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► **To cite this version:**

Susanna Zimmermann. THE REAL PLANE CREMONA GROUP IS A NON-TRIVIAL AMALGAM. 2018. hal-01719064v1

HAL Id: hal-01719064

<https://hal.science/hal-01719064v1>

Preprint submitted on 27 Feb 2018 (v1), last revised 27 Mar 2019 (v2)

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THE REAL PLANE CREMONA GROUP IS A NON-TRIVIAL AMALGAM

SUSANNA ZIMMERMANN

ABSTRACT. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection.

1. INTRODUCTION

The plane Cremona group is the group $\text{Bir}_k(\mathbb{P}^2)$ of birational transformations of \mathbb{P}^2 defined over a field k . [CL2013, Appendix by Y. DE CORNULIER] shows that $\text{Bir}_k(\mathbb{P}^2)$ is not isomorphic to a non-trivial amalgam of two groups if k is algebraically closed. The best one can obtain is that $\text{Bir}_k(\mathbb{P}^2)$ is isomorphic to a non-trivial amalgam modulo one simple relation [B2012, L2010, I1984] or that it is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [W1992]. The world looks different for $k = \mathbb{R}$. We show that $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is indeed isomorphic to a non-trivial amalgamated product of two groups.

The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ contains two groups of Jonquières transformations:

$$\mathcal{J}_* = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1 : 0 : 0] \}$$

$$\mathcal{J}_\circ = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1, \bar{p}_1, p_2, \bar{p}_2 \}$$

where $p_1, p_2 \in \mathbb{P}^2$ are two non-real points such that $p_1, \bar{p}_1, p_2, \bar{p}_2$ are not collinear. The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, \mathcal{J}_* , \mathcal{J}_\circ [BM2014], and the groups \mathcal{J}_* , \mathcal{J}_\circ are not conjugate in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ [Z2015, Proposition 5.3]. Over the field \mathbb{C} , we can send the pencil of conics through four points, no three collinear, onto a pencil of lines through a point, so the complex versions of the two groups are conjugate in $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.

We write $\mathcal{A} := \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and denote by $G_\circ \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by \mathcal{A} and \mathcal{J}_\circ , and by $G_* \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by \mathcal{A} and \mathcal{J}_* .

Theorem 1.1. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the non-trivial amalgamated product of G_* and G_\circ along their intersection.*

Proposition 1.2. *Both G_* and G_\circ have uncountable index in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$*

The intersection $G_* \cap G_\circ$ contains all elements of \mathcal{J}_* with only one real base-point (see remark after Definition 3.2).

The decomposition of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ into an amalgam follows from the structure theorem [Z2015, Theorem 4.4] (Theorem 2.3 below). The task is to prove that G_* and G_\circ are both proper subgroups; this is not straightforward since their complex analogues both coincide with $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$. Many of the ingredients of the proof will be similar to the proof of [Z2015, Theorem 4.4].

2010 *Mathematics Subject Classification.* 14E07; 20F05; 14P99.

The author gratefully acknowledges support by the Swiss National Science Foundation grant P2BSP2_168743.

Corollary 1.3. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on a tree, and all its algebraic subgroups are conjugate to a subgroup of G_* or of G_{\circ} .*

For the subgroups of odd order Corollary 1.3 can also be verified by checking their classification in [Y2016].

ACKNOWLEDGEMENT: I would like to thank Anne Lonjou for asking me whether the real plane Cremona group is isomorphic to a generalised amalgamated product of several groups, and the interesting discussions that followed. I would also like to thank Jérémy Blanc and Yves de Cornulier for helpful remarks, questions and discussions.

2. THE REAL CREMONA GROUP IS AN AMALGAM

In this section we show that $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of G_* and G_{\circ} along their intersection. Non-triviality will be proven in the next section.

Definition 2.1. A *standard quintic transformation* is an element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points in \mathbb{P}^2 . (See [BM2014, Example 3.1] for a definition via blow-ups.)

Remark 2.2. For any standard or special quintic transformation f there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{J}_{\circ}$ [Z2015, Lemma 3.10]. In particular, all standard quintic transformations are contained in G_{\circ} .

We make use of the following structure theorem. Let \mathcal{G} be the free group $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ}$ amalgamated along all pairwise intersections, and let $w: \mathcal{A} \cup \mathcal{J}_* \cup \mathcal{J}_{\circ} \rightarrow \mathcal{G}$ be the canonical word map.

Theorem 2.3 ([Z2015, Theorem 4.4]). *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to \mathcal{G} modulo the following relations:*

(rel. 1) *Let $\theta_1, \theta_2 \in \mathcal{J}_{\circ}$ be standard quintic transformations and $\alpha_1, \alpha_2 \in \mathcal{A}$.*

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$

(rel. 2) *Let $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_{\circ}$ be both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in \mathcal{A}$.*

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$

(rel. 3) *Let $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$ all be of degree 2, or τ_1, τ_2 of degree 2 and τ_3 of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}$.*

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \quad \text{if} \quad \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

Corollary 2.4. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the amalgamated product of G_* and G_{\circ} along their intersection.*

Proof. Let G be the amalgamated product of G_* and G_{\circ} along their intersection. By the universal property of amalgamated products there exists a homomorphism $\psi: G \rightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is the identity map on \mathcal{A} , \mathcal{J}_* and \mathcal{J}_{\circ} . On the other hand, we have a canonical surjective homomorphism $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ} \rightarrow \mathcal{G} \rightarrow G$. We claim that $\mathcal{G} \rightarrow G$ factors through $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By Theorem 2.3 it suffices to check that relations (rel. 1), (rel. 2) and (rel. 3) are satisfied in G .

Relation (rel. 1): standard quintic transformations are contained in G_{\circ} (Remark 2.2). They are not contained G_* because the kernel of the abelianisation of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ contains G_*

but does not contain any of the standard quintic transformations [Z2015, Remark 3.19(3), Proposition 5.3].

Relation (rel. 2) and (rel. 3) are relations among linear, quadratic and cubic transformations. Any transformation of degree ≤ 3 is contained in G_* , some of them in $G_* \cap G_\circ$, and relations (rel. 2) and (rel. 3) hold in G_* .

So, the morphism $\mathcal{G} \rightarrow G$ factors through $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, and the induced homomorphism $\varphi: \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow G$ is the identity on $\mathcal{A}, \mathcal{J}_*, \mathcal{J}_\circ$. Thus $\psi = \varphi^{-1}$. \square

3. THE GROUP G_\circ IS A STRICT SUBGROUP OF $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

This section is devoted to the proof that G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (Corollary 3.8). The key idea is to show that it does not contain any elements of degree 2 having three real base-points - for instance, it does not contain the standard quadratic transformation $[x : y : z] \mapsto [yz : xz : xy]$. To show this, we prove that any element in G_\circ is the composition of elements in $\mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ (see Definition 3.2 for the definition of $\mathcal{H}_* \subset G_\circ \cap \mathcal{J}_*$) such that the successive degree of the composition is non-decreasing (Proposition 3.7)

3.1. Properties of the groups \mathcal{J}_* and \mathcal{J}_\circ . Before we run off to prove our claims, we quickly review the groups \mathcal{J}_* and \mathcal{J}_\circ and some of their properties.

Remark 3.1. The characteristic of an element $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the tuple $(\deg(f); m_1^{e_1}, \dots, m_k^{e_k})$, where m_1, \dots, m_k are the different multiplicities of the base-points of f and e_i the number of base-points of multiplicity m_i .

An element of \mathcal{J}_\circ has characteristic

$$\begin{aligned} & \left(d; \left(\frac{d-1}{2} \right)^4, 2^{\frac{d-1}{2}} \right), \quad \text{if } \deg(f) \text{ is odd} \\ & \left(d; \left(\frac{d}{2} \right)^2, \left(\frac{d-2}{2} \right)^2, 2^{\frac{d-2}{2}}, 1 \right), \quad \text{if } \deg(f) \text{ is even} \end{aligned}$$

Moreover, if $d \geq 2$, $p_1, \bar{p}_1, p_2, \bar{p}_2$ are base-points of multiplicity $\frac{d-1}{2}$ or $\frac{d}{2}$ and $\frac{d-2}{2}$ and all other base-points are of multiplicity 2 or 1 [Z2015, Lemma 3.2].

In particular, \mathcal{J}_\circ contains only quadratic maps with one real and a pair of non-real conjugate base-points.

The group \mathcal{J}_* is isomorphic to $\text{PGL}_2(\mathbb{R}(x)) \rtimes \text{PGL}_2(\mathbb{R})$ because on the chart $z = 1$ all of its elements are of the form

$$f: (x, y) \mapsto \left(\frac{\alpha(y)x + \beta(y)}{\gamma(y)x + \delta(y)}, \frac{ay + b}{cy + d} \right)$$

with $a, b, d, c \in \mathbb{R}$, $ad - bc \neq 0$ and $\alpha, \beta, \delta, \gamma \in \mathbb{R}[y]$, $\alpha\delta - \beta\gamma \neq 0$.

The characteristic of any element of \mathcal{J}_* is $(d; d-1, 1^{2d-2})$, where $[1 : 0 : 0]$ is the base-point of multiplicity $d-1$.

Definition 3.2. We denote by $\mathcal{H} \subset \mathcal{J}_*$ the subgroup of elements having *at most one real* base-point.

If an element of \mathcal{H} is non-linear, all its base-points different from $[1 : 0 : 0]$ are pairs of non-real conjugate points. For any quadratic transformation $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with one real and a pair of non-real conjugate base-points there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{H}$. In fact, the

group \mathcal{H} is generated by its linear and quadratic elements and is hence contained in G_\circ (we can follow the argument of [AC2002, Theorem 8.4.3]).

Definition 3.3. A *special quintic transformation* is an element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points, two of them points in \mathbb{P}^2 and one pair infinitely near one of the other two pairs. (See [BM2014, Example 3.2] for a definition via blow-ups.)

Remark 3.4. (1) A standard (resp. special) quintic transformation f has multiplicity 2 in each of its base-points. In particular, there is no conic containing all its base-points.

(2) The inverse f^{-1} is also a standard (resp. special) quintic transformation and f sends the pencil of conics through two pairs of base-points onto a pencil of conics through two pairs of base-points of f^{-1} .

(3) Pick any three pairs of non-real conjugate points, not all six points contained in one conic. Suppose they are all points in \mathbb{P}^2 (resp. two pairs are points in \mathbb{P}^2 and one pair is in the first neighbourhood of one of the other two pairs). Then there is a standard (resp. special) quintic transformation with these points as its base-points [BM2014, Example 3.1, Example 3.2], [RV2005, §I].

(4) For any standard or special quintic transformation f there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{J}_\circ$ [Z2015, Lemma 3.10]. In particular, all standard and special quintic transformations are contained in G_\circ .

For $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and a point p , let $m_f(p)$ the multiplicity of f in p .

Lemma 3.5 ([Z2015, Lemma 4.8]). *Let $f \in \mathcal{J}_\circ$ be non-linear and $g \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. Suppose that*

$$\deg(fg) \leq \deg(g) \quad (\text{resp. } < \deg(g))$$

Then there exists a base-point $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ of f of multiplicity 2 such that

$$(ineq\ 1) \quad m_{g^{-1}}(p_1) + m_{g^{-1}}(p_2) + m_{g^{-1}}(q) \geq \deg(g) \quad (\text{resp. } > \deg(g))$$

or f has a simple base-point r and there exists $i \in \{1, 2\}$ such that

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(r) \geq \deg(g), \quad \text{where } m_f(p_i) = \deg(f)/2 \quad (\text{resp. } > \deg(g)).$$

Lemma 3.6 ([Z2015, Lemma 4.14]). *In Lemma 3.5 (ineq 1), suppose that q is a real proper point of \mathbb{P}^2 . Then one of the following two assertions holds:*

(1) *there exists $h \in \mathcal{J}_\circ$ of degree 3 with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, q$.*

(2) *$\deg(f)$ is even and there exists $h \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, q and*

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(q) \geq \deg(g), \quad \text{where } m_f(p_i) = \frac{\deg(f)}{2}.$$

3.2. The group G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. The following proposition is the key to proving that G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. The lemmas used in its proof are stated and proven in the next subsection, and to prove them we only use the properties of \mathcal{J}_* , \mathcal{J}_\circ and \mathcal{H} listed in the previous subsection.

Proposition 3.7. *For any $f \in G_\circ$ there exist $g_1, \dots, g_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f = g_m \cdots g_1$ and such that $\deg(g_i \cdots g_1) \leq \deg(g_{i+1} g_i \cdots g_1)$, for $i = 1, \dots, m-1$.*

Proof. We write $f = f_N \cdots f_1$ for some $f_1, \dots, f_N \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$. Define $d_i := \deg(f_i \cdots f_1)$ and

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad n := \max\{i \mid d_i = D\}, \quad k = \sum_{i=1}^n (\deg(f_i) - 1).$$

We claim that there exist $h_1, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that

$$f_{n+1}f_n f_{n-1} = h_k \cdots h_1$$

and the pair (D', k') associated to the composition $f_N \cdots f_{n+2} h_k \cdots h_1 f_{n-2} \cdots f_1$ is strictly smaller than (D, k) with respect to the lexicographic order.

If the claim holds, the proof finishes as follows. The procedure stops at the pair (D'', k'') where $D'' = \deg(f)$ and $f = f'_{n''} \cdots f'_1$. We define

$$I := \max\{i \mid \deg(f'_i \cdots f'_1) > \deg(f'_{i+1} \cdots f'_1)\}, \quad f' := f'_{I+1} \cdots f'_1$$

By construction, we have $\deg(f') < \deg(f)$. We repeat the process for f' . After finitely many steps we will not find such an I anymore, and the composition we have obtained is the one in the statement.

Let us prove the claim. If $D = 1$, all f_i are linear maps, so f is linear, and the claim holds with $m = 1$.

Suppose that $D \geq 2$. Any consecutive letters f_i, f_{i+1} that are both contained in \mathcal{A} or in \mathcal{J}_\circ we replace by their product. This does not increase the pair (D, k) . We can therefore assume that no two consecutive letters are both contained in \mathcal{A} or in \mathcal{J}_\circ . By definition of n , we have $d_n > d_{n+1}$, and in particular,

$$f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{A}, \quad f_n \in \mathcal{A} \setminus \mathcal{J}_\circ, \quad f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{A} \quad \text{and} \quad d_{n-2} \leq d_{n-1} = d_n = D.$$

(1) If $f_{n-1}, f_{n+1} \in \mathcal{H}$, we apply Lemma 3.11 to f_{n-1}, f_n, f_{n+1} . There exist $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{H}$ such that $f_{n+1}f_n f_{n-1} = h_k \cdots h_1$ and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, \\ \deg(h_2 h_1 f_{n-2} \cdots f_1) &= \deg(h_1 f_{n-2} \cdots f_1) \leq D, \\ \deg(h_i \cdots h_2 h_1 f_{n-1} \cdots f_1) &< D, \quad i = 3, \dots, k. \end{aligned}$$

Then $D' \leq D$ and if $D' = D$ then $n' \leq n$ and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

(2) Else, we proceed as follows. Let $m_i(t)$ be the multiplicity of $(f_i \cdots f_1)^{-1}$ in the point t . If $f_{n-1} \in \mathcal{J}_\circ$, then by Lemma 3.5 there exists a base-point $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ of f_{n-1}^{-1} of multiplicity 2 such that

$$m_{n-1}(p_1) + m_{n-1}(p_2) + m_{n-1}(q) \geq D$$

or $\deg(f_{n-1})$ is even and there exists a simple base-point r of f_{n-1}^{-1} such that

$$(1) \quad 2m_{n-1}(p_i) + m_{n-1}(r) \geq D, \quad \text{where } m_{f_{n-1}^{-1}}(p_i) = \deg(f_{n-1})/2.$$

- If q is a non-real point, we can assume that it is a proper point of \mathbb{P}^2 or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$. Since $m_{f_{n-1}^{-1}}(q) = 2$, the points $p_1, \dots, \bar{p}_2, q, \bar{q}$ are not on one conic (Remark 3.1). By Remark 3.4 there exists a standard or special quintic transformation $\theta_{n-1} \in \mathcal{J}_\circ$ with these 6 points as base-points.

- If q is a real point, we can assume that it is a proper point of \mathbb{P}^2 (it cannot be infinitely near one of p_1, \dots, \bar{p}_2). By Lemma 3.6 there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 3 with base-points p_1, \dots, \bar{p}_2, q or there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, q where $m_{f_{n-1}^{-1}}(p_i) = \frac{\deg(f_{n-1})}{2}$ and $2m_{n-1}(p_i) + m_{n-1}(q) \geq D$.

• If there is no such q , we have r as in (1). If r is not a proper point of \mathbb{P}^2 we replace it by the (real) base-point of f_{n-1}^{-1} to which it is infinitely near. This does not change Inequality (1). Remark 3.1 implies that p_i, \bar{p}_i, r are not collinear, so there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, r [Z2015, Lemma 3.4].

In all three cases we define $h_1 := \theta_{n-1}f_{n-1} \in \mathcal{J}_\circ$. Then

$$\deg(h_1f_{n-2} \cdots f_1) = \deg(\theta_{n-1}f_{n-1} \cdots f_1) \leq D \quad \text{and} \quad \deg(\theta_{n-1}f_{n-1}) < \deg(f_{n-1}).$$

We proceed analogously if $f_{n+1} \in \mathcal{J}_\circ$, and find $\theta_{n+1} \in \mathcal{J}_\circ$ satisfying the analogous inequalities with “ $<$ ”, and define $h_k := f_{n+1}\theta_{n+1}^{-1} \in \mathcal{J}_\circ$. Analogously, we have

$$\deg(\theta_{n+1}f_n \cdots f_1) < D, \quad \deg(h_k) < \deg(f_{n+1}).$$

Now, we use Lemma 3.9 and Lemma 3.13. The constructions are summarised in the commutative diagrams below.

If $f_{n-1}, f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$, we apply Lemma 3.9 to $\theta_{n-1}^{-1}, f_n, \theta_{n+1}$. There exist $h_2 \in \mathcal{A}$, $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ$ such that $\theta_{n+1}f_n\theta_{n-1}^{-1} = h_{k-1} \cdots h_2$ and

$$\deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k-1.$$

If $f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{H}$ and $f_{n+1} \in \mathcal{H}$, we apply the last part of Lemma 3.13 to $\theta_{n-1}, f_n, f_{n+1}$. There exist $h_2 \in \mathcal{A}$, $h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f_{n+1}f_n\theta_{n-1} = h_k \cdots h_2$ and

$$\deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k.$$

If $f_{n-1} \in \mathcal{H}$ and $f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$, we apply Lemma 3.13 to $f_{n-1}, f_n, \theta_{n+1}$. There exist $h_1 \in \mathcal{J}_\circ$, $h_2 \in \mathcal{A}$, $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $\theta_{n+1}f_n f_{n-1} = h_{k-1} \cdots h_1$ and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, & \deg(h_1f_{n-2} \cdots f_1) &= \deg(h_2h_1f_{n-2} \cdots f_1) \leq D \\ \deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) &< D, & i &= 3, \dots, k-1. \end{aligned}$$

We claim that in each case, the pair (D', k') is strictly smaller than (D, k) . Indeed, the above conditions imply $D' \leq D$. If $D' = D$ then $n' \leq n$ and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

□

Corollary 3.8. *The group G_\circ contains no maps of degree 2 having more than one real base-point. In particular, $G_\circ \subsetneq \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a strict subgroup.*

Proof. Suppose that G_\circ contains an element τ of degree $\deg(\tau) = 2$ with three real base-points. By Proposition 3.7, there exist $g_1, \dots, g_n \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $\tau = g_n \cdots g_1$ and

$$\deg(g_i \cdots g_1) \leq \deg(g_{i+1} \cdots g_1), \quad i = 1, \dots, n-1.$$

We may suppose that $\deg(g_1) \geq 2$, and then $\deg(g_i \cdots g_1) = 2$ for $i = 1, \dots, n$. This is only possible if

- $\deg(g_1) = 2$,
- $\deg(g_i) = 1, 2, 3, 4$ for $i = 2, \dots, n$,
- if $\deg(g_i) = 2, 3, 4$, then g_i and $(g_{i-1} \cdots g_1)^{-1}$ share 2, 3, 3 base-points of multiplicity:
 - if $\deg(g_i) = 3$, these are a double and two simple base-points
 - if $\deg(g_i) = 4$, these are three double points.

The characteristics in Remark 3.1 and Remark 3.2, and all g_i being contained in $\mathcal{J}_\circ \cup \mathcal{H}$ of degree ≤ 4 implies that the quadratic maps $g_i \cdots g_1$ have two non-real base-points for $i = 1, \dots, n$. This contradicts $\tau = g_n \cdots g_1$ having three real base-points. \square

Proof of Theorem 1.1. By Proposition 2.4 is the amalgamated product of G_* and G_\circ along their intersection. Furthermore, G_* is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by [Z2015, Proposition 5.3], and G_\circ is a strict subgroup by Corollary 3.8. \square

Proof of Proposition 1.2. Let's prove the claim first for G_\circ . We denote by $\sigma: [x : y : z] \mapsto [yz : xz : xy]$ the standard quadratic involution of \mathbb{P}^2 , and define a map

$$\varphi: \{[x : y : z] \mapsto [x + az : y + bz : z] \mid a, b \in \mathbb{R}\} := A \longrightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_\circ, \quad \alpha \mapsto \alpha\sigma G_\circ$$

Note that by Corollary 3.8 the transformation σ is not contained in G_\circ . In particular, $\alpha\sigma G_\circ \neq G_\circ$ for all $\alpha \in A$. We claim that the map φ is injective. Note that for all $\alpha \in A$ the transformation $\sigma\alpha\sigma$ is of degree ≤ 2 . It is linear if and only if σ and $(\alpha\sigma)^{-1}$ have three common base-points, i.e. if and only if $\alpha = \text{Id}$. If $\sigma\alpha\sigma$ is of degree 2 then it has three real base-points. It follows from Corollary 3.8 that that for $\beta, \gamma \in A$ the transformation $\sigma\beta^{-1}\gamma\sigma$ is contained in G_\circ if and only if it is linear, i.e. if and only if $\beta^{-1}\gamma = \text{Id}$. Thus the map φ is injective.

Let's prove the claim for G_* . By [Z2015, Lemma 6.7, Corollary 6.12] we have

$$G_* = \langle \mathcal{J}_*, \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \subset \langle \langle \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \rangle = [\text{Bir}_{\mathbb{R}}(\mathbb{P}^2), \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)].$$

Then there is a surjective map from $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_*$ onto the abelianisation of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is isomorphic to $\bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ [Z2015, Theorem 1.1]. It follows that G_* has uncountable index in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. \square

Proof of Corollary 1.3. By Theorem 1.1, the group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on the Bass-Serre tree T of the amalgam $G_* *_{G_* \cap G_\circ} G_\circ$. Then every element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of finite order has a fixed point on T . It follows that very finite subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ has a fixed point on T [S1980, §I.6.5, Corollary 3], and is in particular conjugate to a subgroup of G_* or G_\circ .

For infinite algebraic subgroups of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, it suffices to check the claim for the maximal ones. Before we list them, we have to introduce some notation. For $n \geq 0$, we denote by \mathbb{F}_n the n -th Hirzebruch surface

$$\mathbb{F}_n = \{[x : y : z][u : v] \in \mathbb{P}_{\mathbb{R}}^2 \times \mathbb{P}_{\mathbb{R}}^1 \mid u^n z = v^n y\}.$$

Note that $\mathbb{F}_0 = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$. By $\mathcal{Q}_{3,1} \subset \mathbb{P}_{\mathbb{R}}^3$ we denote the real hypersurface given by $w^2 = x^2 + y^2 + z^2$. Note that it is isomorphic to the real surface $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, (x, y) \mapsto (\bar{y}, \bar{x}))$. It is obtained by blowing up two conjugate points in \mathbb{P}^2 and contracting the (real) line passing

through them. We denote by $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$ the conic bundle obtained by blowing up a pair of non-real conjugate points on $\mathcal{Q}_{3,1}$.

Note that for $n \geq 0$, the group \mathcal{J}_* is conjugate to the group of birational transformations of \mathbb{F}_n preserving the conic bundle structure $\text{pr}_2: \mathbb{F}_n \rightarrow \mathbb{P}^1$. The group \mathcal{J}_\circ is conjugate to the group of birational transformations of $X_{[2]}$ preserving the conic bundle structure $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$. The infinite maximal algebraic subgroups of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ are conjugate to $\text{Aut}_{\mathbb{R}}(X)$ where X is one of the real rational surfaces in the following list [RZ2016]:

- (1) $X = \mathbb{P}^2$,
- (2) $X = \mathcal{Q}_{3,1}$,
- (3) $X = \mathbb{F}_n$, $n = 0$, $n \geq 2$,
- (4) X is a del Pezzo surface of degree 6 obtained by blowing up a pair of non-real conjugate points on \mathbb{F}_0 ,
- (5) X is a del Pezzo surface of degree 6 obtained by blowing up two real points on \mathbb{F}_0 ,
- (6) $\eta: X \rightarrow X_{[2]}$ is a birational morphism of real conic bundles blowing up $n \geq 1$ pairs with non-real conjugate points on non-real fibres on the non-real conjugate disjoint (-1) -curves of $X_{[2]} \rightarrow \mathcal{Q}_{3,1}$,
- (7) $\eta: X \rightarrow \mathbb{F}_n$ is a birational morphism of real conic bundles blowing up $2n \geq 4$ points on the zero section of self-intersection n .

We have $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset G_* \cap G_\circ$. The groups in (3) and (7) are conjugate to subgroups of G_* , and the groups in (6) conjugate to subgroups of G_\circ .

Via the birational map $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$ that blows up a real point and blows down the strict transforms of the two (real) fibres passing through the point, the group $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ is conjugate to a subgroup of G_* .

Via the birational map $\mathcal{Q}_{3,1} \dashrightarrow \mathbb{P}^2$ that blows up a real point and blows down the strict transforms of the two (non-real conjugate) fibres passing through the point, $\text{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ is conjugate to a subgroup of $G_\circ \cap G_*$.

Group (4): via the blow-down $X \rightarrow \mathbb{F}_0$ of the two exceptional divisors, the group is conjugate to a subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{F}_0)$ generated by a subgroup of $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ and a transformation of \mathbb{F}_0 that is the composition of a transformation preserving the fibrations and the automorphism exchanging the fibrations (see construction in [RZ2016, Proposition 3.5]). Thus group (4) is conjugate to a subgroup of G_* .

Group (5): via the blow-down $X \rightarrow \mathbb{P}^2$ of three disjoint (-1) -curves onto the three coordinate points of \mathbb{P}^2 , the group is conjugate to a subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by a subgroup of $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $[x : y : z] \mapsto [yz : xz : xy]$ [RZ2016, Proposition 3.6]. Since the latter is contained in J_* , group (5) is also conjugate to a subgroup of G_* . \square

3.3. Lowering the degree. In this section we prove the lemmas used in the proof of Proposition 3.7. Let $\varphi \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, $f_1, f_3 \in \mathcal{J}_\circ \cup \mathcal{H}$ and $f_2 \in \mathcal{A}$.

Lemma 3.9. *Suppose that $f_1, f_3 \in \mathcal{J}_\circ$ be standard or special quintic transformations and that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g).$$

Then there exists $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ$ such that $f_3f_2f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2h_1g) < \deg(f_1g), \quad i = 2, \dots, m.$$

Proof. Follows from [Z2015, Lemma 4.10] by taking Λ to be the linear system of $(f_1g)^{-1}$. \square

Lemma 3.10. *Let $f_1, f_3 \in \mathcal{H}$ be of degree 2. Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$ such that $f_3f_2f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2h_1g) < \deg(f_1g), \quad i = 2, \dots, m.$$

Proof. If $\deg(f_3f_2f_1) = 1$, we put $h_m = h_1 = f_3f_2f_1$.

If $\deg(f_3f_2f_1) = 2$, then f_1^{-1} and f_3f_2 have exactly two common base-points, namely their pairs of non-real conjugate base-points. It follows that the quadratic map $f_3f_2f_1$ has two non-real base-points and a real base-point. So there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f_3f_2f_1\alpha \in \mathcal{H}$. The claim follows with $h_1 = \alpha^{-1}$, $h_2 = \beta f_3f_2f_1\alpha$ and $h_m = h_3 = \beta^{-1}$.

If $\deg(f_3f_2f_1) = 3$, then the maps f_1^{-1} and f_3f_2 have exactly one common base-point, namely $[1 : 0 : 0]$. Hence $f_3f_2f_1 \in \mathcal{J}_*$. Since moreover f_1, f_3 have only one real base-point, $f_3f_2f_1$ has only one real base-point. Thus $f_3f_2f_1 \in \mathcal{H}$ and the claim follows with $h_1 = h_m = f_3f_2f_1$.

If $\deg(f_3f_2f_1) = 4$, then the maps f_1^{-1} and f_3f_2 have no common base-points. The following construction is visualised in the diagram below. Let $[1 : 0 : 0], p, \bar{p}$ be the base-points of f_1^{-1} and r, q, \bar{q} the base-points of f_3f_2 , and define $D := \deg(f_1g)$. By $m(t)$ we denote the multiplicity of $(f_1g)^{-1}$ in the point t . The assumptions $\deg(g) \leq D$ and $\deg(f_3f_2f_1g) < D$ imply that

$$(2) \quad m([1 : 0 : 0]) + 2m(p_i) \geq D, \quad m(r) + 2m(q) > D.$$

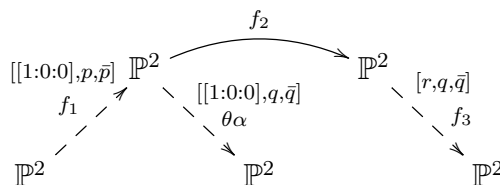
If $m([1 : 0 : 0]) \geq m(r)$, then Inequality (2) implies that

$$m([1 : 0 : 0]) + 2m(q) \geq m(r) + 2m(q) \stackrel{(2)}{>} D,$$

which means that $[1 : 0 : 0], q, \bar{q}$ are not collinear. In particular, there exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has base-points $[1 : 0 : 0], q, \bar{q}$. The above inequality implies

$$\deg(\theta\alpha f_1g) = 2D - m([1 : 0 : 0]) - 2m(q) < D.$$

The maps f_1^{-1} and $\theta\alpha$ have one common base-point, and the maps $\theta\alpha$ and f_3f_2 have two common base-points. We proceed as in (3) and (2).



Suppose that $m([1 : 0 : 0]) < m(r)$. Inequalities (2) imply that

$$m(r) + 2m(p) > m([1 : 0 : 0]) + 2m(p) \geq 2D,$$

so r, p, \bar{p} are not collinear. There exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has these points as base-points. The claim follows as above. \square

Lemma 3.11. *Let $f_1, f_3 \in \mathcal{H}$. Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{H}$, $h_2 \in \mathcal{A}$, $h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and $\deg(h_1) = \deg(f_1) - 1$,

$$\begin{aligned} \deg(h_2 h_1 g) &= \deg(h_1 g) \leq \deg(f_1 g), \\ \deg(h_i \cdots h_3 h_2 h_1 g) &< \deg(f_1 g), \quad i = 3, \dots, m. \end{aligned}$$

If the inequality in the assumption is strict, all inequalities are strict.

Proof. By Remark 3.2, f_1 and f_3 both have exactly one real base-point, and the rest of their base-points are pairs of non-real conjugate points. Let $m(q)$ be the multiplicity of $(f_1 g)^{-1}$ in the point q , and $D := \deg(f_1 g) = \deg(f_2 f_1 g)$. Both f_1 and $f_3 f_2$ have characteristic $(d; d-1, 1^{2d-2})$, hence there exist non-real base-points q_1, q_2 of f_1^{-1} and f_3 , respectively, such that

$$(3) \quad m([1 : 0 : 0]) + 2m(q_1) \geq D, \quad m(f_2^{-1}([1 : 0 : 0])) + 2m(f_2^{-1}(q_2)) > D.$$

We can assume that q_1, q_2 are points in \mathbb{P}^2 or on the exceptional divisor of $[1 : 0 : 0]$ or $f_2^{-1}([1 : 0 : 0])$, respectively. Furthermore, the points $[1 : 0 : 0], q_1, \bar{q}_1$ and the points $f_2^{-1}([1 : 0 : 0]), q_2, \bar{q}_2$ are not collinear because of Bézout theorem.

If q_1, q_2 are both in \mathbb{P}^2 , there exist $\tau_1, \tau_2 \in \mathcal{H}$ of degree 2 with base-points $[1 : 0 : 0], q_1, \bar{q}_1$ and $[1 : 0 : 0], q_2, \bar{q}_2$ respectively. We define $h_1 := \tau_1 f_1 \in \mathcal{H}$ and $h_m := f_3 \tau_2^{-1} \in \mathcal{H}$. By construction we have

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_m) = \deg(f_3) - 1$$

and by Inequalities (3) that

$$\begin{aligned} \deg(h_1 g) &= \deg(\tau_1 f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q_1) \leq D \\ \deg(\tau_2 f_2 f_1 g) &= 2D - m(f_2^{-1}([1 : 0 : 0])) - 2m(f_2^{-1}(q_2)) < D. \end{aligned}$$

We apply Lemma 3.10 to $f_1, \dots, f_{n-2}, h_1, \tau_1, f_2, \tau_2$ to get the claim.

Let us consider what to do if q_1 or q_2 is on the exceptional divisors of $[1 : 0 : 0]$.

If they are both on the exceptional divisors, respectively, then f_1 and f_3 are both of degree at least 3 and

$$2m(q_1) \leq m([1 : 0 : 0]), \quad 2m(f_2^{-1}(q_2)) \leq m(f_2^{-1}([1 : 0 : 0])).$$

Now, Inequalities (3) imply that

$$2m([1 : 0 : 0]) \geq D, \quad 2m(f_2^{-1}([1 : 0 : 0])) > D$$

Since f_1 and f_3 are of degree at least three, this is only possible if $f_2^{-1}([1 : 0 : 0]) = [1 : 0 : 0]$. In particular $f_2 \in \mathcal{H}$. We put $h_1 = h_m = f_3 f_2 f_1 \in \mathcal{H}$.

Let us assume that $f_2 \notin \mathcal{H}$. Then either q_1 or q_2 are points in \mathbb{P}^2 . Suppose that q_1 is in \mathbb{P}^2 and q_2 is on the exceptional divisor of $f_2^{-1}([1 : 0 : 0])$. Then, because of the above reasoning, we have

$$2m(f_2^{-1}([1 : 0 : 0])) > D > 2m([1 : 0 : 0]).$$

Inequalities (3) get us

$$m(f_2^{-1}([1 : 0 : 0])) + 2m(q_1) > m([1 : 0 : 0]) + 2m(q_1) \geq D.$$

Then $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$ are not collinear and all of them are points of \mathbb{P}^2 . There exists $\tau'_2 \in \mathcal{H}$ of degree 2 with base-points $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$. We proceed as in the first case with τ'_2 instead of τ_2 . If q_2 is a point of \mathbb{P}^2 and q_1 is on the exceptional divisor of $[1 : 0 : 0]$, we proceed analogously. \square

Lemma 3.12. *Let $f_1 \in \mathcal{H}$ be of degree 2 and $f_3 \in \mathcal{J}_\circ$ a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1g), \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f_3f_2f_1 = h_m \cdots h_1$ and

$$\begin{aligned} \deg(h_1g) &\leq \deg(f_1g), \\ \deg(h_i \cdots h_3h_2h_1g) &< \deg(f_1g) \quad i = 2, \dots, m. \end{aligned}$$

If we have strict inequality in the assumption, all inequalities are strict.

The same claim holds if $f_1 \in \mathcal{J}_\circ$ is a standard or special quintic transformation and $f_3 \in \mathcal{H}$ is of degree 2.

Proof. By $m(s)$ we denote the multiplicity of $(f_1g)^{-1}$ in s and defined $D := \deg(f_1g)$. Let $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ be the base-points of f_3f_2 and order them such that

$$m(r_1) \geq m(r_2) \geq m(r_3),$$

and r_1 is a proper point of \mathbb{P}^2 and r_2 is either a proper point of \mathbb{P}^2 or in the first neighbourhood of r_1 . Let $[1 : 0 : 0], q, \bar{q}$ be the base-points of f_1^{-1} . Lemma 3.5 and f_1 being quadratic imply respectively that

$$(4) \quad m(r_1) + m(r_2) + m(r_3) > D, \quad m([1 : 0 : 0]) + 2m(q) \geq D.$$

We look at two cases, $q = r_1$ and $q \neq r_1$.

(1) Suppose that $q = r_1$. As r_1, r_2, r_3 are not collinear, at least one of r_2, r_3 is not one a line contracted by f_1^{-1} , say r_i , and it is a point of \mathbb{P}^2 . Then $f_3f_2f_1$ sends the pencil of conics passing through the non-real base-points of f_1 and $f_1^{-1}(r_i), f_1^{-1}(\bar{r}_i)$ onto a pencil of conics through two pairs of non-real conjugate points of f_3^{-1} . Hence, there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f_3f_2f_1\alpha \in \mathcal{J}_\circ$. The claim follows with $h_1 = \alpha^{-1}$, $h_2 = \beta f_3f_2f_1\alpha$ and $h_m = h_3 = \beta^{-1}$.

(2) Suppose that $q \neq r_1$. We look at two cases, depending on the multiplicities $m([1 : 0 : 0]), m(q), m(\bar{q})$.

(2.1) If $m(q) \geq m([1 : 0 : 0])$, then Inequalities (4) and the order of the r_i imply that $m(q) \geq \frac{D}{3}$ and $m(r_1) + m(r_2) > \frac{2D}{3}$, and so

$$m(q) + m(r_1) + m(r_2) > D.$$

In particular, the six points $q, \bar{q}, r_1, \bar{r}_1, r_2, \bar{r}_2$ are not on a conic. Then the chosen order of the points implies that there exists a standard or special quintic transformation $\theta \in \mathcal{J}_\circ$ and $\alpha \in \mathcal{A}$ such that $\theta\alpha$ has these 6 points as base-points. The above inequality implies

$$\deg(\theta\alpha f_1g) = 5D - 4m(q) - 4m(r_1) - 4m(r_2) < D.$$

The maps f_1 and $\theta\alpha$ have two common base-points, and we find the maps h_1, \dots, h_k as in Case 1. The maps $g' := h_k \cdots h_1g, \theta^{-1}, f_2\alpha, f_3$ satisfy the assumptions of Lemma 3.9 with “<”, and we get the maps h_{k+1}, \dots, h_m from there.

(2.2) If $m(q) < m([1 : 0 : 0])$, then Inequalities (4) and the order of the r_i imply that $m([1 : 0 : 0]), m(r_1) \geq \frac{D}{3}$, so

$$m([1 : 0 : 0]) + 2m(r_1) > D$$

In particular, the points $[1 : 0 : 0], r_1, \bar{r}_1$ are not collinear. As they are proper points of \mathbb{P}^2 , there exists $\theta \in \mathcal{J}_\circ$ of degree 2 and $\alpha \in \mathcal{J}_\circ$ such that $\theta\alpha$ has these 3 points as base-points. Moreover, the above inequality implies that

$$\deg(\theta\alpha f_1g) = 2D - m(q) - 2m(r_1) < D.$$

We apply Lemma 3.11 to f_1, α, θ to get h_1, \dots, h_k and then Case 1 to $g' := h_k \cdots h_1 g, \theta^{-1}, f_2 \alpha, f_3$ with “<” to get the rest of the h_i .

The case where f_1 is a standard or special quintic transformation and $f_3 \in \mathcal{H}$ is of degree 2 is done analogously. \square

Lemma 3.13. *Let $f_1 \in \mathcal{H}$ and $f_3 \in \mathcal{J}_\circ$ a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1 g), \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g)$$

Then there exist $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and $\deg(h_1) = \deg(f_1) - 1$,

$$\deg(h_1 g) \leq \deg(f_1 g),$$

$$\deg(h_i \cdots h_3 h_2 h_1 g) < \deg(f_1 g) \quad i = 3, \dots, m.$$

If we have strict inequality in the assumption, all inequalities are strict.

If $f_1 \in \mathcal{J}_\circ$ is a standard or special quintic transformation and $f_3 \in \mathcal{H}$, we have

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g) \quad i = 2, \dots, m.$$

Proof. Let $m(s)$ be the multiplicity of $(f_1 g)^{-1}$ in the point s and define $D := \deg(f_1 g)$. Since f_1 has characteristic $(d; d-1, 1^{2d-2})$ and is contained in \mathcal{H} , there exist a non-real base-point q of f_1^{-1} such that

$$m([1 : 0 : 0]) + 2m(q) \geq D.$$

We can assume that they are points in \mathbb{P}^2 or on the exceptional divisor of $[1 : 0 : 0]$. Furthermore, the points $[1 : 0 : 0], q, \bar{q}$ are not collinear by Bézout theorem.

(1) Suppose that q, \bar{q} are points in \mathbb{P}^2 . Then there exists $\tau \in \mathcal{H}$ of degree 2 such that τ has base-points $[1 : 0 : 0], q, \bar{q}$. We put $h_1 := \tau f_1 \in \mathcal{H}$. Note that

$$\deg(h_1) = \deg(f_1) - 1,$$

$$\deg(h_1 g) = \deg(\tau f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q) \leq D.$$

We apply Lemma 3.12 to $g' := h_1 g$ and τ^{-1}, f_2, f_3 to get h_2, \dots, h_m .

(2) Suppose that q, \bar{q} are on the exceptional divisor of $[1 : 0 : 0]$. Then $m([1 : 0 : 0]) \geq 2m(q)$. Call $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ the base-points of $f_3 f_2$, and suppose that

$$m(r_1) \geq m(r_2) \geq m(r_3)$$

and r_1 is a point in \mathbb{P}^2 . Lemma 3.5 implies that

$$m(r_1) + m(r_2) + m(r_3) > D.$$

We get $m([1 : 0 : 0]) \geq \frac{D}{3}$ and $m(r_1) > \frac{D}{3}$, and so $m([1 : 0 : 0]) + 2m(r_1) > D$. In particular, the points $[1 : 0 : 0], r_1, \bar{r}_1$ are not collinear, and hence there exists $\tau \in \mathcal{H}$ of degree 2 with these three points its base-points. As above we put $h_1 := \tau f_1 \in \mathcal{H}$ and note that

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_1 g) = \deg(\tau f_1 g) < D,$$

and apply Lemma 3.12 to $g' := h_1 g$ and τ^{-1}, f_2, f_3 .

If f_1 is a standard or special quintic transformation and $f_3 \in \mathcal{H}$, we construct h_1, \dots, h_m analogously. \square

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