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Susanna Zimmermann

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# THE REAL PLANE CREMONA GROUP IS A NON-TRIVIAL AMALGAM

SUSANNA ZIMMERMANN

ABSTRACT. We show that the real Cremona group of the plane is a non-trivial amalgam of two groups amalgamated along their intersection.

## 1. INTRODUCTION

The plane Cremona group is the group  $\text{Bir}_k(\mathbb{P}^2)$  of birational transformations of  $\mathbb{P}^2$  defined over a field  $k$ . [CL2013, Appendix by Y. DE CORNULIER] shows that  $\text{Bir}_k(\mathbb{P}^2)$  is not isomorphic to a non-trivial amalgam of two groups if  $k$  is algebraically closed. The best one can obtain is that  $\text{Bir}_k(\mathbb{P}^2)$  is isomorphic to a non-trivial amalgam modulo one simple relation [B2012, L2010, I1984] or that it is isomorphic to a generalised amalgamated product of three groups, amalgamated along all pairwise intersections [W1992]. The world looks different for  $k = \mathbb{R}$ . We show that  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is indeed isomorphic to a non-trivial amalgamated product of two groups.

The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  contains two groups of Jonquières transformations:

$$\mathcal{J}_* = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of lines through } [1 : 0 : 0] \}$$

$$\mathcal{J}_\circ = \{f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \mid f \text{ preserves the pencil of conics through } p_1, \bar{p}_1, p_2, \bar{p}_2 \}$$

where  $p_1, p_2 \in \mathbb{P}^2$  are two non-real points such that  $p_1, \bar{p}_1, p_2, \bar{p}_2$  are not collinear. The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is generated by  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$ ,  $\mathcal{J}_*$ ,  $\mathcal{J}_\circ$  [BM2014], and the groups  $\mathcal{J}_*$ ,  $\mathcal{J}_\circ$  are not conjugate in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  [Z2015, Proposition 5.3]. Over the field  $\mathbb{C}$ , we can send the pencil of conics through four points, no three collinear, onto a pencil of lines through a point, so the complex versions of the two groups are conjugate in  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ .

We write  $\mathcal{A} := \text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and denote by  $G_\circ \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  the subgroup generated by  $\mathcal{A}$  and  $\mathcal{J}_\circ$ , and by  $G_* \subset \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  the subgroup generated by  $\mathcal{A}$  and  $\mathcal{J}_*$ .

**Theorem 1.1.** *The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is the non-trivial amalgamated product of  $G_*$  and  $G_\circ$  along their intersection.*

**Proposition 1.2.** *Both  $G_*$  and  $G_\circ$  have uncountable index in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$*

The intersection  $G_* \cap G_\circ$  contains all elements of  $\mathcal{J}_*$  with only one real base-point (see remark after Definition 3.2).

The decomposition of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  into an amalgam follows from the structure theorem [Z2015, Theorem 4.4] (Theorem 2.3 below). The task is to prove that  $G_*$  and  $G_\circ$  are both proper subgroups; this is not straightforward since their complex analogues both coincide with  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ . Many of the ingredients of the proof will be similar to the proof of [Z2015, Theorem 4.4].

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**Corollary 1.3.** *The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts on a tree, and all its algebraic subgroups are conjugate to a subgroup of  $G_*$  or of  $G_{\circ}$ .*

For the subgroups of odd order Corollary 1.3 can also be verified by checking their classification in [Y2016].

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## 2. THE REAL CREMONA GROUP IS AN AMALGAM

In this section we show that  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is isomorphic to the amalgamated product of  $G_*$  and  $G_{\circ}$  along their intersection. Non-triviality will be proven in the next section.

**Definition 2.1.** A *standard quintic transformation* is an element of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  of degree 5 whose base-points are three pairs of non-real conjugate base-points in  $\mathbb{P}^2$ . (See [BM2014, Example 3.1] for a definition via blow-ups.)

**Remark 2.2.** For any standard or special quintic transformation  $f$  there exist  $\alpha, \beta \in \mathcal{A}$  such that  $\beta f \alpha \in \mathcal{J}_{\circ}$  [Z2015, Lemma 3.10]. In particular, all standard quintic transformations are contained in  $G_{\circ}$ .

We make use of the following structure theorem. Let  $\mathcal{G}$  be the free group  $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ}$  amalgamated along all pairwise intersections, and let  $w: \mathcal{A} \cup \mathcal{J}_* \cup \mathcal{J}_{\circ} \rightarrow \mathcal{G}$  be the canonical word map.

**Theorem 2.3** ([Z2015, Theorem 4.4]). *The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is isomorphic to  $\mathcal{G}$  modulo the following relations:*

**(rel. 1)** *Let  $\theta_1, \theta_2 \in \mathcal{J}_{\circ}$  be standard quintic transformations and  $\alpha_1, \alpha_2 \in \mathcal{A}$ .*

$$w(\alpha_2)w(\theta_1)w(\alpha_1) = w(\theta_2) \quad \text{if} \quad \alpha_2\theta_1\alpha_1 = \theta_2.$$

**(rel. 2)** *Let  $\tau_1, \tau_2 \in \mathcal{J}_* \cup \mathcal{J}_{\circ}$  be both of degree 2 or of degree 3 and  $\alpha_1, \alpha_2 \in \mathcal{A}$ .*

$$w(\tau_1)w(\alpha_1) = w(\alpha_2)w(\tau_2) \quad \text{if} \quad \tau_1\alpha_1 = \alpha_2\tau_2.$$

**(rel. 3)** *Let  $\tau_1, \tau_2, \tau_3 \in \mathcal{J}_*$  all be of degree 2, or  $\tau_1, \tau_2$  of degree 2 and  $\tau_3$  of degree 3, and  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}$ .*

$$w(\tau_2)w(\alpha_1)w(\tau_1) = w(\alpha_3)w(\tau_3)w(\alpha_2) \quad \text{if} \quad \tau_2\alpha_1\tau_1 = \alpha_3\tau_3\alpha_2.$$

**Corollary 2.4.** *The group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is the amalgamated product of  $G_*$  and  $G_{\circ}$  along their intersection.*

*Proof.* Let  $G$  be the amalgamated product of  $G_*$  and  $G_{\circ}$  along their intersection. By the universal property of amalgamated products there exists a homomorphism  $\psi: G \rightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , which is the identity map on  $\mathcal{A}$ ,  $\mathcal{J}_*$  and  $\mathcal{J}_{\circ}$ . On the other hand, we have a canonical surjective homomorphism  $\mathcal{A} * \mathcal{J}_* * \mathcal{J}_{\circ} \rightarrow \mathcal{G} \rightarrow G$ . We claim that  $\mathcal{G} \rightarrow G$  factors through  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . By Theorem 2.3 it suffices to check that relations (rel. 1), (rel. 2) and (rel. 3) are satisfied in  $G$ .

Relation (rel. 1): standard quintic transformations are contained in  $G_{\circ}$  (Remark 2.2). They are not contained  $G_*$  because the kernel of the abelianisation of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  contains  $G_*$

but does not contain any of the standard quintic transformations [Z2015, Remark 3.19(3), Proposition 5.3].

Relation (rel. 2) and (rel. 3) are relations among linear, quadratic and cubic transformations. Any transformation of degree  $\leq 3$  is contained in  $G_*$ , some of them in  $G_* \cap G_\circ$ , and relations (rel. 2) and (rel. 3) hold in  $G_*$ .

So, the morphism  $\mathcal{G} \rightarrow G$  factors through  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , and the induced homomorphism  $\varphi: \text{Bir}_{\mathbb{R}}(\mathbb{P}^2) \rightarrow G$  is the identity on  $\mathcal{A}, \mathcal{J}_*, \mathcal{J}_\circ$ . Thus  $\psi = \varphi^{-1}$ .  $\square$

### 3. THE GROUP $G_\circ$ IS A STRICT SUBGROUP OF $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$

This section is devoted to the proof that  $G_\circ$  is a strict subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  (Corollary 3.8). The key idea is to show that it does not contain any elements of degree 2 having three real base-points - for instance, it does not contain the standard quadratic transformation  $[x : y : z] \mapsto [yz : xz : xy]$ . To show this, we prove that any element in  $G_\circ$  is the composition of elements in  $\mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  (see Definition 3.2 for the definition of  $\mathcal{H}_* \subset G_\circ \cap \mathcal{J}_*$ ) such that the successive degree of the composition is non-decreasing (Proposition 3.7)

**3.1. Properties of the groups  $\mathcal{J}_*$  and  $\mathcal{J}_\circ$ .** Before we run off to prove our claims, we quickly review the groups  $\mathcal{J}_*$  and  $\mathcal{J}_\circ$  and some of their properties.

**Remark 3.1.** The characteristic of an element  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is the tuple  $(\deg(f); m_1^{e_1}, \dots, m_k^{e_k})$ , where  $m_1, \dots, m_k$  are the different multiplicities of the base-points of  $f$  and  $e_i$  the number of base-points of multiplicity  $m_i$ .

An element of  $\mathcal{J}_\circ$  has characteristic

$$\begin{aligned} & \left( d; \left( \frac{d-1}{2} \right)^4, 2^{\frac{d-1}{2}} \right), \quad \text{if } \deg(f) \text{ is odd} \\ & \left( d; \left( \frac{d}{2} \right)^2, \left( \frac{d-2}{2} \right)^2, 2^{\frac{d-2}{2}}, 1 \right), \quad \text{if } \deg(f) \text{ is even} \end{aligned}$$

Moreover, if  $d \geq 2$ ,  $p_1, \bar{p}_1, p_2, \bar{p}_2$  are base-points of multiplicity  $\frac{d-1}{2}$  or  $\frac{d}{2}$  and  $\frac{d-2}{2}$  and all other base-points are of multiplicity 2 or 1 [Z2015, Lemma 3.2].

In particular,  $\mathcal{J}_\circ$  contains only quadratic maps with one real and a pair of non-real conjugate base-points.

The group  $\mathcal{J}_*$  is isomorphic to  $\text{PGL}_2(\mathbb{R}(x)) \rtimes \text{PGL}_2(\mathbb{R})$  because on the chart  $z = 1$  all of its elements are of the form

$$f: (x, y) \mapsto \left( \frac{\alpha(y)x + \beta(y)}{\gamma(y)x + \delta(y)}, \frac{ay + b}{cy + d} \right)$$

with  $a, b, d, c \in \mathbb{R}$ ,  $ad - bc \neq 0$  and  $\alpha, \beta, \delta, \gamma \in \mathbb{R}[y]$ ,  $\alpha\delta - \beta\gamma \neq 0$ .

The characteristic of any element of  $\mathcal{J}_*$  is  $(d; d-1, 1^{2d-2})$ , where  $[1 : 0 : 0]$  is the base-point of multiplicity  $d-1$ .

**Definition 3.2.** We denote by  $\mathcal{H} \subset \mathcal{J}_*$  the subgroup of elements having *at most one real* base-point.

If an element of  $\mathcal{H}$  is non-linear, all its base-points different from  $[1 : 0 : 0]$  are pairs of non-real conjugate points. For any quadratic transformation  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  with one real and a pair of non-real conjugate base-points there exist  $\alpha, \beta \in \mathcal{A}$  such that  $\beta f \alpha \in \mathcal{H}$ . In fact, the

group  $\mathcal{H}$  is generated by its linear and quadratic elements and is hence contained in  $G_\circ$  (we can follow the argument of [AC2002, Theorem 8.4.3]).

**Definition 3.3.** A *special quintic transformation* is an element of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  of degree 5 whose base-points are three pairs of non-real conjugate base-points, two of them points in  $\mathbb{P}^2$  and one pair infinitely near one of the other two pairs. (See [BM2014, Example 3.2] for a definition via blow-ups.)

**Remark 3.4.** (1) A standard (resp. special) quintic transformation  $f$  has multiplicity 2 in each of its base-points. In particular, there is no conic containing all its base-points.

(2) The inverse  $f^{-1}$  is also a standard (resp. special) quintic transformation and  $f$  sends the pencil of conics through two pairs of base-points onto a pencil of conics through two pairs of base-points of  $f^{-1}$ .

(3) Pick any three pairs of non-real conjugate points, not all six points contained in one conic. Suppose they are all points in  $\mathbb{P}^2$  (resp. two pairs are points in  $\mathbb{P}^2$  and one pair is in the first neighbourhood of one of the other two pairs). Then there is a standard (resp. special) quintic transformation with these points as its base-points [BM2014, Example 3.1, Example 3.2], [RV2005, §I].

(4) For any standard or special quintic transformation  $f$  there exist  $\alpha, \beta \in \mathcal{A}$  such that  $\beta f \alpha \in \mathcal{J}_\circ$  [Z2015, Lemma 3.10]. In particular, all standard and special quintic transformations are contained in  $G_\circ$ .

For  $f \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  and a point  $p$ , let  $m_f(p)$  the multiplicity of  $f$  in  $p$ .

**Lemma 3.5** ([Z2015, Lemma 4.8]). *Let  $f \in \mathcal{J}_\circ$  be non-linear and  $g \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . Suppose that*

$$\deg(fg) \leq \deg(g) \quad (\text{resp. } < \deg(g))$$

*Then there exists a base-point  $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$  of  $f$  of multiplicity 2 such that*

$$(ineq\ 1) \quad m_{g^{-1}}(p_1) + m_{g^{-1}}(p_2) + m_{g^{-1}}(q) \geq \deg(g) \quad (\text{resp. } > \deg(g))$$

*or  $f$  has a simple base-point  $r$  and there exists  $i \in \{1, 2\}$  such that*

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(r) \geq \deg(g), \quad \text{where } m_f(p_i) = \deg(f)/2 \quad (\text{resp. } > \deg(g)).$$

**Lemma 3.6** ([Z2015, Lemma 4.14]). *In Lemma 3.5 (ineq 1), suppose that  $q$  is a real proper point of  $\mathbb{P}^2$ . Then one of the following two assertions holds:*

(1) *there exists  $h \in \mathcal{J}_\circ$  of degree 3 with base-points  $p_1, \bar{p}_1, p_2, \bar{p}_2, q$ .*

(2)  *$\deg(f)$  is even and there exists  $h \in \mathcal{J}_\circ$  of degree 2 with base-points  $p_i, \bar{p}_i, q$  and*

$$2m_{g^{-1}}(p_i) + m_{g^{-1}}(q) \geq \deg(g), \quad \text{where } m_f(p_i) = \frac{\deg(f)}{2}.$$

**3.2. The group  $G_\circ$  is a strict subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ .** The following proposition is the key to proving that  $G_\circ$  is a strict subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ . The lemmas used in its proof are stated and proven in the next subsection, and to prove them we only use the properties of  $\mathcal{J}_*$ ,  $\mathcal{J}_\circ$  and  $\mathcal{H}$  listed in the previous subsection.

**Proposition 3.7.** *For any  $f \in G_\circ$  there exist  $g_1, \dots, g_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $f = g_m \cdots g_1$  and such that  $\deg(g_i \cdots g_1) \leq \deg(g_{i+1} g_i \cdots g_1)$ , for  $i = 1, \dots, m-1$ .*

*Proof.* We write  $f = f_N \cdots f_1$  for some  $f_1, \dots, f_N \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$ . Define  $d_i := \deg(f_i \cdots f_1)$  and

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad n := \max\{i \mid d_i = D\}, \quad k = \sum_{i=1}^n (\deg(f_i) - 1).$$

We claim that there exist  $h_1, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that

$$f_{n+1}f_n f_{n-1} = h_k \cdots h_1$$

and the pair  $(D', k')$  associated to the composition  $f_N \cdots f_{n+2} h_k \cdots h_1 f_{n-2} \cdots f_1$  is strictly smaller than  $(D, k)$  with respect to the lexicographic order.

If the claim holds, the proof finishes as follows. The procedure stops at the pair  $(D'', k'')$  where  $D'' = \deg(f)$  and  $f = f'_{n''} \cdots f'_1$ . We define

$$I := \max\{i \mid \deg(f'_i \cdots f'_1) > \deg(f'_{i+1} \cdots f'_1)\}, \quad f' := f'_{I+1} \cdots f'_1$$

By construction, we have  $\deg(f') < \deg(f)$ . We repeat the process for  $f'$ . After finitely many steps we will not find such an  $I$  anymore, and the composition we have obtained is the one in the statement.

Let us prove the claim. If  $D = 1$ , all  $f_i$  are linear maps, so  $f$  is linear, and the claim holds with  $m = 1$ .

Suppose that  $D \geq 2$ . Any consecutive letters  $f_i, f_{i+1}$  that are both contained in  $\mathcal{A}$  or in  $\mathcal{J}_\circ$  we replace by their product. This does not increase the pair  $(D, k)$ . We can therefore assume that no two consecutive letters are both contained in  $\mathcal{A}$  or in  $\mathcal{J}_\circ$ . By definition of  $n$ , we have  $d_n > d_{n+1}$ , and in particular,

$$f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{A}, \quad f_n \in \mathcal{A} \setminus \mathcal{J}_\circ, \quad f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{A} \quad \text{and} \quad d_{n-2} \leq d_{n-1} = d_n = D.$$

(1) If  $f_{n-1}, f_{n+1} \in \mathcal{H}$ , we apply Lemma 3.11 to  $f_{n-1}, f_n, f_{n+1}$ . There exist  $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{H}$  such that  $f_{n+1}f_n f_{n-1} = h_k \cdots h_1$  and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, \\ \deg(h_2 h_1 f_{n-2} \cdots f_1) &= \deg(h_1 f_{n-2} \cdots f_1) \leq D, \\ \deg(h_i \cdots h_2 h_1 f_{n-1} \cdots f_1) &< D, \quad i = 3, \dots, k. \end{aligned}$$

Then  $D' \leq D$  and if  $D' = D$  then  $n' \leq n$  and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

(2) Else, we proceed as follows. Let  $m_i(t)$  be the multiplicity of  $(f_i \cdots f_1)^{-1}$  in the point  $t$ . If  $f_{n-1} \in \mathcal{J}_\circ$ , then by Lemma 3.5 there exists a base-point  $q \notin \{p_1, \bar{p}_1, p_2, \bar{p}_2\}$  of  $f_{n-1}^{-1}$  of multiplicity 2 such that

$$m_{n-1}(p_1) + m_{n-1}(p_2) + m_{n-1}(q) \geq D$$

or  $\deg(f_{n-1})$  is even and there exists a simple base-point  $r$  of  $f_{n-1}^{-1}$  such that

$$(1) \quad 2m_{n-1}(p_i) + m_{n-1}(r) \geq D, \quad \text{where } m_{f_{n-1}^{-1}}(p_i) = \deg(f_{n-1})/2.$$

- If  $q$  is a non-real point, we can assume that it is a proper point of  $\mathbb{P}^2$  or in the first neighbourhood of one of  $p_1, \bar{p}_1, p_2, \bar{p}_2$ . Since  $m_{f_{n-1}^{-1}}(q) = 2$ , the points  $p_1, \dots, \bar{p}_2, q, \bar{q}$  are not on one conic (Remark 3.1). By Remark 3.4 there exists a standard or special quintic transformation  $\theta_{n-1} \in \mathcal{J}_\circ$  with these 6 points as base-points.

- If  $q$  is a real point, we can assume that it is a proper point of  $\mathbb{P}^2$  (it cannot be infinitely near one of  $p_1, \dots, \bar{p}_2$ ). By Lemma 3.6 there exists  $\theta_{n-1} \in \mathcal{J}_\circ$  of degree 3 with base-points  $p_1, \dots, \bar{p}_2, q$  or there exists  $\theta_{n-1} \in \mathcal{J}_\circ$  of degree 2 with base-points  $p_i, \bar{p}_i, q$  where  $m_{f_{n-1}^{-1}}(p_i) = \frac{\deg(f_{n-1})}{2}$  and  $2m_{n-1}(p_i) + m_{n-1}(q) \geq D$ .

• If there is no such  $q$ , we have  $r$  as in (1). If  $r$  is not a proper point of  $\mathbb{P}^2$  we replace it by the (real) base-point of  $f_{n-1}^{-1}$  to which it is infinitely near. This does not change Inequality (1). Remark 3.1 implies that  $p_i, \bar{p}_i, r$  are not collinear, so there exists  $\theta_{n-1} \in \mathcal{J}_\circ$  of degree 2 with base-points  $p_i, \bar{p}_i, r$  [Z2015, Lemma 3.4].

In all three cases we define  $h_1 := \theta_{n-1}f_{n-1} \in \mathcal{J}_\circ$ . Then

$$\deg(h_1f_{n-2} \cdots f_1) = \deg(\theta_{n-1}f_{n-1} \cdots f_1) \leq D \quad \text{and} \quad \deg(\theta_{n-1}f_{n-1}) < \deg(f_{n-1}).$$

We proceed analogously if  $f_{n+1} \in \mathcal{J}_\circ$ , and find  $\theta_{n+1} \in \mathcal{J}_\circ$  satisfying the analogous inequalities with “<”, and define  $h_k := f_{n+1}\theta_{n+1}^{-1} \in \mathcal{J}_\circ$ . Analogously, we have

$$\deg(\theta_{n+1}f_n \cdots f_1) < D, \quad \deg(h_k) < \deg(f_{n+1}).$$

Now, we use Lemma 3.9 and Lemma 3.13. The constructions are summarised in the commutative diagrams below.

If  $f_{n-1}, f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$ , we apply Lemma 3.9 to  $\theta_{n-1}^{-1}, f_n, \theta_{n+1}$ . There exist  $h_2 \in \mathcal{A}$ ,  $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ$  such that  $\theta_{n+1}f_n\theta_{n-1}^{-1} = h_{k-1} \cdots h_2$  and

$$\deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k-1.$$

If  $f_{n-1} \in \mathcal{J}_\circ \setminus \mathcal{H}$  and  $f_{n+1} \in \mathcal{H}$ , we apply the last part of Lemma 3.13 to  $\theta_{n-1}, f_n, f_{n+1}$ . There exist  $h_2 \in \mathcal{A}$ ,  $h_3, \dots, h_k \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $f_{n+1}f_n\theta_{n-1} = h_k \cdots h_2$  and

$$\deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) < D, \quad i = 3, \dots, k.$$

If  $f_{n-1} \in \mathcal{H}$  and  $f_{n+1} \in \mathcal{J}_\circ \setminus \mathcal{H}$ , we apply Lemma 3.13 to  $f_{n-1}, f_n, \theta_{n+1}$ . There exist  $h_1 \in \mathcal{J}_\circ$ ,  $h_2 \in \mathcal{A}$ ,  $h_3, \dots, h_{k-1} \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $\theta_{n+1}f_n f_{n-1} = h_{k-1} \cdots h_1$  and

$$\begin{aligned} \deg(h_1) &= \deg(f_{n-1}) - 1, & \deg(h_1f_{n-2} \cdots f_1) &= \deg(h_2h_1f_{n-2} \cdots f_1) \leq D \\ \deg(h_i \cdots h_3h_2h_1f_{n-2} \cdots f_1) &< D, & i &= 3, \dots, k-1. \end{aligned}$$

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{f_n} & \mathbb{P}^2 \\ \uparrow \mathcal{H} \ni f_{n-1} & & \downarrow \theta_{n+1} \\ \mathbb{P}^2 & \xrightarrow{h_{k-1} \cdots h_1} & \mathbb{P}^2 \\ \text{Lem. 3.13} & & \end{array} \quad \begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{f_n} & \mathbb{P}^2 \\ \uparrow \mathcal{J}_\circ \setminus \mathcal{H} \ni f_{n-1} & & \downarrow \theta_{n-1} \\ \mathbb{P}^2 & \xrightarrow{h_1} & \mathbb{P}^2 \\ \text{deg}(h_1) < \text{deg}(f_{n-1}) & & \end{array} \quad \begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{f_n} & \mathbb{P}^2 \\ \uparrow \mathcal{H} \ni f_{n-1} & & \downarrow \theta_{n-1} \\ \mathbb{P}^2 & \xrightarrow{h_k \cdots h_2} & \mathbb{P}^2 \\ \text{Lem. 3.13} & & \end{array}$$

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{f_n} & \mathbb{P}^2 \\ \uparrow \mathcal{J}_\circ \setminus \mathcal{H} \ni f_{n-1} & & \downarrow \theta_{n+1} \\ \mathbb{P}^2 & \xrightarrow{h_1} & \mathbb{P}^2 \\ \text{deg}(h_1) < \text{deg}(f_{n-1}) & & \end{array} \quad \begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{f_n} & \mathbb{P}^2 \\ \uparrow \theta_{n+1} & & \downarrow \theta_{n+1} \\ \mathbb{P}^2 & \xrightarrow{h_{k-1} \cdots h_2} & \mathbb{P}^2 \\ \text{Lem. 3.9} & & \end{array}$$

We claim that in each case, the pair  $(D', k')$  is strictly smaller than  $(D, k)$ . Indeed, the above conditions imply  $D' \leq D$ . If  $D' = D$  then  $n' \leq n$  and

$$k' = \sum_{i=1}^{n-2} (\deg(f_i) - 1) + (\deg(h_1) - 1) < \sum_{i=1}^{n-1} (\deg(f_i) - 1) = k.$$

□

**Corollary 3.8.** *The group  $G_\circ$  contains no maps of degree 2 having more than one real base-point. In particular,  $G_\circ \subsetneq \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  is a strict subgroup.*

*Proof.* Suppose that  $G_\circ$  contains an element  $\tau$  of degree  $\deg(\tau) = 2$  with three real base-points. By Proposition 3.7, there exist  $g_1, \dots, g_n \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $\tau = g_n \cdots g_1$  and

$$\deg(g_i \cdots g_1) \leq \deg(g_{i+1} \cdots g_1), \quad i = 1, \dots, n-1.$$

We may suppose that  $\deg(g_1) \geq 2$ , and then  $\deg(g_i \cdots g_1) = 2$  for  $i = 1, \dots, n$ . This is only possible if

- $\deg(g_1) = 2$ ,
- $\deg(g_i) = 1, 2, 3, 4$  for  $i = 2, \dots, n$ ,
- if  $\deg(g_i) = 2, 3, 4$ , then  $g_i$  and  $(g_{i-1} \cdots g_1)^{-1}$  share 2, 3, 3 base-points of multiplicity:
  - if  $\deg(g_i) = 3$ , these are a double and two simple base-points
  - if  $\deg(g_i) = 4$ , these are three double points.

The characteristics in Remark 3.1 and Remark 3.2, and all  $g_i$  being contained in  $\mathcal{J}_\circ \cup \mathcal{H}$  of degree  $\leq 4$  implies that the quadratic maps  $g_i \cdots g_1$  have two non-real base-points for  $i = 1, \dots, n$ . This contradicts  $\tau = g_n \cdots g_1$  having three real base-points.  $\square$

*Proof of Theorem 1.1.* By Proposition 2.4 is the amalgamated product of  $G_*$  and  $G_\circ$  along their intersection. Furthermore,  $G_*$  is a strict subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  by [Z2015, Proposition 5.3], and  $G_\circ$  is a strict subgroup by Corollary 3.8.  $\square$

*Proof of Proposition 1.2.* Let's prove the claim first for  $G_\circ$ . We denote by  $\sigma: [x : y : z] \mapsto [yz : xz : xy]$  the standard quadratic involution of  $\mathbb{P}^2$ , and define a map

$$\varphi: \{[x : y : z] \mapsto [x + az : y + bz : z] \mid a, b \in \mathbb{R}\} := A \longrightarrow \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_\circ, \quad \alpha \mapsto \alpha\sigma G_\circ$$

Note that by Corollary 3.8 the transformation  $\sigma$  is not contained in  $G_\circ$ . In particular,  $\alpha\sigma G_\circ \neq G_\circ$  for all  $\alpha \in A$ . We claim that the map  $\varphi$  is injective. Note that for all  $\alpha \in A$  the transformation  $\sigma\alpha\sigma$  is of degree  $\leq 2$ . It is linear if and only if  $\sigma$  and  $(\alpha\sigma)^{-1}$  have three common base-points, i.e. if and only if  $\alpha = \text{Id}$ . If  $\sigma\alpha\sigma$  is of degree 2 then it has three real base-points. It follows from Corollary 3.8 that that for  $\beta, \gamma \in A$  the transformation  $\sigma\beta^{-1}\gamma\sigma$  is contained in  $G_\circ$  if and only if it is linear, i.e. if and only if  $\beta^{-1}\gamma = \text{Id}$ . Thus the map  $\varphi$  is injective.

Let's prove the claim for  $G_*$ . By [Z2015, Lemma 6.7, Corollary 6.12] we have

$$G_* = \langle \mathcal{J}_*, \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \subset \langle \langle \text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \rangle \rangle = [\text{Bir}_{\mathbb{R}}(\mathbb{P}^2), \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)].$$

Then there is a surjective map from  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)/G_*$  onto the abelianisation of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , which is isomorphic to  $\bigoplus_{(0,1]} \mathbb{Z}/2\mathbb{Z}$  [Z2015, Theorem 1.1]. It follows that  $G_*$  has uncountable index in  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ .  $\square$

*Proof of Corollary 1.3.* By Theorem 1.1, the group  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  acts on the Bass-Serre tree  $T$  of the amalgam  $G_* *_{G_* \cap G_\circ} G_\circ$ . Then every element of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  of finite order has a fixed point on  $T$ . It follows that very finite subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  has a fixed point on  $T$  [S1980, §I.6.5, Corollary 3], and is in particular conjugate to a subgroup of  $G_*$  or  $G_\circ$ .

For infinite algebraic subgroups of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , it suffices to check the claim for the maximal ones. Before we list them, we have to introduce some notation. For  $n \geq 0$ , we denote by  $\mathbb{F}_n$  the  $n$ -th Hirzebruch surface

$$\mathbb{F}_n = \{[x : y : z][u : v] \in \mathbb{P}_{\mathbb{R}}^2 \times \mathbb{P}_{\mathbb{R}}^1 \mid u^n z = v^n y\}.$$

Note that  $\mathbb{F}_0 = \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1$ . By  $\mathcal{Q}_{3,1} \subset \mathbb{P}_{\mathbb{R}}^3$  we denote the real hypersurface given by  $w^2 = x^2 + y^2 + z^2$ . Note that it is isomorphic to the real surface  $(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, (x, y) \mapsto (\bar{y}, \bar{x}))$ . It is obtained by blowing up two conjugate points in  $\mathbb{P}^2$  and contracting the (real) line passing



through them. We denote by  $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$  the conic bundle obtained by blowing up a pair of non-real conjugate points on  $\mathcal{Q}_{3,1}$ .

Note that for  $n \geq 0$ , the group  $\mathcal{J}_*$  is conjugate to the group of birational transformations of  $\mathbb{F}_n$  preserving the conic bundle structure  $\text{pr}_2: \mathbb{F}_n \rightarrow \mathbb{P}^1$ . The group  $\mathcal{J}_\circ$  is conjugate to the group of birational transformations of  $X_{[2]}$  preserving the conic bundle structure  $\pi_{[2]}: X_{[2]} \rightarrow \mathbb{P}^1$ . The infinite maximal algebraic subgroups of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  are conjugate to  $\text{Aut}_{\mathbb{R}}(X)$  where  $X$  is one of the real rational surfaces in the following list [RZ2016]:

- (1)  $X = \mathbb{P}^2$ ,
- (2)  $X = \mathcal{Q}_{3,1}$ ,
- (3)  $X = \mathbb{F}_n$ ,  $n = 0$ ,  $n \geq 2$ ,
- (4)  $X$  is a del Pezzo surface of degree 6 obtained by blowing up a pair of non-real conjugate points on  $\mathbb{F}_0$ ,
- (5)  $X$  is a del Pezzo surface of degree 6 obtained by blowing up two real points on  $\mathbb{F}_0$ ,
- (6)  $\eta: X \rightarrow X_{[2]}$  is a birational morphism of real conic bundles blowing up  $n \geq 1$  pairs with non-real conjugate points on non-real fibres on the non-real conjugate disjoint  $(-1)$ -curves of  $X_{[2]} \rightarrow \mathcal{Q}_{3,1}$ ,
- (7)  $\eta: X \rightarrow \mathbb{F}_n$  is a birational morphism of real conic bundles blowing up  $2n \geq 4$  points on the zero section of self-intersection  $n$ .

We have  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2) \subset G_* \cap G_\circ$ . The groups in (3) and (7) are conjugate to subgroups of  $G_*$ , and the groups in (6) conjugate to subgroups of  $G_\circ$ .

Via the birational map  $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$  that blows up a real point and blows down the strict transforms of the two (real) fibres passing through the point, the group  $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$  is conjugate to a subgroup of  $G_*$ .

Via the birational map  $\mathcal{Q}_{3,1} \dashrightarrow \mathbb{P}^2$  that blows up a real point and blows down the strict transforms of the two (non-real conjugate) fibres passing through the point,  $\text{Aut}_{\mathbb{R}}(\mathcal{Q}_{3,1})$  is conjugate to a subgroup of  $G_\circ \cap G_*$ .

Group (4): via the blow-down  $X \rightarrow \mathbb{F}_0$  of the two exceptional divisors, the group is conjugate to a subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{F}_0)$  generated by a subgroup of  $\text{Aut}_{\mathbb{R}}(\mathbb{F}_0)$  and a transformation of  $\mathbb{F}_0$  that is the composition of a transformation preserving the fibrations and the automorphism exchanging the fibrations (see construction in [RZ2016, Proposition 3.5]). Thus group (4) is conjugate to a subgroup of  $G_*$ .

Group (5): via the blow-down  $X \rightarrow \mathbb{P}^2$  of three disjoint  $(-1)$ -curves onto the three coordinate points of  $\mathbb{P}^2$ , the group is conjugate to a subgroup of  $\text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  generated by a subgroup of  $\text{Aut}_{\mathbb{R}}(\mathbb{P}^2)$  and  $[x : y : z] \mapsto [yz : xz : xy]$  [RZ2016, Proposition 3.6]. Since the latter is contained in  $J_*$ , group (5) is also conjugate to a subgroup of  $G_*$ .  $\square$

**3.3. Lowering the degree.** In this section we prove the lemmas used in the proof of Proposition 3.7. Let  $\varphi \in \text{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ ,  $f_1, f_3 \in \mathcal{J}_\circ \cup \mathcal{H}$  and  $f_2 \in \mathcal{A}$ .

**Lemma 3.9.** *Suppose that  $f_1, f_3 \in \mathcal{J}_\circ$  be standard or special quintic transformations and that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g).$$

*Then there exists  $h_1 \in \mathcal{A}$ ,  $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ$  such that  $f_3f_2f_1 = h_m \cdots h_1$  and*

$$\deg(h_i \cdots h_2h_1g) < \deg(f_1g), \quad i = 2, \dots, m.$$

*Proof.* Follows from [Z2015, Lemma 4.10] by taking  $\Lambda$  to be the linear system of  $(f_1g)^{-1}$ .  $\square$

**Lemma 3.10.** *Let  $f_1, f_3 \in \mathcal{H}$  be of degree 2. Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

*Then there exist  $h_1 \in \mathcal{A}$ ,  $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$  such that  $f_3f_2f_1 = h_m \cdots h_1$  and*

$$\deg(h_i \cdots h_2h_1g) < \deg(f_1g), \quad i = 2, \dots, m.$$

*Proof.* If  $\deg(f_3f_2f_1) = 1$ , we put  $h_m = h_1 = f_3f_2f_1$ .

If  $\deg(f_3f_2f_1) = 2$ , then  $f_1^{-1}$  and  $f_3f_2$  have exactly two common base-points, namely their pairs of non-real conjugate base-points. It follows that the quadratic map  $f_3f_2f_1$  has two non-real base-points and a real base-point. So there exist  $\alpha, \beta \in \mathcal{A}$  such that  $\beta f_3f_2f_1\alpha \in \mathcal{H}$ . The claim follows with  $h_1 = \alpha^{-1}$ ,  $h_2 = \beta f_3f_2f_1\alpha$  and  $h_m = h_3 = \beta^{-1}$ .

If  $\deg(f_3f_2f_1) = 3$ , then the maps  $f_1^{-1}$  and  $f_3f_2$  have exactly one common base-point, namely  $[1 : 0 : 0]$ . Hence  $f_3f_2f_1 \in \mathcal{J}_*$ . Since moreover  $f_1, f_3$  have only one real base-point,  $f_3f_2f_1$  has only one real base-point. Thus  $f_3f_2f_1 \in \mathcal{H}$  and the claim follows with  $h_1 = h_m = f_3f_2f_1$ .

If  $\deg(f_3f_2f_1) = 4$ , then the maps  $f_1^{-1}$  and  $f_3f_2$  have no common base-points. The following construction is visualised in the diagram below. Let  $[1 : 0 : 0], p, \bar{p}$  be the base-points of  $f_1^{-1}$  and  $r, q, \bar{q}$  the base-points of  $f_3f_2$ , and define  $D := \deg(f_1g)$ . By  $m(t)$  we denote the multiplicity of  $(f_1g)^{-1}$  in the point  $t$ . The assumptions  $\deg(g) \leq D$  and  $\deg(f_3f_2f_1g) < D$  imply that

$$(2) \quad m([1 : 0 : 0]) + 2m(p_i) \geq D, \quad m(r) + 2m(q) > D.$$

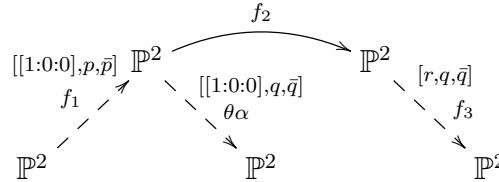
If  $m([1 : 0 : 0]) \geq m(r)$ , then Inequality (2) implies that

$$m([1 : 0 : 0]) + 2m(q) \geq m(r) + 2m(q) \stackrel{(2)}{>} D,$$

which means that  $[1 : 0 : 0], q, \bar{q}$  are not collinear. In particular, there exists  $\alpha \in \mathcal{A}$  and  $\theta \in \mathcal{H}$  of degree 2 such that  $\theta\alpha$  has base-points  $[1 : 0 : 0], q, \bar{q}$ . The above inequality implies

$$\deg(\theta\alpha f_1g) = 2D - m([1 : 0 : 0]) - 2m(q) < D.$$

The maps  $f_1^{-1}$  and  $\theta\alpha$  have one common base-point, and the maps  $\theta\alpha$  and  $f_3f_2$  have two common base-points. We proceed as in (3) and (2).



Suppose that  $m([1 : 0 : 0]) < m(r)$ . Inequalities (2) imply that

$$m(r) + 2m(p) > m([1 : 0 : 0]) + 2m(p) \geq 2D,$$

so  $r, p, \bar{p}$  are not collinear. There exists  $\alpha \in \mathcal{A}$  and  $\theta \in \mathcal{H}$  of degree 2 such that  $\theta\alpha$  has these points as base-points. The claim follows as above.  $\square$

**Lemma 3.11.** *Let  $f_1, f_3 \in \mathcal{H}$ . Suppose that*

$$\deg(g) \leq \deg(f_1g) \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

Then there exist  $h_1 \in \mathcal{H}$ ,  $h_2 \in \mathcal{A}$ ,  $h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{H}$  such that  $f_3 f_2 f_1 = h_m \cdots h_1$  and  $\deg(h_1) = \deg(f_1) - 1$ ,

$$\begin{aligned} \deg(h_2 h_1 g) &= \deg(h_1 g) \leq \deg(f_1 g), \\ \deg(h_i \cdots h_3 h_2 h_1 g) &< \deg(f_1 g), \quad i = 3, \dots, m. \end{aligned}$$

If the inequality in the assumption is strict, all inequalities are strict.

*Proof.* By Remark 3.2,  $f_1$  and  $f_3$  both have exactly one real base-point, and the rest of their base-points are pairs of non-real conjugate points. Let  $m(q)$  be the multiplicity of  $(f_1 g)^{-1}$  in the point  $q$ , and  $D := \deg(f_1 g) = \deg(f_2 f_1 g)$ . Both  $f_1$  and  $f_3 f_2$  have characteristic  $(d; d-1, 1^{2d-2})$ , hence there exist non-real base-points  $q_1, q_2$  of  $f_1^{-1}$  and  $f_3$ , respectively, such that

$$(3) \quad m([1 : 0 : 0]) + 2m(q_1) \geq D, \quad m(f_2^{-1}([1 : 0 : 0])) + 2m(f_2^{-1}(q_2)) > D.$$

We can assume that  $q_1, q_2$  are points in  $\mathbb{P}^2$  or on the exceptional divisor of  $[1 : 0 : 0]$  or  $f_2^{-1}([1 : 0 : 0])$ , respectively. Furthermore, the points  $[1 : 0 : 0], q_1, \bar{q}_1$  and the points  $f_2^{-1}([1 : 0 : 0]), q_2, \bar{q}_2$  are not collinear because of Bézout theorem.

If  $q_1, q_2$  are both in  $\mathbb{P}^2$ , there exist  $\tau_1, \tau_2 \in \mathcal{H}$  of degree 2 with base-points  $[1 : 0 : 0], q_1, \bar{q}_1$  and  $[1 : 0 : 0], q_2, \bar{q}_2$  respectively. We define  $h_1 := \tau_1 f_1 \in \mathcal{H}$  and  $h_m := f_3 \tau_2^{-1} \in \mathcal{H}$ . By construction we have

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_m) = \deg(f_3) - 1$$

and by Inequalities (3) that

$$\begin{aligned} \deg(h_1 g) &= \deg(\tau_1 f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q_1) \leq D \\ \deg(\tau_2 f_2 f_1 g) &= 2D - m(f_2^{-1}([1 : 0 : 0])) - 2m(f_2^{-1}(q_2)) < D. \end{aligned}$$

We apply Lemma 3.10 to  $f_1, \dots, f_{n-2}, h_1, \tau_1, f_2, \tau_2$  to get the claim.

Let us consider what to do if  $q_1$  or  $q_2$  is on the exceptional divisors of  $[1 : 0 : 0]$ .

If they are both on the exceptional divisors, respectively, then  $f_1$  and  $f_3$  are both of degree at least 3 and

$$2m(q_1) \leq m([1 : 0 : 0]), \quad 2m(f_2^{-1}(q_2)) \leq m(f_2^{-1}([1 : 0 : 0])).$$

Now, Inequalities (3) imply that

$$2m([1 : 0 : 0]) \geq D, \quad 2m(f_2^{-1}([1 : 0 : 0])) > D$$

Since  $f_1$  and  $f_3$  are of degree at least three, this is only possible if  $f_2^{-1}([1 : 0 : 0]) = [1 : 0 : 0]$ . In particular  $f_2 \in \mathcal{H}$ . We put  $h_1 = h_m = f_3 f_2 f_1 \in \mathcal{H}$ .

Let us assume that  $f_2 \notin \mathcal{H}$ . Then either  $q_1$  or  $q_2$  are points in  $\mathbb{P}^2$ . Suppose that  $q_1$  is in  $\mathbb{P}^2$  and  $q_2$  is on the exceptional divisor of  $f_2^{-1}([1 : 0 : 0])$ . Then, because of the above reasoning, we have

$$2m(f_2^{-1}([1 : 0 : 0])) > D > 2m([1 : 0 : 0]).$$

Inequalities (3) get us

$$m(f_2^{-1}([1 : 0 : 0])) + 2m(q_1) > m([1 : 0 : 0]) + 2m(q_1) \geq D.$$

Then  $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$  are not collinear and all of them are points of  $\mathbb{P}^2$ . There exists  $\tau'_2 \in \mathcal{H}$  of degree 2 with base-points  $[1 : 0 : 0], f_2(q_1), f_2(\bar{q}_1)$ . We proceed as in the first case with  $\tau'_2$  instead of  $\tau_2$ . If  $q_2$  is a point of  $\mathbb{P}^2$  and  $q_1$  is on the exceptional divisor of  $[1 : 0 : 0]$ , we proceed analogously.  $\square$

**Lemma 3.12.** *Let  $f_1 \in \mathcal{H}$  be of degree 2 and  $f_3 \in \mathcal{J}_\circ$  a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1g), \quad \text{and} \quad \deg(f_3f_2f_1g) < \deg(f_2f_1g)$$

*Then there exist  $h_1 \in \mathcal{A}$ ,  $h_2, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $f_3f_2f_1 = h_m \cdots h_1$  and*

$$\begin{aligned} \deg(h_1g) &\leq \deg(f_1g), \\ \deg(h_i \cdots h_3h_2h_1g) &< \deg(f_1g) \quad i = 2, \dots, m. \end{aligned}$$

*If we have strict inequality in the assumption, all inequalities are strict.*

*The same claim holds if  $f_1 \in \mathcal{J}_\circ$  is a standard or special quintic transformation and  $f_3 \in \mathcal{H}$  is of degree 2.*

*Proof.* By  $m(s)$  we denote the multiplicity of  $(f_1g)^{-1}$  in  $s$  and defined  $D := \deg(f_1g)$ . Let  $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$  be the base-points of  $f_3f_2$  and order them such that

$$m(r_1) \geq m(r_2) \geq m(r_3),$$

and  $r_1$  is a proper point of  $\mathbb{P}^2$  and  $r_2$  is either a proper point of  $\mathbb{P}^2$  or in the first neighbourhood of  $r_1$ . Let  $[1 : 0 : 0], q, \bar{q}$  be the base-points of  $f_1^{-1}$ . Lemma 3.5 and  $f_1$  being quadratic imply respectively that

$$(4) \quad m(r_1) + m(r_2) + m(r_3) > D, \quad m([1 : 0 : 0]) + 2m(q) \geq D.$$

We look at two cases,  $q = r_1$  and  $q \neq r_1$ .

(1) Suppose that  $q = r_1$ . As  $r_1, r_2, r_3$  are not collinear, at least one of  $r_2, r_3$  is not one a line contracted by  $f_1^{-1}$ , say  $r_i$ , and it is a point of  $\mathbb{P}^2$ . Then  $f_3f_2f_1$  sends the pencil of conics passing through the non-real base-points of  $f_1$  and  $f_1^{-1}(r_i), f_1^{-1}(\bar{r}_i)$  onto a pencil of conics through two pairs of non-real conjugate points of  $f_3^{-1}$ . Hence, there exist  $\alpha, \beta \in \mathcal{A}$  such that  $\beta f_3f_2f_1\alpha \in \mathcal{J}_\circ$ . The claim follows with  $h_1 = \alpha^{-1}$ ,  $h_2 = \beta f_3f_2f_1\alpha$  and  $h_m = h_3 = \beta^{-1}$ .

(2) Suppose that  $q \neq r_1$ . We look at two cases, depending on the multiplicities  $m([1 : 0 : 0]), m(q), m(\bar{q})$ .

(2.1) If  $m(q) \geq m([1 : 0 : 0])$ , then Inequalities (4) and the order of the  $r_i$  imply that  $m(q) \geq \frac{D}{3}$  and  $m(r_1) + m(r_2) > \frac{2D}{3}$ , and so

$$m(q) + m(r_1) + m(r_2) > D.$$

In particular, the six points  $q, \bar{q}, r_1, \bar{r}_1, r_2, \bar{r}_2$  are not on a conic. Then the chosen order of the points implies that there exists a standard or special quintic transformation  $\theta \in \mathcal{J}_\circ$  and  $\alpha \in \mathcal{A}$  such that  $\theta\alpha$  has these 6 points as base-points. The above inequality implies

$$\deg(\theta\alpha f_1g) = 5D - 4m(q) - 4m(r_1) - 4m(r_2) < D.$$

The maps  $f_1$  and  $\theta\alpha$  have two common base-points, and we find the maps  $h_1, \dots, h_k$  as in Case 1. The maps  $g' := h_k \cdots h_1g, \theta^{-1}, f_2\alpha, f_3$  satisfy the assumptions of Lemma 3.9 with “<”, and we get the maps  $h_{k+1}, \dots, h_m$  from there.

(2.2) If  $m(q) < m([1 : 0 : 0])$ , then Inequalities (4) and the order of the  $r_i$  imply that  $m([1 : 0 : 0]), m(r_1) \geq \frac{D}{3}$ , so

$$m([1 : 0 : 0]) + 2m(r_1) > D$$

In particular, the points  $[1 : 0 : 0], r_1, \bar{r}_1$  are not collinear. As they are proper points of  $\mathbb{P}^2$ , there exists  $\theta \in \mathcal{J}_\circ$  of degree 2 and  $\alpha \in \mathcal{J}_\circ$  such that  $\theta\alpha$  has these 3 points as base-points. Moreover, the above inequality implies that

$$\deg(\theta\alpha f_1g) = 2D - m(q) - 2m(r_1) < D.$$

We apply Lemma 3.11 to  $f_1, \alpha, \theta$  to get  $h_1, \dots, h_k$  and then Case 1 to  $g' := h_k \cdots h_1 g, \theta^{-1}, f_2 \alpha, f_3$  with “<” to get the rest of the  $h_i$ .

The case where  $f_1$  is a standard or special quintic transformation and  $f_3 \in \mathcal{H}$  is of degree 2 is done analogously.  $\square$

**Lemma 3.13.** *Let  $f_1 \in \mathcal{H}$  and  $f_3 \in \mathcal{J}_\circ$  a standard or special quintic transformation. Suppose that*

$$\deg(g) \leq \deg(f_1 g), \quad \text{and} \quad \deg(f_3 f_2 f_1 g) < \deg(f_2 f_1 g)$$

*Then there exist  $h_1 \in \mathcal{H}, h_2 \in \mathcal{A}, h_3, \dots, h_m \in \mathcal{A} \cup \mathcal{J}_\circ \cup \mathcal{H}$  such that  $f_3 f_2 f_1 = h_m \cdots h_1$  and  $\deg(h_1) = \deg(f_1) - 1$ ,*

$$\deg(h_1 g) \leq \deg(f_1 g),$$

$$\deg(h_i \cdots h_3 h_2 h_1 g) < \deg(f_1 g) \quad i = 3, \dots, m.$$

*If we have strict inequality in the assumption, all inequalities are strict.*

*If  $f_1 \in \mathcal{J}_\circ$  is a standard or special quintic transformation and  $f_3 \in \mathcal{H}$ , we have*

$$\deg(h_i \cdots h_2 h_1 g) < \deg(f_1 g) \quad i = 2, \dots, m.$$

*Proof.* Let  $m(s)$  be the multiplicity of  $(f_1 g)^{-1}$  in the point  $s$  and define  $D := \deg(f_1 g)$ . Since  $f_1$  has characteristic  $(d; d-1, 1^{2d-2})$  and is contained in  $\mathcal{H}$ , there exist a non-real base-point  $q$  of  $f_1^{-1}$  such that

$$m([1 : 0 : 0]) + 2m(q) \geq D.$$

We can assume that they are points in  $\mathbb{P}^2$  or on the exceptional divisor of  $[1 : 0 : 0]$ . Furthermore, the points  $[1 : 0 : 0], q, \bar{q}$  are not collinear by Bézout theorem.

(1) Suppose that  $q, \bar{q}$  are points in  $\mathbb{P}^2$ . Then there exists  $\tau \in \mathcal{H}$  of degree 2 such that  $\tau$  has base-points  $[1 : 0 : 0], q, \bar{q}$ . We put  $h_1 := \tau f_1 \in \mathcal{H}$ . Note that

$$\deg(h_1) = \deg(f_1) - 1,$$

$$\deg(h_1 g) = \deg(\tau f_1 g) = 2D - m([1 : 0 : 0]) - 2m(q) \leq D.$$

We apply Lemma 3.12 to  $g' := h_1 g$  and  $\tau^{-1}, f_2, f_3$  to get  $h_2, \dots, h_m$ .

(2) Suppose that  $q, \bar{q}$  are on the exceptional divisor of  $[1 : 0 : 0]$ . Then  $m([1 : 0 : 0]) \geq 2m(q)$ . Call  $r_1, \bar{r}_1, r_2, \bar{r}_2, r_3, \bar{r}_3$  the base-points of  $f_3 f_2$ , and suppose that

$$m(r_1) \geq m(r_2) \geq m(r_3)$$

and  $r_1$  is a point in  $\mathbb{P}^2$ . Lemma 3.5 implies that

$$m(r_1) + m(r_2) + m(r_3) > D.$$

We get  $m([1 : 0 : 0]) \geq \frac{D}{3}$  and  $m(r_1) > \frac{D}{3}$ , and so  $m([1 : 0 : 0]) + 2m(r_1) > D$ . In particular, the points  $[1 : 0 : 0], r_1, \bar{r}_1$  are not collinear, and hence there exists  $\tau \in \mathcal{H}$  of degree 2 with these three points its base-points. As above we put  $h_1 := \tau f_1 \in \mathcal{H}$  and note that

$$\deg(h_1) = \deg(f_1) - 1, \quad \deg(h_1 g) = \deg(\tau f_1 g) < D,$$

and apply Lemma 3.12 to  $g' := h_1 g$  and  $\tau^{-1}, f_2, f_3$ .

If  $f_1$  is a standard or special quintic transformation and  $f_3 \in \mathcal{H}$ , we construct  $h_1, \dots, h_m$  analogously.  $\square$

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SUSANNA ZIMMERMANN, UNIVERSITÉ TOULOUSE PAUL SABATIER, INSTITUT DE MATHÉMATIQUES, 118 ROUTE DE NARBONNE, 31062 TOULOUSE CEDEX 9

*E-mail address:* `susanna.zimmermann@math.univ-toulouse.fr`