ preserving 8 points in general position is the identity. By definition the tree $\mathcal{T}_b$ is the orbit of the edge from $(\mathbb{P}^2, \mathrm{id})$ to (S_b, η_b), under the action of the subgroup $G_b = \langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2), b \rangle$. Therefore the group G_b acts on the tree $\mathcal{T}_b$ with fundamental domain a single edge, with stabilizer of vertices $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and $\langle b \rangle$, and trivial stabilizer for the entire edge. By Theorem 5.1 (or more precisely

by using the special convention for two subgroups, see Remark 5.2), it follows that G_b is the free product

$$G_b \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) * \langle b \rangle,$$

and $\mathcal{T}_b$ is the associated Bass-Serre tree.

(2) By applying the universal property of the free product to the two morphisms $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) \to \{0\} \subset \mathbf{Z}/2\mathbf{Z}$ and $\langle b \rangle \xrightarrow{\simeq} \mathbf{Z}/2\mathbf{Z}$, we get a surjective morphism

$$\varphi_b \colon G_b \to \mathbf{Z}/2\mathbf{Z}.$$

By Theorem A, we have an isomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \simeq \ast_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)} G_i$ where $I = \mathcal{B} \cup \{e\}$. By the universal property of the amalgamated product, applied to the trivial morphism from G_e and the collection of morphisms $G_b \to \ast_{b \in \mathcal{B}} \mathbf{Z}/2\mathbf{Z}$ that send G_b to the corresponding factor of the free product via φ_b, we get a surjective morphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \to \ast_{b \in \mathcal{B}} \mathbf{Z}/2\mathbf{Z}$.

(3) The result is immediate by composing the above morphism with the abelianization morphism

$$\mathop{\ast}_{\mathcal{B}} \mathbf{Z}/2\mathbf{Z} \to \bigoplus_{\mathcal{B}} \mathbf{Z}/2\mathbf{Z}. \qquad \square$$

Remark 5.10. Another way to express point (2) of Theorem C is that we have isomorphisms

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)/\langle\langle G_e \rangle\rangle \xrightarrow{\simeq} \mathop{\ast}_{b \in \mathcal{B}} \big(G_b/\langle\langle \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) \rangle\rangle\big) \xrightarrow{\simeq} \mathop{\ast}_{b \in \mathcal{B}} \mathbf{Z}/2\mathbf{Z}.$$

References

[BKS04] T. Bauer, A. Küronya & T. Szemberg. Zariski chambers, volumes, and stable base loci. *J. Reine Angew. Math.*, 576:209–233, 2004.

[Can17] S. Cantat. The Cremona group. *To appear in Proceedings of 2015 Summer Institute on Algebraic Geometry, AMS Proceedings of Symposia in Pure Mathematics*, 2017.

[CL13] S. Cantat & S. Lamy. Normal subgroups in the Cremona group. *Acta Math.*, 210(1):31–94, 2013. With an appendix by Yves de Cornulier.

[Dea81] R. A. Dean. A rational polynomial whose group is the quaternions. *Amer. Math. Monthly*, 88(1):42–45, 1981.

[Deb01] O. Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[HM13] C. D. Hacon & J. McKernan. The Sarkisov program. *J. Algebraic Geom.*, 22(2):389–405, 2013.

[IKT93] V. A. Iskovskikh, F. K. Kabdykairov & S. L. Tregub. Relations in a two-dimensional Cremona group over a perfect field. *Izv. Ross. Akad. Nauk Ser. Mat.*, 57(3):3–69, 1993. Translation in Russian Acad. Sci. Izv. Math. 42 (1994), no. 3, 427–478.

[Isk91] V. Iskovskikh. Generators in the two-dimensional Cremona group over a nonclosed field. Translation of the 1991 paper from Trudy Mat. Inst. Steklov. pages 173–188. 1991.

[Isk96] V. A. Iskovskikh. Factorization of birational mappings of rational surfaces from the point of view of Mori theory. *Uspekhi Mat. Nauk*, 51(4(310)):3–72, 1996.

[Kal13] A.-S. Kaloghiros. Relations in the Sarkisov program. *Compos. Math.*, 149(10):1685–1709, 2013.

[KKL14] A.-S. Kaloghiros, A. Küronya & V. Lazić. Finite generation and geography of models. In *Minimal models and extremal rays*, Advanced Studies in Pure Mathematics. Mathematical Society of Japan, Tokyo., 2014.

[KM98] J. Kollár & S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Lie13] C. Liedtke. Algebraic surfaces in positive characteristic. In *Birational geometry, rational curves, and arithmetic*, pages 229–292. Springer, New York, 2013.

[Liu02] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Lon16] A. Lonjou. Non simplicité du groupe de Cremona sur tout corps. *Ann. Inst. Fourier (Grenoble)*, 66(5):2021–2046, 2016.

[Mat02] K. Matsuki. *Introduction to the Mori program.* Universitext. Springer-Verlag, New York, 2002.

[Mor96] P. Morandi. *Field and Galois theory*, volume 167 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1996.

[Mun84] J. R. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[Rei97] M. Reid. Chapters on algebraic surfaces. In *Complex algebraic geometry (Park City, UT, 1993)*, volume 3 of *IAS/Park City Math. Ser.*, pages 3–159. Amer. Math. Soc., Providence, RI, 1997.

[RS62] J. B. Rosser & L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[San81] J.-J. Sansuc. Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. *J. Reine Angew. Math.*, 327:12–80, 1981.

[Ser03] J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[Sil09] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[Ter99] H. Terakawa. The d-very ampleness on a projective surface in positive characteristic. *Pacific J. Math.*, 187(1):187–199, 1999.

[Zim15] S. Zimmermann. The abelianisation of the real Cremona group. *arXiv:1510.08705*, 2015.

Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France

E-mail address: `slamy@math.univ-toulouse.fr`

E-mail address: `susanna.zimmermann@math.univ-toulouse.fr`