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THE REAL PLANE CREMONA GROUP IS A NON-TRIVIAL AMALGAM

SUSANNA ZIMMERMANN

ABSTRACT. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection.

1. INTRODUCTION

The plane Cremona group is the group $\text{Bir}_k(\mathbb{P}^2)$ of birational transformations of \mathbb{P}^2 defined over a field k . [CL2013, Appendix by Y. DE CORNULIER] shows that $\text{Bir}_k(\mathbb{P}^2)$ is not isomorphic to a non-trivial amalgam of two groups if k is algebraically closed. The best one can obtain is that $\text{Bir}_k(\mathbb{P}^2)$ is isomorphic to a non-trivial amalgam modulo one simple relation [B2012, L2010, I1984] or that it is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [W1992]. The world looks different for $k = \mathbb{R}$. We show that $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is indeed isomorphic to a non-trivial amalgamated product of two groups.

The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ contains two groups of Jonquières transformations:

$$\mathcal{J}_* = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1 : 0 : 0] \}$$

$$\mathcal{J}_\circ = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1, \bar{p}_1, p_2, \bar{p}_2 \}$$

where $p_1, p_2 \in \mathbb{P}^2$ are two non-real points such that $p_1, \bar{p}_1, p_2, \bar{p}_2$ are not collinear. The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is generated by $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$, \mathcal{J}_* , \mathcal{J}_\circ [BM2014], and the groups \mathcal{J}_* , \mathcal{J}_\circ are not conjugate in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ [Z2015, Proposition 5.3]. Over the field \mathbb{C} , we can send the pencil of conics through four points, no three collinear, onto a pencil of lines through a point, so the complex versions of the two groups are conjugate in $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$.

We write $\mathcal{A} := \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and denote by $G_\circ \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by \mathcal{A} and \mathcal{J}_\circ , and by $G_* \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ the subgroup generated by \mathcal{A} and \mathcal{J}_* .

Theorem 1.1. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the non-trivial amalgamated product of G_* and G_\circ along their intersection.*

Proposition 1.2. *Both G_* and G_\circ have uncountable index in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$*

The intersection $G_* \cap G_\circ$ contains all elements of \mathcal{J}_* with only one real base-point (see remark after Definition 3.2).

The decomposition of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ into an amalgam follows from the structure theorem [Z2015, Theorem 4.4] (Theorem 2.3 below). The task is to prove that G_* and G_\circ are both proper subgroups; this is not straightforward since their complex analogues both coincide with $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$. Many of the ingredients of the proof will be similar to the proof of [Z2015, Theorem 4.4].

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Corollary 1.3. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on a tree, and all its algebraic subgroups are conjugate to a subgroup of G_* or of G_{\circ} .*

For the subgroups of odd order Corollary 1.3 can also be verified by checking their classification in [Y2016].

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2. THE REAL CREMONA GROUP IS AN AMALGAM

In this section we show that $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to the amalgamated product of G_* and G_{\circ} along their intersection. Non-triviality will be proven in the next section.

Definition 2.1. A *standard quintic transformation* is an element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points in \mathbb{P}^2 . (See [BM2014, Example 3.1] for a definition via blow-ups.)

Remark 2.2. For any standard or special quintic transformation f there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{J}_{\circ}$ [Z2015, Lemma 3.10]. In particular, all standard quintic transformations are contained in G_{\circ} .

We make use of the following structure theorem. Let \mathcal{G} be the free group $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ}$ amalgamated along all pairwise intersections, and let $w: \mathcal{A} \cup \mathcal{J}_* \cup \mathcal{J}_{\circ} \rightarrow \mathcal{G}$ be the canonical word map.

Theorem 2.3 ([Z2015, Theorem 4.4]). *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is isomorphic to \mathcal{G} modulo the following relations:*

(rel. 1) *Let $\theta_1, \theta_2 \in \mathcal{J}_{\circ}$ be standard quintic transformations and $\alpha_1, \alpha_2 \in \mathcal{A}$.*

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$

(rel. 2) *Let $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_{\circ}$ be both of degree 2 or of degree 3 and $\alpha_1, \alpha_2 \in \mathcal{A}$.*

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$

(rel. 3) *Let $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$ all be of degree 2, or τ_1, τ_2 of degree 2 and τ_3 of degree 3, and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}$.*

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \quad \text{if} \quad \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

Corollary 2.4. *The group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the amalgamated product of G_* and G_{\circ} along their intersection.*

Proof. Let G be the amalgamated product of G_* and G_{\circ} along their intersection. By the universal property of amalgamated products there exists a homomorphism $\psi: G \rightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is the identity map on \mathcal{A} , \mathcal{J}_* and \mathcal{J}_{\circ} . On the other hand, we have a canonical surjective homomorphism $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ} \rightarrow \mathcal{G} \rightarrow G$. We claim that $\mathcal{G} \rightarrow G$ factors through $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. By Theorem 2.3 it suffices to check that relations (rel. 1), (rel. 2) and (rel. 3) are satisfied in G .

Relation (rel. 1): standard quintic transformations are contained in G_{\circ} (Remark 2.2). They are not contained G_* because the kernel of the abelianisation of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ contains G_*

but does not contain any of the standard quintic transformations [Z2015, Remark 3.19(3), Proposition 5.3].

Relation (rel. 2) and (rel. 3) are relations among linear, quadratic and cubic transformations. Any transformation of degree ≤ 3 is contained in G_* , some of them in $G_* \cap G_\circ$, and relations (rel. 2) and (rel. 3) hold in G_* .

So, the morphism $\mathcal{G} \rightarrow G$ factors through $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, and the induced homomorphism $\varphi: \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow G$ is the identity on $\mathcal{A}, \mathcal{J}_*, \mathcal{J}_\circ$. Thus $\psi = \varphi^{-1}$. \square

3. THE GROUP G_\circ IS A STRICT SUBGROUP OF $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

This section is devoted to the proof that G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ (Corollary 3.8). The key idea is to show that it does not contain any elements of degree 2 having three real base-points - for instance, it does not contain the standard quadratic transformation $[x : y : z] \mapsto [yz : xz : xy]$. To show this, we prove that any element in G_\circ is the composition of elements in $\mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ (see Definition 3.2 for the definition of $\mathcal{H}_* \subset G_\circ \cap \mathcal{J}_*$) such that the successive degree of the composition is non-decreasing (Proposition 3.7)

3.1. Properties of the groups \mathcal{J}_* and \mathcal{J}_\circ . Before we run off to prove our claims, we quickly review the groups \mathcal{J}_* and \mathcal{J}_\circ and some of their properties.

Remark 3.1. The characteristic of an element $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is the tuple $(\deg(f); m_1^{e_1}, \dots, m_k^{e_k})$, where m_1, \dots, m_k are the different multiplicities of the base-points of f and e_i the number of base-points of multiplicity m_i .

An element of \mathcal{J}_\circ has characteristic

$$\begin{aligned} & \left(d; \left(\frac{d-1}{2} \right)^4, 2^{\frac{d-1}{2}} \right), \quad \text{if } \deg(f) \text{ is odd} \\ & \left(d; \left(\frac{d}{2} \right)^2, \left(\frac{d-2}{2} \right)^2, 2^{\frac{d-2}{2}}, 1 \right), \quad \text{if } \deg(f) \text{ is even} \end{aligned}$$

Moreover, if $d \geq 2$, $p_1, \bar{p}_1, p_2, \bar{p}_2$ are base-points of multiplicity $\frac{d-1}{2}$ or $\frac{d}{2}$ and $\frac{d-2}{2}$ and all other base-points are of multiplicity 2 or 1 [Z2015, Lemma 3.2].

In particular, \mathcal{J}_\circ contains only quadratic maps with one real and a pair of non-real conjugate base-points.

The group \mathcal{J}_* is isomorphic to $\text{PGL}_2(\mathbb{R}(x)) \rtimes \text{PGL}_2(\mathbb{R})$ because on the chart $z = 1$ all of its elements are of the form

$$f: (x, y) \mapsto \left(\frac{\alpha(y)x + \beta(y)}{\gamma(y)x + \delta(y)}, \frac{ay + b}{cy + d} \right)$$

with $a, b, d, c \in \mathbb{R}$, $ad - bc \neq 0$ and $\alpha, \beta, \delta, \gamma \in \mathbb{R}[y]$, $\alpha\delta - \beta\gamma \neq 0$.

The characteristic of any element of \mathcal{J}_* is $(d; d-1, 1^{2d-2})$, where $[1 : 0 : 0]$ is the base-point of multiplicity $d-1$.

Definition 3.2. We denote by $\mathcal{H} \subset \mathcal{J}_*$ the subgroup of elements having *at most one real* base-point.

If an element of \mathcal{H} is non-linear, all its base-points different from $[1 : 0 : 0]$ are pairs of non-real conjugate points. For any quadratic transformation $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ with one real and a pair of non-real conjugate base-points there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{H}$. In fact, the

group \mathcal{H} is generated by its linear and quadratic elements and is hence contained in G_\circ (we can follow the argument of [AC2002, Theorem 8.4.3]).

Definition 3.3. A *special quintic transformation* is an element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of degree 5 whose base-points are three pairs of non-real conjugate base-points, two of them points in \mathbb{P}^2 and one pair infinitely near one of the other two pairs. (See [BM2014, Example 3.2] for a definition via blow-ups.)

Remark 3.4. (1) A standard (resp. special) quintic transformation f has multiplicity 2 in each of its base-points. In particular, there is no conic containing all its base-points.

(2) The inverse f^{-1} is also a standard (resp. special) quintic transformation and f sends the pencil of conics through two pairs of base-points onto a pencil of conics through two pairs of base-points of f^{-1} .

(3) Pick any three pairs of non-real conjugate points, not all six points contained in one conic. Suppose they are all points in \mathbb{P}^2 (resp. two pairs are points in \mathbb{P}^2 and one pair is in the first neighbourhood of one of the other two pairs). Then there is a standard (resp. special) quintic transformation with these points as its base-points [BM2014, Example 3.1, Example 3.2], [RV2005, §I].

(4) For any standard or special quintic transformation f there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f \alpha \in \mathcal{J}_\circ$ [Z2015, Lemma 3.10]. In particular, all standard and special quintic transformations are contained in G_\circ .

For $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ and a point p , let $m_f(p)$ the multiplicity of f in p .

Lemma 3.5 ([Z2015, Lemma 4.8]). *Let $f \in \mathcal{J}_\circ$ be non-linear and $g \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. Suppose that*

$$\deg(fg) \leq \deg(g) \quad (\text{resp. } < \deg(g))$$

Then there exists a base-point $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ of f of multiplicity 2 such that

$$(ineq\ 1) \quad m_{g^{-1}}(p_1) + m_{g^{-1}}(p_2) + m_{g^{-1}}(q) \geq \deg(g) \quad (\text{resp. } > \deg(g))$$

or f has a simple base-point r and there exists $i \in \{1, 2\}$ such that

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(r) \geq \deg(g), \quad \text{where } m_f(p_i) = \deg(f)/2 \quad (\text{resp. } > \deg(g)).$$

Lemma 3.6 ([Z2015, Lemma 4.14]). *In Lemma 3.5 (ineq 1), suppose that q is a real proper point of \mathbb{P}^2 . Then one of the following two assertions holds:*

(1) *there exists $h \in \mathcal{J}_\circ$ of degree 3 with base-points $p_1, \bar{p}_1, p_2, \bar{p}_2, q$.*

(2) *$\deg(f)$ is even and there exists $h \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, q and*

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(q) \geq \deg(g), \quad \text{where } m_f(p_i) = \frac{\deg(f)}{2}.$$

3.2. The group G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. The following proposition is the key to proving that G_\circ is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. The lemmas used in its proof are stated and proven in the next subsection, and to prove them we only use the properties of \mathcal{J}_* , \mathcal{J}_\circ and \mathcal{H} listed in the previous subsection.

Proposition 3.7. *For any $f \in G_\circ$ there exist $g_1, \dots, g_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f = g_m \cdots g_1$ and such that $\deg(g_i \cdots g_1) \leq \deg(g_{i+1}g_i \cdots g_1)$, for $i = 1, \dots, m-1$.*

Proof. We write $f = f_N \cdots f_1$ for some $f_1, \dots, f_N \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$. Define $d_i := \deg(f_i \cdots f_1)$ and

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad n := \max\{i \mid d_i = D\}, \quad k = \sum_{i=1}^n (\deg(f_i) - 1).$$

We claim that there exist $h_1, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that

$$f_{n+1}f_nf_{n-1} = h_k \cdots h_1$$

and the pair (D', k') associated to the composition $f_N \cdots f_{n+2}h_k \cdots h_1f_{n-2} \cdots f_1$ is strictly smaller than (D, k) with respect to the lexicographic order.

If the claim holds, the proof finishes as follows. The procedure stops at the pair (D'', k'') where $D'' = \deg(f)$ and $f = f'_{n''} \cdots f'_1$. We define

$$I := \max\{i \mid \deg(f'_i \cdots f'_1) > \deg(f'_{i+1} \cdots f'_1)\}, \quad f' := f'_{I+1} \cdots f'_1$$

By construction, we have $\deg(f') < \deg(f)$. We repeat the process for f' . After finitely many steps we will not find such an I anymore, and the composition we have obtained is the one in the statement.

Let us prove the claim. If $D = 1$, all f_i are linear maps, so f is linear, and the claim holds with $m = 1$.

Suppose that $D \geq 2$. Any consecutive letters f_i, f_{i+1} that are both contained in \mathcal{A} or in \mathcal{J}_\circ we replace by their product. This does not increase the pair (D, k) . We can therefore assume that no two consecutive letters are both contained in \mathcal{A} or in \mathcal{J}_\circ . By definition of n , we have $d_n > d_{n+1}$, and in particular,

$$f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{A}, \quad f_n \in \mathcal{A} \setminus \mathcal{J}_\circ, \quad f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{A} \quad \text{and} \quad d_{n-2} \leq d_{n-1} = d_n = D.$$

(1) If $f_{n-1}, f_{n+1} \in \mathcal{H}$, we apply Lemma 3.11 to f_{n-1}, f_n, f_{n+1} . There exist $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{H}$ such that $f_{n+1}f_nf_{n-1} = h_k \cdots h_1$ and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, \\ \deg(h_2h_1f_{n-2} \cdots f_1) &= \deg(h_1f_{n-2} \cdots f_1) \leq D, \\ \deg(h_i \cdots h_2h_1f_{n-1} \cdots f_1) &< D, \quad i = 3, \dots, k. \end{aligned}$$

Then $D' \leq D$ and if $D' = D$ then $n' \leq n$ and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

(2) Else, we proceed as follows. Let $m_i(t)$ be the multiplicity of $(f_i \cdots f_1)^{-1}$ in the point t . If $f_{n-1} \in \mathcal{J}_\circ$, then by Lemma 3.5 there exists a base-point $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$ of f_{n-1}^{-1} of multiplicity 2 such that

$$m_{n-1}(p_1) + m_{n-1}(p_2) + m_{n-1}(q) \geq D$$

or $\deg(f_{n-1})$ is even and there exists a simple base-point r of f_{n-1}^{-1} such that

$$(1) \quad 2m_{n-1}(p_i) + m_{n-1}(r) \geq D, \quad \text{where } m_{f_{n-1}^{-1}}(p_i) = \deg(f_{n-1})/2.$$

• If q is a non-real point, we can assume that it is a proper point of \mathbb{P}^2 or in the first neighbourhood of one of $p_1, \bar{p}_1, p_2, \bar{p}_2$. Since $m_{f_{n-1}^{-1}}(q) = 2$, the points $p_1, \dots, \bar{p}_2, q, \bar{q}$ are not on one conic (Remark 3.1). By Remark 3.4 there exists a standard or special quintic transformation $\theta_{n-1} \in \mathcal{J}_\circ$ with these 6 points as base-points.

• If q is a real point, we can assume that it is a proper point of \mathbb{P}^2 (it cannot be infinitely near one of p_1, \dots, \bar{p}_2). By Lemma 3.6 there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 3 with base-points p_1, \dots, \bar{p}_2, q or there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, q where $m_{f_{n-1}^{-1}}(p_i) = \frac{\deg(f_{n-1})}{2}$ and $2m_{n-1}(p_i) + m_{n-1}(q) \geq D$.

• If there is no such q , we have r as in (1). If r is not a proper point of \mathbb{P}^2 we replace it by the (real) base-point of f_{n-1}^{-1} to which it is infinitely near. This does not change Inequality (1). Remark 3.1 implies that p_i, \bar{p}_i, r are not collinear, so there exists $\theta_{n-1} \in \mathcal{J}_\circ$ of degree 2 with base-points p_i, \bar{p}_i, r [Z2015, Lemma 3.4].

In all three cases we define $h_1 := \theta_{n-1}f_{n-1} \in \mathcal{J}_\circ$. Then

$$\deg(h_1 f_{n-2} \cdots f_1) = \deg(\theta_{n-1} f_{n-1} \cdots f_1) \leq D \quad \text{and} \quad \deg(\theta_{n-1} f_{n-1}) < \deg(f_{n-1}).$$

We proceed analogously if $f_{n+1} \in \mathcal{J}_\circ$, and find $\theta_{n+1} \in \mathcal{J}_\circ$ satisfying the analogous inequalities with “ $<$ ”, and define $h_k := f_{n+1}\theta_{n+1}^{-1} \in \mathcal{J}_\circ$. Analogously, we have

$$\deg(\theta_{n+1} f_n \cdots f_1) < D, \quad \deg(h_k) < \deg(f_{n+1}).$$

Now, we use Lemma 3.9 and Lemma 3.13. The constructions are summarised in the commutative diagrams below.

If $f_{n-1}, f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$, we apply Lemma 3.9 to $\theta_{n-1}^{-1}, f_n, \theta_{n+1}$. There exist $h_2 \in \mathcal{A}$, $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ$ such that $\theta_{n+1} f_n \theta_{n-1}^{-1} = h_{k-1} \cdots h_2$ and

$$\deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k-1.$$

If $f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{H}$ and $f_{n+1} \in \mathcal{H}$, we apply the last part of Lemma 3.13 to $\theta_{n-1}, f_n, f_{n+1}$. There exist $h_2 \in \mathcal{A}$, $h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f_{n+1} f_n \theta_{n-1} = h_k \cdots h_2$ and

$$\deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k.$$

If $f_{n-1} \in \mathcal{H}$ and $f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$, we apply Lemma 3.13 to $f_{n-1}, f_n, \theta_{n+1}$. There exist $h_1 \in \mathcal{J}_\circ$, $h_2 \in \mathcal{A}$, $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $\theta_{n+1} f_n f_{n-1} = h_{k-1} \cdots h_1$ and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, \quad \deg(h_1 f_{n-2} \cdots f_1) = \deg(h_2 h_1 f_{n-2} \cdots f_1) \leq D \\ \deg(h_i \cdots h_3 h_2 h_1 f_{n-2} \cdots f_1) &< D, \quad i = 3, \dots, k-1. \end{aligned}$$

The diagrams illustrate the construction of h_i and the degree bounds. Each diagram shows a sequence of maps between \mathbb{P}^2 . The first diagram shows a path from \mathbb{P}^2 to \mathbb{P}^2 via f_n , with intermediate maps $h_{k-1} \cdots h_1$ and h_k . The second diagram shows a path from \mathbb{P}^2 to \mathbb{P}^2 via f_n , with intermediate maps h_1 and $h_k \cdots h_2$. The third diagram shows a path from \mathbb{P}^2 to \mathbb{P}^2 via f_n , with intermediate maps h_1 and $h_{k-1} \cdots h_2$. The diagrams are labeled with 'Lem. 3.13' and 'Lem. 3.9'.

We claim that in each case, the pair (D', k') is strictly smaller than (D, k) . Indeed, the above conditions imply $D' \leq D$. If $D' = D$ then $n' \leq n$ and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

□

Corollary 3.8. *The group G_\circ contains no maps of degree 2 having more than one real base-point. In particular, $G_\circ \subsetneq \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a strict subgroup.*

Proof. Suppose that G_\circ contains an element τ of degree $\deg(\tau) = 2$ with three real base-points. By Proposition 3.7, there exist $g_1, \dots, g_n \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $\tau = g_n \cdots g_1$ and

$$\deg(g_i \cdots g_1) \leq \deg(g_{i+1} \cdots g_1), \quad i = 1, \dots, n-1.$$

We may suppose that $\deg(g_1) \geq 2$, and then $\deg(g_i \cdots g_1) = 2$ for $i = 1, \dots, n$. This is only possible if

- $\deg(g_1) = 2$,
- $\deg(g_i) = 1, 2, 3, 4$ for $i = 2, \dots, n$,
- if $\deg(g_i) = 2, 3, 4$, then g_i and $(g_{i-1} \cdots g_1)^{-1}$ share 2, 3, 3 base-points of multiplicity:
 if $\deg(g_i) = 3$, these are a double and two simple base-points
 if $\deg(g_i) = 4$, these are three double points.

The characteristics in Remark 3.1 and Remark 3.2, and all g_i being contained in $\mathcal{J}_\circ \cup \mathcal{H}$ of degree ≤ 4 implies that the quadratic maps $g_i \cdots g_1$ have two non-real base-points for $i = 1, \dots, n$. This contradicts $\tau = g_n \cdots g_1$ having three real base-points. \square

Proof of Theorem 1.1. By Proposition 2.4 is the amalgamated product of G_* and G_\circ along their intersection. Furthermore, G_* is a strict subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ by [Z2015, Proposition 5.3], and G_\circ is a strict subgroup by Corollary 3.8. \square

Proof of Proposition 1.2. Let's prove the claim first for G_\circ . We denote by $\sigma: [x : y : z] \mapsto [yz : xz : xy]$ the standard quadratic involution of \mathbb{P}^2 , and define a map

$$\varphi: \{[x : y : z] \mapsto [x + az : y + bz : z] \mid a, b \in \mathbb{R}\} := A \longrightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_\circ, \quad \alpha \mapsto \alpha\sigma G_\circ.$$

Note that by Corollary 3.8 the transformation σ is not contained in G_\circ . In particular, $\alpha\sigma G_\circ \neq G_\circ$ for all $\alpha \in A$. We claim that the map φ is injective. Note that for all $\alpha \in A$ the transformation $\sigma\alpha\sigma$ is of degree ≤ 2 . It is linear if and only if σ and $(\alpha\sigma)^{-1}$ have three common base-points, i.e. if and only if $\alpha = \text{Id}$. If $\sigma\alpha\sigma$ is of degree 2 then it has three real base-points. It follows from Corollary 3.8 that for $\beta, \gamma \in A$ the transformation $\sigma\beta^{-1}\gamma\sigma$ is contained in G_\circ if and only if it is linear, i.e. if and only if $\beta^{-1}\gamma = \text{Id}$. Thus the map φ is injective.

Let's prove the claim for G_* . By [Z2015, Lemma 6.7, Corollary 6.12] we have

$$G_* = \langle \mathcal{J}_*, \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \subset \langle \langle \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \rangle = [\text{Bir}_{\mathbb{R}}(\mathbb{P}^2), \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)].$$

Then there is a surjective map from $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_*$ onto the abelianisation of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, which is isomorphic to $\bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$ [Z2015, Theorem 1.1]. It follows that G_* has uncountable index in $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$. \square

Proof of Corollary 1.3. By Theorem 1.1, the group $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ acts on the Bass-Serre tree T of the amalgam $G_* *_{G_* \cap G_\circ} G_\circ$. Then every element of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ of finite order has a fixed point on T . It follows that every finite subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ has a fixed point on T [S1980, §I.6.5, Corollary 3], and is in particular conjugate to a subgroup of G_* or G_\circ .

For infinite algebraic subgroups of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, it suffices to check the claim for the maximal ones. Before we list them, we have to introduce some notation. For $n \geq 0$, we denote by \mathbb{F}_n the n -th Hirzebruch surface

$$\mathbb{F}_n = \{[x : y : z][u : v] \in \mathbb{P}_{\mathbb{R}}^2 \times \mathbb{P}_{\mathbb{R}}^1 \mid u^n z = v^n y\}.$$

Note that $\mathbb{F}_0 = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$. By $\mathcal{Q}_{3,1} \subset \mathbb{P}_{\mathbb{R}}^3$ we denote the real hypersurface given by $w^2 = x^2 + y^2 + z^2$. Note that it is isomorphic to the real surface $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, (x, y) \mapsto (\bar{y}, \bar{x}))$. It is obtained by blowing up two conjugate points in \mathbb{P}^2 and contracting the (real) line passing

through them. We denote by $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$ the conic bundle obtained by blowing up a pair of non-real conjugate points on $\mathcal{Q}_{3,1}$.

Note that for $n \geq 0$, the group \mathcal{J}_* is conjugate to the group of birational transformations of \mathbb{F}_n preserving the conic bundle structure $\text{pr}_2: \mathbb{F}_n \rightarrow \mathbb{P}^1$. The group \mathcal{J}_\circ is conjugate to the group of birational transformations of $X_{[2]}$ preserving the conic bundle structure $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$. The infinite maximal algebraic subgroups of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ are conjugate to $\text{Aut}_{\mathbb{R}}(X)$ where X is one of the real rational surfaces in the following list [RZ2016]:

- (1) $X = \mathbb{P}^2$,
- (2) $X = \mathcal{Q}_{3,1}$,
- (3) $X = \mathbb{F}_n$, $n = 0$, $n \geq 2$,
- (4) X is a del Pezzo surface of degree 6 obtained by blowing up a pair of non-real conjugate points on \mathbb{F}_0 ,
- (5) X is a del Pezzo surface of degree 6 obtained by blowing up two real points on \mathbb{F}_0 ,
- (6) $\eta: X \rightarrow X_{[2]}$ is a birational morphism of real conic bundles blowing up $n \geq 1$ pairs with non-real conjugate points on non-real fibres on the non-real conjugate disjoint (-1) -curves of $X_{[2]}$ that are the exceptional divisors of $X_{[2]} \rightarrow \mathcal{Q}_{3,1}$,
- (7) $\eta: X \rightarrow \mathbb{F}_n$ is a birational morphism of real conic bundles blowing up $2n \geq 4$ points on the zero section of self-intersection n .

We have $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset G_* \cap G_\circ$. The groups in (3) and (7) are conjugate to subgroups of G_* , and the groups in (6) conjugate to subgroups of G_\circ .

Via the birational map $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$ that blows up a real point and blows down the strict transforms of the two (real) fibres passing through the point, the group $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ is conjugate to a subgroup of G_* .

Via the birational map $\mathcal{Q}_{3,1} \dashrightarrow \mathbb{P}^2$ that blows up a real point and blows down the strict transforms of the two (non-real conjugate) fibres passing through the point, $\text{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$ is conjugate to a subgroup of $G_\circ \cap G_*$.

Group (4): via the blow-down $X \rightarrow \mathbb{F}_0$ of the two exceptional divisors, the group is conjugate to a subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{F}_0)$ generated by a subgroup of $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$ and a transformation of \mathbb{F}_0 that is the composition of a transformation preserving the fibrations and the automorphism exchanging the fibrations (see construction in [RZ2016, Proposition 3.5]). Thus group (4) is conjugate to a subgroup of G_* .

Group (5): via the blow-down $X \rightarrow \mathbb{P}^2$ of three disjoint (-1) -curves onto the three coordinate points of \mathbb{P}^2 , the group is conjugate to a subgroup of $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ generated by a subgroup of $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ and $[x : y : z] \mapsto [yz : xz : xy]$ [RZ2016, Proposition 3.6]. Since the latter is contained in J_* , group (5) is also conjugate to a subgroup of G_* . \square

3.3. Lowering the degree. In this section we prove the lemmas used in the proof of Proposition 3.7. Let $\varphi \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$, $f_1, f_3 \in \mathcal{J}_\circ \cup \mathcal{H}$ and $f_2 \in \mathcal{A}$.

Lemma 3.9. *Suppose that $f_1, f_3 \in \mathcal{J}_\circ$ be standard or special quintic transformations and that*

$$\deg(g) \leq \deg(f_1 g) \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g).$$

Then there exists $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g), \quad i = 2, \dots, m.$$

Proof. Follows from [Z2015, Lemma 4.10] by taking Λ to be the linear system of $(f_1 g)^{-1}$. \square

Lemma 3.10. *Let $f_1, f_3 \in \mathcal{H}$ be of degree 2. Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$ such that $f_3f_2f_1 = h_m \cdots h_1$ and

$$\deg(h_i \cdots h_2h_1g) < \deg(f_1g), \quad i = 2, \dots, m.$$

Proof. If $\deg(f_3f_2f_1) = 1$, we put $h_m = h_1 = f_3f_2f_1$.

If $\deg(f_3f_2f_1) = 2$, then f_1^{-1} and f_3f_2 have exactly two common base-points, namely their pairs of non-real conjugate base-points. It follows that the quadratic map $f_3f_2f_1$ has two non-real base-points and a real base-point. So there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f_3f_2f_1\alpha \in \mathcal{H}$. The claim follows with $h_1 = \alpha^{-1}$, $h_2 = \beta f_3f_2f_1\alpha$ and $h_m = h_3 = \beta^{-1}$.

If $\deg(f_3f_2f_1) = 3$, then the maps f_1^{-1} and f_3f_2 have exactly one common base-point, namely $[1 : 0 : 0]$. Hence $f_3f_2f_1 \in \mathcal{J}_*$. Since moreover f_1, f_3 have only one real base-point, $f_3f_2f_1$ has only one real base-point. Thus $f_3f_2f_1 \in \mathcal{H}$ and the claim follows with $h_1 = h_m = f_3f_2f_1$.

If $\deg(f_3f_2f_1) = 4$, then the maps f_1^{-1} and f_3f_2 have no common base-points. The following construction is visualised in the diagram below. Let $[1 : 0 : 0], p, \bar{p}$ be the base-points of f_1^{-1} and r, q, \bar{q} the base-points of f_3f_2 , and define $D := \deg(f_1g)$. By $m(t)$ we denote the multiplicity of $(f_1g)^{-1}$ in the point t . The assumptions $\deg(g) \leq D$ and $\deg(f_3f_2f_1g) < D$ imply that

$$(2) \quad m([1 : 0 : 0]) + 2m(p_i) \geq D, \quad m(r) + 2m(q) > D.$$

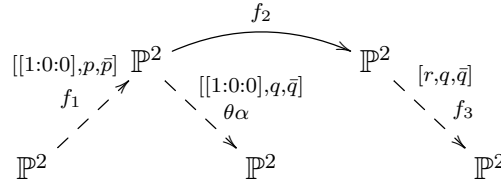
If $m([1 : 0 : 0]) \geq m(r)$, then Inequality (2) implies that

$$m([1 : 0 : 0]) + 2m(q) \geq m(r) + 2m(q) \stackrel{(2)}{>} D,$$

which means that $[1 : 0 : 0], q, \bar{q}$ are not collinear. In particular, there exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has base-points $[1 : 0 : 0], q, \bar{q}$. The above inequality implies

$$\deg(\theta\alpha f_1g) = 2D - m([1 : 0 : 0]) - 2m(q) < D.$$

The maps f_1^{-1} and $\theta\alpha$ have one common base-point, and the maps $\theta\alpha$ and f_3f_2 have two common base-points. We proceed as in (3) and (2).



Suppose that $m([1 : 0 : 0]) < m(r)$. Inequalities (2) imply that

$$m(r) + 2m(p) > m([1 : 0 : 0]) + 2m(p) \geq 2D,$$

so r, p, \bar{p} are not collinear. There exists $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{H}$ of degree 2 such that $\theta\alpha$ has these points as base-points. The claim follows as above. \square

Lemma 3.11. *Let $f_1, f_3 \in \mathcal{H}$. Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{H}$, $h_2 \in \mathcal{A}$, $h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and $\deg(h_1) = \deg(f_1) - 1$,

$$\begin{aligned} \deg(h_2 h_1 g) &= \deg(h_1 g) \leq \deg(f_1 g), \\ \deg(h_i \cdots h_3 h_2 h_1 g) &< \deg(f_1 g), \quad i = 3, \dots, m. \end{aligned}$$

If the inequality in the assumption is strict, all inequalities are strict.

Proof. By Remark 3.2, f_1 and f_3 both have exactly one real base-point, and the rest of their base-points are pairs of non-real conjugate points. Let $m(q)$ be the multiplicity of $(f_1 g)^{-1}$ in the point q , and $D := \deg(f_1 g) = \deg(f_2 f_1 g)$. Both f_1 and $f_3 f_2$ have characteristic $(d; d-1, 1^{2d-2})$, hence there exist non-real base-points q_1, q_2 of f_1^{-1} and f_3 , respectively, such that

$$(3) \quad m([1 : 0 : 0]) + 2m(q_1) \geq D, \quad m(f_2^{-1}([1 : 0 : 0])) + 2m(f_2^{-1}(q_2)) > D.$$

We can assume that q_1, q_2 are points in \mathbb{P}^2 or on the exceptional divisor of $[1 : 0 : 0]$ or $f_2^{-1}([1 : 0 : 0])$, respectively. Furthermore, the points $[1 : 0 : 0], q_1, \bar{q}_1$ and the points $f_2^{-1}([1 : 0 : 0]), q_2, \bar{q}_2$ are not collinear because of Bézout theorem.

If q_1, q_2 are both in \mathbb{P}^2 , there exist $\tau_1, \tau_2 \in \mathcal{H}$ of degree 2 with base-points $[1 : 0 : 0], q_1, \bar{q}_1$ and $[1 : 0 : 0], q_2, \bar{q}_2$ respectively. We define $h_1 := \tau_1 f_1 \in \mathcal{H}$ and $h_m := f_3 \tau_2^{-1} \in \mathcal{H}$. By construction we have

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_m) = \deg(f_3) - 1$$

and by Inequalities (3) that

$$\begin{aligned} \deg(h_1 g) &= \deg(\tau_1 f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q_1) \leq D \\ \deg(\tau_2 f_2 f_1 g) &= 2D - m(f_2^{-1}([1 : 0 : 0])) - 2m(f_2^{-1}(q_2)) < D. \end{aligned}$$

We apply Lemma 3.10 to $f_1, \dots, f_{n-2}, h_1, \tau_1, f_2, \tau_2$ to get the claim.

Let us consider what to do if q_1 or q_2 is on the exceptional divisors of $[1 : 0 : 0]$.

If they are both on the exceptional divisors, respectively, then f_1 and f_3 are both of degree at least 3 and

$$2m(q_1) \leq m([1 : 0 : 0]), \quad 2m(f_2^{-1}(q_2)) \leq m(f_2^{-1}([1 : 0 : 0])).$$

Now, Inequalities (3) imply that

$$2m([1 : 0 : 0]) \geq D, \quad 2m(f_2^{-1}([1 : 0 : 0])) > D$$

Since f_1 and f_3 are of degree at least three, this is only possible if $f_2^{-1}([1 : 0 : 0]) = [1 : 0 : 0]$. In particular $f_2 \in \mathcal{H}$. We put $h_1 = h_m = f_3 f_2 f_1 \in \mathcal{H}$.

Let us assume that $f_2 \notin \mathcal{H}$. Then either q_1 or q_2 are points in \mathbb{P}^2 . Suppose that q_1 is in \mathbb{P}^2 and q_2 is on the exceptional divisor of $f_2^{-1}([1 : 0 : 0])$. Then, because of the above reasoning, we have

$$2m(f_2^{-1}([1 : 0 : 0])) > D > 2m([1 : 0 : 0]).$$

Inequalities (3) get us

$$m(f_2^{-1}([1 : 0 : 0])) + 2m(q_1) > m([1 : 0 : 0]) + 2m(q_1) \geq D.$$

Then $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$ are not collinear and all of them are points of \mathbb{P}^2 . There exists $\tau'_2 \in \mathcal{H}$ of degree 2 with base-points $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$. We proceed as in the first case with τ'_2 instead of τ_2 . If q_2 is a point of \mathbb{P}^2 and q_1 is on the exceptional divisor of $[1 : 0 : 0]$, we proceed analogously. \square

Lemma 3.12. *Let $f_1 \in \mathcal{H}$ be of degree 2 and $f_3 \in \mathcal{J}_o$ a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1g), \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist $h_1 \in \mathcal{A}$, $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_o \cup \mathcal{H}$ such that $f_3f_2f_1 = h_m \cdots h_1$ and

$$\deg(h_1g) \leq \deg(f_1g),$$

$$\deg(h_i \cdots h_3h_2h_1g) < \deg(f_1g) \quad i = 2, \dots, m.$$

If we have strict inequality in the assumption, all inequalities are strict.

The same claim holds if $f_1 \in \mathcal{J}_o$ is a standard or special quintic transformation and $f_3 \in \mathcal{H}$ is of degree 2.

Proof. By $m(s)$ we denote the multiplicity of $(f_1g)^{-1}$ in s and defined $D := \deg(f_1g)$. Let $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ be the base-points of f_3f_2 and order them such that

$$m(r_1) \geq m(r_2) \geq m(r_3),$$

and r_1 is a proper point of \mathbb{P}^2 and r_2 is either a proper point of \mathbb{P}^2 or in the first neighbourhood of r_1 . Let $[1 : 0 : 0], q, \bar{q}$ be the base-points of f_1^{-1} . Lemma 3.5 and f_1 being quadratic imply respectively that

$$(4) \quad m(r_1) + m(r_2) + m(r_3) > D, \quad m([1 : 0 : 0]) + 2m(q) \geq D.$$

We look at two cases, $q = r_1$ and $q \neq r_1$.

(1) Suppose that $q = r_1$. As r_1, r_2, r_3 are not collinear, at least one of r_2, r_3 is not one a line contracted by f_1^{-1} , say r_i , and it is a point of \mathbb{P}^2 . Then $f_3f_2f_1$ sends the pencil of conics passing through the non-real base-points of f_1 and $f_1^{-1}(r_i), f_1^{-1}(\bar{r}_i)$ onto a pencil of conics through two pairs of non-real conjugate points of f_3^{-1} . Hence, there exist $\alpha, \beta \in \mathcal{A}$ such that $\beta f_3f_2f_1\alpha \in \mathcal{J}_o$. The claim follows with $h_1 = \alpha^{-1}$, $h_2 = \beta f_3f_2f_1\alpha$ and $h_m = h_3 = \beta^{-1}$.

(2) Suppose that $q \neq r_1$. We look at two cases, depending on the multiplicities $m([1 : 0 : 0]), m(q), m(\bar{q})$.

(2.1) If $m(q) \geq m([1 : 0 : 0])$, then Inequalities (4) and the order of the r_i imply that $m(q) \geq \frac{D}{3}$ and $m(r_1) + m(r_2) > \frac{2D}{3}$, and so

$$m(q) + m(r_1) + m(r_2) > D.$$

In particular, the six points $q, \bar{q}, r_1, \bar{r}_1, r_2, \bar{r}_2$ are not on a conic. Then the chosen order of the points implies that there exists a standard or special quintic transformation $\theta \in \mathcal{J}_o$ and $\alpha \in \mathcal{A}$ such that $\theta\alpha$ has these 6 points as base-points. The above inequality implies

$$\deg(\theta\alpha f_1g) = 5D - 4m(q) - 4m(r_1) - 4m(r_2) < D.$$

The maps f_1 and $\theta\alpha$ have two common base-points, and we find the maps h_1, \dots, h_k as in Case 1. The maps $g' := h_k \cdots h_1g, \theta^{-1}, f_2\alpha, f_3$ satisfy the assumptions of Lemma 3.9 with “<”, and we get the maps h_{k+1}, \dots, h_m from there.

(2.2) If $m(q) < m([1 : 0 : 0])$, then Inequalities (4) and the order of the r_i imply that $m([1 : 0 : 0]), m(r_1) \geq \frac{D}{3}$, so

$$m([1 : 0 : 0]) + 2m(r_1) > D$$

In particular, the points $[1 : 0 : 0], r_1, \bar{r}_1$ are not collinear. As they are proper points of \mathbb{P}^2 , there exists $\theta \in \mathcal{J}_o$ of degree 2 and $\alpha \in \mathcal{J}_o$ such that $\theta\alpha$ has these 3 points as base-points. Moreover, the above inequality implies that

$$\deg(\theta\alpha f_1g) = 2D - m(q) - 2m(r_1) < D.$$

We apply Lemma 3.11 to f_1, α, θ to get h_1, \dots, h_k and then Case 1 to $g' := h_k \cdots h_1 g, \theta^{-1}, f_2 \alpha, f_3$ with “<” to get the rest of the h_i .

The case where f_1 is a standard or special quintic transformation and $f_3 \in \mathcal{H}$ is of degree 2 is done analogously. \square

Lemma 3.13. *Let $f_1 \in \mathcal{H}$ and $f_3 \in \mathcal{J}_\circ$ a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1 g), \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g)$$

Then there exist $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ such that $f_3 f_2 f_1 = h_m \cdots h_1$ and $\deg(h_1) = \deg(f_1) - 1$,

$$\deg(h_1 g) \leq \deg(f_1 g),$$

$$\deg(h_i \cdots h_3 h_2 h_1 g) < \deg(f_1 g) \quad i = 3, \dots, n.$$

If we have strict inequality in the assumption, all inequalities are strict.

If $f_1 \in \mathcal{J}_\circ$ is a standard or special quintic transformation and $f_3 \in \mathcal{H}$, we have

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g) \quad i = 2, \dots, m.$$

Proof. Let $m(s)$ be the multiplicity of $(f_1 g)^{-1}$ in the point s and define $D := \deg(f_1 g)$. Since f_1 has characteristic $(d; d-1, 1^{2d-2})$ and is contained in \mathcal{H} , there exist a non-real base-point q of f_1^{-1} such that

$$m([1 : 0 : 0]) + 2m(q) \geq D.$$

We can assume that they are points in \mathbb{P}^2 or on the exceptional divisor of $[1 : 0 : 0]$. Furthermore, the points $[1 : 0 : 0], q, \bar{q}$ are not collinear by Bézout theorem.

(1) Suppose that q, \bar{q} are points in \mathbb{P}^2 . Then there exists $\tau \in \mathcal{H}$ of degree 2 such that τ has base-points $[1 : 0 : 0], q, \bar{q}$. We put $h_1 := \tau f_1 \in \mathcal{H}$. Note that

$$\deg(h_1) = \deg(f_1) - 1,$$

$$\deg(h_1 g) = \deg(\tau f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q) \leq D.$$

We apply Lemma 3.12 to $g' := h_1 g$ and τ^{-1}, f_2, f_3 to get h_2, \dots, h_m .

(2) Suppose that q, \bar{q} are on the exceptional divisor of $[1 : 0 : 0]$. Then $m([1 : 0 : 0]) \geq 2m(q)$. Call $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$ the base-points of $f_3 f_2$, and suppose that

$$m(r_1) \geq m(r_2) \geq m(r_3)$$

and r_1 is a point in \mathbb{P}^2 . Lemma 3.5 implies that

$$m(r_1) + m(r_2) + m(r_3) > D.$$

We get $m([1 : 0 : 0]) \geq \frac{D}{3}$ and $m(r_1) > \frac{D}{3}$, and so $m([1 : 0 : 0]) + 2m(r_1) > D$. In particular, the points $[1 : 0 : 0], r_1, \bar{r}_1$ are not collinear, and hence there exists $\tau \in \mathcal{H}$ of degree 2 with these three points its base-points. As above we put $h_1 := \tau f_1 \in \mathcal{H}$ and note that

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_1 g) = \deg(\tau f_1 g) < D,$$

and apply Lemma 3.12 to $g' := h_1 g$ and τ^{-1}, f_2, f_3 .

If f_1 is a standard or special quintic transformation and $f_3 \in \mathcal{H}$, we construct h_1, \dots, h_m analogously. \square

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