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► **To cite this version:**

Susanna Zimmermann. The real plane Cremona group is an amalgamated product. *Annales de l'Institut Fourier*, 2021, 71 (3), pp.1023-1045. 10.5802/aif.3415/. hal-01719064v2

**HAL Id: hal-01719064**

**<https://hal.science/hal-01719064v2>**

Submitted on 27 Mar 2019

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# THE REAL PLANE CREMONA GROUP IS AN AMALGAMATED PRODUCT

SUSANNA ZIMMERMANN

ABSTRACT. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection and give an alternative proof of its abelianisation.

## 1. INTRODUCTION

The plane Cremona group is the group  $\text{Bir}_k(\mathbb{P}^2)$  of birational transformations of  $\mathbb{P}^2$  defined over a field  $k$ . For algebraically closed fields  $k$ , the Noether-Castelnuovo theorem [5] shows that  $\text{Bir}_k(\mathbb{P}^2)$  is generated by  $\text{Aut}_k(\mathbb{P}^2)$  and the subgroup preserving the pencil of lines through  $[1 : 0 : 0]$ . It implies that the normal subgroup generated by  $\text{Aut}_k(\mathbb{P}^2)$  is equal to  $\text{Bir}_k(\mathbb{P}^2)$ . Furthermore, [4, Appendix by CORNULIER] shows that  $\text{Bir}_k(\mathbb{P}^2)$  is not isomorphic to a non-trivial amalgam of two groups. However, it is isomorphic to a non-trivial amalgam modulo one simple relation [1, 10, 7], and it is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [16]. For  $k = \mathbb{R}$ , the group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and the two subgroups

$$\mathcal{J}_* = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1 : 0 : 0]\}$$

$$\mathcal{J}_\circ = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1, \bar{p}_1, p_2, \bar{p}_2\}$$

where  $p_1, p_2 \in \mathbb{P}^2$  are two fixed non-real points such that  $p_1, \bar{p}_1, p_2, \bar{p}_2$  are not collinear [3, Theorem 1.1]. Over  $\mathbb{C}$ , the analog of  $\mathcal{J}_\circ$  is conjugate to  $\mathcal{J}_*$  since a pencil of conics through four points in  $\mathbb{P}^2$  in general position can be sent onto a pencil of lines through point.

We define  $\mathcal{G}_\circ \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  to be the subgroup generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and  $\mathcal{J}_\circ$ , and by  $\mathcal{G}_* \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  the subgroup generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and  $\mathcal{J}_*$ . Then  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is generated by  $\mathcal{G}_* \cup \mathcal{G}_\circ$ , and the intersection  $\mathcal{G}_* \cap \mathcal{G}_\circ$  contains the subgroup  $\mathcal{H}$  generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and the involution  $[x : y : z] \mapsto [xz : yz : x^2 + y^2]$ , which is contained  $\mathcal{J}_\circ \cap \mathcal{J}_*$ .

**Theorem 1.1.** *We have  $\mathcal{G}_* \cap \mathcal{G}_\circ = \mathcal{H}$ , it is a proper subgroup of  $\mathcal{G}_\circ$  and  $\mathcal{G}_*$  and*

$$\text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \simeq \mathcal{G}_\circ \underset{\mathcal{H}}{*} \mathcal{G}_*.$$

*Moreover, both  $\mathcal{G}_*$  and  $\mathcal{G}_\circ$  have uncountable index in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ .*

The action of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  on the Bass-Serre tree associated to the amalgamated product yields the following:

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2010 *Mathematics Subject Classification.* 14E07; 20F05; 14P99.

During this work, the author was supported by the Swiss National Science Foundation, by Projet PEPS 2018 JC/JC and by ANR Project FIBALGA ANR-18-CE40-0003-01.

**Corollary 1.2.** *The group  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts on a tree, and all its algebraic subgroups are conjugate to a subgroup of  $\mathcal{G}_*$  or of  $\mathcal{G}_o$ .*

For the finite subgroups of odd order Corollary 1.2 can also be verified by checking their classification in [17].

An earlier version of this article used an explicit presentation of  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  and of  $\mathcal{G}_o$  in terms of generators and generating relations, the first of which is proven in [18, Theorem 4.4], and the second was proven analogously in the earlier version of this article. The present version does not use either presentation. Instead, we look at the groupoid  $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$  of birational maps between rational real Mori fibre spaces of dimension 2. It contains  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  as subgroupoid, is generated by Sarkisov links and the elementary relations are a set of generating relations [11, Theorem 3.1], see also Theorem 2.5. This information is encoded in a square complex on which  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts [11, §2–3], see also § 2. The groups  $\mathcal{G}_o$ ,  $\mathcal{G}_*$  and  $\mathcal{H}$  will turn out to be stabilisers of the union of elementary discs containing vertices marked with elements of  $\mathcal{G}_o$ ,  $\mathcal{G}_*$  and  $\mathcal{G}_o \cap \mathcal{G}_*$ .

This allows us to provide a new proof of the abelianisation of  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  given in [18]. Using the presentation of  $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2)$  we can construct a surjective homomorphism of groupoids  $\mathrm{Sar}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ , whose restriction onto  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is surjective as well and coincides with its abelianisation by construction. This will give an alternative proof of the following statement.

**Theorem 1.3.** [18, Theorem 1.1(1)&(3)] *There is a surjective homomorphism of groups*

$$\Phi: \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

*such that its restriction to  $\mathcal{J}_o$  is surjective and  $\mathcal{G}_* \subset \ker(\Phi) = [\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2), \mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$ , where the right hand side is also equal to the normal subgroup generated by  $\mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ .*

ACKNOWLEDGEMENT: I would like to thank Anne Lonjou for asking me whether the real plane Cremona group is isomorphic to a generalised amalgamated product of several groups, and the interesting discussions that followed. I would also like to thank Stéphane Lamy for discussions on the square complex, and Jérémy Blanc, Yves de Cornulier for helpful remarks, questions and discussions.

## 2. A SQUARE COMPLEX ASSOCIATED TO THE CREMONA GROUP

In this section we recall the square complex constructed in [11], whose vertices are marked rational surfaces, its edges of length two are Sarkisov links between them and discs relations between Sarkisov links.

By a surface  $S$ , we mean a smooth projective surface defined over  $\mathbb{R}$ , and by  $S_{\mathbb{C}}$  the same surface but defined over  $\mathbb{C}$ . We define the Néron-Severi space  $N^1(S_{\mathbb{C}})$  as the space of  $\mathbb{R}$ -divisors  $N^1(S_{\mathbb{C}}) := \mathrm{Div}(S_{\mathbb{C}}) \otimes \mathbb{R} / \equiv$ . The action of the Galois group  $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$  on  $N^1(S_{\mathbb{C}})$  factors through a finite group, and we denote by  $N^1(S)$  the subspace of the  $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -invariant classes. Since we only consider  $\mathbb{R}$ -rational surfaces and  $\mathbb{C}[S_{\mathbb{C}}]^* = \mathbb{C}^*$ ,  $N^1(S)$  is also the space of classes of divisors defined over  $\mathbb{R}$  (see for instance [14, Lemma 6.3(iii)]). The dimension of  $N^1(S_{\mathbb{C}})$  is the Picard rank of  $S$  and is denoted by  $\rho(S)$ . If not stated otherwise, all morphisms are defined over  $\mathbb{R}$ .

## 2.1. Rank $r$ fibrations.

**Definition 2.1.** Let  $S$  be a smooth rational surface, and  $r \geq 1$  an integer. We say that  $S$  is a *rank  $r$  fibration* if there exists a morphism  $\pi: S \rightarrow B$ , where  $B$  is a point or  $B = \mathbb{P}^1$ , with relative Picard number  $\rho(S/B) = r$ , and such that the anticanonical divisor  $-K_S$  is  $\pi$ -ample.

The last condition means that for any curve  $C$  contracted to a point by  $\pi$ , we have  $K_S \cdot C < 0$ . Observe that the condition on the Picard number is that  $\rho(S) = r$  if  $B$  is a point, and  $\rho(S) = r + 1$  if  $B = \mathbb{P}^1$ . If  $S$  is a rank  $r$  fibration, we will write  $S/B$  if we want to emphasize the basis of the fibration, and  $S^r$  when we want to emphasize the rank.

An isomorphism between two fibrations  $S/B$  and  $S'/B'$  (necessarily of the same rank  $r$ ) is an isomorphism  $S \xrightarrow{\cong} S'$  such that there exists an isomorphism on the bases (necessarily uniquely defined) that makes the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\cong} & S' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{\cong} & B' \end{array}$$

Observe that the definition of a rank  $r$  fibration puts together several notions. If  $B$  is a point, then  $S$  is just a del Pezzo surface of Picard rank  $r$  (over the base field  $\mathbb{R}$ ). If  $B$  is a curve, then  $S$  is just a conic bundle of relative Picard rank  $r$ : a general fiber is isomorphic to a smooth plane conic over  $\mathbb{R}$ , and over  $\mathbb{C}$  any singular fiber is the union of two  $(-1)$ -curves secant at one point. Remark also that being a rank 1 fibrations is equivalent to being a (smooth) rational Mori fibre space of dimension 2.

As the following examples make it clear, there are sometimes several choices for a structure of rank  $r$  fibration on a given surface, that may even correspond to distinct ranks.

**Example 2.2.** (1)  $\mathbb{P}^2$  with the morphism  $\mathbb{P}^2 \rightarrow \text{pt}$ , or the Hirzebruch surface  $\mathbb{F}_n$  with the morphism  $\mathbb{F}_n \rightarrow \mathbb{P}^1$ , are rank 1 fibrations.

(2)  $\mathbb{F}_1$  with the morphism  $\mathbb{F}_1 \rightarrow \text{pt}$  is a rank 2 fibration. Idem for  $\mathbb{F}_0 \rightarrow \text{pt}$ . The blow-up  $S^2 \rightarrow \mathbb{F}_n \rightarrow \mathbb{P}^1$  of a Hirzebruch surface in a real point or a pair of non-real conjugate point such that each of them is in a distinct fiber, is a rank 2 fibration over  $\mathbb{P}^1$ .

(3) The blow-up of two distinct real points on  $\mathbb{P}^2$ , or of two points of  $\mathbb{F}_n$  not lying on the same fiber, gives examples of rank 3 fibrations, with morphisms to the point or to  $\mathbb{P}^1$  respectively.

(4) The quadric surface  $\mathcal{Q}$  obtained by blowing up  $\mathbb{P}^2$  in a pair of non-real conjugate points and then contracting the strict transform of the line passing through them is a del Pezzo surface of degree 8 of Picard rank  $\rho(\mathcal{Q}) = 1$ , and so  $\mathcal{Q} \rightarrow \text{pt}$  is a rank 1 fibration.

(5) The blow-up  $\mathcal{CB}_6$  of a pair of non-real conjugate points on  $\mathcal{Q}$  is a del Pezzo surface of degree 6 of Picard rank  $\rho(\mathcal{CB}_6) = 2$ . It has a conic bundle structure  $\mathcal{CB}_6 \rightarrow \mathbb{P}^1$ , and so  $\mathcal{CB}_6/\text{pt}$  is a rank 2 fibration while  $\mathcal{CB}_6/\mathbb{P}^1$  is a rank 1 fibration.

(6) The blow-up of a real point in  $\mathcal{CB}_6$  yields a rank 2 fibration  $\mathcal{CB}_5/\mathbb{P}^1$ . Note that  $\mathcal{CB}_5$  is the blow-up of  $\mathbb{P}^2$  in two pairs of non-real conjugate points and the fibres of  $\mathcal{CB}_5/\mathbb{P}^1$  are the strict transforms of conics passing through the two pairs.

We call *marking* on a rank  $r$  fibration  $S/B$  a choice of a birational map  $\varphi: S \dashrightarrow \mathbb{P}^2$ . We say that two marked fibrations  $\varphi: S/B \dashrightarrow \mathbb{P}^2$  and  $\varphi': S'/B' \dashrightarrow \mathbb{P}^2$  are *equivalent* if  $\varphi'^{-1} \circ \varphi: S/B \rightarrow S'/B'$  is an isomorphism of fibrations. We denote by  $(S/B, \varphi)$  an equivalence class under this relation.

If  $S'/B'$  and  $S/B$  are marked fibrations of respective rank  $r'$  and  $r$ , we say that  $S'/B'$  *factorizes* through  $S/B$  if the birational map  $S' \rightarrow S$  induced by the markings is a morphism, and moreover there exists a (uniquely defined) morphism  $B \rightarrow B'$  such that the following diagram commutes:

$$\begin{array}{ccc} S' & \xrightarrow{\pi'} & B' \\ & \searrow & \nearrow \\ & S \xrightarrow{\pi} & B \end{array}$$

In fact if  $B' = \text{pt}$  the last condition is empty, and if  $B' \simeq \mathbb{P}^1$  it means that  $S' \rightarrow S$  is a morphism of fibration over a common basis  $\mathbb{P}^1$ . Note that  $r' \geq r$ .

**2.2. Square complex.** We define a 2-dimensional complex  $\mathcal{X}$  as follows. Vertices are equivalence classes of marked rank  $r$  fibrations, with  $3 \geq r \geq 1$ . We put an oriented edge from  $(S'/B', \varphi')$  to  $(S/B, \varphi)$  if  $S'/B'$  factorizes through  $S/B$ . If  $r' > r$  are the respective ranks of  $S'/B'$  and  $S/B$ , we say that the edge has type  $r', r$ . For each triplets of pairwise linked vertices  $(S''^3/B'', \varphi'')$ ,  $(S'^2/B', \varphi')$ ,  $(S^1/B, \varphi)$ , we glue a triangle. In this way we obtain a 2-dimensional simplicial complex  $\mathcal{X}$ .

**Lemma 2.3.** [11, Lemma 2.3] *For each edge of type 3, 1 from  $S''/B''$  to  $S/B$ , there exist exactly two triangles that admit this edge as a side.*

In view of the lemma, by gluing all the pairs of triangles along edges of type 3,1, and keeping only edges of types 3,2 and 2,1, we obtain a square complex that we still denote  $\mathcal{X}$ . When drawing subcomplexes of  $\mathcal{X}$  we will often drop part of the information which is clear by context, about the markings, the equivalence classes and/or the fibration. For instance  $S/B$  must be understood as  $(S/B, \varphi)$  for an implicit marking  $\varphi$ , and  $(\mathbb{P}^2, \varphi)$  as  $(\mathbb{P}^2/\text{pt}, \varphi)$ . The Cremona group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts on  $\mathcal{X}$  by

$$f \cdot (S/B, \varphi) := (S/B, f \circ \varphi).$$

**2.3. Sarkisov links and elementary relations.** In this section we show that the complex  $\mathcal{X}$  encodes the notion of Sarkisov links (or *links* for short), and of elementary relation between them.

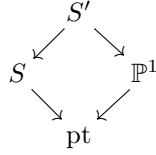
First we rephrase the usual notion of Sarkisov links between 2-dimensional Mori fiber spaces. Let  $(S/B, \varphi)$ ,  $(S'/B', \varphi')$  be two marked rank 1 fibrations. We say that the induced birational map  $S \dashrightarrow S'$  is a *Sarkisov link* if there exists a marked rank 2 fibration  $S''/B''$  that factorizes through both  $S/B$  and  $S'/B'$ . Equivalently, the vertices corresponding to  $S/B$  and  $S'/B'$  are at distance 2 in the complex  $\mathcal{X}$ , with middle vertex  $S''/B''$ :

$$\begin{array}{ccc} & S''/B'' & \\ & \swarrow & \searrow \\ S/B & & S'/B' \end{array}$$

This definition is in fact equivalent to the usual definition of a link of type I, II, III or IV from  $S/B$  to  $S'/B'$  (see [9, Definition 2.14] for the definition in arbitrary

dimension). Below we recall these definitions in the context of surfaces, in terms of commutative diagrams where each morphism has relative Picard number 1 (such a diagram corresponds to a “two rays game”), and we give some examples. Remark that these diagrams are not part of the complex  $\mathcal{X}$ : in each case, the corresponding subcomplex of  $\mathcal{X}$  is just a path of two edges, as described above.

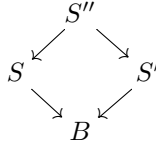
- Type I:  $B$  is a point,  $B' \simeq \mathbb{P}^1$ , and  $S' \rightarrow S$  is the blow-up of a real point or a pair of non-real conjugate points such that we have a diagram



Then we take  $S''/B'' := S'/\text{pt}$ . We also refer to the map  $S \dashrightarrow S'$  as link of type I.

Examples are given by the blow-up of a real point on  $S' = \mathbb{F}_1 \rightarrow \mathbb{P}^2 = S$  or the blow-up of a pair of non-real conjugate points  $S' = \mathcal{CB}_6 \rightarrow \mathcal{Q} = S$ . The fibration  $\mathbb{F}_1/\mathbb{P}^1$  corresponds to the lines through the point, and the fibration  $\mathcal{CB}_6/\mathbb{P}^1$  corresponds to a pencil of conics in  $\mathbb{P}^2$  passing through two pairs of non-real conjugate points.

- Type II:  $B = B'$ , and there exist two blow-ups  $S'' \rightarrow S$  and  $S'' \rightarrow S'$  that fit into a diagram of the form:

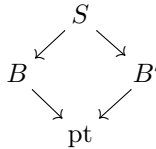


Then we take  $S''/B'' := S''/B$ . We also refer to the map  $S \dashrightarrow S'$  as link of type II.

An example is given by  $S = \mathbb{P}^2$ ,  $S''$  the blow-up of a pair of non-real conjugate points and  $S' = \mathcal{Q}$ . Other examples are the blow-up of a conic bundle in a real point, or in a pair of non-real conjugate points not contained in the same fibre, followed by the contraction of the strict transform of the fibre(s) containing it (them).

- Type III: symmetric situation of a link of type I. We refer to the map  $S' \rightarrow S$  as link of type III.

- Type IV:  $(S, \varphi)$  and  $(S', \varphi')$  are equal as marked surfaces, but the fibrations to  $B$  and  $B'$  are distinct. In this situation  $B$  and  $B'$  must be isomorphic to  $\mathbb{P}^1$ , and we have a diagram



Then we take  $S''/B'' := S/\text{pt}$ .

For rational surfaces, a type IV link always corresponds to the two rulings on  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ , that is,  $S/B = \mathbb{F}_0/\mathbb{P}^1$  is one of the rulings,  $S'/B' = \mathbb{F}_0/\mathbb{P}^1$  the other one, and  $S''/B'' = \mathbb{F}_0/\text{pt}$ . See [8, Theorem 2.6 (iv)] for other examples in the context of non-rational surfaces.

A *path of links* is a finite sequence of marked rank 1 fibrations

$$(S_0/B_0, \varphi_0), \dots, (S_n/B_n, \varphi_n),$$

such that for all  $0 \leq i \leq n-1$ , the induced map  $g_i: S_i/B_i \dashrightarrow S_{i+1}/B_{i+1}$  is a Sarkisov link.

**Proposition 2.4.** [11, Proposition 2.6] *Let  $(S'/B, \varphi)$  be a marked rank 3 fibration. Then there exist finitely many squares in  $\mathcal{X}$  with  $S'$  as a corner, and the union of these squares is a subcomplex of  $\mathcal{X}$  homeomorphic to a disk with center corresponding to  $S'$ .*

In the situation of Proposition 2.4, by going around the boundary of the disc we obtain a path of Sarkisov links whose composition is the identity (or strictly speaking, an automorphism). We say that this path is an *elementary relation* between links, coming from  $S^3/B$ . More generally, any composition of links that corresponds to a loop in the complex  $\mathcal{X}$  is called a *relation* between Sarkisov links.

**Theorem 2.5.** [11, Proposition 3.14, Proposition 3.15]

- (1) *Any birational map between rank 1 fibrations is a composition of links, and in particular the complex  $\mathcal{X}$  is connected.*
- (2) *Any relation between links is a composition of elementary relations, and in particular  $\mathcal{X}$  is simply connected.*

The first part of Theorem 2.5 can also be found in [8, Theorem 2.5]. In fact, a relative version can be extracted from the classification of links in [8, Theorem 2.6].

**Proposition 2.6.** *Let  $S/\mathbb{P}^1$  and  $S'/\mathbb{P}^1$  be rank 1 fibrations. Any birational map  $f: S \dashrightarrow S'$  over  $\mathbb{P}^1$  is a composition of links of rank 1 fibrations over  $\mathbb{P}^1$ , that is, a composition of Sarkisov links of type II over  $\mathbb{P}^1$ . In particular:*

- (1) *Any element of  $\mathcal{J}_*$  is conjugate to a composition of automorphisms of some Hirzebruch surface and links of type II between them.*
- (2) *Any element of  $\mathcal{J}_\circ$  is conjugate to a composition automorphisms of  $\mathcal{CB}_6$  and links  $\mathcal{CB}_6 \dashrightarrow \mathcal{CB}_6$  of type II.*

**2.4. Elementary discs.** We call the disc with center a rank 3 fibration from Proposition 2.4 an *elementary disc*. In this section, we classify them and therewith obtain an explicit list of elementary relations among rank 1 fibrations.

**Lemma 2.7.** *Any edge of  $\mathcal{X}$  is contained in a square. In particular,  $\mathcal{X}$  is the union of elementary discs.*

*Proof.* Let  $e$  be an edge of  $\mathcal{X}$ . If it is an edge between a rank 2 fibration and a rank 3 fibration, let  $e'$  be an edge from the rank 2 fibration to a rank 1 fibration. By [11, Lemma 2.3], there is a unique square in  $\mathcal{X}$  with  $e$  and  $e'$  among its edges. Suppose that  $e$  is the edge between a rank 1 fibration  $S/B$  and a rank 2 fibration  $S'/B'$ . If  $S' = S$  (and so  $B = \mathbb{P}^1$ ,  $B' = \text{pt}$ ), then  $S$  is a del Pezzo surface with  $\rho(S) = 2$  endowed with a conic bundle structure  $S/\mathbb{P}^1$ . In particular,  $S = \mathbb{F}_n$ ,  $n \in \{0, 1\}$  or  $S = \mathcal{CB}_6$ . Let  $S'' \rightarrow S$  be the blow-up of a general point. Then  $S''$  is also a del Pezzo surface, and so  $S''/\text{pt}$  is a rank 3 fibration and the square

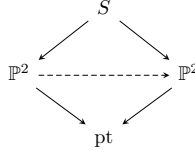
$$\begin{array}{ccc} S/\text{pt} & \longleftarrow & S''/\text{pt} \\ \downarrow e & & \downarrow \\ S/\mathbb{P}^1 & \longleftarrow & S''/\mathbb{P}^1 \end{array}$$

contained in  $\mathcal{X}$ . Suppose that  $S' \rightarrow S$  is the blow-up of a real point or a pair of non-real conjugate points (and so  $B = B'$ ). Note that if  $B = \text{pt}$ , the surface  $S'$  is a del Pezzo surface of degree  $\leq 6$ . The blow-up  $S'' \rightarrow S'$  of a general point yields a rank 3 fibration  $S''/B$  which factorises through  $S/B$ . The two blow-ups commute and  $\mathcal{X}$  contains a square of the form

$$\begin{array}{ccc} S'/B & \longleftarrow & S''/B \\ \downarrow e & & \downarrow \\ S/B & \longleftarrow & \tilde{S}/B \end{array}$$

It follows that  $\mathcal{X}$  is the union of squares. The claim now follows from Proposition 2.4. □

In general, Lemma 2.7 is not true for an arbitrary perfect field  $k$ . Indeed, if  $k$  has an extension of degree 8, then a Bertini involution whose set of base-points is an orbit of 8 conjugate points is a link of type II



with  $S/\text{pt}$  a del Pezzo surface of degree 1, and whose corresponding two edges in  $\mathcal{X}$  are not contained in any square [11, Lemma 4.3].

In what follows,  $X_d$  is a del Pezzo surface of degree  $d = (K_{X_d})^2$ . A arrow marked by 1 is the blow-up of a real point and an arrow marked with 2 is the blow-up of a pair of non-real conjugate points.

**Proposition 2.8.** *Any elementary disc in  $\mathcal{X}$  is one of the discs  $\mathcal{D}_1, \dots, \mathcal{D}_6$  in Figures 4 – 1, where the arrows are marked by 1 (or 2) if it is the blow-up of a real point (or a pair of non-real conjugate points).*

*Proof.* Let  $\mathcal{D}$  be an elementary disc and let  $S/B$  be the rank 3 fibration that is its center, and let  $S'/B'$  be a rank 1 fibration through which the rank 3 fibration  $S/B$  factors.

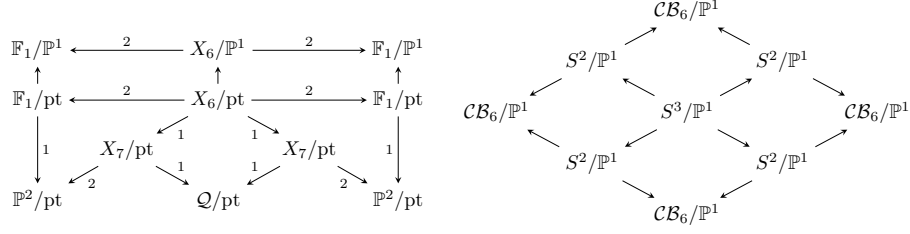
(a) Suppose that  $B = \text{pt}$ . Then  $S$  is a del Pezzo surface with  $\rho(S) = 3$ , and hence  $S'/\text{pt}$  is a del Pezzo surface as well.

If  $B' = \mathbb{P}^1$ , then  $S' \simeq \mathbb{F}_n$ ,  $n \in \{0, 1\}$  or  $S' \simeq \mathcal{CB}_6$ , and  $S \rightarrow S' \rightarrow \mathbb{P}^1$  is a rank 2 fibration. Then  $S \rightarrow S'$  is the blow-up of one real point or one pair of non-real conjugate points. If  $S' = \mathbb{F}_0$ , then  $\mathcal{D} = \mathcal{D}_5$ . If  $S' = \mathbb{F}_1$ , then  $\mathcal{D} = \mathcal{D}_1$  or  $\mathcal{D} = \mathcal{D}_5$ . If  $S' = \mathcal{CB}_6$  then  $\mathcal{D} = \mathcal{D}_3$  or  $\mathcal{D} = \mathcal{D}_4$ .

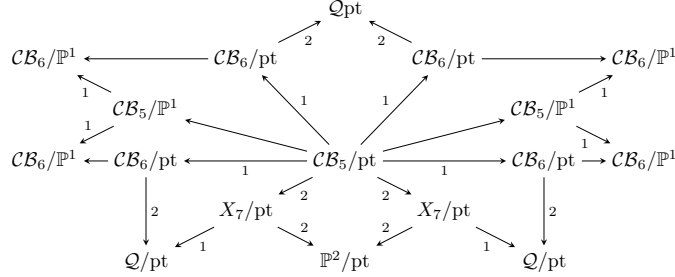
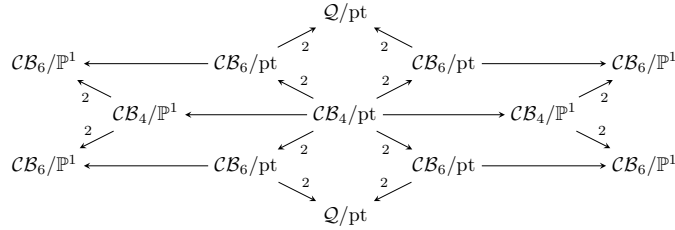
If  $B' = \text{pt}$ , then  $S' = \mathbb{P}^2$  or  $S' = \mathcal{Q}$ , and  $\pi: S \rightarrow S'$  is the blow-up of two points (two real points, two pairs of non-real conjugate points or a real point and a pair of non-real points) in general position. If  $S' = \mathbb{P}^2$  then  $\mathcal{D}$  is one of the discs  $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5$ . Note that in the discs  $\mathcal{D}_1$  and  $\mathcal{D}_3$  we cover also the case where  $S' = \mathcal{Q}$  and  $\pi$  blows up two points, at least one of which is a real point. If  $S' = \mathcal{Q}$  and  $\pi$  blows up two pairs of non-real conjugate points,  $\mathcal{D} = \mathcal{D}_4$ .

(b) Suppose that  $B = \mathbb{P}^1$ . Then  $B' = \mathbb{P}^1$ ,  $S' = \mathbb{F}_n$  or  $S' = \mathcal{CB}_6$  and  $\pi: S \rightarrow S'$  is a blow-up in two points (again two real points, two pairs of non-real conjugate points or a real point and a pair of non-real points). Furthermore, any rank 1

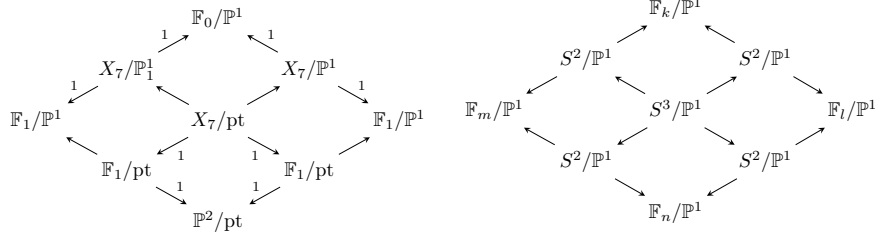


FIGURE 1. Disc  $\mathcal{D}_1$  on the left,  $\mathcal{D}_2$  on the right.

fibration in the elementary disc is of the form  $S''/\mathbb{P}^1$ . If  $S' = \mathcal{CB}_6$ , then  $S$  has two singular fibres whose  $\mathbb{C}$ -components are conjugate, and hence all rank 1 fibrations in the elementary discs are isomorphic to  $\mathcal{CB}_6$ . The hypothesis that  $-K_S$  is  $\pi$ -ample implies that the  $\mathbb{C}$ -components of the points blown up by  $\pi$  are on pairwise distinct smooth fibres. Thus  $\mathcal{D} = \mathcal{D}_2$  or  $\mathcal{D} = \mathcal{D}_6$ .  $\square$

FIGURE 2. Disc  $\mathcal{D}_3$ .FIGURE 3. Disc  $\mathcal{D}_4$ .

**Lemma 2.9.** *Let  $i, j \in \{1, \dots, 6\}$  and  $\mathcal{D}_i \neq \mathcal{D}_j$ . If they intersect non-trivially, their intersection is a single vertex or is a boundary segment of the form  $S/B \leftarrow S''/B'' \rightarrow S'/B'$ , where  $S/B$  and  $S'/B'$  are rank 1 fibrations.*

FIGURE 4. Disc  $\mathcal{D}_5$  on the left, disc  $\mathcal{D}_6$  on the right.

*Proof.* Suppose that  $\mathcal{D}_i \cap \mathcal{D}_j$  is non-empty and does not consist of one vertex. Then it contains at least one edge  $e$ . If it is in the relative interior of  $\mathcal{D}_i$ , the center of  $\mathcal{D}_i$  is a vertex of  $e$  and hence  $\mathcal{D}_i = \mathcal{D}_j$ . Suppose that  $e$  is on the relative boundary of  $\mathcal{D}_i$ , i.e. is of the form  $S''/B'' \rightarrow S/B$  with vertices  $S''/B''$  and  $S/B$  a rank 2 and rank 1 fibration, respectively. Following the two-rays game, there is a unique edge  $S''/B'' \rightarrow S'/B'$  to another rank 1 fibration  $S'/B'$ , which is hence contained in both  $\mathcal{D}_i$  and  $\mathcal{D}_j$ . If  $\mathcal{D}_i \cap \mathcal{D}_j$  contains another edge in the relative boundary of  $\mathcal{D}_i$ , then the list of elementary discs  $\mathcal{D}_1, \dots, \mathcal{D}_6$  in Proposition 2.8 implies that  $\mathcal{D}_i = \mathcal{D}_j$ .  $\square$

### 3. PROOF OF THE MAIN THEOREMS

**3.1. The group  $\mathcal{G}_* \cap \mathcal{G}_\circ$ .** We denote by  $\mathcal{H} \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  the subgroup generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and the quadratic involution  $\sigma: [x : y : z] \mapsto [xz : yz : x^2 + y^2]$ . Equivalently, it is generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and the quadratic maps in  $\mathcal{J}_\circ \cap \mathcal{J}_*$  [18, Lemma 3.2]. In particular,  $\mathcal{H} \subseteq \mathcal{G}_\circ \cap \mathcal{G}_*$ . We will prove equality in this section.

- Remark 3.1.**
- (1) For any quadratic map  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  with one real base-point and a pair of non-real conjugate base-points there exist  $\alpha, \beta \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  such that  $\alpha f \beta = \sigma$ .
  - (2) The path along the boundary of an elementary disc  $\mathcal{D}_1$  connecting its two vertices  $\mathbb{P}^1/\text{pt}$  corresponds to a quadratic map with one real and two non-real conjugate base-points. It is thus contained in  $\mathcal{H}$  and hence also in  $\mathcal{G}_\circ \cap \mathcal{G}_*$ .
  - (3) We can write  $\sigma = \varphi_2 \circ \varphi_1$  where  $\varphi_1: \mathbb{P}^2 \dashrightarrow \mathcal{Q}$  and  $\varphi_2: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  are two links of type II. It follows that any element of  $\mathcal{H}$  is a composition of links  $\mathbb{P}^2 \dashrightarrow \mathcal{Q}$  and  $\mathcal{Q} \dashrightarrow \mathbb{P}^2$  of type II.

**Definition 3.2.**

- (1) We call  $\mathcal{X}_\circ \subset \mathcal{X}$  the union of the  $\mathcal{G}_\circ$ -orbits of all discs  $\mathcal{D}_1, \dots, \mathcal{D}_4$  containing a vertex  $(\mathcal{Q}/\text{pt}, \tau)$  or  $(\mathcal{CB}_6/\mathbb{P}^1, \tau \circ \pi)$  with  $\tau: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  is a link of type II and  $\pi: \mathcal{CB}_6 \rightarrow \mathcal{Q}$  a link of type III.
- (2) We call  $\mathcal{X}_* \subset \mathcal{X}$  the union of  $\mathcal{G}_*$ -orbits of all discs  $\mathcal{D}_1, \mathcal{D}_5, \mathcal{D}_6$  containing a vertex  $(\mathbb{F}_n/\mathbb{P}^1, \pi_n)$  for some  $n \geq 0$ , with  $\pi_1: \mathbb{F}_1 \rightarrow \mathbb{P}^2$  a link of type III and  $\pi_n = \pi_1 \circ \varphi_n$ ,  $n \neq 1$ , where  $\varphi_n: \mathbb{F}_n \dashrightarrow \mathbb{F}_1$  is a composition of Sarkisov links of type II between Hirzebruch surfaces.

As the type of elementary disc is invariant by the action of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , any elementary disc contained in  $\mathcal{X}_\circ$  is one of type  $\mathcal{D}_1, \dots, \mathcal{D}_4$ , and any elementary disc contained in  $\mathcal{X}_*$  is one of type  $\mathcal{D}_1, \mathcal{D}_5$  or  $\mathcal{D}_6$ .

**Lemma 3.3.**

- (1) The set of vertices  $(\mathbb{P}^2, g)$  of  $\mathcal{X}_\circ$  is the  $\mathcal{G}_\circ$ -orbit of  $(\mathbb{P}^2, \text{id})$ .
- (2) The set of vertices  $(\mathbb{P}^2, g)$  of  $\mathcal{X}_*$  is the  $\mathcal{G}_*$ -orbit of  $(\mathbb{P}^2, \text{id})$ .
- (3) We have

$$\mathcal{X} = \bigcup_{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)} f(\mathcal{X}_\circ) \cup \bigcup_{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)} f(\mathcal{X}_*).$$

- (4) The sets  $\mathcal{X}_\circ$  and  $\mathcal{X}_*$  are connected.

(5) For any  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , the set  $\mathcal{X}_\circ \cap f(\mathcal{X}_*)$  is the union of all elementary discs  $\mathcal{D}_1$  containing a vertex  $(\mathbb{P}^2, g)$  with  $g \in \mathcal{G}_\circ$  and a vertex  $(\mathbb{P}^2, f \circ g')$  with  $g' \in \mathcal{G}_*$ .

(6) The set  $\mathcal{X}_\circ \cap \mathcal{X}_*$  is the union of all elementary discs  $\mathcal{D}_1$  containing a vertex  $(\mathbb{P}^2, g)$  with  $g \in \mathcal{G}_\circ \cap \mathcal{G}_*$ .

*Proof.* (1) For any  $g \in \mathcal{G}_\circ$  we have  $(\mathbb{P}^2, g) = g(\mathbb{P}^2, \text{id})$ . Let  $\tau: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  be a link of type II. There is an elementary disc  $\mathcal{D}_1$  containing  $(\mathbb{P}^2, \text{id})$  and  $(\mathcal{Q}, \tau)$ . Then  $g\mathcal{D}_1$  is an elementary disc containing  $(\mathbb{P}^2, g)$  and  $(\mathcal{Q}, g \circ \tau)$ , and it follows that  $(\mathbb{P}^2, g) \in \mathcal{X}_\circ$ . For the converse, if  $(\mathbb{P}^2, f)$  is contained in  $\mathcal{X}_\circ$ , it is the vertex of an elementary disc  $\mathcal{D}_i$ ,  $i \in \{1, 3\}$  contained  $\mathcal{X}_\circ$ . There is  $g \in \mathcal{G}_\circ$  such that  $\mathcal{D}_i = g\mathcal{D}'_i$  is the image by  $g$  of another elementary disc of type  $\mathcal{D}'_i$  containing a vertex  $(\mathcal{Q}, \tau)$  where  $\tau: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  is a link of type II. We can choose the vertex  $(\mathcal{Q}, \tau)$  inside  $\mathcal{D}'_i$  such that there are two consecutive edges connecting it to the vertex  $(\mathbb{P}^2, g^{-1} \circ f) \in \mathcal{D}'_i$ . The two edges correspond to a link  $\tau': \mathcal{Q} \dashrightarrow \mathbb{P}^2$  of type II. In particular,  $\tau \circ \tau' = g^{-1} \circ f$ . Since  $\tau \circ \tau' \in \mathcal{G}_\circ$ , it follows that  $f \in \mathcal{G}_\circ$ .

(2) Is proven analogously to (1) with  $\mathbb{F}_1$  instead of  $\mathcal{Q}$  and links  $\pi_1: \mathbb{F}_1 \rightarrow \mathbb{P}^2$  of type I and their inverse instead of links  $\tau: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  of type II.

(3) The complex  $\mathcal{X}$  is the union of elementary discs by Lemma 2.7, hence the orbit of  $\mathcal{X}_\circ$  and  $\mathcal{X}_*$  by  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  covers  $\mathcal{X}$ .

(4) Let  $i \in \{*, \circ\}$ . It suffices to check that for any vertex  $(\mathbb{P}^2, f)$  of  $\mathcal{X}_i$  there is a path in  $\mathcal{X}_i$  from  $(\mathbb{P}^2, f)$  to  $(\mathbb{P}^2, \text{id})$ . (1)&(2) imply that  $f \in \mathcal{G}_i$ . We write  $f = \alpha_n f_n \alpha_{n-1} \cdots f_1 \alpha_0$ , where  $\alpha_j \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ ,  $f_j \in \mathcal{J}_i$ . Proposition 2.6 implies that each  $f_j$  is conjugate by  $\psi_i: \mathbb{P}^2 \dashrightarrow S_i$  to a composition of links of type II preserving the associated conic bundle  $S_i/\mathbb{P}^1$ , where  $\psi_*: \mathbb{P}^2 \dashrightarrow \mathbb{F}_1 = S_*$  is the blow-up of  $[1 : 0 : 0]$  and  $\psi_\circ: \mathbb{P}^2 \dashrightarrow \mathcal{Q} \dashrightarrow \mathcal{CB}_6 = S_\circ$  the composition a link of type II and of type I blowing up  $[1 : i : 0], [0 : 1 : i]$  and their conjugates. This yields a decomposition of  $f$  into links which corresponds to a path from  $(\mathbb{P}^2, \text{id})$  to  $(\mathbb{P}^2, f)$  along the boundary of elementary discs contained in  $\mathcal{X}_i$ .

(5) Lemma 2.9 and Proposition 2.8 imply that the intersection  $\mathcal{X}_\circ \cap f(\mathcal{X}_*)$  consists of elementary discs of type  $\mathcal{D}_1$ . By (1)&(2) such a disc contains a vertex  $(\mathbb{P}^2, g)$  with  $g \in \mathcal{G}_\circ$  and a vertex  $(\mathbb{P}^2, f \circ g')$  with  $g' \in \mathcal{G}_*$ , and any elementary disc  $\mathcal{D}_1$  with this property is contained  $\mathcal{X}_* \cap f(\mathcal{X}_\circ)$ .

(6) By (5) for  $f = \text{id}$ , the set  $\mathcal{X}_\circ \cap \mathcal{X}_*$  is the union of elementary discs  $\mathcal{D}_1$  containing a vertex  $(\mathbb{P}^2, g)$  with  $g \in \mathcal{G}_\circ$  and a vertex  $(\mathbb{P}^2, g')$  with  $g' \in \mathcal{G}_*$ . Remark 3.1(2) implies that  $g, g' \in \mathcal{G}_* \cap \mathcal{G}_\circ$ .  $\square$

**Lemma 3.4.** *We have the following stabilisers under the action of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  on  $\mathcal{X}$ :*

- (1)  $\text{Stab}(\mathcal{X}_\circ) = \mathcal{G}_\circ$ ,
- (2)  $\text{Stab}(\mathcal{X}_*) = \mathcal{G}_*$ ,

$$(3) \text{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*) = \mathcal{G}_* \cap \mathcal{G}_\circ.$$

*Proof.* (1)&(2) Let  $i \in \{\circ, *\}$ . As  $\mathcal{X}_i$  is the  $\mathcal{G}_i$ -orbit of certain elementary discs, we have  $\mathcal{G}_i \subset \text{Stab}(\mathcal{X}_i)$ . Let  $f \in \text{Stab}(\mathcal{X}_i)$  and  $(\mathbb{P}^2, g) \in \mathcal{X}_i$ . Then  $(\mathbb{P}^2, f \circ g) \in \mathcal{X}_i$ , thus  $f \circ g \in \mathcal{G}_i$  by Lemma 3.3(1), and so  $f \in \mathcal{G}_i$ .

(3) The inclusion  $\mathcal{G}_\circ \cap \mathcal{G}_* \subset \text{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*)$  follows from (1)&(2). Let  $f \in \text{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*)$  and  $(\mathbb{P}^2, g) \in \mathcal{X}_\circ \cap \mathcal{X}_*$ . Then  $(\mathbb{P}^2, g \circ f) \in \mathcal{X}_\circ \cap \mathcal{X}_*$ . By Lemma 3.3(6), we have  $g, g \circ f \in \mathcal{G}_* \cap \mathcal{G}_\circ$ , and so  $f \in \mathcal{G}_\circ \cap \mathcal{G}_*$ .  $\square$

**Lemma 3.5.** *Let  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . Then the following hold:*

- (1) *Let  $i \in \{\circ, *\}$ . If  $\mathcal{X}_i$  and  $f(\mathcal{X}_i)$  intersect, then they are equal.*
- (2)  *$\mathcal{X}_* \cap \mathcal{X}_\circ$  is connected.*
- (3)  *$\text{Stab}(\mathcal{X}_\circ \cap \mathcal{X}_*) = \mathcal{G}_\circ \cap \mathcal{G}_* = \mathcal{H}$ .*

*Proof.* (1) Suppose that  $\mathcal{X}_i \cap f(\mathcal{X}_i)$  is non-empty. As the type of elementary disc is invariant by the action of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , the set  $\mathcal{X}_i \cap f(\mathcal{X}_i)$  contains a vertex  $(\mathbb{P}^2, h)$  for some  $h \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . By Lemma 3.3(1)&(2) we have  $h \in \mathcal{G}_i$  and there exists  $g \in \mathcal{G}_i$  such that  $h = f \circ g$ . It follows that  $f \in \mathcal{G}_i$ . As  $\mathcal{G}_i \subset \text{Stab}(\mathcal{X}_i)$ , we have  $f(\mathcal{X}_i) = \mathcal{X}_i$ .

(2)&(3) Let  $g \in \mathcal{G}_* \cap \mathcal{G}_\circ$  and write  $g = g_1 \circ \cdots \circ g_n = f_1 \circ \cdots \circ f_m$  for some  $g_i \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_*$  and  $f_i \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \cup \mathcal{J}_\circ$ . Then the relation

$$g_n^{-1} \circ \cdots \circ g_1^{-1} \circ f_1 \circ \cdots \circ f_m = \text{id}$$

induces a loop  $\gamma$  in  $\mathcal{X}$  starting at  $(\mathbb{P}^2, \text{id})$  and passing through  $(\mathbb{P}^2, g_n^{-1})$ ,  $(\mathbb{P}^2, g_n^{-1} \circ g_{n-1}^{-1})$ ,  $\dots$ ,  $(\mathbb{P}^2, g_n^{-1} \circ \cdots \circ g_1^{-1}) = (\mathbb{P}^2, g^{-1})$ ,  $(\mathbb{P}^2, g^{-1} \circ f_1)$ ,  $\dots$ ,  $(\mathbb{P}^2, g^{-1} \circ f_1 \circ \cdots \circ f_{m-1})$ ,  $(\mathbb{P}^2, g^{-1} \circ f) = (\mathbb{P}^2, \text{id})$ . We can suppose that  $\gamma(\frac{1}{2}) = (\mathbb{P}^2, g^{-1})$ . Since  $\mathcal{X}_*$  and  $\mathcal{X}_\circ$  are connected by Lemma 3.3(4) and  $\mathcal{X}$  is simply connected by Proposition 2.5(2), we can assume that  $\gamma$  that  $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_*$  and  $\gamma([\frac{1}{2}, 1]) \subset g^{-1}(\mathcal{X}_\circ) = \mathcal{X}_\circ$  (Lemma 3.4(1)). The loop  $\gamma$  is the boundary of a disc  $D$  in  $\mathcal{X}$ , and by Proposition 2.5(2) the disc  $D$  is a finite union of elementary discs. In particular,  $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_*$  passes through elementary discs contained in  $\mathcal{X}_*$  and  $\gamma([\frac{1}{2}, 1]) \subset \mathcal{X}_\circ$  passes through elementary discs contained in  $\mathcal{X}_\circ$ . The complex  $\mathcal{X}$  is covered by its subcomplexes  $f(\mathcal{X}_i)$ ,  $i \in \{*, \circ\}$  by Lemma 3.3(3), hence so is the disc  $D$ . It follows that  $D$  contains a finite connected union  $U$  of elementary discs, all contained in the union of sets of the form  $f(\mathcal{X}_\circ) \cap f'(\mathcal{X}_*)$ ,  $f, f' \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , and which contains  $(\mathbb{P}^2, \text{id})$  and  $(\mathbb{P}^2, g^{-1})$ . Lemma 3.3(5) implies that  $U$  is covered by elementary discs of type  $\mathcal{D}_1$ . The path from  $(\mathbb{P}^2, \text{id})$  to  $(\mathbb{P}^2, g^{-1})$  along the boundary of these discs induces a decomposition of  $g^{-1}$  into elements of  $\mathcal{H}$  (Remark 3.1(2)). This yields  $\mathcal{G}_* \cap \mathcal{G}_\circ \subset \mathcal{H}$ . Now (3) follows from Lemma 3.4(3). This and Lemma 3.3(6) imply (2).  $\square$

**3.2. The index of the subgroup  $\mathcal{G}_\circ$  and the properness of  $\mathcal{H}$  in  $\mathcal{G}_*$ .** We now prove that  $\mathcal{G}_*$  is not contained in  $\mathcal{G}_\circ$  by showing that quadratic maps having three real base-points in  $\mathbb{P}^2$  are in  $\mathcal{G}_* \setminus \mathcal{G}_\circ$ , and as consequence that the set  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2) / \mathcal{G}_\circ$  is uncountable.

Recall that any quadratic map  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  with three real base-points in  $\mathbb{P}^2$  is contained in  $\mathcal{G}_*$ , as there exist  $\alpha, \beta \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  such that  $\alpha f \beta$  is given by

$$\alpha f \beta: [x : y : z] \mapsto [yz : xz : xy]$$

and so  $\alpha f \beta \in \mathcal{J}_*$ .

**Lemma 3.6.** *The group  $\mathcal{G}_\circ$  does not contain any quadratic map with three real base-points in  $\mathbb{P}^2$ . In particular,  $\mathcal{H} \subsetneq \mathcal{G}_*$ .*

*Proof.* Let  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  be a quadratic map with three real base-points  $p_1, p_2, p_3$  in  $\mathbb{P}^2$  and suppose that  $f \in \mathcal{G}_o$ . Then  $f \in \mathcal{G}_o \cap \mathcal{G}_* = \mathcal{H}$  by Lemma 3.5(3) and there is a decomposition  $f = \psi_r \cdots \psi_1$  into links  $\psi_i: \mathbb{P}^2 \dashrightarrow \mathcal{Q}$  and  $\psi_j: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  of type II (Remark 3.1(3)). On the other hand, there is a decomposition of  $f = \varphi_4 \cdots \varphi_1$  as follows: The link  $\varphi_1: \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$  is the blow-up of  $p_1$  of  $f$ ,  $\varphi_2: \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$  is the link of type II blowing up of  $p_2$  and contracting the line strict transform of the line through  $p_1, p_2$ , and  $\varphi_3: \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$  is a link of type II blowing up  $p_3$  and contracting the strict transform of the line through  $p_1, p_3$ , and  $\varphi_4: \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$  is a link of type III contracting the strict transform of the line through  $p_2, p_3$ . This corresponds to a path along the boundary of the two discs  $\mathcal{D}_5$  and  $\mathcal{D}'_5$  with respective center  $(X_7, \varphi)$  and  $(X_7, \varphi' \circ f^{-1})$ , where  $\varphi: X_7 \dashrightarrow \mathbb{P}^2$  is the blow-up of the points  $p_1, p_2$  and  $\varphi': X_7 \dashrightarrow \mathbb{P}^2$  is the blow-up of the two base-points of  $f^{-1}$  which are the images of the lines through  $p_2, p_3$  and  $p_1, p_3$ . The relation

$$\text{id} = \psi_1^{-1} \cdots \psi_r^{-1} \varphi_4 \cdots \varphi_1$$

corresponds to a loop  $\gamma$  in  $\mathcal{X}$  which is the border of a disc  $D \subset \mathcal{X}$  that is a finite union of elementary discs (Theorem 2.5). We can assume that  $\gamma(\frac{1}{2}) = (\mathbb{P}^2, f^{-1})$  and that  $\gamma([0, \frac{1}{2}]) \subset \mathcal{X}_o \cap \mathcal{X}_*$ . It is therefore covered by elementary discs of the form  $\mathcal{D}_1$  (Lemma 3.3(6)). Furthermore,  $\gamma([\frac{1}{2}, 1]) \subset \mathcal{D}_5 \cup \mathcal{D}'_5 \subset \mathcal{X}_*$  by construction of  $\gamma$ . The discs  $\mathcal{D}_5$  and  $\mathcal{D}'_5$  intersect only elementary discs of type  $\mathcal{D}_1$  (they do so in a vertex  $\mathbb{P}^2/\text{pt}$  or in exactly two connected edges corresponding to a link of type I) and elementary discs type  $\mathcal{D}_6$ , which in turn only intersect other discs of type  $\mathcal{D}_6$  or discs of type  $\mathcal{D}_5$ . It follows that there is a connected union  $V \subset D$  of discs of type  $\mathcal{D}_1$  that contains  $(\mathbb{P}^2, \text{id})$  and  $(\mathbb{P}^2, f^{-1})$  such that the subset of  $\tilde{D}$  bounded by  $V$  and  $\gamma([\frac{1}{2}, 1])$  is only covered by discs of type  $\mathcal{D}_1, \mathcal{D}_5$  and  $\mathcal{D}_6$ . We can then assume that  $\gamma([0, \frac{1}{2}]) \subset V$  and that the set  $\tilde{D}$  is homeomorphic to a disc. As discs of type  $\mathcal{D}_6$  only intersect other discs  $\mathcal{D}_6$  or discs of type  $\mathcal{D}_5$ , it follows that there is a connected union  $\tilde{V} \subset \tilde{D}$  of discs  $\mathcal{D}_5$  containing  $(\mathbb{P}^2, \text{id})$  and  $(\mathbb{P}^2, f^{-1})$  intersecting the discs of type  $\mathcal{D}_1$  in  $V$  along their boundary. We obtain a path along boundaries of discs of type  $\mathcal{D}_1$  of links of type I and type III from  $(\mathbb{P}^2, \text{id})$  to  $(\mathbb{P}^2, f^{-1})$ . This is impossible as  $f^{-1}$  has not such decomposition. It follows that  $f \notin \mathcal{G}_o$ . The rest of the claim follows from the fact that  $\mathcal{H} = \mathcal{G}_* \cap \mathcal{G}_o$  (Lemma 3.5(3)).  $\square$

**Lemma 3.7.** *The group  $\mathcal{G}_o$  has uncountable index in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ .*

*Proof.* Denote by  $\sigma: [x : y : z] \mapsto [yz : xz : xy]$  the standard quadratic involution of  $\mathbb{P}^2$ , and define the group

$$A := \{[x : y : z] \mapsto [x + az : y + bz : z] \mid a, b \in \mathbb{R}\} \subset \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$$

Consider the map between sets

$$\psi: A \longrightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_o, \quad \alpha \mapsto (\alpha \circ \sigma)\mathcal{G}_o$$

We now prove that it is injective, which will yield the claim. First, note that  $\sigma \circ \alpha \circ \sigma$  is of degree  $\leq 2$  for all  $\alpha \in A$  since  $\sigma \circ \alpha$  and  $\sigma$  have two common base-points, namely  $[1 : 0 : 0]$  and  $[0 : 0 : 1]$ . Then  $\sigma \circ \alpha \circ \sigma \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  if and only if  $\sigma$  and  $\sigma \circ \alpha$  have three common base-points. This is the case if and only if  $\alpha = \text{Id}$ , as  $\alpha \in A$ . Moreover, if  $\sigma \circ \alpha \circ \sigma$  is of degree 2, it has three real base-points.

Now, let  $\beta, \gamma \in A$  such that  $(\beta \circ \sigma)\mathcal{G}_o = (\gamma \circ \sigma)\mathcal{G}_o$ . Then  $\sigma \circ (\beta^{-1} \circ \gamma) \circ \sigma \in \mathcal{G}_o$ . It follows from Lemma 3.6 and the above that  $\sigma \circ \beta^{-1} \circ \gamma \circ \sigma \in \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ , which is the case if and only if  $\beta^{-1} \circ \gamma = \text{id}$ . Thus the map  $\psi$  is injective.  $\square$

**3.3. The index of the subgroup  $\mathcal{G}_\circ$  and a quotient of  $\text{Bir}_\mathbb{R}(\mathbb{P}^2)$ .** We recall the construction of a surjective homomorphism

$$\psi: \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

given in [18, §3.2.1, §3.2.2].

**Definition–Lemma 3.8** ([18, §3.2.1, §3.2.2]). Any element  $f \in \mathcal{J}_\circ$  is conjugate to a birational map  $\hat{f}: \mathcal{CB}_6/\mathbb{P}^1 \dashrightarrow \mathcal{CB}_6/\mathbb{P}^1$  preserving the conic bundle structure. Such a map  $\hat{f}$  has a decomposition  $\hat{f} = \varphi_s \cdots \varphi_1$  into links  $\mathcal{CB}_6 \dashrightarrow \mathcal{CB}_6$  of type II (Proposition 2.6), and such a link does not blow up any point on a singular fibre of  $\mathcal{CB}_6/\mathbb{P}^1$ . For  $j = 1, \dots, s$ , let  $C_j$  be a (real or non-real) fibre of  $\mathcal{CB}_6/\mathbb{P}^1$  contracted by  $\varphi_j$  and  $[a_j + ib_j : 1] \in \mathbb{P}^1$  its image by the projection onto  $\mathbb{P}^1$ . We define  $v_j = 1 - \frac{|a_j|}{a_j^2 + b_j^2} \in (0, 1]$  if  $a_j + b_j \notin \mathbb{R}$ , and  $v_j = 0$  otherwise. Then

$$\psi: \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}, \quad f \mapsto \sum_{j=1}^s e_{v_j}$$

is a surjective homomorphism of groups whose kernel contains all elements of  $\mathcal{J}_\circ$  of degree  $\leq 4$ .

**Remark 3.9.** A standard quintic transformation  $f \in \text{Bir}_\mathbb{R}(\mathbb{P}^2)$  is a transformation of degree 5 having three pairs of non-real conjugate base-points in  $\mathbb{P}^2$ , and no three of the six base-points are collinear (see [3, Example] or [13, §1] for equivalent definitions). In particular, the six points are not on one conic, and  $f$  sends the pencil of conics through two of the pairs of base-points of  $f$  onto a pencil of conics through two pairs of base-points of  $f^{-1}$ . As two non-collinear pairs of non-real conjugate points can be sent by an automorphism of  $\mathbb{P}^2$  onto any other two non-collinear pairs of non-real conjugate points, there are  $\alpha, \beta \in \text{Aut}_\mathbb{R}(\mathbb{P}^2)$  such that  $g := \alpha \circ f \circ \beta \in \mathcal{J}_\circ$ . In particular,  $f \in \mathcal{G}_\circ$ . Moreover, it is conjugate to a link of type II of the conic bundle  $\mathcal{CB}_6/\mathbb{P}^1$ . Moreover, the image  $\psi(g)$  of  $g$  under the homomorphism  $\psi: \mathcal{J}_\circ \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  from Definition-Lemma 3.8 is a generator of a  $\mathbb{Z}/2\mathbb{Z}$ .

We denote by  $\text{Sar}_\mathbb{R}(\mathbb{P}^2)$  the set of birational transformations between rank 1 fibrations. It is a groupoid and contains  $\text{Bir}_\mathbb{R}(\mathbb{P}^2)$  as subgroupoid.

**Remark 3.10.** By Theorem 2.5, the groupoid  $\text{Sar}_\mathbb{R}(\mathbb{P}^2)$  is generated by links and any relation in  $\text{Sar}_\mathbb{R}(\mathbb{P}^2)$  is a product of conjugates of elementary relations.

**Proposition 3.11** (cf. [18, Proposition 5.3]). *The homomorphism  $\psi: \mathcal{J}_\circ \rightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  lift to a surjective homomorphism*

$$\Psi: \text{Bir}_\mathbb{R}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

whose kernel contains  $\mathcal{G}_*$ .

*Proof.* It suffices to construct a homomorphism of groupoids

$$\Psi: \text{Sar}_\mathbb{R}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$$

whose restriction to its subgroup  $\mathcal{J}_o$  is exactly  $\psi: \mathcal{J}_o \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  and whose kernel contains  $\mathcal{G}_*$ .

Let  $\varphi: \mathcal{CB}_6/\mathbb{P}^1 \dashrightarrow \mathcal{CB}_6/\mathbb{P}^1$  be a link of type II blowing up a pair of non-real conjugate points. Let  $\tau_1: \mathbb{P}^2 \dashrightarrow \mathcal{Q}$  be a link of type II blowing up  $[1 : i : 0], [1 : -i : 0]$ , and  $\tau_2: \mathcal{Q} \dashrightarrow \mathcal{CB}_6$  the link of type I blowing up  $\tau_1([0 : 1 : i]), \tau_1([0 : 1 : -i])$ . Then  $f := \tau_1^{-1} \circ \tau_2^{-1} \circ \varphi \circ \tau_2 \circ \tau_1: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  is a standard quintic transformation contained in  $\mathcal{J}_o$  and we define  $\Psi(\varphi) := \psi(f)$ . For any other link  $\varphi$  we define  $\Psi(\varphi) := 0$ . Note that by Definition–Lemma 3.8 and Remark 3.9, we have  $\Psi(f) = \psi(f)$  for any  $f \in \mathcal{J}_o$ . By Remark 3.10 it remains to check that any relation in  $\text{Sar}_{\mathbb{R}}(\mathbb{P}^2)$  is sent onto zero by  $\Psi$ . Any relation in  $\text{Sar}_{\mathbb{R}}(\mathbb{P}^2)$  is the composition of conjugates of elementary relations. Let  $\varphi_r \circ \dots \circ \varphi_1$  be an elementary relation. We can assume that one of the  $\varphi_i$  is a link of type II of  $\mathcal{CB}_6/\mathbb{P}^1$ , as otherwise the relation is sent onto zero by definition of  $\Psi$ . The elementary relation corresponds to an elementary disc  $\mathcal{D}$  in  $\mathcal{X}$  (Theorem 2.5(2)) and  $\varphi_i$  corresponds to a segment on the boundary of the elementary disc. The list of elementary discs in Proposition 2.8 yields that  $\mathcal{D}$  is one of  $\mathcal{D}_2, \mathcal{D}_3$  or  $\mathcal{D}_4$ . The boundary of each such disc is conjugate via  $\tau_1$  or  $\tau_1\tau_2$  to a relation inside the group  $\mathcal{J}_o$ , so its image by  $\psi$  is zero (as  $\psi$  is a homomorphism of groups). Thus its image by  $\Psi$  is zero. Note that  $\mathcal{G}_*$  is contained in  $\ker(\Psi)$  by definition of  $\Psi$ .  $\square$

The proof of Proposition 3.11 uses the principal idea of the construction of the quotients constructed in [2]; instead of working with a presentation of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  in terms of generators and generating relations, we pass to such a presentation of the groupoid of birational transformations between rational rank 1 fibrations (rational Mori fibre spaces) and construct a homomorphism from this groupoid to  $\bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ . In [2] one has then to prove in a second step that it is surjective. Here, this is not necessary, as we know that by construction its restriction to  $\mathcal{J}_o$  is surjective. The proof of Proposition 3.11 given in [18, Proposition 5.2] uses a presentation of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  in terms of generators and relations. The proof given here is independent from it.

For a perfect field with an extension of degree 8 (for instance  $k = \mathbb{Q}$ ), there is a surjective homomorphism  $\text{Bir}_k(\mathbb{P}^2) \rightarrow *_I \mathbb{Z}/2\mathbb{Z}$  with  $|I| \geq |k|$  [11]. It is induced by the free product structure  $\text{Bir}_k(\mathbb{P}^2) \simeq G * (*_I \mathbb{Z}/2\mathbb{Z})$ , where the factors  $\mathbb{Z}/2\mathbb{Z}$  are generated by Bertini involutions whose set of base-points consist of one orbit of eight conjugate points and  $G$  is generated by the other generators given in [6] and  $\text{Aut}_k(\mathbb{P}^2)$ . The free product structure is proven by constructing a quotient of  $\mathcal{X}$  on which  $\text{Bir}_k(\mathbb{P}^2)$  acts and by showing that it is its Bass-Serre tree. So at its heart, also the construction of the homomorphism from  $\text{Bir}_k(\mathbb{P}^2) \rightarrow *_I \mathbb{Z}/2\mathbb{Z}$  uses the presentation of  $\text{Sar}_k(\mathbb{P}^2)$  in terms of generators and generating relations.

**Corollary 3.12.** *The group  $\mathcal{G}_*$  does not contain any standard quintic transformations (in particular,  $\mathcal{H} \subsetneq \mathcal{G}_o$ ) and the index of  $\mathcal{G}_*$  in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is uncountable.*

*Proof.* Let  $\Psi: \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  be the surjective homomorphism from Proposition 3.11 and  $f \in \mathcal{G}_o$  a standard quintic transformation. It is conjugate to a standard quintic transformation in  $\mathcal{J}_o$  (Remark 3.9), so  $\Psi(f) = \psi(f) \neq 0$  by Proposition 3.11 and Lemma 3.8. Therefore  $f \notin \ker \Psi$ , while  $\mathcal{G}_* \subset \ker \Psi$ . It follows that  $\mathcal{G}_* \cap \mathcal{G}_o$  is a proper subgroup of  $\mathcal{G}_o$ . Moreover,  $\Psi$  induces a surjective map  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_* \longrightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  and hence the quotient  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/\mathcal{G}_*$  is uncountable.  $\square$

**3.4. Proof of the main results.** We finally assemble the proofs of Theorem 1.1, Corollary 1.2 and Theorem 1.3.

*Proof of Theorem 1.1.* The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is generated by  $\mathcal{G}_*$  and  $\mathcal{G}_o$ . Let  $f_n \circ \dots \circ f_1 = \text{id}_{\mathbb{P}^2}$  be a relation in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  with  $f_i \in \mathcal{G}_* \cup \mathcal{G}_o$ . By Theorem 2.5(1) for each  $i = 1, \dots, r$  there are links  $\varphi_{i1}, \dots, \varphi_{ir_i}$  such that  $f_i = \varphi_{ir_i} \circ \dots \circ \varphi_{i1}$ . Then

$$\varphi_{nr_n} \circ \dots \circ \varphi_{n1} \circ \dots \circ \varphi_{1r_1} \circ \dots \circ \varphi_{11} = \text{id}_{\mathbb{P}^2}$$

is a relation inside the groupoid  $\text{Sar}_{\mathbb{R}}(\mathbb{P}^2)$  and is thus a composition of conjugates of elementary relations (Remark 3.10), and each elementary relation corresponds to a loop around the boundary of an elementary disc (Theorem 2.5). Elementary discs are classified in Proposition 2.8. The boundary of discs  $\mathcal{D}_1, \mathcal{D}_3$  and  $\mathcal{D}_5$  respectively corresponds to a relation in  $\mathcal{G}_o \cap \mathcal{G}_*$ , in  $\mathcal{G}_o$  and in  $\mathcal{G}_*$ , respectively. We attach to the boundary of a disc  $\mathcal{D}_2$  a segment  $e$  of the form  $\mathcal{Q}/\text{pt} \leftarrow X_7/\text{pt} \rightarrow \mathbb{P}^2/\text{pt}$ . To the boundary of a disc  $\mathcal{D}_4$  we attach first the segment  $\mathcal{CB}_6/\mathbb{P}^1 \leftarrow \mathcal{CB}_6/\text{pt} \rightarrow \mathcal{Q}/\text{pt}$  and at that one the segment  $e$ . This yields that the elementary relation associate to the disc is conjugate to a relation in  $\mathcal{G}_o$ . To the boundary of a disc  $\mathcal{D}_6$  we attach the segment corresponding to a composition of links of type II (or an isomorphism if  $n = 1$ ) and a link of type III  $\mathbb{F}_n \dashrightarrow \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ . Then the elementary relation associated to  $\mathcal{D}_6$  is conjugate to a relation in  $\mathcal{G}_*$ .

It follows that the relation  $f_n \circ \dots \circ f_1 = \text{id}$  is conjugate to a composition of conjugates of relations in  $\mathcal{G}_*$  or  $\mathcal{G}_o$  or  $\mathcal{G}_o \cap \mathcal{G}_*$ . This implies that

$$\text{Bir}_{\mathbb{R}}(\mathbb{P}^2) = \mathcal{G}_* \underset{\mathcal{G}_* \cap \mathcal{G}_o}{*} \mathcal{G}_o.$$

Moreover, we have  $\mathcal{G}_* \cap \mathcal{G}_o = \mathcal{H}$  by Lemma 3.5(3), and it is a proper subgroup of  $\mathcal{G}_*$  and  $\mathcal{G}_o$  by Lemma 3.6 and Corollary 3.12. So the amalgamated structure on  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is nontrivial. Finally, the index of  $\mathcal{G}_*$  is uncountable by Lemma 3.7 and the index of  $\mathcal{G}_o$  is uncountable by Corollary 3.12.  $\square$

*Theorem 1.3.* The homomorphism  $\Psi: \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  from Proposition 3.11 coincides with the one given in [18, Proposition 5.3] since its restriction to  $\mathcal{J}_o$  is the surjective homomorphism  $\psi: \mathcal{J}_o \rightarrow \bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  and its kernel contains  $\mathcal{G}_*$  by construction, hence it also contains  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and  $\mathcal{J}_*$ . The kernel of  $\Psi$  is computed in [18, §6] by using [18, §2–3] and is equal to  $[\text{Bir}_{\mathbb{R}}(\mathbb{P}^2), \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)]$  and to the normal subgroup generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ .  $\square$

*Proof of Corollary 1.2.* By Theorem 1.1, the group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts on the Bass-Serre tree  $T$  of the amalgamated product  $\mathcal{G}_* \underset{\mathcal{G}_* \cap \mathcal{G}_o}{*} \mathcal{G}_o$ . Then every element of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  of finite order has a fixed point on  $T$ . It follows that every finite subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  has a fixed point on  $T$  [15, §I.6.5, Corollary 3], and is in particular conjugate to a subgroup of  $\mathcal{G}_*$  or of  $\mathcal{G}_o$ . For infinite algebraic subgroups of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , it suffices to check the claim for the maximal algebraic subgroups of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . By [12, Theorem 1.1], the infinite maximal algebraic subgroups of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  are conjugate to the group  $\text{Aut}_{\mathbb{R}}(X)$  of real automorphisms of a real rational surface  $X$  from the following list:

- (1)  $X = \mathbb{P}^2$ ,
- (2)  $X = \mathcal{Q}$ ,
- (3)  $X = \mathbb{F}_n$ ,  $n = 0$ ,  $n \geq 2$ ,
- (4)  $X$  is a del Pezzo surface of degree 6 with a birational morphism  $X \longrightarrow \mathbb{F}_0$  blowing-up a pair of non-real conjugate points.



- (5)  $X$  is a del Pezzo surface of degree 6 with a birational morphism  $X \rightarrow \mathbb{F}_0$  blowing-up two real points on  $\mathbb{F}_0$ ,
- (6) There is a birational morphism  $X/\mathbb{P}^1 \rightarrow \mathcal{CB}_6/\mathbb{P}^1$  of conic bundles blowing up  $n \geq 1$  pairs of non-real conjugate points on non-real fibres on the pair of non-real conjugate disjoint  $(-1)$ -curves of  $\mathcal{CB}_6$  (the exceptional divisors of the link  $\mathcal{CB}_6 \rightarrow \mathcal{Q}$  of type III),
- (7) There is a birational morphism  $X \rightarrow \mathbb{F}_n$  of conic bundles blowing up  $2n \geq 4$  points on the zero section of self-intersection  $n$ .

(1)&(2)&(3) We have  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset \mathcal{H} = \mathcal{G}_* \cap \mathcal{G}_o$ . The group  $\text{Aut}_{\mathbb{R}}(\mathcal{Q})$  is conjugate to a subgroup of  $\mathcal{G}_o$  via a link  $f: \mathcal{Q} \dashrightarrow \mathbb{P}^2$  of type II, and  $\text{Aut}_{\mathbb{R}}(\mathbb{F}_n)$  is conjugate to a subgroup of  $\mathcal{G}_*$  via a (possibly empty) composition of links of type II and one link of type III  $\mathbb{F}_n \dashrightarrow \mathbb{F}_1 \rightarrow \mathbb{P}^2$ .

(4) The surface  $X$  contains exactly three pairs of non-real conjugate disjoint  $(-1)$ -curves. The group  $\text{Aut}_{\mathbb{R}}(X)$  is generated by the lift of a subgroup of  $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$  and two elements, each of which descends via the contraction  $X \rightarrow \mathbb{F}_0$  of one pair of  $(-1)$ -curves to birational maps of  $\mathbb{F}_0$  preserving one of the two fibrations  $\mathbb{F}_0/\mathbb{P}^1$  [12, Proposition 3.5(2)&(3)]. It follows that  $\text{Aut}_{\mathbb{R}}(X)$  is conjugate to a subgroup of  $\mathcal{G}_*$ .

(5) The surface  $X$  contains exactly six real  $(-1)$ -curves. Via the blow-down  $\eta: X \rightarrow \mathbb{P}^2$  of three disjoint ones, the group  $\text{Aut}_{\mathbb{R}}(X)$  is conjugate to a subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  generated by the of  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  preserving the images of the  $(-1)$ -curves and by  $[x : y : z] \mapsto [yz : xz : xy]$  [12, Proposition 3.6(2)&(3)]. So,  $\eta \text{Aut}_{\mathbb{R}}(X) \eta^{-1} \subset \mathcal{G}_*$ .

(6) The group  $\text{Aut}_{\mathbb{R}}(X)$  is generated by the lift of a subgroup of  $\text{Aut}_{\mathbb{R}}(\mathcal{Q})$  by  $X \rightarrow \mathcal{CB}_6 \rightarrow \mathcal{Q}$  and by elements descending via  $X \rightarrow \mathcal{CB}_6$  to birational maps of  $\mathcal{CB}_6$  preserving the conic bundle structure [12, Proposition 4.5(1)&(2)]. It follows that  $\text{Aut}_{\mathbb{R}}(X)$  is conjugate to a subgroup of  $\mathcal{G}_o$ .

(7) The group  $\text{Aut}_{\mathbb{R}}(X)$  is generated by the lift of a subgroups of  $\text{Aut}_{\mathbb{R}}(\mathbb{F}_n)$  by  $X \rightarrow \mathbb{F}_n$  and by elements descending via  $X \rightarrow \mathbb{F}_n$  to birational maps of  $\mathbb{F}_n$  preserving the conic bundle structure [12, Proposition 4.8(1)&(2)]. It follows that  $\text{Aut}_{\mathbb{R}}(X)$  is conjugate to a subgroup of  $\mathcal{G}_*$ .  $\square$

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