Invasion of an empty habitat by two competitors: spreading properties of monostable two-species competition–diffusion systems

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INVASION OF AN EMPTY HABITAT BY TWO COMPETITORS: SPREADING PROPERTIES OF MONOSTABLE TWO-SPECIES COMPETITION–DIFFUSION SYSTEMS

LÉO GIRARDIN[1], KING-YEUNG LAM[2]

Abstract. This paper is concerned with some spreading properties of monostable Lotka–Volterra two-species competition–diffusion systems when the initial values are null or exponentially decaying in a right half-line. Thanks to a careful construction of super-solutions and sub-solutions, we improve previously known results and settle open questions. In particular, we show that if the weaker competitor is also the faster one, then it is able to evade the stronger and slower competitor by invading first into unoccupied territories. The pair of speeds depends on the initial values. If these are null in a right half-line, then the first speed is the KPP speed of the fastest competitor and the second speed is given by an exact formula depending on the first speed and on the minimal speed of traveling waves connecting the two semi-extinct equilibria. Furthermore, the unbounded set of pairs of speeds achievable with exponentially decaying initial values is characterized, up to a negligible set.

Contents

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1. Introduction

In this paper, we are interested in some spreading properties of the classical monostable Lotka–Volterra two-species competition–diffusion system

$$\begin{cases} \partial_t u - \partial_{xx} u = u\left(1-u-av\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t v - d\partial_{xx} v = rv\left(1-v-bu\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u\left(0,x\right) = u_0\left(x\right) & \text{for all } x\in\mathbb{R} \\ v\left(0,x\right) = v_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases} \tag{1.1}$$

with $d>0$, $a\in(0,1)$, $b>1$, $r>0$ and $u_0, v_0 \in \mathscr{C}\left(\mathbb{R},[0,1]\right)\setminus\{0\}$. The assumptions on a and b mean that u and v are respectively the stronger and the weaker competitor.

Recall from the classical literature [1, 12, 21] that the scalar Fisher-KPP equation

$$\begin{cases} \partial_t w - \delta\partial_{xx} w = \rho w\left(1-w\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ w\left(0,x\right) = w_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases}$$

with $\delta,\rho>0$ and $w_0\in\mathscr{C}_b\left(\mathbb{R}\right)$ with nonempty support included in $(-\infty,0]$ has the following spreading property: there exists a unique $c_{KPP}>0$ satisfying

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|<ct} \left|w\left(t,x\right)-1\right| = 0 \text{ for each } c < c_{KPP} \\ \lim\limits_{t\to+\infty} \sup\limits_{ct<x} \left|w\left(t,x\right)\right| = 0 \text{ for each } c > c_{KPP} \end{cases}.$$

These asymptotics describe the invasion of the unstable state 0 by the stable state 1 and c_{KPP} is consequently referred to as the spreading speed of this invasion. Furthermore, c_{KPP} coincides with the minimal speed of the traveling wave solutions, which are particular entire solutions of the form $w:(t,x)\mapsto\varphi\left(x-ct\right)$ with $\varphi\geq 0$, $\varphi\left(-\infty\right)=1$ and $\varphi\left(+\infty\right)=0$. A striking result is the so-called *linear determinacy* property: there exists such a pair (φ,c) if and only if the linear equation

$$-\delta\varphi'' - c\varphi' = \rho\varphi,$$

namely, the linearization at $\varphi=0$ of the semilinear equation satisfied by φ, admits a positive solution in $\mathbb{R}$. Consequently, $c_{KPP}=2\sqrt{\rho\delta}$. As far as the system (1.1)

is concerned, this result shows that in the absence of the competitor, u and v respectively spread at speed 2 and $2\sqrt{rd}$.

Recall also from the collection of works due to Lewis, Li and Weinberger [22, 24] that the competition–diffusion system

$$\begin{cases} \partial_t u - \partial_{xx} u = u\left(1-u-av\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ \partial_t v - d\partial_{xx} v = rv\left(1-v-bu\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u\left(0,x\right) = \tilde{u}_0\left(x\right) & \text{for all } x\in\mathbb{R} \\ v\left(0,x\right) = 1-\tilde{v}_0\left(x\right) & \text{for all } x\in\mathbb{R} \end{cases} \tag{1.2}$$

with $\tilde{u}_0$ and $\tilde{v}_0$ compactly supported and $\tilde{u}_0$ nonnegative nonzero, has an analogous spreading property: there exists a unique $c_{LLW}>0$ satisfying

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|<ct} \left(\left|u\left(t,x\right)-1\right|+\left|v\left(t,x\right)\right|\right) = 0 \text{ for each } c<c_{LLW} \\ \lim\limits_{t\to+\infty} \sup\limits_{ct<|x|} \left(\left|u\left(t,x\right)\right|+\left|v\left(t,x\right)-1\right|\right) = 0 \text{ for each } c>c_{LLW} \end{cases}$$

and describing the invasion of the unstable state $(0,1)$ by the stable state $(1,0)$. As in the KPP case, the spreading speed c_{LLW} is the minimal speed of the monotonic traveling wave solutions; linearizing at $(0,1)$, it is easily deduced that the linear speed is $2\sqrt{1-a}$ and that $c_{LLW}\geq 2\sqrt{1-a}$. However, contrarily to the KPP case, the converse inequality $c_{LLW}\leq 2\sqrt{1-a}$ is only sometimes true. More precisely,

- on one hand, according to Lewis–Li–Weinberger [22], linear determinacy holds if
$$d\leq 2 \text{ and } \frac{ab-1}{1-a}\leq \frac{1}{r}\left(2-d\right),$$
a result which was later on improved by Huang [17] who established that the weaker condition
$$\frac{\left(2-d\right)\left(1-a\right)+r}{rb}\geq \max\left(a,\frac{d-2}{2\left|d-1\right|}\right)$$
is sufficient;
- but on the other hand, Huang–Han [18] constructed explicit counter-examples where $c_{LLW}>2\sqrt{1-a}$.

Anyways, a rough upper estimate of c_{LLW} can be obtained by comparison with the KPP equation satisfied by u in the absence of v: $c_{LLW}\leq 2$ (the competition always slows down the invasion of u). The strict inequality $c_{LLW}<2$ is expected but, as far as we know, cannot be established easily. Of course, it automatically holds in case of linear determinacy.

We focus now on the system (1.1) and observe that, when u_0 and v_0 are both null or exponentially decaying in $[0,+\infty)$, the long-time behavior in $(0,+\infty)$ is unclear. It is the purpose of this paper to address this question.

If $rd>1$ and u_0 and v_0 are compactly supported, then for all small $\epsilon>0$, one expects the following statements to hold:

$$\lim_{t\to+\infty} \sup_{|x|<(c_{LLW}-\epsilon)t} \left(\left|u\left(t,x\right)-1\right|+\left|v\left(t,x\right)\right|\right) = 0,$$

$$\lim_{t\to+\infty} \sup_{(2+\epsilon)t<|x|<(2\sqrt{rd}-\epsilon)t} \left(\left|u\left(t,x\right)\right|+\left|v\left(t,x\right)-1\right|\right) = 0,$$

$$\lim_{t\to+\infty} \sup_{(2\sqrt{rd}+\epsilon)t<|x|} \left(\left|u\left(t,x\right)\right|+\left|v\left(t,x\right)\right|\right) = 0.$$

This fact, which we are going to prove in the forthcoming pages (see Proposition 3.1) by adapting very slightly arguments from the related literature and is therefore not really new, basically means the following: the empty space is first invaded by the faster competitor v at speed $2\sqrt{rd}$ and then the replacement of v by the stronger competitor u occurs somewhere in the area $c_{LLW} \leq \frac{x}{t} \leq 2$. In particular, as far as spreading speeds are concerned, the first invasion ($(0,0)$ by $(0,1)$) is not influenced by the second invasion ($(0,1)$ by $(1,0)$): the competition exerted by the exponential tail of u in the area $2 < \frac{x}{t}$ is negligible.

It is then natural to investigate whether the converse statement is true: is the second invasion influenced by the first one? Is it possible to show that the speed c_2 of the second invasion is exactly c_{LLW}, or is there on the contrary a possibility of acceleration, namely $c_2 > c_{LLW}$?

Previous works on the spreading properties of the system (1.1) with u_0 and v_0 supported in $(-\infty, 0]$ are due to Carrère [3] and Lin and Li [26]. Carrère studied the bistable case ($a > 1$, $b > 1$). She proved that the second invasion admits a single spreading speed which is the speed of the unique bistable traveling wave connecting $(0,1)$ to $(1,0)$: the two invasions are indeed independent. We point out that the bistable case is easier to handle (based on the uniqueness of traveling wave speed and profile in the bistable case, the arguments used on the left of the second transition can be used again on its right). As a matter of fact, Lin and Li investigated the monostable case with stable coexistence ($a < 1$, $b < 1$) and compactly supported initial data but were unable to determine the second speed. All three monostable cases (stable coexistence, stable $(1,0)$, stable $(0,1)$) being handled quite similarly (see Lewis–Li–Weinberger [22] for instance), the technical obstacles they encountered should not depend on the sign of $b - 1$.

In the present paper, we adopt a different point of view: we aim directly for the construction of (almost) optimal pairs of super-solutions and sub-solutions. This point of view is highly fruitful. On one hand, it brings forth a complete spreading result when the support of u_0 is included in a left half-line (namely, u_0 is Heavyside-like or compactly supported) and v_0 is compactly supported. This result shows that, surprisingly, acceleration of the second front does indeed occur in some cases.

Heuristically, there are three spreading mechanisms involved:

(1) u invading a hostile environment where $v = 1$ at speed $2\sqrt{1-a}$;
(2) u chasing the competitor v at speed c_{LLW};
(3) u invading an environment where $v = \mathbf{1}_{\{x \leq 2\sqrt{rd}t\}}$ at some speed c.

While both c_{LLW} and c are greater than or equal to $2\sqrt{1-a}$ (by the comparison principle), the sign of $c - c_{LLW}$ can vary. One of our main results (Theorem 1.1) states that the actual invasion speed of u is the maximum of c_{LLW} and c.

The problem we consider in this paper was in fact already considered by Shigesada and Kawasaki in 1997 [33], where they illustrated numerically the hair-trigger effect (the fact that a small number of the weaker competitor eventually reaches the range front of the other species and establishes a KPP-type wave into the open space). They also gave practical estimates of the respective spreading speeds of the species, based on linearizations. Our work takes into account the possibility of failure of linear determinacy, and discovers additionally the possibility of a further accelerated invasion. Therefore, we have completely settled the mathematical questions raised by their study. Let us also point out here that, inspired by the study

of Shigesada and Kawasaki, Li [23] very recently addressed similar questions in the framework of integro-difference systems.

Our approach also delivers a general existence and nonexistence result related to *propagating terraces* (succession of compatible traveling waves with decreasingly ordered speeds, first described by Fife and McLeod [11]) having the unstable steady state $(0,1)$ as intermediate steady state and corresponding to exponentially decaying initial data. As far as scalar terraces for reaction–diffusion equations are concerned, Ducrot, Giletti and Matano [9] showed quite generically that all intermediate states are stable from below (see also Poláčik [29] for a complete account in the general setting). In more sophisticated contexts (reaction–diffusion systems, nonlocal equations, etc.), propagating terraces with unstable intermediate states are observed numerically (see Nadin–Perthame–Tang [28] for the nonlocal KPP equation, Faye–Holzer [10] for a different two-component reaction–diffusion system). Rigorous analytical studies are however very difficult and have only been carried out in simple cases. For instance, a closely related paper due to Iida, Lui and Ninomiya [19] studied a monostable system of cooperatively coupled KPP equations. The comparison with that paper shows well the value of the present paper: Iida–Lui–Ninomiya only considered Heavyside-like or compactly supported initial data and obtained that all spreading speeds are in fact scalar KPP speeds (independent invasions). In this regard, the present paper is, to the best of our knowledge, unprecedented.

Our approach relies heavily upon the comparison principle. Therefore it might be appropriate for cooperative systems of arbitrary size with couplings more sophisticated than the Lotka–Volterra one considered in Iida–Lui–Ninomiya [19]. Let us point out right now that a fully coupled cooperative system, namely a cooperative system where the positivity of any one component implies the positivity of all the others, cannot admit propagating terrace solutions. Unfortunately, our approach cannot be adapted to settings devoid of comparison principle.

Finally, let us point out that our forthcoming results would still hold true if $u(1-u)$ and $rv(1-v)$ were replaced by more general KPP reaction terms. In order to ease the reading, however, we focus on the traditional logistic form.

1.1. **Main results.** Define the auxiliary function

$$f: \begin{array}{ccc} \left[2\sqrt{1-a},+\infty\right) & \to & \left(2\sqrt{a},2\left(\sqrt{1-a}+\sqrt{a}\right)\right] \\ c & \mapsto & c-\sqrt{c^2-4(1-a)}+2\sqrt{a} \end{array}. \tag{1.3}$$

This function is decreasing and bijective and satisfies in particular

$$f(2)=2,$$

$$f^{-1}:\tilde{c}\mapsto\frac{\tilde{c}}{2}-\sqrt{a}+\frac{2(1-a)}{\tilde{c}-2\sqrt{a}}.$$

1.1.1. *Spreading properties of initially localized solutions.*

Theorem 1.1. *Let $u_0\in\mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with support included in a left half-line and $v_0\in\mathscr{C}(\mathbb{R},[0,1])\setminus\{0\}$ with compact support. Let (u,v) be the solution of equation* (1.1)*.*

(1) Assume $2\sqrt{rd}<2$. Then

$$\lim_{t\to+\infty}\sup_{x\geq 0}|v(t,x)|=0,$$

$$\lim_{t\to+\infty} \sup_{0\le x<(2-\varepsilon)t} |u(t,x)-1| = 0 \text{ for each } \varepsilon \in (0,2),$$
$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x} |u(t,x)| = 0 \text{ for each } \varepsilon > 0.$$

(2) Assume $2\sqrt{rd} \in (2, f(c_{LLW}))$ *and define*

$$c_{acc} = f^{-1}\left(2\sqrt{rd}\right) = \sqrt{rd} - \sqrt{a} + \frac{1-a}{\sqrt{rd}-\sqrt{a}} \in (c_{LLW}, 2).$$

Then

$$\lim_{t\to+\infty} \sup_{0\le x<(c_{acc}-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for each } \varepsilon \in (0, c_{acc}),$$

$$\lim_{t\to+\infty} \sup_{(c_{acc}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)| + |v(t,x)-1|) = 0 \text{ for each } \varepsilon \in \left(0, \frac{2\sqrt{rd}-c_{acc}}{2}\right),$$
$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} (|u(t,x)| + |v(t,x)|) = 0 \text{ for each } \varepsilon > 0.$$

(3) Assume $2\sqrt{rd} \ge f(c_{LLW})$. *Then*

$$\lim_{t\to+\infty} \sup_{0\le x<(c_{LLW}-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for each } \varepsilon \in (0, c_{LLW}),$$

$$\lim_{t\to+\infty} \sup_{(c_{LLW}+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)| + |v(t,x)-1|) = 0 \text{ for each } \varepsilon \in \left(0, \frac{2\sqrt{rd}-c_{LLW}}{2}\right),$$
$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x} (|u(t,x)| + |v(t,x)|) = 0 \text{ for each } \varepsilon > 0.$$

In the first case, v goes extinct. In the second case, v invades first at speed $2\sqrt{rd}$ and is then replaced by u at speed $c_{acc} > c_{LLW}$. In the third case, v invades first at speed $2\sqrt{rd}$ and is then replaced by u at speed c_{LLW}.

Notice that the limits above are chiefly concerned with $x \ge 0$. This is intentional, for the sake of brevity and clarity. In $(-\infty, 0)$, two behaviors are possible, depending on whether u_0 is compactly supported or Heavyside-like. In the former case, all inequalities above hold with x replaced by $|x|$ (and this claim is proved simply by symmetry). In the latter case, (u, v) converges uniformly to $(1, 0)$ in $(-\infty, 0)$ (and this claim can be proved by a standard comparison argument).

1.1.2. *The set of admissible pairs of speeds for more general initial data.* Define the auxiliary function

$$\begin{array}{rccc} \lambda_v : & \left[2\sqrt{rd}, +\infty\right) & \to & \left(0, \sqrt{\frac{r}{d}}\right] \\ & c & \mapsto & \frac{1}{2d}\left(c - \sqrt{c^2 - 4rd}\right) \end{array}.$$

Theorem 1.2. *Let* $c_1 \in \left[2\sqrt{rd}, +\infty\right)$ *and* $c_2 \in [c_{LLW}, c_1]$. *Let* (u, v) *be a solution of (1.1) such that*

$$c_2 = \sup\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{0\le x\le ct} (|u(t,x)-1| + |v(t,x)|) = 0\right\}$$

and such that at least one of the following two properties holds true:

(1) $x \mapsto v(0,x)\, e^{\lambda_v(c_1)x}$ *is bounded in* $\mathbb{R}$*; or*

(2) c_1 *satisfies*

$$c_1 \geq \inf\left\{c>0 \mid \lim_{t\to+\infty}\sup_{x\geq ct}|v(t,x)|=0\right\}.$$

Then $c_2 \geq f^{-1}(c_1)$.

The assumption on c_2 basically means that u spreads at speed c_2. However, in general, the spreading speed is ill-defined: the *minimal spreading speed* of u,

$$\sup\left\{c>0 \mid \lim_{t\to+\infty}\sup_{0\leq x\leq ct}|u(t,x)-1|=0\right\},$$

might very well be smaller than its *maximal spreading speed*,

$$\inf\left\{c>0 \mid \lim_{t\to+\infty}\sup_{0\leq x\leq ct}|u(t,x)|=0\right\}.$$

On this problem, we refer to Hamel –Nadin [16].

The properties (1) and (2) above are more or less equivalent. Indeed, on one hand, (1) directly implies (2) by standard comparison; on the other hand, if (2) holds, then for all $\lambda \in (0,\lambda_v(c_1))$, there exists T_λ such that $x \mapsto v(T_\lambda,x)\,\mathrm{e}^{\lambda x}$ is bounded in $\mathbb{R}$. However the proof of the latter implication is difficult. In fact, instead of establishing it, we will directly prove the result in each case. We emphasize that although (2) might be easier to understand in that it directly relates c_1 to the spreading of v, (1) has the advantage of being easier to apply since it only requires knowledge of the initial condition.

In short, this theorem means that if v spreads no faster than c_1 and if u spreads at speed c_2, then $c_2 \geq f^{-1}(c_1)$. The next theorem shows the sharpness of this threshold: any $c_2 > f^{-1}(c_1)$ can actually be achieved.

Theorem 1.3. *Let* $c_1 \in \left(2\sqrt{rd},+\infty\right)$ *and* $c_2 \in (c_{LLW},c_1)$.

Assume $c_1 > f(c_2)$. *Then there exists* $(u_{c_1,c_2},v_{c_1,c_2}) \in \mathscr{C}\left(\mathbb{R},[0,1]^2\right)$ *such that the solution* (u,v) *of (1.1) with initial value* $(u_0,v_0)=(u_{c_1,c_2},v_{c_1,c_2})$ *satisfies*

$$\lim_{t\to+\infty}\sup_{x<(c_2-\varepsilon)t}(|u(t,x)-1|+|v(t,x)|)=0 \text{ for each } \varepsilon\in(0,c_2),$$

$$\lim_{t\to+\infty}\sup_{(c_2+\varepsilon)t<x<(c_1-\varepsilon)t}(|u(t,x)|+|v(t,x)-1|)=0 \text{ for each } \varepsilon\in\left(0,\frac{c_1-c_2}{2}\right),$$

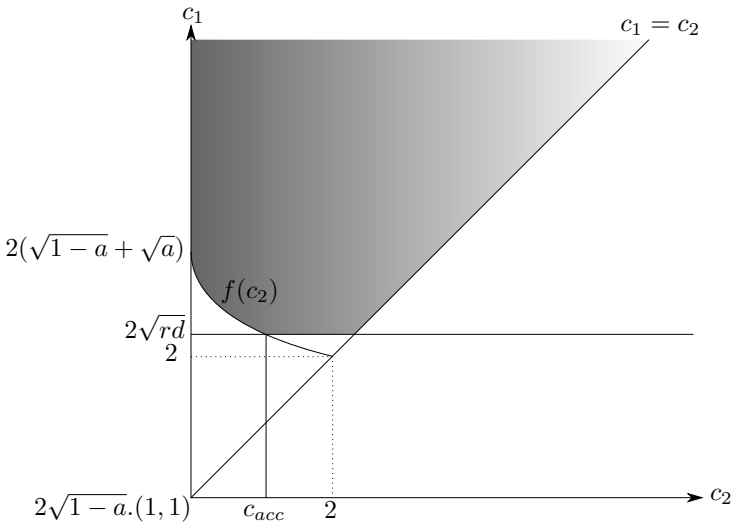
$$\lim_{t\to+\infty}\sup_{(c_1+\varepsilon)t<x}(|u(t,x)|+|v(t,x)|)=0 \text{ for each } \varepsilon>0.$$

Let us point out that this solution (u,v) is not a proper propagating terrace in the sense of Ducrot–Giletti–Matano [9]: the locally uniform convergence of the profiles is missing (as in Carrère [3]).

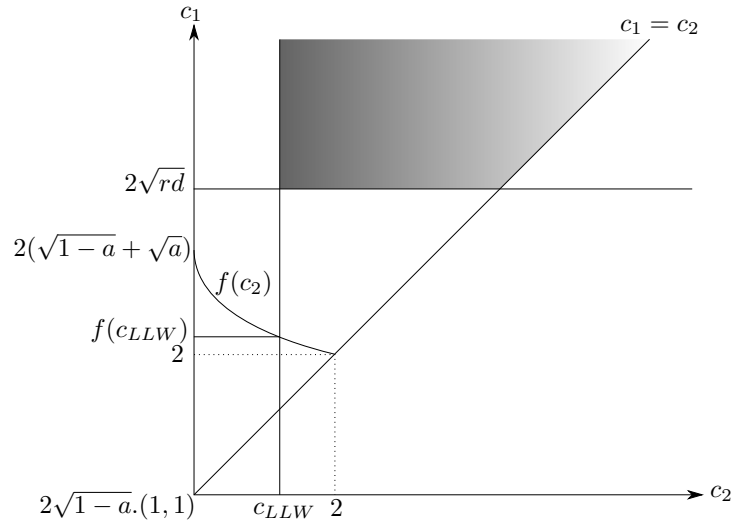
The fact that the set of admissible speeds is not always the maximal set

$$\left\{(c_1,c_2)\in\left[2\sqrt{rd},+\infty\right)\times[c_{LLW},+\infty) \mid c_1>c_2\right\}.$$

settles completely a question raised by the first author [14].



(A) $c_{LLW} = 2\sqrt{1-a}$, $2\sqrt{rd} < f(c_{LLW})$. In this case Theorem 1.1(2) applies with $(c_1, c_2) = (2\sqrt{rd}, c_{acc})$, where $c_{acc} > c_{LLW}$.



(B) $c_{LLW} > 2\sqrt{1-a}$, $2\sqrt{rd} > f(c_{LLW})$. In this case Theorem 1.1(3) applies with $(c_1, c_2) = (2\sqrt{rd}, c_{LLW})$.

FIGURE 1.1. Examples of sets of admissible pairs of speeds (c_1, c_2)

1.1.3. *The super-solutions and sub-solutions.* The preceding theorems will be proved thanks to the following three propositions, which are of independent interest and concern existence results for super-solutions and sub-solutions (the precise definition of these will be recalled in the next section).

Let

$$\begin{array}{rccc}\lambda: & \left[2\sqrt{1-a},+\infty\right) & \to & \left(0,\sqrt{1-a}\right] \\ & c & \mapsto & \frac{1}{2}\left(c-\sqrt{c^2-4\left(1-a\right)}\right)\end{array},$$

$$\Lambda:(c,\tilde{c})\mapsto\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)}\right).$$

The domain of Λ is the set of all $(c,\tilde{c})$ such that $c\geq 2\sqrt{1-a}$ and $\tilde{c}\geq\max\left(c,f\left(c\right)\right)$.

For all $c\geq c_{LLW}$ and $\tilde{c}\geq\max\left(c,f\left(c\right)\right)$, $\overline{w_{c,\tilde{c}}}$ denotes the function

$$\overline{w_{c,\tilde{c}}}:(t,x)\mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t)}.$$

Proposition 1.4. *Let $c_1\geq 2\sqrt{rd}$, $c_2\geq c_{LLW}$ and assume $c_2<c_1<f\left(c_2\right)$.*

There exist $c>c_2$, $\tilde{c}\in\left(c_1,f\left(c\right)\right)$, $L>0$ and $\delta^\star>0$ such that, for all $\delta\in(0,\delta^\star)$, all $\kappa\in\left(0,\min\left(\frac{1-a}{2},\frac{\delta}{2}\right)\right)$ and all $\zeta>L$, there exists $R_\delta>0$ and a sub-solution $\left(\underline{u_{\delta,\zeta,\kappa}},\overline{v_{\delta,\zeta}}\right)$ of (1.1) satisfying the following properties:

(1) $\underline{u_{\delta,\zeta,\kappa}}\left(0,x\right)\leq 1-a$ for all $x\in\mathbb{R}$;

(2) the support of $x\mapsto\underline{u_{\delta,\zeta,\kappa}}\left(0,x\right)$ is included in $[0,L+\zeta+2R_\delta]$;

(3) $\underline{u_{\delta,\zeta,\kappa}}\left(0,x\right)\leq\kappa$ for all $x\in[L,L+\zeta+2R_\delta]$;

(4) there exists $X>0$ such that $\underline{u_{\delta,\zeta,\kappa}}$ satisfies

$$\partial_t\underline{u_{\delta,\zeta,\kappa}}-\partial_{xx}\underline{u_{\delta,\zeta,\kappa}}\leq(1-\delta)\,\underline{u_{\delta,\zeta,\kappa}}\text{ in }\left\{(t,x)\in[0,+\infty)\times\mathbb{R}\mid x>X+\tilde{c}t\right\}.$$

(5) there exists $C_\delta>0$ depending only on δ such that

$$\overline{v}_{\delta,\zeta}\left(0,x\right)\geq\min\left(1,C_\delta e^{-\lambda_v(\tilde{c})(x-\zeta)}\right)\text{ for all }x\in\mathbb{R};$$

(6) the following spreading property holds true:

$$\lim_{t\to+\infty}\sup_{L\leq x<(c-\varepsilon)t}\left|\underline{u_{\delta,\zeta,\kappa}}\left(t,x\right)-\frac{1-a}{2}\right|=0\text{ for all }\varepsilon\in(0,c).$$

Proposition 1.5. *Let $c_2\in\left(\max\left(c_{LLW},f^{-1}\left(2\sqrt{rd}\right)\right),2\right)$.*

There exists $\delta^\star>0$ and $\left(c_1^\delta,c_2^\delta\right)_{\delta\in(0,\delta^\star)}$ such that

$$c_2<c_2^\delta<c_1^\delta<2\sqrt{rd}\text{ for all }\delta\in(0,\delta^\star),$$

$$\lim_{\delta\to 0}\left(c_1^\delta,c_2^\delta\right)=\left(2\sqrt{rd},c_2\right),$$

and, for all $\delta\in(0,\delta^\star)$, there exists a super-solution $\left(\overline{u_\delta},\underline{v_\delta}\right)$ of (1.1) satisfying the following properties:

(1) there exists $y_0\in\mathbb{R}$ such that, for all $y\geq y_0$ and $t\geq 0$,

$$\overline{u_\delta}\left(0,x-y-\frac{\left(\Lambda\left(c_2,2\sqrt{rd}\right)\right)^2+1}{\Lambda\left(c_2,2\sqrt{rd}\right)}t\right)\geq\min\left(1,\overline{w_{c_2,2\sqrt{rd}}}\left(t,x\right)\right)\text{ for all }x\in\mathbb{R};$$

(2) $x\mapsto\underline{v_\delta}\left(0,x\right)$ is compactly supported;

(3) $\underline{v_\delta\left(0,x\right)}\leq 1-\delta$ for all $x\in\mathbb{R}$;

(4) the following spreading property holds true:

$$\lim_{t\to+\infty}\sup_{\left(c_2^\delta+\varepsilon\right)t<x<\left(c_1^\delta-\varepsilon\right)t}\left(\left|\overline{u_\delta}\left(t,x\right)\right|+\left|\underline{v_\delta}\left(t,x\right)-(1-2\delta)\right|\right)=0\text{ for all }\varepsilon\in\left(0,\frac{c_1^\delta-c_2^\delta}{2}\right).$$

Proposition 1.6. *Let $c_1 > 2\sqrt{rd}$, $c_2 > c_{LLW}$ and assume $c_1 > \max(c_2, f(c_2))$. There exists $\delta^\star > 0$ and $\left(c_2^\delta\right)_{\delta\in(0,\delta^\star)}$ such that*

$$c_2^\delta > c_2 \text{ for all } \delta \in (0,\delta^\star),$$

$$\lim_{\delta\to 0} c_2^\delta = c_2,$$

and, for all $\delta \in (0,\delta^\star)$, there exists a super-solution of (1.1) $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ and a sub-solution of (1.1) $\left(\underline{u_\delta}, \overline{v_\delta}\right)$ satisfying the following properties:

(1) there exists $y_0 \in \mathbb{R}$ such that, for all $y \geq y_0$ and $t \geq 0$,

$$\overline{u_\delta}\left(0, x - y - \frac{\left(\Lambda(c_2,c_1)\right)^2 + 1}{\Lambda(c_2,c_1)} t\right) \geq \min\left(1, \overline{w_{c_2,c_1}}(t,x)\right) \text{ for all } x \in \mathbb{R};$$

(2) the support of $\underline{v_\delta}$ is a right half-line and there exists $(y,z) \in \mathbb{R}^2$ such that

$$\frac{1}{2} \leq e^{\lambda_v(c_1)(x-c_1 t)} \underline{v_\delta}(0, x-y) \leq 1 \text{ for all } t \geq 0 \text{ and } x \geq z;$$

(3) $\underline{u_\delta}(0,x) \leq \overline{u_\delta}(0,x)$ and $\underline{v_\delta}(0,x) \leq \overline{v_\delta}(0,x)$ for all $x \in \mathbb{R}$;

(4) the following spreading properties hold true:

$$\lim_{t\to+\infty} \sup_{x<(c_2-\varepsilon)t} \left|\underline{u_\delta}(t,x) - (1-a)\right| = 0 \text{ for all } \varepsilon \in (0, c_2),$$

$$\lim_{t\to+\infty} \sup_{\left(c_2^\delta+\varepsilon\right)t<x<(c_1-\varepsilon)t} \left(|\overline{u_\delta}(t,x)| + \left|\underline{v_\delta}(t,x) - (1-2\delta)\right|\right) = 0 \text{ for all } \varepsilon \in \left(0, \frac{c_1 - c_2^\delta}{2}\right).$$

$$\lim_{t\to+\infty} \sup_{(c_1+\varepsilon)t<x} \left(|\overline{u_\delta}(t,x)| + |\overline{v_\delta}(t,x)|\right) = 0 \text{ for all } \varepsilon > 0,$$

The forms of the super- and sub-solutions of Proposition 1.4 and Proposition 1.5 are illustrated in Figure 4.4 and Figure 4.3 respectively. Those of Proposition 1.6 are illustrated in Figure 4.1 and Figure 4.2.

1.2. **The quantities $f(c_2)$, $\lambda(c_2)$, $\Lambda(c_2,c_1)$.** Let us explain by a heuristic argument how these three quantities come out naturally in the problem and what is their ecological meaning.

Assume that v invades the uninhabited territory at some speed $c_1 \geq 2\sqrt{rd}$ and that u chases v at some speed $c_2 \in [c_{LLW}, c_1)$. In the area where $v \simeq 1$, u looks like the exponential tail of the monostable traveling wave connecting $(0,1)$ to $(1,0)$ at speed c_2, that is

$$u(t,x) \simeq \mathrm{e}^{-\lambda(c_2)(x-c_2 t)}.$$

Accordingly, in a neighborhood of $x = \tilde{c}t$ with $\tilde{c} \in (c_2, c_1)$, we can observe non-negligible quantities only if we consider the rescaled function

$$w : (t,x) \mapsto u(t,x)\,\mathrm{e}^{\lambda(c_2)(x-c_2 t)}$$

instead of u itself.

Yet, in a neighborhood of $x = \tilde{c}t$ with $\tilde{c} > c_1$, where $(u,v) \simeq (0,0)$, w satisfies at the first order

$$\partial_t w - \partial_{xx} w = \left(1 + \lambda(c_2)\left(\tilde{c} - c_2\right)\right) w$$

whence the exponential ansatz $w(t,x) = \mathrm{e}^{-\Lambda(x-\tilde{c}t)}$ leads to the equation

$$\Lambda^2 - \tilde{c}\Lambda + \left(1 + \lambda(c_2)\left(\tilde{c} - c_2\right)\right) = 0.$$

The minimal zero of this equation being precisely

$$\Lambda(c_2,\tilde{c}) = \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(\lambda(c_2)(\tilde{c}-c_2)+1)}\right),$$

we deduce then that $\tilde{c}$ has to satisfy

$$\tilde{c}^2 - 4(\lambda(c_2)(\tilde{c}-c_2)+1) \geq 0$$

that is $\tilde{c} \geq f(c_2)$. Passing to the limit $\tilde{c} \to c_1$, we find indeed $c_1 \geq f(c_2)$.

1.3. **Organization of the paper.** In Section 2, we recall the comparison principle for (1.1) and define super-solutions and sub-solutions.

In Section 3, we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 assuming Proposition 1.4, Proposition 1.5 and Proposition 1.6 are true.

In Section 4, we prove Proposition 1.4, Proposition 1.5 and Proposition 1.6. These constructions are rather delicate and require several objects and preliminary lemmas, which we summarize in a table at the beginning of Subsection 4.1.

In Section 5, we comment on the results and provide some future perspectives.

2. Competitive comparison principle

2.1. **Competitive comparison principle.** In what follows, vectors in $\mathbb{R}^2$ are always understood as column vectors.

We define the competitive ordering $\preceq$ in $\mathbb{R}^2$ as follows: for all $(u_1,v_1) \in \mathbb{R}^2$, $(u_2,v_2) \in \mathbb{R}^2$,

$$(u_1,v_1) \preceq (u_2,v_2) \text{ if } u_1 \leq u_2 \text{ and } v_1 \geq v_2.$$

The strict competitive ordering $\prec$ is defined by

$$(u_1,v_1) \prec (u_2,v_2) \text{ if } u_1 < u_2 \text{ and } v_1 > v_2.$$

We define also the operators

$$P : (u,v) \mapsto \partial_t(u,v) - \operatorname{diag}(1,d)\,\partial_{xx}(u,v),$$

$$F : (u,v) \mapsto \begin{pmatrix} u(1-u-av) \\ rv(1-v-bu) \end{pmatrix}.$$

With these notations, (1.1) can be written as

$$\begin{cases} P(u,v) = F(u,v) & \text{in } (0,+\infty)\times\mathbb{R} \\ (u,v)(0,x) = (u_0,v_0)(x) & \text{for all } x \in \mathbb{R} \end{cases}.$$

Definition 2.1. A classical super-solution of (1.1) is a pair

$$(\overline{u},\underline{v}) \in \mathscr{C}^1\left((0,+\infty),\mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)\right) \cap \mathscr{C}\left([0,+\infty)\times\mathbb{R},[0,1]^2\right)$$

satisfying

$$P(\overline{u},\underline{v}) \succeq F(\overline{u},\underline{v}) \text{ in } (0,+\infty)\times\mathbb{R}.$$

A classical sub-solution of (1.1) is a pair

$$(\underline{u},\overline{v}) \in \mathscr{C}^1\left((0,+\infty),\mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)\right) \cap \mathscr{C}\left([0,+\infty)\times\mathbb{R},[0,1]^2\right)$$

satisfying

$$P(\underline{u},\overline{v}) \preceq F(\underline{u},\overline{v}) \text{ in } (0,+\infty)\times\mathbb{R}.$$

The unbounded domain $(0,+\infty)\times\mathbb{R}$ can be replaced in the above definition by a bounded parabolic cylinder $(0,T)\times(-R,R)$. In such a case, the required regularity is $\mathscr{C}^1\left((0,T),\mathscr{C}^2\left((-R,R),[0,1]^2\right)\right) \cap \mathscr{C}\left([0,T]\times[-R,R],[0,1]^2\right)$.

We also recall that it is possible to extend the theory of super- and sub-solutions to Sobolev spaces. The full extension is outside of the scope of this reminder, however a very partial extension will be necessary later on. More precisely, we will use the following definition and theorem. In what follows, $\circ$ denotes the Hadamard product $(u_1, v_1) \circ (u_2, v_2) = (u_1 u_2, v_1 v_2)$.

Definition 2.2. A generalized super-solution of (1.1) is a pair

$$(\overline{u}, \underline{v}) \in \mathscr{C}^{0,1}\left((0,+\infty) \times \mathbb{R}, [0,1]^2\right)$$

satisfying, for all $(U,V) \in \mathscr{D}\left((0,+\infty) \times \mathbb{R}, [0,1]^2\right)$,

$$\int \partial_t (\overline{u}, \underline{v}) \circ (U,V) + \operatorname{diag}(1,d)\, \partial_x (\overline{u}, \underline{v}) \circ \partial_x (U,V) \succeq \int F(\overline{u}, \underline{v}) \circ (U,V).$$

A generalized sub-solution of (1.1) is a pair

$$(\underline{u}, \overline{v}) \in \mathscr{C}^{0,1}\left((0,+\infty) \times \mathbb{R}, [0,1]^2\right)$$

satisfying, for all $(U,V) \in \mathscr{D}\left((0,+\infty) \times \mathbb{R}, [0,1]^2\right)$,

$$\int \partial_t (\underline{u}, \overline{v}) \circ (U,V) + \operatorname{diag}(1,d)\, \partial_x (\underline{u}, \overline{v}) \circ \partial_x (U,V) \preceq \int F(\underline{u}, \overline{v}) \circ (U,V).$$

Again, the unbounded domain $(0,+\infty) \times \mathbb{R}$ can be replaced by a bounded parabolic cylinder $(0,T) \times (-R,R)$. The following important theorem, that will be used many times thereafter, actually uses the local definition.

Theorem 2.3. *Let $R > 0$, $T > 0$, $Q = (0,T) \times (-R,R)$ and*

$$(\overline{u}_1, \overline{u}_2, \underline{v}_1, \underline{v}_2) \in \mathscr{C}^1\left([0,T], \mathscr{C}^2\left([-R,R],[0,1]^4\right)\right) \cap \mathscr{C}\left([0,T] \times [-R,R], [0,1]^4\right).$$

1. *Assume that $(\overline{u}_1, \underline{v}_1)$ and $(\overline{u}_1, \underline{v}_2)$ are both classical super-solutions in Q. Then $(\overline{u}_1, \max(\underline{v}_1, \underline{v}_2))$ is a generalized super-solution in Q.*
2. *Assume that $(\overline{u}_1, \underline{v}_1)$ and $(\overline{u}_2, \underline{v}_1)$ are both classical super-solutions in Q. Then $(\min(\overline{u}_1, \overline{u}_2), \underline{v}_1)$ is a generalized super-solution in Q.*

Remark. We state this theorem in a bounded parabolic cylinder in order to be able to construct later on more complex super- and sub-solutions, for instance super-solutions $(\overline{u}, \underline{v})$ with $\overline{u}$ of the form

$$\overline{u}(t,x) = \begin{cases} \overline{u}_1(t,x) & \text{if } x < x(t) \\ \overline{u}_2(t,x) & \text{if } x \in [x(t), y(t)] \\ \overline{u}_3(t,x) & \text{if } x > y(t) \end{cases},$$

where $x(t) < y(t)$ and $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_3$ are such that $\overline{u}_1(t,x) \leq \overline{u}_2(t,x)$ if $x < x(t)$, $\overline{u}_2(t,x) \leq \overline{u}_1(t,x)$ in a right-sided neighborhood of $x(t)$, $\overline{u}_2(t,x) \leq \overline{u}_3(t,x)$ in a left-sided neighborhood of $y(t)$ and $\overline{u}_3(t,x) \leq \overline{u}_2(t,x)$ if $x > y(t)$. Although we do not have any global information on $\overline{u}_1 - \overline{u}_2$, $\overline{u}_1 - \overline{u}_3$ and $\overline{u}_2 - \overline{u}_3$, the local theorem shows that the construction is still valid.

Proof. Since the second statement is proved similarly, we only prove the first one.

For simplicity, we only consider the special case where $\Gamma = (\underline{v}_1 - \underline{v}_2)^{-1}(\{0\})$ is a smooth hypersurface, which is always satisfied for our purposes. A proof that does not require such a regularity assumption can be found for instance in [32].

Define $\underline{v} = \max(\underline{v}_1, \underline{v}_2)$ and let $(U,V) \in \mathscr{D}\left(\overline{Q}, [0,1]^2\right)$. On one hand,

$$\partial_t \overline{u}_1 - \partial_{xx}\overline{u}_1 \geq F_1(\overline{u}_1, \underline{v})$$

is satisfied in the classical sense (using for instance $-a\overline{u}_1\underline{v}_1 \geq -a\overline{u}_1\underline{v}$). On the other hand, we have assumed that $\Gamma = (\underline{v}_1 - \underline{v}_2)^{-1}(\{0\})$ is a smooth hypersurface, so that we may integrate by parts. Denoting $Q_1 = (\underline{v}_1 - \underline{v}_2)^{-1}([0,1])$, $Q_2 = (\underline{v}_2 - \underline{v}_1)^{-1}([0,1])$, ν the outward unit normal to Q_1, we find $\Gamma = \partial Q_1 \backslash \partial Q = \partial Q_2 \backslash \partial Q$ and $(\partial_x \underline{v}_1 - \partial_x \underline{v}_2)\nu \leq 0$ on Γ, whence

$$\begin{aligned}
&\int_Q \partial_t \underline{v} V + d\partial_x \underline{v} \partial_x V \\
&= \int_{Q_1} \partial_t \underline{v}_1 V + d\partial_x \underline{v}_1 \partial_x V + \int_{Q_2} \partial_t \underline{v}_2 V + d\partial_x \underline{v}_2 \partial_x V \\
&= \int_{Q_1} (\partial_t \underline{v}_1 - d\partial_{xx}\underline{v}_1) V + \int_{\partial Q_1} \partial_x \underline{v}_1 V \nu + \int_{Q_2} (\partial_t \underline{v}_2 - d\partial_{xx}\underline{v}_2) V + \int_{\partial Q_2} \partial_x \underline{v}_1 V (-\nu) \\
&\leq \int_{Q_1} F_2(\overline{u}_1, \underline{v}_1) V + \int_{Q_2} F_2(\overline{u}_1, \underline{v}_2) V + \int_\Gamma (\partial_x \underline{v}_1 - \partial_x \underline{v}_2) V \nu \\
&\leq \int_Q F_2(\overline{u}_1, \underline{v}) V.
\end{aligned}$$

This completes the proof. □

An inversion of the roles yields a similar statement on sub-solutions.

Theorem 2.4. *Let $R > 0$, $T > 0$, $Q = (0,T) \times (-R,R)$ and*

$$(\underline{u}_1, \underline{u}_2, \overline{v}_1, \overline{v}_2) \in \mathscr{C}^1\left([0,T], \mathscr{C}^2\left([-R,R],[0,1]^4\right)\right) \cap \mathscr{C}\left([0,T] \times [-R,R], [0,1]^4\right).$$

(1) Assume that $(\underline{u}_1, \overline{v}_1)$ and $(\underline{u}_1, \overline{v}_2)$ are both classical sub-solutions in Q. Then $(\underline{u}_1, \min(\overline{v}_1, \overline{v}_2))$ is a generalized sub-solution in Q.

(2) Assume that $(\underline{u}_1, \overline{v}_1)$ and $(\underline{u}_2, \overline{v}_1)$ are both classical sub-solutions in Q. Then $(\max(\underline{u}_1, \underline{u}_2), \overline{v}_1)$ is a generalized sub-solution in Q.

Since a classical super- or sub-solution is *a fortiori* a generalized super- or sub-solution respectively, from now on, we omit the adjectives classical and generalized and always have in mind the generalized notion.

The comparison principle for (1.1), directly derived from the comparison principle for cooperative systems (see Protter–Weinberger [30]) via the transformation $w = 1 - v$, reads as follows.

Theorem 2.5. *Let $(\overline{u}, \underline{v})$ and $(\underline{v}, \overline{u})$ be respectively a super-solution and a sub-solution of (1.1). Assume that*

$$(\overline{u}, \underline{v})(0,x) \succeq (\underline{u}, \overline{v})(0,x) \text{ for all } x \in \mathbb{R}.$$

Then

$$(\overline{u}, \underline{v}) \succeq (\underline{u}, \overline{v}) \text{ in } [0,+\infty) \times \mathbb{R}.$$

Furthermore, if there exists $(T,x) \in (0,+\infty) \times \mathbb{R}$ such that $\overline{u}(T,x) = \underline{u}(T,x)$ or $\underline{v}(T,x) = \overline{v}(T,x)$, then

$$(\overline{u}, \underline{v}) = (\underline{u}, \overline{v}) \text{ in } [0,T] \times \mathbb{R}.$$

In other words, $(\overline{u},\underline{v}) \succ (\underline{u},\overline{v})$ holds at $t=0$ if and only if it holds at all $t \geq 0$.

Finally, we recall an important existence result that will be used later on.

Theorem 2.6. *Let $(\overline{u},\underline{v})$ and $(\underline{v},\overline{u})$ be respectively a super-solution and a sub-solution of (1.1). Assume that for some $(u_0,v_0) \in C(\mathbb{R},[0,1]^2)$ we have*

$$(\overline{u},\underline{v})(0,x) \succeq (u_0,v_0)(x) \succeq (\underline{u},\overline{v})(0,x) \quad \text{for all } x \in \mathbb{R},$$

then the solution (u,v) of (1.1) with initial data (u_0,v_0) satisfies

$$(\overline{u},\underline{v}) \succeq (u,v) \succeq (\underline{u},\overline{v}) \text{ in } [0,+\infty)\times\mathbb{R}.$$

3. Proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In this section, we assume Proposition 1.4, Proposition 1.5 and Proposition 1.6 are true.

3.1. **Proof of Theorem 1.2.**

Proof. Let $c_1 \geq 2\sqrt{rd}$ and $c_2 \geq c_{LLW}$ such that $c_1 \geq c_2$ and $c_1 < f(c_2)$.

First, we consider the case where $x \mapsto v_0(x)\,\mathrm{e}^{\lambda_v(c_1)x}$ is globally bounded. By contradiction, assume the existence of a solution (u,v) such that both the boundedness of $x \mapsto v_0(x)\,\mathrm{e}^{\lambda_v(c_1)x}$ and the equality

$$c_2 = \sup\left\{c>0 \mid \lim_{t\to+\infty}\sup_{0\leq x\leq ct}\left(|u(t,x)-1|+|v(t,x)|\right)=0\right\}$$

are true.

Define c, $\tilde{c}$, $\delta^\star$, $\delta=\frac{\delta^\star}{2}$, R_δ, L, $\underline{u_{\delta,\zeta,\kappa}}$, $\overline{v_{\delta,\eta}}$ as in Proposition 1.4. Note that $c>c_2$, $\tilde{c}>c_1$.

In view of the equality satisfied by c_2, there exists $T \geq \frac{2L}{c_2}$ such that, for all $x \in \left[0,\frac{c_2}{2}T\right]$, $u(T,x) \geq 1-\frac{a}{2}$.

We claim that $(t,x) \mapsto v(t,x)\,\mathrm{e}^{\lambda_v(c_1)(x-c_1t)}$ is globally bounded in $[0,+\infty)\times\mathbb{R}$. To see this, it suffices to observe that, by definition of $\lambda_v(c_1)$, $Ce^{-\lambda_v(c_1)(x-c_1t)}$ is a supersolution of the equation of v for any constant $C>0$. Hence standard comparison implies that

$$v(t,x) \leq \left(\sup_{x\in\mathbb{R}} v(0,x)e^{\lambda_v(c_1)x}\right)e^{-\lambda_v(c_1)(x-c_1t)}.$$

Since $c_1<\tilde{c}$ and λ_v is decreasing, we have $\lambda_v(c_1)>\lambda_v(\tilde{c})$. Hence, there exists $\zeta>L$ such that

$$v(T,x) \leq \overline{v_{\delta,\zeta}}(0,x) \text{ for all } x\in\mathbb{R}.$$

Now, we fix

$$\kappa = \frac{1}{2}\min\left(\min\left(\frac{1-a}{2},\frac{\delta}{2}\right), \min_{x\in[L,L+\zeta+2R_\delta]} u(T,x)\right).$$

It follows that

$$u(T,x) \geq \begin{cases} 1-a & \text{for } x\in[0,L],\\ \kappa & \text{for } x\in(L,L+\zeta+2R_\delta],\\ 0 & \text{for } x\in\mathbb{R}\setminus[0,L+\zeta+2R_\delta],\end{cases}$$

whence $u(T,x) \geq \underline{u_{\delta,\zeta,\kappa}}(0,x)$ for $x\in\mathbb{R}$. Then

$$(\underline{u},\overline{v}) : (t,x) \mapsto \left(\underline{u_{\delta,\zeta,\kappa}}(t-T,x),\overline{v}(t-T,x)\right)$$

is a sub-solution of (1.1) which satisfies $(\underline{u},\overline{v}) \preceq (u,v)$ at $t=T$, whence by the comparison principle of Theorem 2.5 it satisfies the same inequality at any time $t \geq T$.

Now, due to the spreading property satisfied by $\underline{u}$, for all $\varepsilon \in (0,c)$, there exists $T_\varepsilon \geq T$ such that, for all $t \geq T_\varepsilon$,

$$\inf_{L \leq x < (c-\varepsilon)t} u(t,x) \geq \frac{1-a}{4}.$$

Thus, by comparison of (u,v) with the spatially homogeneous sub-solution $\left(\underline{U},\overline{V}\right)$ satisfying the system

$$\begin{cases} \underline{U}'(t) = \underline{U}(t)\left(1-\underline{U}(t)-a\overline{V}(t)\right) & \text{for all } t \in (T_\varepsilon,+\infty) \\ \overline{V}'(t) = r\overline{V}(t)\left(1-\overline{V}(t)-b\underline{U}(t)\right) & \text{for all } t \in (T_\varepsilon,+\infty) \\ \underline{U}(T_\varepsilon) = \frac{1-a}{4} \\ \overline{V}(T_\varepsilon) = 1 \end{cases},$$

whose convergence to $(1,0)$ is well-known, we find

$$\lim_{t\to+\infty} \sup_{0 \leq x < (c-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in (0,c).$$

This means $c_2 \geq c$, and directly contradicts the chose of $c > c_2$ made at the beginning of the proof.

Next, we consider the case where

$$c_1 \geq \inf\left\{ c>0 \mid \lim_{t\to+\infty} \sup_{x \geq ct} |v(t,x)| = 0 \right\}.$$

Since the proof is mostly the same, we only sketch it. Again, we argue by contradiction and use Proposition 1.4. Using the assumption on the spreading of v, we can establish the following estimate:

$$v(t,x-\hat{c}t) \leq \mathbf{1}_{y \leq y_0}(x-\hat{c}t) + \frac{\delta}{2a}\mathbf{1}_{y \geq y_0}(x-\hat{c}t),$$

for some $y_0 \in \mathbb{R}$ and with $\hat{c} = \frac{\tilde{c}+c_1}{2}$. Thanks to this, we can directly use $\eta \underline{u}_{\delta,\zeta,\kappa}$, for some small $\eta > 0$, as sub-solution for u and deduce a contradiction. We point out that in this case, we do not use the competitive comparison principle but instead use the scalar one. □

3.2. **Proof of Theorem 1.1.**

3.2.1. *Hair-trigger effect and extinction.*

Proposition 3.1. *Let $u_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with support included in a left half-line and $v_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with compact support. Let (u,v) be the solution of 1.1.*

(1) If $2\sqrt{rd} > 2$, then

$$\lim_{t\to+\infty} \sup_{0 \leq x < (c_{LLW}-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in (0, c_{LLW}),$$

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t < x} (|u(t,x)| + |v(t,x)|) = 0 \text{ for all } \varepsilon > 0,$$

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t < x < \left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)| + |v(t,x)-1|) = 0 \text{ for all } \varepsilon \in \left(0, \frac{2\sqrt{rd}-c_{LLW}}{2}\right).$$

(2) If $2\sqrt{rd} < 2$, *then*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |v(t,x)| = 0,$$

$$\lim_{t\to+\infty} \sup_{0\le x<(2-\varepsilon)t} |u(t,x) - 1| = 0 \text{ for all } \varepsilon \in (0,2),$$

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t<x} |u(t,x)| = 0 \text{ for all } \varepsilon > 0.$$

Remark. The inequality regarding $(2+\varepsilon)t < x < \left(2\sqrt{rd} - \varepsilon t\right)$ is by far the more interesting and the less trivial. It basically means that u does not exert any competition far ahead of its own territory. It was first proved by Ducrot, Giletti and Matano [8] in the case of predator–prey interactions (the conclusion being then that no predation occurs far ahead of the territory of the predator), and by Lin and Li [26] in case of two-species competition (the conclusion being that the region of coexistence falls behind the territory where the faster diffuser dominates). The proof of Ducrot *et al.* was sufficiently robust and generic to be reused by Carrère [3] in the bistable competitive case and to be reused again here, in the monostable case. Although it would certainly be interesting to write the result of Ducrot *et al.* in the most general form possible (with more than two species and minimal assumptions on the interactions), this is far beyond the scope of this paper. Therefore we simply adapt the main idea of their proof.

Proof. First, applying the comparison principle with the solution of

$$\begin{cases} \partial_t u_{KPP} - \partial_{xx} u_{KPP} = u_{KPP}(1 - u_{KPP}) & \text{in } (0,+\infty)\times\mathbb{R} \\ u_{KPP}(0,x) = u_0(x) & \text{for all } x \in \mathbb{R} \end{cases},$$

we find directly $u \le u_{KPP}$, whence

$$\lim_{t\to+\infty} \sup_{x>(2+\varepsilon)t} u(t,x) = 0 \text{ for all } \varepsilon > 0.$$

Similarly,

$$\lim_{t\to+\infty} \sup_{x>\left(2\sqrt{rd}+\varepsilon\right)t} v(t,x) = 0 \text{ for all } \varepsilon > 0.$$

Furthermore, (u,v) satisfies also $(u,v) \succeq (u_{LLW}, v_{LLW})$, where (u_{LLW}, v_{LLW}) is the solution of (1.1) with initial data $(u_0, 1)$, and by Lewis–Li–Weinberger [22], this yields

$$\lim_{t\to+\infty} \sup_{0\le x<(c_{LLW}-\varepsilon)t} |u(t,x) - 1| + |v(t,x)| = 0 \text{ for all } \varepsilon \in (0, c_{LLW}).$$

Next, let us prove that if $2\sqrt{rd} < 2$ and provided

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t<x<(2-\varepsilon)t} (|u(t,x) - 1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in \left(0, \frac{2-2\sqrt{rd}}{2}\right),$$

then in fact the above limit can be reinforced as

$$\lim_{t\to+\infty} \sup_{0\le x<(2-\varepsilon)t} (|u(t,x) - 1| + |v(t,x)|) = 0 \text{ for all } \varepsilon \in (0,2).$$

Let $\varepsilon \in \left(0, \frac{2-2\sqrt{rd}}{3}\right)$. It is well-known (see Du–Lin [4, 5]) that there exists a unique solution of

$$\begin{cases} -\underline{\varphi}'' = \underline{\varphi}\left(1-a-\underline{\varphi}\right) & \text{in } (0,+\infty) \\ \underline{\varphi}(0) = 0 & \\ \underline{\varphi}(x) > 0 & \text{for all } x > 0 \end{cases}.$$

Furthermore, this solution is increasing in $(0,+\infty)$ and converges to $1-a$ at $+\infty$. In view of the assumption on the limit of (u,v) in $\left(2\sqrt{rd}+\varepsilon\right)t < x < (2-\varepsilon)t$, there exists $T \geq 0$ and $x_0 > 0$ such that

$$u(t, y+(2-2\varepsilon)t) > 1-a > \underline{\varphi}(y+(2-2\varepsilon)t - x_0) \text{ for } (y,t) \in \{0\} \times [T,+\infty),$$

$$u(t, y+(2-2\varepsilon)t) \geq \underline{\varphi}(y+(2-2\varepsilon)t - x_0) \text{ for } (y,t) \in (-\infty, 0] \times \{T\},$$

$$\underline{\varphi}(y+(2-2\varepsilon)t - x_0) > 0 \text{ when } (y,t) = (0,T).$$

Let $\tilde{u}(t,y) = u(t, y+(2-2\varepsilon)t)$ and $\underline{\tilde{u}}(t,y) = \underline{\varphi}(y+(2-2\varepsilon)t - x_0)$. Then they satisfy for all $(t,y) \in (0,+\infty) \times \mathbb{R}$,

$$\partial_t \underline{\tilde{u}} - \partial_{xx}\underline{\tilde{u}} - (2-2\varepsilon)\partial_x \underline{\tilde{u}} - \underline{\tilde{u}}(1-\underline{\tilde{u}} - av) \leq 0 = \partial_{xx}\tilde{u} - (2-2\varepsilon)\partial_x \tilde{u} - \tilde{u}(1-\tilde{u}-av),$$

it follows by virtue of the scalar comparison principle and of a change of variable $y = x - (2-2\varepsilon)t$ that

$$\underline{u}(t,x) \leq u(t,x) \text{ for all } t \geq T \text{ and } x \leq (2-2\varepsilon)t.$$

Consequently,

$$\liminf_{t\to+\infty} \inf_{\varepsilon t < x < (2-2\varepsilon)t} u(t,x) \geq 1-a,$$

whence there exists $T' \geq 0$ such that

$$\inf_{\varepsilon t < x < (2-2\varepsilon)t} u(t,x) \geq \frac{1-a}{2} > 0 \text{ for all } t \geq T'.$$

By virtue of Theorem 2.5, the solution (U,V) of (1.1) with constant initial values $\left(\frac{1-a}{2}, 1\right)$ satisfies

$$(U,V)(t-T', x) \preceq (u,v)(t,x) \text{ for all } t \geq T' \text{ and } \varepsilon t < x < (2-2\varepsilon)t.$$

Since (U,V) coincides with the solution of the ODE system

$$\begin{cases} U' = U(1-U-aV) \\ V' = rV(1-V-bU) \\ (U,V)(0) = \left(\frac{1-a}{2}, 1\right) \end{cases},$$

standard theory on such systems shows that (U,V) converges to $(1,0)$, whence (u,v) itself converges to $(1,0)$ uniformly in $\varepsilon t < x < (2-2\varepsilon)t$. Recalling that we also have the estimate

$$\lim_{t\to+\infty} \sup_{0 \leq x < (c_{LLW}-\varepsilon)t} |u(t,x)-1| + |v(t,x)| = 0 \text{ for all } \varepsilon \in (0, c_{LLW}),$$

the claim is proved.

It now remains to prove the most difficult part, namely

$$\lim_{t\to+\infty} \sup_{(2+\varepsilon)t < x < \left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)| + |v(t,x)-1|) = 0, \quad \text{if } 2\sqrt{rd} > 2,$$

and

$$\lim_{t\to+\infty} \sup_{\left(2\sqrt{rd}+\varepsilon\right)t < x < (2-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0, \quad \text{if } 2\sqrt{rd} < 2.$$

Since this is a symmetric statement and since the forthcoming proof does not rely upon the assumptions $a < 1$ and $b > 1$, we only do the case $2\sqrt{rd} > 2$ (when v spreads faster than u) and the proof will be valid for the other case (when u spreads faster than v) as well.

Step 1: Let $\bar{u} : (t,x) \mapsto \min\left(1, e^{-(x-2t-x_1)}\right)$, where x_1 is chosen such that $\bar{u}(0,x) \geq u(0,x)$ for all $x \in \mathbb{R}$. Then, by standard scalar comparison,

$$u(t,x) \leq \bar{u}(t,x) \qquad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Step 2: We show that for each $c \in (2, 2\sqrt{rd})$, there exist positive constants δ, x_2, η_1, R such that

(3.1)

$$v(t', x + x_2 + ct) \geq \eta_1 \qquad \text{for all } t \geq 1, x \in (-2R, 2R) \text{ and } t' \in [(1-\delta)t, (1+\delta)t].$$

To show (3.1), fix $c \in \left(2, 2\sqrt{dr}\right)$ and fix δ so small that

$$2 < \frac{c}{1+\delta} < \frac{c}{1-\delta} < 2\sqrt{dr}.$$

Let $\eta > 0$, $R > 0$, $x_2 \in \mathbb{R}$, $\tilde{c} \in \left[\frac{c}{1+\delta}, \frac{c}{1-\delta}\right]$ and define

$$\underline{v}^{\tilde{c}} : (t,x) \mapsto \eta e^{-\frac{\tilde{c}}{2d}(x-\tilde{c}t)} \psi_{4R}(x - \tilde{c}t - x_2),$$

where $(\lambda_{4R}, \psi_{4R})$ is the Dirichlet principal eigenpair defined by

$$\begin{cases} -d\psi_{4R}'' = \lambda_{4R}\psi_{4R} & \text{in } (-4R, 4R) \\ \psi_{4R}(\pm 4R) = 0 & \\ \psi_{4R}(x) > 0 & \text{for all } x \in (-4R, 4R) \\ \max \psi_{4R} = 1 & \end{cases}.$$

The principal eigenvalue λ_{4R} is positive, vanishes as $R \to +\infty$ and ψ_{4R} is extended into $\mathbb{R}$ by setting $\psi_{4R}(x) = 0$ if $|x| > 4R$.

Obviously,

$$\partial_t \underline{v}^{\tilde{c}} - d\partial_{xx}\underline{v}^{\tilde{c}} - r\underline{v}^{\tilde{c}}\left(1 - \underline{v}^{\tilde{c}} - bu\right) \leq \partial_t \underline{v}^{\tilde{c}} - d\partial_{xx}\underline{v}^{\tilde{c}} - r\underline{v}^{\tilde{c}}\left(1 - \underline{v}^{\tilde{c}} - b\overline{u}\right),$$

whence the right-hand side above divided by $\eta e^{-\frac{\tilde{c}}{2d}(x-\tilde{c}t)}$ is *a fortiori* smaller than or equal to

$$\begin{aligned} &\frac{\tilde{c}^2}{2d}\psi_{4R} - \tilde{c}\psi_{4R}' - d\left(\psi_{4R}'' + \frac{\tilde{c}^2}{4d^2}\psi_{4R} - \frac{\tilde{c}}{d}\psi_{4R}'\right) - r\psi_{4R}\left(1 - \underline{v}^{\tilde{c}} - b\overline{u}\right) \\ &\leq \left(\frac{\tilde{c}^2}{4d} + \lambda_{4R} - r + r\left(\underline{v}^{\tilde{c}} + b\overline{u}\right)\right)\psi_{4R} \\ &\leq \left(\lambda_{4R} + r\left(\underline{v}^{\tilde{c}} + b\overline{u} - \gamma\right)\right)\psi_{4R}, \end{aligned}$$

where the last inequality holds provided we choose the constant $\gamma > 0$ so small that

$$2\sqrt{r(1-\gamma)d} > \frac{c}{1-\delta} \geq \tilde{c}.$$

Therefore, by choosing R so large that $\lambda_{4R} < r\frac{\gamma}{4}$, x_2 so large that

$$\bar{u}(t,x) \leq \frac{\gamma}{4b} \text{ for all } t \geq 0 \text{ and } x \geq 2t + x_2 - 4R$$

(which is possible by Step 1), and η so small that

$$\eta \sup_{\tilde{c} \in \left[\frac{c}{1+\delta}, \frac{c}{1-\delta}\right]} \sup_{\xi \in (-4R+x_2, 4R+x_2)} \left(e^{-\frac{\tilde{c}}{2d}\xi}\psi_{4R}(\xi - x_2)\right) \leq \frac{\gamma}{4},$$

$$\eta \sup_{\hat{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]} \left(\mathrm{e}^{-\frac{\hat{c}}{2d}(x-\hat{c})}\psi_{4R}\left(x-\hat{c}-x_2\right)\right) \leq v\left(1,x\right) \text{ for all } x\in\mathbb{R}$$

we deduce that $\underline{v}^{\tilde{c}}$ is a sub-solution for the single parabolic equation satisfied by v.

By scalar comparison, $v(t,x)\geq \underline{v}^{\tilde{c}}(t,x)$ for all $t\geq 1$ and $x\in\mathbb{R}$. It follows then that

$$\begin{aligned} v\left(\frac{c}{\tilde{c}}t, x+x_2+ct\right) &\geq \underline{v}^{\tilde{c}}\left(\frac{c}{\tilde{c}}t, x+x_2+ct\right) \\ &= \eta e^{-\frac{\tilde{c}}{2d}(x+x_2)}\psi_{4R}(x) \\ &\geq \eta e^{-\frac{\tilde{c}}{2d}x_2}e^{-\frac{\tilde{c}}{d}R}\min_{[-2R,2R]}\psi_{4R} \end{aligned}$$

for all $t\geq 1$ and $x\in[-2R,2R]$. Noticing that the last expression on the right-hand side above is constant and denoting

$$\eta_1 = \eta \min_{[-2R,2R]}\psi_{4R} \inf_{\hat{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]} \left(e^{-\frac{\hat{c}}{2d}x_2}e^{-\frac{\hat{c}}{d}R}\right),$$

we may take the infimum over all $\tilde{c}\in\left[\frac{c}{1+\delta},\frac{c}{1-\delta}\right]$ and obtain indeed (3.1).

Step 3: We are now in position to show that, for any small $\varepsilon>0$,

$$\lim_{t\to+\infty}\sup_{(2+\varepsilon)t<x<(2\sqrt{rd}-\varepsilon)t}\left|v\left(t,x\right)-1\right| = 0.$$

Assume by contradiction the existence of sequences $(t_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ and of $c\in(2,2\sqrt{rd})$ such that, as $n\to+\infty$, $t_n\to+\infty$, $\frac{x_n}{t_n}\to c$ and $\limsup v\left(t_n,x_n\right)<1$.

Denote $(c_n)_{n\in\mathbb{N}} = \left(\frac{x_n}{t_n}\right)_{n\in\mathbb{N}}$, assume without loss of generality that $\left|\frac{c}{c_n}-1\right| < \delta/2$, where $\delta=\delta(c)$ is specified in Step 2.

For all $n\in\mathbb{N}$, define $\tau_n = \frac{c_n}{c}t_n = \frac{x_n}{c}$ and

$$v_n : (t,x)\mapsto v\left(t+\frac{c}{c_n}\tau_n, x+x_2+c\tau_n\right).$$

By Step 2 (with $t'=t+\frac{c}{c_n}\tau_n$ and $t=\tau_n$), we deduce that

$$v_n(t,x)\geq\eta_1 \text{ if } |x|\leq 2R \text{ and } \left|t+\left(\frac{c}{c_n}-1\right)\tau_n\right| < \delta\tau_n,$$

and hence (using $\left|\frac{c}{c_n}-1\right|<\frac{\delta}{2}$) if $|x-x_2|\leq 2R$ and $|t|<\frac{\delta}{2}\tau_n$.

By classical parabolic estimates (see Lieberman [25]), $(v_n)_{n\in\mathbb{N}}$ converges up to a diagonal extraction in $\mathscr{C}_{loc}\left(\mathbb{R}^2,[0,1]\right)$ to a limit v_∞ which satisfies (using the fact that $u(t+\frac{c}{c_n}\tau_n, x+c\tau_n)\to 0$ in $\mathscr{C}_{loc}\left(\mathbb{R}^2,[0,1]\right)$ by Step 1, since $(c\tau_n)/(\frac{c}{c_n}\tau_n) = c_n\geq\frac{c}{1+\delta}>2$ for all n)

$$\partial_t v_\infty - d\partial_{xx}v_\infty - rv_\infty\left(1-v_\infty\right) = 0 \text{ in } \mathbb{R}^2$$

and, in view of the above estimates,

$$v_\infty\left(t,x+x_2\right)\geq\eta_1 \text{ for all } t\in\mathbb{R} \text{ and } x\in[-2R,2R].$$

By standard classification of the entire solutions of the KPP equation, this implies $v_\infty\equiv 1$. In particular,

$$v(t_n,x_n) = v\left(\frac{c}{c_n}\tau_n, c\tau_n\right) = v_n(0,-x_2)\to 1.$$

This directly contradicts $\limsup v(t_n, x_n) < 1$. □

In view of Proposition 3.1, in order to prove Theorem 1.1, we only have to prove that for each sufficiently small $\varepsilon > 0$,

$$\lim_{t\to+\infty} \sup_{x<(c^\star-\varepsilon)t} (|u(t,x)-1| + |v(t,x)|) = 0,$$

$$\lim_{t\to+\infty} \sup_{(c^\star+\varepsilon)t<x<\left(2\sqrt{rd}-\varepsilon\right)t} (|u(t,x)| + |v(t,x)-1|) = 0,$$

where

$$c^\star = \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right).$$

3.2.2. *Proof of Theorem 1.1.* We begin with an algebraic lemma.

Lemma 3.2. *Let $c_2 \geq 2\sqrt{1-a}$ and $c_1 > c_2$ such that $c_1 \geq f(c_2)$. Then*

$$\frac{(\Lambda(c_2,c_1))^2 + 1}{\Lambda(c_2,c_1)} < c_1.$$

Proof. First, $\Lambda(c_2,c_1)$ is well-defined as $c_1 \geq \max\{c_2, f(c_2)\}$. Noticing that

$$(\Lambda(c_2,c_1))^2 - c_1\Lambda(c_2,c_1) + \lambda(c_2)(c_1-c_2) + 1 = 0,$$

we find that the claimed inequality is equivalent to $-\lambda(c_2)(c_1-c_2) < 0$. □

Now we prove the remaining part of Theorem 1.1. Assume $2\sqrt{rd} > 2$, define $c^\star$ as above, fix $u_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with support included in a left half-line and $v_0 \in \mathscr{C}(\mathbb{R},[0,1]) \setminus \{0\}$ with compact support and let (u,v) be the solution of equation (1.1).

Proof. By virtue of Theorem 1.2 and Proposition 3.1,

$$\sup\left\{c > 0 \mid \lim_{t\to+\infty} \sup_{x\leq ct} (|u(t,x)-1| + |v(t,x)|) = 0\right\} \geq c^\star.$$

It remains to verify that the quantity

$$\overline{c} = \inf\left\{0 < c < 2\sqrt{rd} \mid \lim_{t\to+\infty} \sup_{ct\leq x\leq \frac{2\sqrt{rd}+c}{2}t} (|u(t,x)| + |v(t,x)-1|) = 0\right\}$$

satisfies $\overline{c} \leq c^\star$. Notice that by Proposition 3.1, $\overline{c} \leq 2$.

Assume by contradiction $\overline{c} \in (c^\star, 2]$ and let $c_2 \in (c^\star, \overline{c})$. Define $\delta^\star$ as in Proposition 1.5, let $\delta \in (0, \delta^\star)$ so small that $c_2^\delta < \overline{c}$ and define subsequently $(\overline{u}_\delta, \underline{v}_\delta)$.

By standard comparison,

$$u(t,x) \leq \min\left(1, \overline{w_{c_2,2\sqrt{rd}}}(t,x)\right) \text{ for all } (t,x) \in (0,+\infty)\times\mathbb{R}.$$

By virtue of Proposition 1.5, there exists $y_0 \in \mathbb{R}$ such that, for all $y \geq y_0$ and $t \geq 0$,

$$\overline{u_\delta}\left(0, x-y-\frac{\left(\Lambda\left(c_2,2\sqrt{rd}\right)\right)^2+1}{\Lambda\left(c_2,2\sqrt{rd}\right)}t\right) \geq \min\left(1, \overline{w_{c_2,2\sqrt{rd}}}(t,x)\right) \text{ for all } x \in \mathbb{R},$$

Since

$$c_2 < \overline{c} \leq 2 < 2\sqrt{rd},$$

$$c_2 > c^\star = \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right) \geq \max\left(2\sqrt{1-a}, f^{-1}\left(2\sqrt{rd}\right)\right),$$

Lemma 3.2 yields

$$\frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} < 2\sqrt{rd}.$$

Choose $c > 0$ such that

$$\max\left(\frac{1}{2}\left(2\sqrt{rd} + \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)}\right), 2\right) < c < 2\sqrt{rd}.$$

By virtue of Proposition 1.5, $x \mapsto \underline{v_\delta}\left(0, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} t\right)$ is compactly supported for all $y \geq y_0$ and $t \geq 0$. Since also $2 < c < 2\sqrt{rd}$, by virtue of Proposition 3.1, there exists $T_0 \geq 0$ such that, for all $T \geq T_0$,

$$\underline{v_\delta}\left(0, x - cT\right) \leq v\left(T, x\right) \text{ for all } x \in \mathbb{R}.$$

Now, relating the parameters y and T as follows,

$$cT = y + \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T, \quad \text{where we have } c > \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)},$$

we find the existence of $y \geq y_0$ and $T \geq T_0$ such that

$$\underline{v_\delta}\left(0, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T\right) \leq v\left(T, x\right) \text{ for all } x \in \mathbb{R}.$$

Then

$$(\overline{u}, \underline{v}) : (t, x) \mapsto \left(\overline{u_\delta}, \underline{v_\delta}\right)\left(t - T, x - y - \frac{\left(\Lambda\left(c_2, 2\sqrt{rd}\right)\right)^2 + 1}{\Lambda\left(c_2, 2\sqrt{rd}\right)} T\right)$$

is a super-solution of (1.1) which satisfies $(u, v) \preceq (\overline{u}, \underline{v})$ at $t = T$, whence by the comparison principle of Theorem 2.5 it satisfies the same inequality at any time $t \geq T$.

A contradiction follows from Proposition 1.5 and $c_2^\delta < \bar{c}$, as in the proof of Theorem 1.2. □

3.3. **Proof of Theorem 1.3.** Let $c_1 > 2\sqrt{rd}$, $c_2 > c_{LLW}$ and assume $c_1 > \max\left(c_2, f\left(c_2\right)\right)$.

Proof. Fix $\delta^\star$, $\delta = \frac{\delta^\star}{2}$ and c_2^δ, and define the super- and sub-solutions $(\overline{u_\delta}, \underline{v_\delta})$ and $(\underline{u_\delta}, \overline{v_\delta})$ as in Proposition 1.6.

First, let $(u_0, v_0) \in \mathscr{C}\left(\mathbb{R}, [0,1]^2\right)$ be a pair such that

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(0, x) \preceq \left(u_0, v_0\right)(x) \preceq \left(\overline{u_\delta}, \underline{v_\delta}\right)(0, x) \text{ for all } x \in \mathbb{R}$$

and satisfying also

$$u_0\left(x\right) \leq \overline{w_{c_2, c_1}}\left(0, x\right) \text{ for all } x \in \mathbb{R}.$$

By virtue of Theorem 2.6, there exists a (unique) solution (u, v) of (1.1) such that

$$(u,v)(0,x) = (u_0,v_0)(x) \text{ for all } x \in \mathbb{R},$$

$$\left(\underline{u_\delta}, \overline{v_\delta}\right)(t,x) \preceq (u,v)(t,x) \preceq \left(\overline{u_\delta}, \underline{v_\delta}\right)(t,x) \text{ for all } t \in (0,+\infty) \text{ and } x \in \mathbb{R},$$

$$u(t,x) \leq \min\left(1, \overline{w_{c_2,c_1}}(t,x)\right) \text{ for all } t \in (0,+\infty) \text{ and } x \in \mathbb{R}.$$

Next, in view of the spreading properties of the super-solution and the sub-solution and thanks to the comparison argument with the ODE system detailed in the proof of Theorem 1.2, it only remains to show that the quantity

$$\bar{c} = \inf\left\{ c > 0 \mid \lim_{t\to+\infty} \sup_{ct \leq x \leq \frac{c_1+c}{2}t} \left(|u(t,x)| + |v(t,x) - 1|\right) = 0 \right\}$$

satisfies $\bar{c} \leq c_2$.

Now, the choice of super- and sub-solutions above proves that $\bar{c} \in [c_2, c_2^\delta]$. Suppose to the contrary that $\bar{c} > c_2$. Then we can fix a sufficiently small $\delta' \in (0,\delta)$ such that $c_2^{\delta'} \in (c_2, \bar{c})$. (This is possible since $c_2^{\delta'} \searrow c_2$ as $\delta' \searrow 0$, by Proposition 1.6.) Then, thanks to

- the estimate $u \leq \min\left(1, \overline{w_{c_2,c_1}}\right)$,
- Lemma 3.2 which controls from above the speed of $\overline{w_{c_2,c_1}}$,
- the control from below of the exponential tail of v,

we can use the super-solution $\left(\overline{u_{\delta'}}, \underline{v_{\delta'}}\right)$ associated with a sufficiently small $\delta' \in (0,\delta)$ as barrier after some large time $T_{\delta'}$ to slow down the invasion of u in an impossible way. More precisely, just as in the proof of Theorem 1.1, there exist large T' and y_0 such that, for all $x \in \mathbb{R}$,

$$u(T',x) \leq \min\left(1, \overline{w_{c_2,c_1}}(T',x)\right) \leq \overline{u_{\delta'}}(0, x - y_0)$$

and

$$v(T',x) \geq \underline{v_{\delta'}}(0, x - y_0).$$

This implies that

$$(u(t,x), v(t,x)) \preceq \left(\overline{u_{\delta'}}(t - T', x - y_0), \underline{v_{\delta'}}(t - T', x - y_0)\right) \text{ for all } t \geq T' \text{ and } x \in \mathbb{R}$$

and $\bar{c} \leq c_2^{\delta'}$, which is a contradiction. This ends the proof. □

4. Proofs of Proposition 1.4, Proposition 1.5 and Proposition 1.6

4.1. **Several useful objects.** In this subjection, we will define components which will be used for our later constructions. For ease of reading, we suggest the readers to skip Subsection 4.1 and only refer to it when a specific object is being used.

List of Objects			
Object(s)	*Defined in*	*Used in*	*Property*
$f(c)$	Sect. 1.1, (1.3)		$f(c) = c + 2\sqrt{a}$ $-\sqrt{c^2-4(1-a)}$
c_{acc}	Theorem 1.1(2)		$c_{acc} > c_{LLW}$
$\lambda(c)$	Sect. 1.1.3		$\lambda(c) = \lambda_\delta(c)\big\vert_{\delta=0}$
$\Lambda(c,\tilde{c})$	Sect. 1.1.3; Lemma 3.2		$\Lambda(c,\tilde{c}) = \Lambda_\delta(c,\tilde{c})\big\vert_{\delta=0}$
$\overline{w_{c,\tilde{c}}}$	Sect. 1.1.3		
$\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}$	Proposition 1.4	Sect. 3.1	Fig. 4.4
$\overline{u_\delta}, \underline{v_\delta}$	Proposition 1.5	Sect. 3.2.2	Fig. 4.3
$(\overline{u_\delta}, \underline{v_\delta}), (\underline{u_\delta}, \overline{v_\delta})$	Proposition 1.6	Sect. 3.3	Fig. 4.1 and Fig. 4.2
$\lambda_v(c)$	Sect. 4.1.1	Sect. 4.3.2 and 4.5.1	$d\lambda^2 - c\lambda + r = 0$
a_δ	Sect. 4.1.2		$a_\delta \to a$ as $\delta \to 0$
$\lambda_\delta(c)$	Sect. 4.1.3		$\lambda^2 - c\lambda + (1-a_\delta) = 0$
c^δ_{LLW}	Sect. 4.1.4; Lemma 4.2		Minimal wave speed of $P(u,v) = F_\delta(u,v)$
F_δ	Sect. 4.1.4		
$(\overline{\varphi_{\delta,c}}, \underline{\psi_{\delta,c}})$	Sect. 4.1.5	Sect. 4.3.1 and 4.4.1	Monotone wave profile of $P(u,v) = F_\delta(u,v)$
$\underline{\theta_{\delta,c,A}}$	Sect. 4.1.6; Lemma 4.10	Sect. 4.3.1 and 4.4.1	(4.9)
$\underline{\omega_{\delta,R}}$, R^ω_δ, $x_{\delta,R}$	Sect. 4.1.7; Lemma 4.3	Sect. 4.4.1	(4.1)
$\underline{\pi_{\delta,c,h}}$, $h^\star$	Sect. 4.1.8; Lemma 4.9	Sect. 4.3.1	(4.2)
$\underline{\beta_{c,B,\eta}}, \xi_\beta, K_\beta$	Sect. 4.1.9; Lemma 4.8	Sect. 4.3.1	Exp. decay of $\underline{v}$ at $+\infty$
$\underline{\alpha_l}, L_\alpha, x_l$	Sect. 4.1.10; Lemma 4.4	Sect. 4.5.1	(4.3)
$\underline{\chi_c}$	Sect. 4.1.11	Sect. 4.3.2 and 4.5.1	(4.4)
$\underline{f_\delta}(c), \Lambda_\delta(c,\tilde{c})$	Sect. 4.1.12	Beginning, Sect. 4.4	(4.5)
$\overline{w_{\delta,c,\tilde{c}}}$	Sect. 4.1.13; Lemma 4.5	Sect. 4.3.1 and 4.4.1	(4.6)
$\underline{w_{c,\tilde{c},A,\eta}}, X_w$	Sect. 4.1.14; Lemma 4.7	Sect. 4.3.2	(4.8)
$\underline{z_{\delta,c,\tilde{c}}}, R_z$	Sect. 4.1.15; Lemma 4.6	Sect. 4.5.1	(4.7)
$\lambda^{-\infty}(c)$	Lemma 4.1		

List of Intersection Points			
Symbol	*Defined in*	*Used in*	*Relation*
$x_0(t)$, ζ_0	Lemma 4.11	Sect. 4.3.2	$\underline{\chi_c}(x_0(t) - ct + \zeta_0) = \underline{w_{c,\tilde{c},A,\eta}}(t, x_0(t))$
$\xi_{1,\kappa}, \zeta_{1,\kappa}$, A_κ	Lemma 4.12	Sect. 4.3.1 and Sect. 4.4.1	$\underline{\theta_{\delta,c,A_\kappa}}(\xi_{1,\kappa}) = \underline{\psi_{\delta,c}}(\xi_{1,\kappa} - \zeta_{1,\kappa})$
$x_2(t), \zeta_2$	Lemma 4.13	Sect. 4.3.1 and Sect. 4.4.1	$\overline{\varphi_{\delta,c}}(x_2(t) - ct) = \overline{w_{\delta,c,\tilde{c}}}(t, x_2(t) - \zeta_2)$
$x_3(t), \zeta_3$	Lemma 4.14	Sect. 4.4.1	$\underline{\psi_{\delta,c}}(\hat{x}_3(t) - ct)$ $= \underline{\omega_{\delta,R_\delta}}(\hat{x}_3(t) - (2\sqrt{r(1-2\delta)}\bar{d} - \delta)t - \hat{\zeta}_3)$
	Lemma 4.15	Sect. 4.3.1	$\underline{\psi_{\delta,c}}(x_3(t) - ct) = \underline{\pi_{\delta,\tilde{c},h}}(x_3(t) - \tilde{c}t - \zeta_3)$
ξ_4, ζ_4	Lemma 4.16	Sect. 4.3.1	$\underline{\pi_{\delta,c,h}}(\xi_4) = \underline{\beta_{c,B,\eta}}(\xi_4 - \zeta_4)$
$x_{0,\kappa}(t), \zeta_0$	Lemma 4.17	Sect. 4.5.1	$\underline{\alpha_L}(x_{0,\kappa}(t)) = \underline{\chi_c}(x_{0,\kappa}(t) - ct - \zeta_0)$
$x_1(t)$	Lemma 4.18	Sect. 4.5.1	$\underline{\chi_c}(x_1(t) - ct) = \frac{\chi_c(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)} \underline{z_{c,\tilde{c},\delta}}(t, x_1(t) - \zeta)$

4.1.1. λ_v. The function λ_v is defined as

$$\begin{array}{rccc} \lambda_v: & \left[2\sqrt{rd}, +\infty\right) & \to & \left(0, \sqrt{\frac{r}{d}}\right] \\ & c & \mapsto & \frac{1}{2d}\left(c - \sqrt{c^2 - 4rd}\right) \end{array}.$$

4.1.2. a_δ. For all $\delta \in \left[0, \frac{1}{2}\right)$, we denote

$$a_\delta = \frac{(1-2\delta)\, a}{1+\delta}.$$

Notice that $a_0 = a$ and that $\delta \mapsto a_\delta$ is decreasing.

4.1.3. λ_δ. For all $\delta \in \left[0, \frac{1}{2}\right]$, the function λ_δ is defined as

$$\begin{array}{rccc} \lambda_\delta : & \left[2\sqrt{1-a_\delta}, +\infty\right) & \to & \left(0, \sqrt{1-a_\delta}\right] \\ & c & \mapsto & \frac{1}{2}\left(c - \sqrt{c^2 - 4\,(1-a_\delta)}\right) \end{array}.$$

The family $(\lambda_\delta)_{\delta \in \left[0, \frac{1}{2}\right]}$ is continuous and increasing in δ. Note that $\lambda_0 = \lambda$, the latter being introduced in Subsection 1.1.3.

4.1.4. c^δ_{LLW}. For all $\delta \in \left[0, \frac{1}{2}\right)$, c^δ_{LLW} denotes the minimal wave speed of the problem $P(u,v) = F_\delta(u,v)$, where

$$F_\delta : (u,v) \mapsto \begin{pmatrix} u\,(1+\delta-u-av) \\ rv\,(1-2\delta-v-bu) \end{pmatrix}.$$

Notice that (u,v) is a solution of $P(u,v) = F_\delta(u,v)$ if and only if

$$(U,V) : (t,x) \mapsto \left(\frac{u}{1+\delta}, \frac{v}{1-2\delta}\right)\left(\frac{t}{1+\delta}, \frac{x}{\sqrt{1+\delta}}\right)$$

is a solution of

$$P(U,V) = \begin{pmatrix} U\left(1 - U - \frac{(1-2\delta)a}{1+\delta} V\right) \\ \frac{(1-2\delta)r}{1+\delta} V\left(1 - V - \frac{(1+\delta)b}{1-2\delta} U\right) \end{pmatrix}.$$

Therefore

$$c^\delta_{LLW} = \sqrt{1+\delta}\,\hat{c}^\delta_{LLW},$$

where $\hat{c}^\delta_{LLW}$ is the minimal wave speed of equation (1.2) where (r,a,b) is replaced by $\left(\frac{(1-2\delta)r}{1+\delta}, a_\delta, \frac{(1+\delta)b}{1-2\delta}\right)$. As such, c^δ_{LLW} satisfies

$$2\sqrt{(1-a_\delta)\,(1+\delta)} \le c^\delta_{LLW} \le 2\sqrt{1+\delta}.$$

4.1.5. $\left(\overline{\varphi_{\delta,c}}, \underline{\psi_{\delta,c}}\right)$. For all $\delta \in \left[0, \frac{1}{2}\right)$ and $c \ge c^\delta_{LLW}$, $\left(\overline{\varphi_{\delta,c}}, \underline{\psi_{\delta,c}}\right)$ denotes a componentwise monotonic profile of traveling wave with speed c for the problem $P(u,v) = F_\delta(u,v)$, connecting $(1+\delta, 0)$ to $(0, 1-2\delta)$ and satisfying the normalization $\underline{\psi_{\delta,c}}(0) = \frac{1-2\delta}{2}$.

The existence of such a profile is well-known (and proved for instance in [24]). In fact, in the appendix, we will prove that any profile of traveling wave is componentwise monotonic and show that the condition

$$\frac{c + \sqrt{c^2 + 4rd}}{2d} \ge \frac{c - \sqrt{c^2 - 4\,(1-a)}}{2}$$

implies the uniqueness, up to translation, of the profile associated with a particular speed $c \ge c_{LLW}$. However these results are not actually required here. What is required indeed is the forthcoming Lemma 4.1.

4.1.6. $\underline{\theta_{\delta,c,A}}$. For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c^{\delta}_{LLW}$ and $A > 0$, define the function

$$\underline{\theta_{\delta,c,A}} : \xi \mapsto A\mathrm{e}^{\frac{1}{2d}\left(\sqrt{c^2+4rd(b-1+\delta)}-c\right)(\xi-\xi_\theta)} - \mathrm{e}^{\frac{1}{2d}\left(-\sqrt{c^2+4rd(b-1+\delta)}-c\right)(\xi-\xi_\theta)}$$

where the constant

$$\xi_\theta = \frac{d\ln A}{\sqrt{c^2+4rd\left(b-1+\delta\right)}}$$

is fixed so that $\underline{\theta_{\delta,c,A}}\left(0\right) = 0$. This function is increasing in $\mathbb{R}$.

4.1.7. $\underline{\omega_{\delta,R}}$ *and* R^{ω}_{δ}. For all $\delta \in [0,1)$ and all $R > 0$ large enough, $\underline{\omega_{\delta,R}} : [-R,R] \to [0,+\infty)$ denotes the unique nonnegative nonzero solution of

$$\begin{cases} -d\omega'' - \left(2\sqrt{r\left(1-\delta\right)d} - \delta\right)\omega' = r\omega\left(1-\delta-\omega\right) & \text{in } (-R,R) \\ \omega\left(\pm R\right) = 0 \end{cases}. \tag{4.1}$$

It is well-known that this problem admits a solution if and only if R is larger than or equal to some $R^{\omega}_{\delta} > 0$.

We extend the definition of $\underline{\omega_{\delta,R}}$ into the whole real line by setting $\underline{\omega_{\delta,R}}\left(\xi\right) = 0$ if $|\xi| > R$.

4.1.8. $\underline{\pi_{\delta,c,h}}$. For all $\delta \in [0,1)$, $c \geq 2\sqrt{rd}$ and all $h \in \mathbb{R}$, $\underline{\pi_{\delta,c,h}}$ denotes

$$\underline{\pi_{\delta,c,h}} : (\xi) \mapsto \underline{\pi_{\delta,c}}(\xi) + h\xi, \tag{4.2}$$

where $\underline{\pi_{\delta,c}}$ denotes the unique (decreasing) profile of traveling wave solution of

$$\partial_t v - d\partial_{xx} v = rv\left(1-\delta-v\right)$$

connecting 0 to $1-\delta$ at speed c and satisfying $\underline{\pi_{\delta,c}}\left(0\right) = \frac{1-\delta}{2}$.

4.1.9. $\underline{\beta_{c,B,\eta}}$. For all $c > 2\sqrt{rd}$, $\eta \in \left(0, \frac{1}{d}\sqrt{c^2-4rd}\right)$ and $B > 0$, $\underline{\beta_{c,B,\eta}}$ denotes

$$\underline{\beta_{c,B,\eta}} : \xi \mapsto \max\left(0, \mathrm{e}^{-\lambda_v(c)(\xi+\xi_\beta)} - K_\beta \mathrm{e}^{-(\lambda_v(c)+\eta)(\xi+\xi_\beta)}\right),$$

where the constants

$$K_\beta = \frac{r\left(1+bB\right)}{\eta\left(\sqrt{c^2-4rd}-d\eta\right)}$$

and $\xi_\beta = \frac{\ln K_\beta}{\eta}$ are fixed so that $\underline{\beta_{c,B,\eta}}$ is positive in $(0,+\infty)$ and null elsewhere.

4.1.10. $\underline{\alpha_l}$ *and* L_α. Similarly to the construction of $\underline{\omega_{\delta,R}}$ and R^{ω}_{δ}, we define $\underline{\alpha_l} : \mathbb{R} \to [0,+\infty)$ and $L_\alpha > 0$ such that, for all $l \geq L_\alpha$,

$$\begin{cases} -\underline{\alpha_l}'' = \underline{\alpha_l}\left(1-a-\underline{\alpha_l}\right) & \text{in } (0,l) \\ \underline{\alpha_l}\left(0\right) = \underline{\alpha_l}\left(l\right) = 0 & \\ \underline{\alpha_l}\left(x\right) = 0 & \text{if } \left|x-\frac{l}{2}\right| > \frac{l}{2} \end{cases}. \tag{4.3}$$

4.1.11. $\underline{\chi_c}$. For all $c \geq 2\sqrt{\frac{1-a}{2}}$, $\underline{\chi_c}$ denotes the unique (decreasing) profile of traveling wave solution of

$$\partial_t u - \partial_{xx} u = u\left(\frac{1-a}{2} - u\right) \tag{4.4}$$

connecting 0 to $\frac{1-a}{2}$ at speed c and satisfying $\underline{\chi_c}\left(0\right) = \frac{1-a}{4}$.

4.1.12. *$f_\delta(c)$ and $\Lambda_\delta(c,\tilde{c})$.* For all $\delta \in \left[0, \frac{1}{2}\right)$, f_δ denotes the function

$$\begin{array}{rccc} f_\delta: & \left[2\sqrt{1-a_\delta}, +\infty\right) & \to & \left(0, 2\left(\sqrt{1-a_\delta}+\sqrt{a_\delta}\right)\right] \\ & c & \mapsto & c-\sqrt{c^2-4(1-a_\delta)}+2\sqrt{a_\delta} \end{array}.$$

Notice right now that provided $\tilde{c} - f_\delta(c) > -4\sqrt{a_\delta}$, $\tilde{c} - f_\delta(c)$ has exactly the sign of $\tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1)$. Indeed, by the fact that $(\lambda_\delta(c))^2 - c\lambda_\delta(c) + 1 - a_\delta = 0$,

$$\begin{aligned} \tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1) &= (\tilde{c}-2\lambda_\delta(c))^2 - 4\left(1-\lambda_\delta(c)c+(\lambda_\delta(c))^2\right) \\ &= \left(\tilde{c}-c+\sqrt{c^2-4(1-a_\delta)}\right)^2 - 4a_\delta \\ &= (\tilde{c}-f_\delta(c)+2\sqrt{a_\delta})^2 - (2\sqrt{a_\delta})^2 \\ &= (\tilde{c}-f_\delta(c))(\tilde{c}-f_\delta(c)+4\sqrt{a_\delta}). \end{aligned}$$

For all $\delta \in \left[0, \frac{1}{2}\right)$, Λ_δ denotes the function

$$\Lambda_\delta : (c,\tilde{c}) \mapsto \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(\lambda_\delta(c)(\tilde{c}-c)+1)}\right).$$

Its domain is the set of all $(c,\tilde{c})$ such that $c \geq 2\sqrt{1-a_\delta}$ and $\tilde{c} \geq \max(c, f_\delta(c))$ and it is decreasing with respect to both c and $\tilde{c}$. As a function of c only, with a fixed $\tilde{c}$, it bijectively maps $\left[2\sqrt{1-a_\delta}, +\infty\right)$ onto

$$\left(\frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4(a_\delta+1)}\right), \frac{1}{2}\left(\tilde{c} - \sqrt{\tilde{c}^2 - 4\left(\tilde{c}\sqrt{1-a_\delta}+2a_\delta-1\right)}\right)\right].$$

The family $(\Lambda_\delta)_{\delta \in \left[0,\frac{1}{2}\right)}$ is increasing. Recalling the earlier definition of Λ, we find $\Lambda_0 = \Lambda$.

Finally, by construction, for all $(c,\tilde{c})$ such that $c \geq 2\sqrt{1-a_\delta}$ and $\tilde{c} \geq \max(c, f_\delta(c))$, $\Lambda_\delta(c,\tilde{c})$ satisfies

$$(\Lambda_\delta(c,\tilde{c}))^2 - \tilde{c}\Lambda_\delta(c,\tilde{c}) + \lambda_\delta(c)(\tilde{c}-c) + 1 = 0. \tag{4.5}$$

4.1.13. *$\overline{w_{\delta,c,\tilde{c}}}$.* For all $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c^\delta_{LLW}$ and $\tilde{c} \geq \max(c, f_\delta(c))$, $\overline{w_{\delta,c,\tilde{c}}}$ denotes the function

$$\overline{w_{\delta,c,\tilde{c}}} : (t,x) \mapsto \mathrm{e}^{-\lambda_\delta(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})(x-\tilde{c}t)}.$$

In view of (4.5),

$$\overline{w_{\delta,c,\tilde{c}}}(t,x) = \mathrm{e}^{\left((\Lambda_\delta(c,\tilde{c}))^2+1\right)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})x} \text{ for all } (t,x) \in [0,+\infty)\times\mathbb{R}.$$

Recalling the earlier definition of $\overline{w_{c,\tilde{c}}}$, we find $\overline{w_{c,\tilde{c}}} = \overline{w_{0,c,\tilde{c}}}$.

4.1.14. *$\underline{w_{c,\tilde{c},A,\eta}}$.* For all $c \geq c_{LLW}$, $\tilde{c} \geq c$ such that $\tilde{c} > f(c)$, $\eta \in \left(0, \min\left(\Lambda(c,\tilde{c}), \sqrt{\tilde{c}^2 - 4(\lambda(c)(\tilde{c}-c)+1)}\right)\right)$ and $A > 0$, $\underline{w_{c,\tilde{c},A,\eta}}$ denotes

$$\underline{w_{c,\tilde{c},A,\eta}} : (t,x) \mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\max\left(0, \mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right),$$

where

$$K_w = \max\left(1, \frac{1+aA}{\eta\left(\sqrt{\tilde{c}^2-4(\lambda(c)(\tilde{c}-c)+1)}-\eta\right)}\right) = \max\left(1, \frac{1+aA}{\eta(\tilde{c}-\eta-2\Lambda_0(c,\tilde{c}))}\right)$$

and $x_w = \frac{\ln K_w}{\eta}$ is fixed so that, for all $t \geq 0$, $x \mapsto \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ is positive in $(\tilde{c}t, +\infty)$, null elsewhere, increasing in $\left(\tilde{c}t, \frac{\ln(\Lambda(c,\tilde{c})+\eta)-\ln(\Lambda(c,\tilde{c}))}{\eta} + \tilde{c}t\right)$ and decreasing in $\left(\frac{\ln(\Lambda(c,\tilde{c})+\eta)-\ln(\Lambda(c,\tilde{c}))}{\eta} + \tilde{c}t, +\infty\right)$. Hereafter, the point where the global maximum is attained at $t = 0$ is denoted X_w.

4.1.15. $\underline{z_{\delta,c,\tilde{c}}}$. For all $c \geq c_{LLW}$, $\tilde{c} \in (f(c) - 4\sqrt{a}, f(c))$ and $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4(\lambda(c)(\tilde{c}-c)+1)\right)\right)$, $\underline{z_{c,\tilde{c},\delta}}$ denotes the function defined by

$$\underline{z_{\delta,c,\tilde{c}}}(t,x) = \begin{cases} e^{-\lambda(c)(\tilde{c}-c)t} e^{-\frac{\tilde{c}}{2}(x-\tilde{c}t)} \sin\left(\frac{\pi}{2R_z}(x-\tilde{c}t)\right) & \text{if } x - \tilde{c}t \in [0, 2R_z] \\ 0 & \text{otherwise} \end{cases}.$$

where

$$R_z = \frac{\pi}{\sqrt{-\tilde{c}^2 + 4(\lambda(c)(\tilde{c}-c)+1-\delta)}}.$$

Hereafter, the point where the global maximum is attained at $t = 0$ is denoted X_z.

4.2. **Several useful lemmas.**

Lemma 4.1. *Let $c > c_{LLW}$ and (φ, ψ) be a profile of traveling wave solution of (1.1) with speed c.*

Then there exist $A > 0$ and $B > 0$ such that

$$\varphi(\xi) = Ae^{-\lambda(c)\xi} + h.o.t. \text{ as } \xi \to +\infty$$

and

$$\psi(\xi) = Be^{\lambda^{-\infty}(c)\xi} + h.o.t. \text{ as } \xi \to -\infty$$

where

$$\lambda^{-\infty}(c) = \frac{1}{2d}\left(\sqrt{c^2 + 4rd(b-1)} - c\right),$$

The proof of this lemma is quite lengthy. Therefore it is postponed to the appendix (see Corollary A.3 and Corollary A.6).

Lemma 4.2. *The function $\delta \mapsto c^\delta_{LLW}$ is continuous and nondecreasing in $\left[0, \frac{1}{2}\right)$.*

Proof. Recalling that

$$c^\delta_{LLW} = \sqrt{1+\delta}\hat{c}^\delta_{LLW},$$

where $\hat{c}^\delta_{LLW}$ is the minimal wave speed of the system (1.2) where (r, a, b) is replaced by $\left(\frac{(1-2\delta)r}{1+\delta}, a_\delta, \frac{(1+\delta)b}{1-2\delta}\right)$, the continuity of $\delta \mapsto c^\delta_{LLW}$ follows directly from the theorem due to Kan-on [20] establishing the continuity of the spreading speed of (1.2) with respect to the parameters (r, a, b).

The monotonicity follows from the comparison principle. □

Lemma 4.3. *Let $\delta \in [0, 1)$. Then for all $R \geq R^\omega_\delta$,*

$$\max_{[-R,R]} \underline{\omega_{\delta,R}} < 1 - \delta.$$

Furthermore, if $\delta > 0$, then there exists $R_\delta \geq R^\omega_\delta$ such that, for all $R \geq R_\delta$,

$$\max_{[-R,R]} \underline{\omega_{\delta,R}} \geq 1 - 2\delta,$$

and there exists a unique $x_{\delta,R} \in (-R, R)$ such that $\underline{\omega_{\delta,R}}$ is increasing in $[-R, x_{\delta,R}]$, decreasing in $[x_{\delta,R}, R]$ and maximal at $x_{\delta,R}$.

Proof. The first inequality follows very classically from the first and second order conditions at any local maximum and from the strong maximum principle.

The second inequality comes from the locally uniform convergence of $\underline{\omega_{\delta,R}}$ to $1-\delta$ as $R \to +\infty$. This fact is also well-known and its proof is not detailed here.

Finally, the piecewise strict monotonicity comes from the inequality

$$-d\underline{\omega_{\delta,R}}'' - \left(2\sqrt{r\left(1-\delta\right)d} - \delta\right)\underline{\omega_{\delta,R}}' > 0 \text{ in } \left(-R,R\right),$$

which implies the nonexistence of local minima. □

The function $\underline{\alpha_l}$ satisfies of course a similar property.

Lemma 4.4. *For all $l \geq L_\alpha$,*

$$\max_{[0,l]} \underline{\alpha_l} < 1-a.$$

Furthermore, there exists $L \geq L_\alpha$ such that, for all $l \geq L$,

$$\max_{[0,l]} \underline{\alpha_l} \geq \frac{1-a}{2},$$

and there exists a unique $x_l \in (-l,l)$ such that $\underline{\alpha_l}$ is increasing in $[0,x_l]$, decreasing in $[x_l,l]$ and maximal at x_l.

Lemma 4.5. *For all $\delta \in \left[0,\frac{1}{2}\right)$, $c \geq c^\delta_{LLW}$ and $\tilde{c} \geq \max\left(c, f_\delta\left(c\right)\right)$, $\overline{w_{\delta,c,\tilde{c}}}$ satisfies*

$$\partial_t \overline{w_{\delta,c,\tilde{c}}} - \partial_{xx}\overline{w_{\delta,c,\tilde{c}}} = \overline{w_{\delta,c,\tilde{c}}} \text{ in } [0,+\infty)\times\mathbb{R}.$$

Proof. The following equality being straightforward,

(4.6)
$$\partial_t \overline{w_{\delta,c,\tilde{c}}} - \partial_{xx}\overline{w_{\delta,c,\tilde{c}}} - \overline{w_{\delta,c,\tilde{c}}} = \left(-\lambda_\delta\left(c\right)\left(\tilde{c}-c\right) + \tilde{c}\Lambda_\delta\left(c,\tilde{c}\right) - \left(\Lambda_\delta\left(c,\tilde{c}\right)\right)^2 - 1\right)\overline{w_{\delta,c}},$$

the conclusion follows from (4.5). □

Quite similarly, we have the following lemma.

Lemma 4.6. *For all $c \geq c_{LLW}$, $\tilde{c} \in \left(f\left(c\right) - 4\sqrt{a}, f\left(c\right)\right)$ and $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)\right)\right)$, $\underline{z_{\delta,c,\tilde{c}}}$ satisfies*

$$\partial_t \underline{z_{\delta,c,\tilde{c}}} - \partial_{xx}\underline{z_{\delta,c,\tilde{c}}} = \left(1-\delta\right)\underline{z_{\delta,c,\tilde{c}}} \text{ in } [0,+\infty)\times\mathbb{R}. \tag{4.7}$$

Proof. It suffices to verify

$$-\lambda\left(c\right)\left(\tilde{c}-c\right) + \frac{\tilde{c}^2}{4} + \left(\frac{\pi}{2R_z}\right)^2 - \left(1-\delta\right) = 0,$$

which, in view of the definition of R_z, is equivalent to

$$-\lambda\left(c\right)\left(\tilde{c}-c\right) + \frac{\tilde{c}^2}{4} + \left(\frac{-\tilde{c}^2 + 4\lambda\left(c\right)\left(\tilde{c}-c\right) + 4\left(1-\delta\right)}{4}\right) - \left(1-\delta\right) = 0.$$

The last statement obviously holds. □

Lemma 4.7. *For all $c \geq c_{LLW}$, $\tilde{c} \geq c$ such that $\tilde{c} > f\left(c\right)$, $\eta \in \left(0, \min\left(\Lambda\left(c,\tilde{c}\right), \sqrt{\tilde{c}^2 - 4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)}\right)\right)$ and $A > 0$, the function $\underline{w_{c,\tilde{c},A,\eta}}$ satisfies, for all $\sigma \geq \eta$,*

$$\partial_t \underline{w_{c,\tilde{c},A,\eta}} - \partial_{xx}\underline{w_{c,\tilde{c},A,\eta}} \leq \underline{w_{c,\tilde{c},A,\eta}}\left(1 - \underline{w_{c,\tilde{c},A,\eta}} - aAe^{-\sigma\left(x-\tilde{c}t+x_w\right)}\right). \tag{4.8}$$

Remark. The above inequality is to be understood in the weak sense associated with generalized sub-solutions.

Proof. For $x - \tilde{c}t < 0$, $\underline{w_{c,\tilde{c},A,\eta}}$ is trivial and the inequality obviously holds. We focus on the case $x - \tilde{c}t > 0$, where $\underline{w_{c,\tilde{c},A,\eta}}$ reduces to

$$(t,x) \mapsto \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w \mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right).$$

First, differentiating, we find:

$$\begin{aligned}\partial_t \underline{w_{c,\tilde{c},A,\eta}}(t,x) = &-\lambda(c)(\tilde{c}-c)\,\underline{w_{c,\tilde{c},A,\eta}}(t,x)\\ &+ \tilde{c}\Lambda(c,\tilde{c})\,\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\\ &- K_w\left(\tilde{c}\left(\Lambda(c,\tilde{c})+\eta\right)\right)\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)},\end{aligned}$$

$$\partial_{xx}\underline{w_{c,\tilde{c},A,\eta}} = \mathrm{e}^{-\lambda(c)(\tilde{c}-c)t}\left(\left(\Lambda(c,\tilde{c})\right)^2\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} - K_w\left(\Lambda(c,\tilde{c})+\eta\right)^2\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}\right),$$

so that the auxiliary function

$$Q : (t,x) \mapsto \mathrm{e}^{\lambda(c)(\tilde{c}-c)t}\left(-\partial_t \underline{w_{c,\tilde{c},A,\eta}} + \partial_{xx}\underline{w_{c,\tilde{c},A,\eta}} + \underline{w_{c,\tilde{c},A,\eta}}\right)(t,x)$$

satisfies

$$\begin{aligned}Q(t,x) = &\left(\lambda(c)(\tilde{c}-c) - \tilde{c}\Lambda(c,\tilde{c}) + \left(\Lambda(c,\tilde{c})\right)^2 + 1\right)\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\\ &- K_w\left(\lambda(c)(\tilde{c}-c) - \tilde{c}\left(\Lambda(c,\tilde{c})+\eta\right) + \left(\Lambda(c,\tilde{c})+\eta\right)^2 + 1\right)\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}.\end{aligned}$$

Using (4.5), it follows

$$Q(t,x) = K_w\eta\left(\tilde{c} - \eta - 2\Lambda(c,\tilde{c})\right)\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)},$$

that is, recalling the definition of $\Lambda_\delta(c,\tilde{c})$ as well as that of K_w,

$$Q(t,x) \geq (1+aA)\,\mathrm{e}^{-(\Lambda(c,\tilde{c})+\eta)(x-\tilde{c}t+x_w)}.$$

Next, getting rid of all the negative terms and using $\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t} \leq 1$, we find

$$\mathrm{e}^{\lambda(c)(\tilde{c}-c)t}\underline{w_{c,\tilde{c},A,\eta}}\left(\underline{w_{c,\tilde{c},A,\eta}} + aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right) \leq \mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} + aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right)$$

Finally, using $x > \tilde{c}t$, $x_w = \frac{1}{\eta}\ln K_w \geq 0$ as well as the assumption $0 < \eta \leq \min\{\Lambda(c,\tilde{c}),\sigma\}$, we find

$$\begin{aligned}\mathrm{e}^{\eta(x-\tilde{c}t+x_w)}\left(\mathrm{e}^{-\Lambda(c,\tilde{c})(x-\tilde{c}t+x_w)} + aA\mathrm{e}^{-\sigma(x-\tilde{c}t+x_w)}\right) &\leq \mathrm{e}^{-(\Lambda(c,\tilde{c})-\eta)x_w} + aA\mathrm{e}^{-(\sigma-\eta)x_w}\\ &\leq 1 + aA\end{aligned}$$

and the proof is therefore ended. □

With an analogous proof, we obtain directly the following lemma.

Lemma 4.8. *For all $c > 2\sqrt{rd}$, $\eta \in \left(0, \min\left(\lambda_v(c), \frac{1}{d}\sqrt{c^2-4rd}\right)\right)$ and $B > 0$, $\underline{\beta_{c,B,\eta}}$ satisfies, for all $\sigma \geq \eta$,*

$$-d\underline{\beta_{c,B,\eta}}'' - c\underline{\beta_{c,B,\eta}}' \leq r\underline{\beta_{c,B,\eta}}\left(1 - \underline{\beta_{c,B,\eta}} - bBe^{-\sigma(\xi+\xi_\beta)}\right) \text{ in } (\mathbb{R}\setminus\{0\}).$$

Lemma 4.9. *For all $\delta \in (0,1)$, $c > 2\sqrt{rd}$ and $h > 0$, $\underline{\pi_{\delta,c,h}}$ satisfies*

$$-d\underline{\pi_{\delta,c,h}}'' - c\underline{\pi_{\delta,c,h}}' \leq r\underline{\pi_{\delta,c,h}}\left(1 - \delta - \underline{\pi_{\delta,c,h}}\right) \text{ in } \left[-\sqrt{\frac{c}{rh}}, \sqrt{\frac{c}{rh}}\right].$$

Furthermore, there exists $h^\star > 0$ such that, for all $h \in (0,h^\star]$,

$$\max_{\left[-\sqrt{\frac{c}{rh}},0\right]} \underline{\pi_{\delta,c,h}} \geq 1 - 2\delta,$$

$$\max_{\left[-\sqrt{\frac{c}{rh}},0\right]} \underline{\pi_{\delta,c,h}} > \max\left(\underline{\pi_{\delta,c,h}}\left(0\right), \underline{\pi_{\delta,c,h}}\left(-\sqrt{\frac{c}{rh}}\right)\right).$$

Remark. It should be achievable to prove that the global maximum of $\underline{\pi_{\delta,c,h}}$ in $\left(-\sqrt{\frac{c}{rh}},0\right)$ is actually unique and that $\underline{\pi_{\delta,c,h}}$ is increasing in $(-\infty,\xi^\star)$ and decreasing in $(\xi^\star,0)$ but this is really unnecessary for our purpose.

Proof. Recalling that $\underline{\pi_{\delta,c,h}}(\xi) = \underline{\pi_{\delta,c}}(\xi) + h\xi$ (see Subsection 4.1.8), we have

$$\begin{aligned}
&- d\underline{\pi_{\delta,c,h}}''\left(\xi\right) - c\underline{\pi_{\delta,c,h}}'\left(\xi\right)\\
&= r\underline{\pi_{\delta,c}}\left(\xi\right)\left(1-\delta-\underline{\pi_{\delta,c}}\left(\xi\right)\right) - ch\\
&= r\underline{\pi_{\delta,c,h}}\left(\xi\right)\left(1-\delta-\underline{\pi_{\delta,c,h}}\left(\xi\right)\right) - hr\left(\xi\left(1-\delta-\underline{\pi_{\delta,c}}\left(\xi\right)\right) + \frac{c}{r} - \underline{\pi_{\delta,c,h}}\left(\xi\right)\xi\right)\\
&= r\underline{\pi_{\delta,c,h}}\left(\xi\right)\left(1-\delta-\underline{\pi_{\delta,c,h}}\left(\xi\right)\right) - hr\left(-h\xi^2 + \left(1-\delta-2\underline{\pi_{\delta,c}}\left(\xi\right)\right)\xi + \frac{c}{r}\right).
\end{aligned}$$

It is easily verified that, in $\left[-\sqrt{\frac{c}{rh}},\sqrt{\frac{c}{rh}}\right]$,

$$-h\xi^2 + \left(1-\delta-2\underline{\pi_{\delta,c}}\left(\xi\right)\right)\xi + \frac{c}{r} > -h\xi^2 + \frac{c}{r} \geq 0,$$

where we used the facts

$$\underline{\pi_{\delta,c,h}} > \frac{1-\delta}{2} \quad \text{for } \xi<0, \quad \text{and} \quad \underline{\pi_{\delta,c,h}} < \frac{1-\delta}{2} \quad \text{for } \xi>0.$$

And the stated differential inequality is established.

The maximum of $\underline{\pi_{\delta,c,h}}$ in $\left[-\sqrt{\frac{c}{rh}},0\right]$ is larger than or equal to

$$\underline{\pi_{\delta,c,h}}\left(-\sqrt{\frac{c}{rh}}\right) = \underline{\pi_{\delta,c}}\left(-\sqrt{\frac{c}{rh}}\right) - \sqrt{\frac{ch}{r}},$$

which is itself larger than or equal to $1-2\delta$ if h is small enough.

Finally, since $\underline{\pi_{\delta,c}}'\left(-\sqrt{\frac{c}{rh}}\right)$ vanishes exponentially as $h\to 0$,

$$\underline{\pi_{\delta,c,h}}'\left(-\sqrt{\frac{c}{rh}}\right) = \underline{\pi_{\delta,c}}'\left(-\sqrt{\frac{c}{rh}}\right) + h > 0, \quad \text{and} \quad \underline{\pi_{\delta,c,h}}'(0) = \underline{\pi_{\delta,c}}'(0) + h < 0,$$

for all sufficiently small h. This implies that the values at $\xi=0$ and $-\sqrt{\frac{c}{rh}}$ are smaller than the aforementioned maximum. □

Lemma 4.10. *For all $\delta\in\left[0,\frac{1}{2}\right)$, $c\geq c^\delta_{LLW}$ and $A>0$, $\underline{\theta_{\delta,c,A}}$ satisfies*

$$- d\underline{\theta_{\delta,c,A}}'' - c\underline{\theta_{\delta,c,A}}' - r\underline{\theta_{\delta,c,A}}\left(1-\delta-b\right) = 0 \text{ in } \mathbb{R}. \tag{4.9}$$

Proof. Note that $\underline{\theta_{\delta,c,A}}$ is a linear combination of $\xi\mapsto \mathrm{e}^{\frac{1}{2d}\left(\pm\sqrt{c^2+4rd(b-1+\delta)}-c\right)\xi}$, where $\frac{1}{2d}\left(\pm\sqrt{c^2+4rd(b-1+\delta)}-c\right)$ are the two distinct roots of the characteristic polynomial associated with the above linear ODE (4.9). □

Lemma 4.11. *For all $c>2\sqrt{1-a}$, $\tilde{c}\geq c$ such that*

$$\tilde{c} > f\left(c\right), \quad \eta\in\left(0,\min\left(\sqrt{\tilde{c}^2-4\left(\lambda\left(c\right)\left(\tilde{c}-c\right)+1\right)}\right),\lambda_v\left(\tilde{c}\right)\right), \quad \text{and} \quad A>0,$$

there exists $\zeta_0\in\mathbb{R}$ such that the equation

$$\underline{\chi_c}\left(x-ct+\zeta_0\right) = \underline{w_{c,\tilde{c},A,\eta}}\left(t,x\right)$$

admits for all $t\geq 0$ an isolated solution $x_0\left(t\right)\in\mathbb{R}$ such that

(1) $\underline{\chi_c}(x-ct+\zeta_0) > \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ *in a left-sided neighborhood of* $x_0(t)$*;*
(2) $\underline{\chi_c}(x-ct+\zeta_0) < \underline{w_{c,\tilde{c},A,\eta}}(t,x)$ *in a right-sided neighborhood of* $x_0(t)$*;*
(3) $\tilde{c}t < x_0(t) < X_w + \tilde{c}t$.

Furthermore, $x_0 \in \mathscr{C}^1([0,+\infty),(0,+\infty))$.

Proof. Recall from standard results on the KPP equation that, since $c > 2\sqrt{1-a}$, there exists $\zeta_{0,1} \in \mathbb{R}$ such that

$$\underline{\chi_c}(x+\zeta_{0,1}) \sim \mathrm{e}^{-\lambda(c)x} \text{ as } x \to +\infty.$$

Hence there exists $\zeta_0 \in \mathbb{R}$ such that, for all $x \geq 0$,

$$\underline{\chi_c}(x+\zeta_0) \leq \frac{1}{2} e^{-\lambda(c)x} \max_{y\in\mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y) \leq \frac{1}{2} \max_{y\in\mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y)$$

with $\max\limits_{y\in\mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,y)$ uniquely attained at X_w.

From the intermediate value theorem and the respective strict monotonicities of $\underline{\chi_c}$ in $\mathbb{R}$ and $x \mapsto \underline{w_{c,\tilde{c},A,\eta}}(0,x)$ in $[0,X_w]$, it clearly follows that $\underline{\chi_c}(x+\zeta_0) = \underline{w_{c,\tilde{c},A,\eta}}(0,x)$ admits a unique solution $x_0(0)$ in $(0,X_w)$.

Next, to define in the same way $x_0(t)$, it suffices to verify that for all $t > 0$,

$$\underline{w_{c,\tilde{c},A,\eta}}(t, X_w + \tilde{c}t) > \underline{\chi_c}(X_w + (\tilde{c}-c)t + \zeta_0).$$

Since $X_w + \tilde{c}t \geq 0$, it is *a fortiori* sufficient to verify that for all $t \geq 0$,

$$\mathrm{e}^{-\lambda(c)(\tilde{c}-c)t} \max_{x\in\mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,x) > \frac{1}{2} \max_{x\in\mathbb{R}} \underline{w_{c,\tilde{c},A,\eta}}(0,x)\, \mathrm{e}^{-\lambda(c)(X_w+(\tilde{c}-c)t)}.$$

This inequality reduces in fact to $2 > \mathrm{e}^{-\lambda(c)X_w}$, which holds as $\lambda(c)$ and X_w are both positive. The existence of $x_0(t)$ for all $t > 0$ follows.

Finally, the regularity of x_0 follows from the aforementioned monotonicities and the implicit function theorem. □

Lemma 4.12. *For all* $\delta \in \left[0, \frac{1}{2}\right)$, $c \geq c^{\delta}_{LLW}$ *and* $\kappa \in (0,\delta]$, *there exists* $\zeta_{1,\kappa} \in \mathbb{R}$ *and* $A_\kappa > 0$ *such that the equation*

$$\underline{\theta_{\delta,c,A_\kappa}}(\xi) = \underline{\psi_{\delta,c}}(\xi - \zeta_{1,\kappa})$$

admits an isolated solution $\xi_{1,\kappa} \in \mathbb{R}$ *such that*

(1) $\underline{\theta_{\delta,c,A_\kappa}}(\xi) > \underline{\psi_{\delta,c}}(\xi-\zeta_{1,\kappa})$ *for* ξ *in a left-sided neighborhood of* $\xi_{1,\kappa}$*;*
(2) $\underline{\theta_{\delta,c,A_\kappa}}(\xi) < \underline{\psi_{\delta,c}}(\xi-\zeta_{1,\kappa})$ *for* ξ *in a right-sided neighborhood of* $\xi_{1,\kappa}$*;*
(3) $\underline{\psi_{\delta,c}}(\xi_{1,\kappa}-\zeta_{1,\kappa}) \leq \kappa$*;*
(4) $\zeta_{1,\kappa} - \xi_{1,\kappa} \to +\infty$ *as* $\kappa \to 0$.

Proof. Let δ, c and κ be given as in the statement, define

$$\lambda^{-\infty} = \frac{1}{2d}\left(\sqrt{c^2 + 4rd(b-1+(b+2)\delta)} - c\right),$$
$$\lambda^{+}_{\theta} = \frac{1}{2d}\left(\sqrt{c^2 + 4rd(b-1+\delta)} - c\right),$$
$$\lambda^{-}_{\theta} = \frac{1}{2d}\left(-\sqrt{c^2 + 4rd(b-1+\delta)} - c\right),$$
$$\xi_\theta = \frac{d\ln A}{\sqrt{c^2+4rd(b-1+\delta)}} = \frac{\ln A}{\lambda^{+}_{\theta} - \lambda^{-}_{\theta}},$$

and notice that

$$\lambda^{-}_{\theta} < 0 < \lambda^{+}_{\theta} < \lambda^{-\infty}.$$

Let $\tilde{\kappa} \in (0, \kappa]$ such that $(1-\tilde{\kappa})\,\lambda^{-\infty} > \lambda_\theta^+$.
In view of Lemma 4.1,

$$\lim_{\xi \to -\infty} \left(\frac{\underline{\psi_{\delta,c}}'(\xi)}{\underline{\psi_{\delta,c}}(\xi)} \right) = \lambda^{-\infty}.$$

Therefore, by monotonicity of $\underline{\psi_{\delta,c}}$, there exists $\zeta_\kappa \in \mathbb{R}$ such that for all $\xi \leq 0$,

$$\underline{\psi_{\delta,c}}(\xi - \zeta_\kappa) \leq \kappa,$$

$$\left(1 - \frac{\tilde{\kappa}}{2}\right) \lambda^{-\infty} \leq \frac{\underline{\psi_{\delta,c}}'(\xi - \zeta_\kappa)}{\underline{\psi_{\delta,c}}(\xi - \zeta_\kappa)} \leq \left(1 + \frac{\tilde{\kappa}}{2}\right) \lambda^{-\infty}.$$

Note that $\zeta_\kappa \to +\infty$ as $\kappa \to 0$. It remains to find $A > 0$, $\zeta_1 > \zeta_\kappa$ and $\xi_1 \in (0, \zeta_1 - \zeta_\kappa]$ such that

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(\xi_1 - \zeta_1),$$

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi_1)}{\underline{\theta_{\delta,c,A}}(\xi_1)} \leq (1-\tilde{\kappa})\,\lambda^{-\infty}.$$

For all $\xi \in \mathbb{R}$,

$$\underline{\theta_{\delta,c,A}}'(\xi) = A\lambda_\theta^+ \mathrm{e}^{\lambda_\theta^+(\xi - \xi_\theta)} - \lambda_\theta^- \mathrm{e}^{\lambda_\theta^-(\xi-\xi_\theta)} > 0,$$

whence for all $\xi > 0$ the condition

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi)}{\underline{\theta_{\delta,c,A}}(\xi)} < (1-\tilde{\kappa})\,\lambda^{-\infty}$$

holds true if and only if

$$(1-\tilde{\kappa})\,\lambda^{-\infty} > \lambda_\theta^+ + \frac{\lambda_\theta^+ - \lambda_\theta^-}{A\mathrm{e}^{(\lambda_\theta^+ - \lambda_\theta^-)(\xi - \xi_\theta)} - 1},$$

that is if and only if

$$A\mathrm{e}^{(\lambda_\theta^+ - \lambda_\theta^-)(\xi-\xi_\theta)} - 1 > \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+},$$

that is if and only if $\xi > \xi_1$ where

$$\xi_1 = \max\left(0, \xi_\theta + \frac{1}{\lambda_\theta^+ - \lambda_\theta^-}\left(\ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right) - \ln A\right)\right).$$

In view of the definition of ξ_θ,

$$\begin{aligned}\xi_1 &= \max\left(0, \frac{1}{\lambda_\theta^+ - \lambda_\theta^-} \ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right)\right) \\ &= \frac{1}{\lambda_\theta^+ - \lambda_\theta^-} \ln\left(1 + \frac{\lambda_\theta^+ - \lambda_\theta^-}{(1-\tilde{\kappa})\,\lambda^{-\infty} - \lambda_\theta^+}\right).\end{aligned}$$

In particular, $\xi_1 > 0$ does not depend on A and, by construction, we have

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi)}{\underline{\theta_{\delta,c,A}}(\xi)} < (1-\tilde{\kappa})\,\lambda^{-\infty} \text{ for all } \xi > \xi_1,$$

$$\frac{\underline{\theta_{\delta,c,A}}'(\xi_1)}{\underline{\theta_{\delta,c,A}}(\xi_1)} = (1-\tilde{\kappa})\,\lambda^{-\infty}.$$

Now, the function $\underline{\theta_{\delta,c,A}}$ is increasing with $\underline{\theta_{\delta,c,A}}(0)=0$ and

$$\begin{aligned}\underline{\theta_{\delta,c,A}}(\xi_1) &= A\mathrm{e}^{\lambda_\theta^+(\xi_1-\xi_\theta)} - \mathrm{e}^{\lambda_\theta^-(\xi_1-\xi_\theta)}\\ &= A^{1-\frac{\lambda_\theta^+}{\lambda_\theta^+-\lambda_\theta^-}}\mathrm{e}^{\lambda_\theta^+\xi_1} - A^{-\frac{\lambda_\theta^-}{\lambda_\theta^+-\lambda_\theta^-}}\mathrm{e}^{\lambda_\theta^-\xi_1}\\ &= A^{\frac{-\lambda_\theta^-}{\lambda_\theta^+-\lambda_\theta^-}}\left(\mathrm{e}^{\lambda_\theta^+\xi_1}-\mathrm{e}^{\lambda_\theta^-\xi_1}\right).\end{aligned}$$

As a function of A, this quantity is increasing (recall $\lambda_\theta^- < 0$) and vanishes as $A\to 0$. We fix now A such that

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(-\zeta_\kappa) \le \kappa.$$

Defining $\zeta_1 = \xi_1 + \zeta_\kappa > \zeta_\kappa$, we obtain indeed

$$\underline{\theta_{\delta,c,A}}(\xi_1) = \underline{\psi_{\delta,c}}(\xi_1-\zeta_1) \le \kappa,$$

$$\begin{aligned}\underline{\theta_{\delta,c,A}}'(\xi_1) &= (1-\tilde{\kappa})\lambda^{-\infty}\underline{\theta_{\delta,c,A}}(\xi_1)\\ &= (1-\tilde{\kappa})\lambda^{-\infty}\underline{\psi_{\delta,c}}(\xi_1-\zeta_1)\\ &< \underline{\psi_{\delta,c}}'(\xi_1-\zeta_1),\end{aligned}$$

as well as the limit

$$\lim_{\kappa\to 0}(\zeta_1-\xi_1) = \lim_{\kappa\to 0}\zeta_\kappa = +\infty.$$

This completes the proof.

□

Lemma 4.13. *There exists $\delta_0 \in \left(0,\frac{1}{2}\right)$ such that, for all $\delta\in[0,\delta_0)$, $c > c_{LLW}^\delta$ and $\tilde{c}\ge \max(c, f_\delta(c))$, there exists $\zeta_2\in\mathbb{R}$ such that the equation*

$$\overline{\varphi_{\delta,c}}(x-ct) = \overline{w_{\delta,c,\tilde{c}}}(t, x-\zeta_2)$$

admits for all $t\ge 0$ an isolated solution $x_2(t)\in\mathbb{R}$ such that

(1) $\overline{\varphi_{\delta,c}}(x-ct) > \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)$ for all $x\in(x_2(t),+\infty)$;
(2) $\overline{\varphi_{\delta,c}}(x-ct) < \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)$ for all $x\in(-\infty,x_2(t))$;
(3) $\overline{\varphi_{\delta,c,\tilde{c}}}(x_2(t)-ct)\le\frac{\delta}{b}$.

Furthermore,

(1) $x_2\in\mathscr{C}^1([0,+\infty),(\zeta_2,+\infty))$;
(2) $x_2(t) = \tilde{c}t + O(1)$ as $t\to+\infty$.

Remark. As $\delta\to 0$, $f_\delta(c)\to f(c)$. It can be verified that $(f_\delta(c))_{\delta\in\left[0,\frac{1}{2}\right)}$ is increasing, so that the convergence occurs from above.

Proof. Recall from Lemma 4.1 that there exists $\zeta\in\mathbb{R}$ such that,

$$\overline{\varphi_{\delta,c}}(\xi-\zeta)\sim \mathrm{e}^{-\lambda_\delta(c)\xi} \text{ as } \xi\to+\infty.$$

Hence, by the intermediate value theorem, for each $t\ge 0$ and each $\zeta_2\in\mathbb{R}$, the equation

$$\begin{aligned}\overline{\varphi_{\delta,c}}(x-ct) &= \overline{w_{\delta,c,\tilde{c}}}(t,x-\zeta_2)\\ &= \mathrm{e}^{-\lambda_\delta(c)(\tilde{c}-c)t}\mathrm{e}^{-\Lambda_\delta(c,\tilde{c})(x-\zeta_2-\tilde{c}t)}\end{aligned}$$

admits at least one solution $x(t)$ provided $\Lambda_\delta(c,\tilde{c}) > \lambda_\delta(c)$. This inequality is true indeed, since it is equivalent to

$$\tilde{c} - \sqrt{\tilde{c}^2 - 4\left(\lambda_\delta(c)(\tilde{c}-c)+1\right)} > 2\lambda_\delta(c),$$

that is to

$$\tilde{c}^2 - 4\lambda_\delta(c)\tilde{c} + 4\left(\lambda_\delta(c)\right)^2 > \tilde{c}^2 - 4\left(\lambda_\delta(c)(\tilde{c}-c)+1\right),$$

that is to

$$\left(\lambda_\delta(c)\right)^2 - c\lambda_\delta(c) + 1 > 0,$$

that is (recalling that $\lambda_\delta(c)$ is characterized by $\left(\lambda_\delta(c)\right)^2 - c\lambda_\delta(c) + 1 - a_\delta = 0$) to the obviously true following inequality,

$$a_\delta > 0.$$

Since $\overline{\varphi_{\delta,c}}(\xi) < 1+\delta$ for all $\xi \in \mathbb{R}$, any such solution satisfies

$$-\ln(1+\delta) < \lambda_\delta(c)(\tilde{c}-c)t + \Lambda_\delta(c,\tilde{c})\left(x(t) - \zeta_2 - \tilde{c}t\right),$$

that is

$$x(t) > \zeta_2 + \left(\tilde{c} - \frac{\lambda_\delta(c)(\tilde{c}-c) + \ln(1+\delta)/t}{\Lambda_\delta(c,\tilde{c})}\right)t.$$

By

$$\lim_{\delta'\to 0}\left(\tilde{c} - \frac{\lambda_{\delta'}(c)(\tilde{c}-c) + \ln(1+\delta')/t}{\Lambda_{\delta'}(c,\tilde{c})}\right) = \tilde{c} - \frac{\lambda(c)(\tilde{c}-c)}{\Lambda(c,\tilde{c})}$$

uniformly for $t \geq 1$, and, due to the preceding observation,

$$\frac{\lambda(c)(\tilde{c}-c)}{\Lambda(c,\tilde{c})} < \tilde{c} - c,$$

we deduce that $x(t) > \zeta_2 + ct$ provided δ is small enough. Therefore the set of solutions is bounded from below and admit an infimum $I(t) > \zeta_2 + ct$. Back to the exponential estimates, it is also clear that the set of solutions is bounded from above and admits therefore a supremum $S(t)$.

Recall that the asymptotic estimate for $\overline{\varphi_{\delta,c}}$ can be differentiated. Setting $g : (t,x) \mapsto \overline{\varphi_{\delta,c}}(x-ct) - \overline{w_{\delta,c,\tilde{c}}}(t, x-\zeta_2)$, we find that for any $t \geq 0$ and any solution $x(t) \in [I(t), S(t)]$,

$$\partial_x g(t, x(t)) = \overline{\varphi_{\delta,c}}(x(t)-ct)\left(\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(x(t)-ct) + \Lambda_\delta(c,\tilde{c})\right).$$

Since

$$\lim_{\xi\to+\infty}\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(\xi) + \Lambda_\delta(c,\tilde{c}) = -\lambda_\delta(c) + \Lambda_\delta(c,\tilde{c}) < 0,$$

we can choose ζ_2 large enough so that $\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(\xi) < 0$ for all $\xi \geq \zeta_2$. Since $x(t) - ct > \zeta_2$ for all $t \geq 0$, we deduce

$$\left(\frac{\overline{\varphi_{\delta,c}}'}{\overline{\varphi_{\delta,c}}}\right)(x(t)-ct) + \Lambda_\delta(c,\tilde{c}) < 0,$$

whence g is decreasing with respect to x in a neighborhood of $x(t)$. This implies directly the uniqueness of $x(t)$, namely $I(t) = S(t)$. From now on, we denote this unique solution $x_2(t)$. Of course, the regularity of x_2 follows directly from the implicit function theorem. The above yields that $x_2(t) - ct \geq \zeta_2$ for all $t \geq 0$.

Provided ζ_2 is large enough, for all $\xi \geq \zeta + \zeta_2$,

$$(1-\delta)\,\mathrm{e}^{-\lambda_\delta(c)\xi} \leq \overline{\varphi_{\delta,c}}(\xi-\zeta) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)\xi}.$$

At $\xi = x_2(t) - ct + \zeta \geq \zeta + \zeta_2$, this reads

$$(1-\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)} \leq \overline{w_{\delta,c,\tilde{c}}}\,(t, x_2(t) - \zeta_2) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)},$$

that is

$$\begin{aligned}\ln(1-\delta) - \lambda_\delta(c)(x_2(t) - ct + \zeta) \leq -\lambda_\delta(c)(\tilde{c}-c)\,t - \Lambda_\delta(c,\tilde{c})(x_2(t) - \zeta_2 - \tilde{c}t)\\ \leq \ln(1+\delta) - \lambda_\delta(c)(x_2(t) - ct + \zeta).\end{aligned}$$

The first inequality yields

$$x_2(t) \leq \frac{\tilde{c}\,(\Lambda_\delta(c,\tilde{c}) - \lambda_\delta(c))\,t - \ln(1-\delta) + \lambda_\delta(c)\,\zeta + \Lambda_\delta(c,\tilde{c})\,\zeta_2}{\Lambda_\delta(c,\tilde{c}) - \lambda_\delta(c)}$$

and the second inequality yields

$$x_2(t) \geq \frac{\tilde{c}\,(\Lambda_\delta(c,\tilde{c}) - \lambda_\delta(c))\,t - \ln(1+\delta) + \lambda_\delta(c)\,\zeta + \Lambda_\delta(c,\tilde{c})\,\zeta_2}{\Lambda_\delta(c,\tilde{c}) - \lambda_\delta(c)}.$$

Together these two estimates give that the asymptotic speed of x_2 is exactly $\tilde{c}$.

Finally, using once again $x_2(t) - ct \geq \zeta_2$, we find

$$\overline{\varphi_{\delta,c}}\,(x_2(t) - ct) \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(x_2(t)-ct+\zeta)} \leq (1+\delta)\,\mathrm{e}^{-\lambda_\delta(c)(\zeta+\zeta_2)},$$

and the inequality

$$\overline{\varphi_{\delta,c}}\,(x_2(t) - ct) \leq \frac{\delta}{b} \text{ for all } t \geq 0$$

is indeed satisfied provided ζ_2 is large enough. □

Thanks again to the intermediate value theorem and the implicit function theorem, we can similarly establish the following lemmas. Since they involve the quantities L, x_L and $h^\star$, we recall that these are defined in Lemma 4.4 and Lemma 4.9 respectively.

Lemma 4.14. *There exists $\delta_1 \in \left(0, \frac{1}{2}\right)$ such that, for all $\delta \in (0, \delta_1)$, $c \in \left[c^\delta_{LLW}, 2\right)$ and $\zeta_3 \in \mathbb{R}$, the equation*

$$\underline{\psi_{\delta,c}}\,(x - ct) = \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)\,d} - \delta\right)t - \zeta_3\right)$$

admits for all $t \geq 0$ an isolated solution $x_3(t) \in \mathbb{R}$ such that

1. *$\underline{\psi_{\delta,c}}\,(x - ct) > \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)\,d} - \delta\right)t - \zeta_3\right)$ for x in a left-sided neighborhood of $x_3(t)$;*
2. *$\underline{\psi_{\delta,c}}\,(x - ct) < \underline{\omega_{\delta,R_\delta}}\left(x - \left(2\sqrt{r(1-2\delta)\,d} - \delta\right)t - \zeta_3\right)$ for x in a right-sided neighborhood of $x_3(t)$;*
3. *for all $t \geq 0$,*

$$-R_\delta < x_3(t) - \left(2\sqrt{r(1-2\delta)\,d} - \delta\right)t - \zeta_3 < x_{\delta,R_\delta}.$$

Furthermore, $x_3 \in \mathscr{C}^1([0, +\infty), \mathbb{R})$.

Lemma 4.15. *For all $\delta \in \left(0, \frac{1}{2}\right)$, $c \geq c^\delta_{LLW}$, $\tilde{c} > 2\sqrt{rd}$ such that $\tilde{c} \geq c$ and $h \in (0, h^\star)$, there exists $\zeta_3^0 \in \mathbb{R}$ such that, for all $\zeta_3 \geq \zeta_3^0$, the equation*

$$\underline{\psi_{\delta,c}}\,(x - ct) = \underline{\pi_{\delta,\tilde{c},h}}\,(x - \tilde{c}t - \zeta_3)$$

admits for all $t \geq 0$ an isolated solution $x_3(t) \in \mathbb{R}$ such that

1. *$\underline{\psi_{\delta,c}}\,(x - ct) > \underline{\pi_{\delta,\tilde{c},h}}\,(x - \tilde{c}t - \zeta_3)$ for x in a left-sided neighborhood of $x_3(t)$;*

(2) $\underline{\psi_{\delta,c}}(x-ct) < \underline{\pi_{\delta,\tilde{c},h}}(x-\tilde{c}t-\zeta_3)$ *for* x *in a right-sided neighborhood of* $x_3(t)$;
(3) *for all* $t \geq 0$,

$$-\sqrt{\frac{c}{rh}} < x_3(t) - \tilde{c}t - \zeta_3 < 0.$$

Furthermore, $x_3 \in \mathscr{C}^1([0,+\infty),\mathbb{R})$.

Remark. We have to point out here that the preceding two lemmas defining x_3 will never be used concurrently and no conflict of notation will occur. Lemma 4.14 will be used only in the proof of Proposition 1.5 whereas Lemma 4.15 will be used only in the proof of Proposition 1.6. In other words, going back to Theorem 1.1 and Theorem 1.3, they corresponds to different values of c_1: Lemma 4.14 corresponds to $c_1 = 2\sqrt{rd}$ whereas Lemma 4.15 corresponds to $c_1 > 2\sqrt{rd}$.

Lemma 4.16. *For all* $\delta \in \left[0, \frac{1}{2}\right)$, $c > 2\sqrt{rd}$, $\eta \in \left(0, \frac{1}{d}\sqrt{c^2-4rd}\right)$, $B > 0$, *there exists* $\zeta_4 \in \mathbb{R}$ *such that the equation*

$$\underline{\pi_{\delta,c,h}}(\xi) = \underline{\beta_{c,B,\eta}}(\xi - \zeta_4)$$

admits an isolated solution $\xi_4 \in \mathbb{R}$ *such that*

(1) $\underline{\pi_{\delta,c,h}}(\xi+\zeta_4) > \underline{\beta_{c,B,\eta}}(\xi)$ *for* ξ *in a left-sided neighborhood of* ξ_4;
(2) $\underline{\pi_{\delta,c,h}}(\xi+\zeta_4) < \underline{\beta_{c,B,\eta}}(\xi)$ *for* ξ *in a right-sided neighborhood of* ξ_4;
(3) $\xi_4 > 0$.

Lemma 4.17. *For all* $c > 2\sqrt{1-a}$, *there exists* $\zeta_0 \in \mathbb{R}$ *such that, for all* $\kappa \in \left(0, \frac{1-a}{2}\right)$, *the equation*

$$\underline{\alpha_L}(x) = \underline{\chi_c}(x - ct - \zeta_0)$$

admits for all $t \geq 0$ *a minimal solution* $x_{0,\kappa}(t) \in \mathbb{R}$ *such that*

(1) $\underline{\alpha_L}(x) > \underline{\chi_c}(x-ct-\zeta_0)$ *for* x *in a left-sided neighborhood of* $x_{0,\kappa}(t)$;
(2) $\underline{\chi_c}(x_{0,\kappa}(0) - \zeta_0) = \kappa$;
(3) $x_L < x_{0,\kappa}(t) < L$.

Furthermore, $x_{0,\kappa} \in \mathscr{C}^1([0,+\infty),(x_L,L))$.

Notice that in the above lemma, $x_{0,\kappa}(0) = \left(\underline{\chi_c}\right)^{-1}(\kappa) + \zeta_0$.

Lemma 4.18. *For all* $c > 2\sqrt{1-a}$, $\tilde{c} \geq c$ *such that* $\tilde{c} \in (f(c) - 4\sqrt{a}, f(c))$, $\delta \in \left[0, \frac{1}{4}\left(-\tilde{c}^2 + 4(\lambda(c)(\tilde{c}-c)+1)\right)\right)$ *and* $\zeta > \left(\underline{\chi_c}\right)^{-1}\left(\frac{\delta}{2}\right)$, *the equation*

$$\underline{\chi_c}(x-ct) = \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)}\underline{z_{c,\tilde{c},\delta}}(t,x-\zeta)$$

admits for all $t \geq 0$ *an isolated solution* $x_1(t) \in \mathbb{R}$ *such that*

(1) $\underline{\chi_c}(x-ct) > \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)}\underline{z_{c,\tilde{c},\delta}}(t,x-\zeta)$ *for* x *in a left-sided neighborhood of* $x_1(t)$;
(2) $\underline{\chi_c}(x-ct) < \frac{\underline{\chi_c}(\zeta)}{\underline{z_{c,\tilde{c},\delta}}(0,X_z)}\underline{z_{c,\tilde{c},\delta}}(t,x-\zeta)$ *for* x *in a right-sided neighborhood of* $x_1(t)$;
(3) $\tilde{c}t + \zeta < x_1(t) < X_z + \tilde{c}t + \zeta$.

Furthermore, $x_1 \in \mathscr{C}^1([0,+\infty),(\zeta,+\infty))$.

Remark. Similarly to the third interface x_3 which is defined in two separate lemmas, the zeroth interface is defined concurrently by Lemma 4.11 and Lemma 4.17 and the first interface is defined concurrently by Lemma 4.12 and Lemma 4.18. Lemma 4.11 will be used only in the proof of Proposition 1.6, Lemma 4.12 will be used only in the proof of Proposition 1.6 and in that of Proposition 1.5, Lemma 4.17 and Lemma 4.18 will be used only in the proof of Proposition 1.4.

There exists a small $\delta^\star \in \left(0, \frac{1}{2}\right)$ such that all the lemmas of this subsection involving a parameter δ can be applied in the range $\delta \in (0, \delta^\star)$. By construction, all the objects depending on δ defined in the preceding subsection are also well-defined in this range.

4.3. **Construction of the super-solutions and sub-solutions for Theorem 1.3.** In this subsection, we prove Proposition 1.6.

Let $c_1 > 2\sqrt{rd}$ and $c_2 > c_{LLW}$ such that $c_1 > c_2$ and $c_1 > f(c_2)$. In order to construct a satisfying approximated speed $c_2^\delta \simeq c_2$, we need to find c_2^δ such that:

(1) $c_2^\delta > c_{LLW}^\delta$;
(2) $c_2^\delta \to c_2$ as $\delta \to 0$;
(3) $c_2 < c_2^\delta < c_1$;
(4) $f_\delta\left(c_2^\delta\right) < c_1$
(5) $\Lambda_\delta\left(c_2^\delta, c_1\right)$ is well-defined;
(6) $\Lambda_\delta\left(c_2^\delta, c_1\right) \leq \Lambda\left(c_2, c_1\right)$.

The condition (6) above is equivalent to $\lambda_\delta\left(c_2^\delta\right)\left(c_1 - c_2^\delta\right) \leq \lambda\left(c_2\right)\left(c_1 - c_2\right)$, that is to

$$\frac{\lambda_\delta\left(c_2^\delta\right)}{\lambda\left(c_2\right)} \leq \frac{\left(c_1 - c_2\right)}{\left(c_1 - c_2^\delta\right)},$$

with a right-hand side necessarily larger than 1 provided (3) above is satisfied. Since the function $(\delta, c) \mapsto \lambda_\delta(c)$ is increasing with respect to c and decreasing with respect to δ, the sign of $\lambda_\delta\left(c_2^\delta\right) - \lambda\left(c_2\right)$ is unclear if we only assume $c_2^\delta > c_2$. Hence some care is needed and we cannot simply take a rough approximation like $c_2^\delta = c_2 + \delta$.

In fact, since $a_\delta < a$ and $\lambda(c_2) < \sqrt{1-a}$, we can choose $\delta \in (0, \delta^\star)$ such that

$$\lambda\left(c_2\right) < \sqrt{1 - a_\delta}.$$

Consequently, the following quantity is well-defined:

$$c_2^\delta = \left(\lambda_\delta^{-1} \circ \lambda\right)\left(c_2\right).$$

Since λ and λ_δ are both decreasing functions and $\lambda\left(c_2\right) < \lambda_\delta\left(c_2\right)$, it follows that $c_2^\delta > c_2$, whence

$$\begin{aligned} 4\left(\lambda_\delta\left(c_2^\delta\right)\left(c_1 - c_2^\delta\right) + 1\right) &= 4\left(\lambda\left(c_2\right)\left(c_1 - c_2^\delta\right) + 1\right) \\ &< 4\left(\lambda\left(c_2\right)\left(c_1 - c_2\right) + 1\right) \\ &< c_1^2, \end{aligned}$$

where the last inequality is due to $c_1 - f\left(c_2\right) > 0$ (see also Subsection 4.1.12). By continuity, we can further assume that δ is so small that

$$\begin{cases} c_{LLW} \leq c_{LLW}^\delta < c_2^\delta \\ c_2 < c_2^\delta < c_1 \\ -4\sqrt{a_\delta} < c_1 - f_\delta\left(c_2^\delta\right) \end{cases}.$$

It follows then, from Subsection 4.1.12, that

$$f_\delta\left(c_2^\delta\right) < f\left(c_2\right) < c_1,$$

whence the quantity $\Lambda_\delta\left(c_2^\delta, c_1\right)$ is well-defined. By definition, it satisfies

$$\begin{aligned}\Lambda_\delta\left(c_2^\delta, c_1\right) &= \frac{1}{2}\left(c_1 - \sqrt{c_1^2 - 4\left(\lambda_\delta\left(c_2^\delta\right)\left(c_1 - c_2^\delta\right) + 1\right)}\right)\\ &= \frac{1}{2}\left(c_1 - \sqrt{c_1^2 - 4\left(\lambda\left(c_2\right)\left(c_1 - c_2^\delta\right) + 1\right)}\right)\\ &< \frac{1}{2}\left(c_1 - \sqrt{c_1^2 - 4\left(\lambda\left(c_2\right)\left(c_1 - c_2\right) + 1\right)}\right),\end{aligned}$$

so that $\Lambda_\delta\left(c_2^\delta, c_1\right) < \Lambda\left(c_2, c_1\right)$.

4.3.1. *Super-solution.* The pair $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ is defined by (see Figure 4.2)

$$\overline{u_\delta}\left(t,x\right) = \begin{cases}\min\left(1, \overline{\varphi_{\delta,c_2^\delta}}\left(x - c_2^\delta t - \zeta_{1,\kappa}\right)\right) & \text{if } x < x_2\left(t\right) + \zeta_{1,\kappa}\\ \overline{w_{\delta,c_2^\delta,c_1}}\left(t, x - \zeta_{1,\kappa} - \zeta_2\right) & \text{if } x \geq x_2\left(t\right) + \zeta_{1,\kappa}\end{cases},$$

$$\underline{v_\delta}\left(t,x\right) = \begin{cases}\max\left(0, \theta_{\delta,c_2^\delta,A_\kappa}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_{1,\kappa} + c_2^\delta t\\ \underline{\psi_{\delta,c_2^\delta}}\left(x - c_2^\delta t - \zeta_{1,\kappa}\right) & \text{if } x \in \left[\xi_{1,\kappa} + c_2^\delta t, x_3\left(t\right) + \zeta_{1,\kappa}\right)\\ \underline{\pi_{\delta,c_1,h}}\left(x - c_1 t - \zeta_{1,\kappa} - \zeta_3\right) & \text{if } x \in \left[x_3\left(t\right) + \zeta_{1,\kappa}, \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3\right)\\ \underline{\beta_{c_1,B,\eta_\beta}}\left(x - c_1 t - \zeta_{1,\kappa} - \zeta_3 - \zeta_4\right) & \text{if } x \geq \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3\end{cases},$$

where

- $\kappa \in (0, \delta]$ is fixed so small that $\zeta_{1,\kappa} - \xi_{1,\kappa} + x_2\left(0\right)$ is large enough so that for all $t \geq 0$, $\xi_{1,\kappa} + c_2^\delta t < x_2\left(t\right) + \zeta_{1,\kappa}$ (see Lemma 4.12(4) and Lemma 4.13 and use $x_2(t) \geq c_1 t + O(1) \geq c_2^\delta t + O(1)$);
- ζ_3 is fixed so large that, for all $t \geq 0$, $x_2\left(t\right) < x_3\left(t\right)$ (by Lemma 4.13 Lemma 4.15, $x_2(t) - c_1 t$ and $x_3(t) - c_1 t$ are both bounded uniformly in $t \geq 0$, whence we can translate $x_3(t)$ to the right by increasing ζ_3);
- $h = \frac{h^\star}{2}$;
- $\eta_\beta = \frac{1}{2}\min\left(\frac{1}{d}\sqrt{c_1^2 - 4rd}, \Lambda_\delta\left(c_2^\delta, c_1\right)\right)$;
- $B = \mathrm{e}^{\Lambda_\delta\left(c_2^\delta, c_1\right)\xi_\beta} 2\overline{u_\delta}\left(0, \zeta_{1,\kappa} + \zeta_3 + \zeta_4\right)$.

The inequality

$$x_3\left(t\right) + \zeta_{1,\kappa} < \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3$$

is guaranteed by Lemma 4.15 and Lemma 4.16 which respectively show that $x_3\left(t\right) < c_1 t + \zeta_3$ and $\xi_4 > 0$. In conclusion, we have

$$\xi_{1,\kappa} + c_2^\delta t < x_2\left(t\right) + \zeta_{1,\kappa} < x_3\left(t\right) + \zeta_{1,\kappa} < \xi_4 + c_1 t + \zeta_{1,\kappa} + \zeta_3 \quad \text{for all } t \geq 0,$$

i.e. $\underline{v_\delta}$ is well-defined for all $t \geq 0$.

4.3.2. *Sub-solution.* First define the pair $(\underline{u}, \overline{v})$ by (see Figure 4.1)

$$\underline{u}\left(t,x\right) = \begin{cases}\underline{\chi_{c_2}}\left(x - c_2 t + \zeta_0\right) & \text{if } x < x_0\left(t\right)\\ \underline{w_{c_2,c_1,A,\eta_w}}\left(t,x\right) & \text{if } x \geq x_0\left(t\right)\end{cases},$$

$$\overline{v}\left(t,x\right) = \min\left(1, \mathrm{e}^{-\lambda_v\left(c_1\right)\left(x - c_1 t\right)}\right),$$

where $\eta_w = \frac{1}{2}\min\left(\sqrt{c_1^2 - 4\left(\lambda\left(c_2\right)\left(c_1 - c_2\right) + 1\right)}, \lambda_v\left(c_1\right)\right)$.

The function $\underline{u}$ depends on a constant $A > 0$ which will be fixed later on.

4.3.3. *Up to some translations, the sub-solution $(\underline{u}, \overline{v})$ is initially smaller than the super-solution $(\overline{u_\delta}, \underline{v_\delta})$.* First, let $\underline{V} : \mathbb{R} \to [0,1]$ be the smallest nonincreasing continuous function such that

$$\underline{v_\delta}(0,x) \leq \underline{V}(x) \text{ for all } x \in \mathbb{R}$$

and let $\zeta_5 \in \mathbb{R}$ such that, for all $t \geq 0$, $x \mapsto \underline{v_\delta}(t, x + c_1 t)$ is $\mathscr{C}^1$ and nonincreasing in $(\zeta_5, +\infty)$. The existence of ζ_5 follows from the fact that the last discontinuity of $\partial_x \underline{v_\delta}$ and the last local maximum of $\underline{v_\delta}$ move both at most at speed c_1. The limit of $\underline{V}$ at $-\infty$ is smaller than 1 and $\underline{V}(x) = \underline{v_\delta}(0,x)$ if $x > \zeta_5$. Therefore, since $\overline{v}$ and $\underline{v_\delta}$ have the same exponential decay at $+\infty$, there exists $\zeta_6 \geq \zeta_5$ such that:

(1) for all $t \geq 0$, $x \mapsto \underline{v_\delta}(t, x + c_1 t)$ is $\mathscr{C}^1$ and nonincreasing in $(\zeta_6, +\infty)$;
(2) for all $x \in \mathbb{R}$, $\underline{v_\delta}(0,x) \leq \underline{V}(x) \leq \overline{v}(0, x - \zeta_6)$.

Notice that with this definition of ζ_5 and ζ_6, the irregularity of $\overline{v}$ is initially on the right of the last irregularity of $\underline{v_\delta}$. Since the distance between these two points is nondecreasing with respect to t, it is bounded from below by the initial distance.

Next, quite similarly, we define $\zeta_7 \in \mathbb{R}$ such that

$$\underline{u}(0, x + \zeta_7) \leq \overline{u_\delta}(0,x) \text{ for all } x \in \mathbb{R}.$$

The irregularity of $\underline{u}$ moves faster than the first irregularity of $\overline{u_\delta}$ (as $c_1 > c_2^\delta$), whence it is impossible to guarantee that they stay ordered. This is not a major issue but some additional care will be required later on. Still, without loss of generality, we assume that ζ_7 is so large that the irregularity of $\underline{u}$ and the second (last) irregularity of $\overline{u_\delta}$, which both move at speed c_1, stay ordered.

4.3.4. *Cleansing.* Now that all required translations are done, we fix

$$A = 2\mathrm{e}^{\lambda_v(c_1)x_w}\mathrm{e}^{\lambda_v(c_1)(\zeta_6+\zeta_7)},$$

and thus there remains only one parameter: δ.

From now on, all the subscripts referring to fixed parameters are omitted. Furthermore, since all the properties of the functions $\underline{\chi}$, $\underline{w}$, $\overline{w_\delta}$, $\left(\overline{\varphi_\delta}, \underline{\psi_\delta}\right)$, $\underline{\theta_\delta}$, $\underline{\omega_\delta}$, $\underline{\pi_\delta}$, $\underline{\beta}$ we are interested in are invariant by translation, we assume that these functions were correctly normalized from the beginning, so that $\zeta_{1,\kappa} = \zeta_3 = \zeta_4 = \zeta_7 = 0$, and we fix $x_0(0) = 0$. Similarly, we define $C_\delta = \mathrm{e}^{\lambda_v(c_1)\zeta_6} > 0$ so that $\overline{v_\delta}(t,x) = \min\left(1, C_\delta \mathrm{e}^{-\lambda_v(c_1)(x-c_1 t)}\right)$ and x_w and ξ_β are redefined so that Lemma 4.7 and Lemma 4.8 stay true as stated. Notice that $\underline{\chi}$, $\underline{w}$, $\underline{\beta}$, $\underline{u}$ and $\overline{v}$ now depend on δ because of these various normalizations (and consequently these notations come with a subscript δ from now on).

To summarize, the super- and sub-solutions are now defined as follows:

$$\underline{u_\delta}(t,x) = \begin{cases} \underline{\chi_\delta}(x - c_2 t) & \text{if } x < x_0(t) \\ \underline{w_\delta}(t,x) & \text{if } x \geq x_0(t) \end{cases},$$

$$\overline{v_\delta}(t,x) = \min\left(1, C_\delta \mathrm{e}^{-\lambda_v(c_1)(x-c_1 t)}\right),$$

$$\overline{u_\delta}(t,x) = \begin{cases} \min\left(1, \overline{\varphi_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < x_2(t) \\ \overline{w_\delta}(t,x) & \text{if } x \geq x_2(t) \end{cases},$$

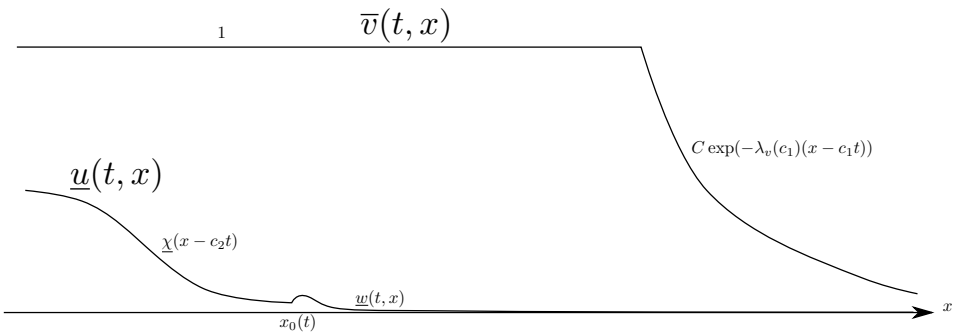


FIGURE 4.1. Sub-solution $\left(\underline{u_\delta}, \overline{v_\delta}\right)$ for Theorem 1.3

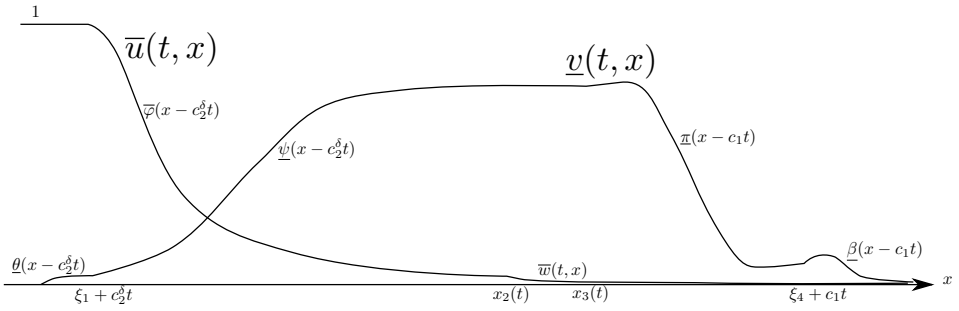


FIGURE 4.2. Super-solution $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ for Theorem 1.3

$$\underline{v_\delta}(t,x) = \begin{cases} \max\left(0, \underline{\theta_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_1 + c_2^\delta t \\ \underline{\psi_\delta}\left(x - c_2^\delta t\right) & \text{if } x \in \left[\xi_1 + c_2^\delta t, x_3(t)\right) \\ \underline{\pi_\delta}(x - c_1 t) & \text{if } x \in \left[x_3(t), \xi_4 + c_1 t\right) \\ \underline{\beta_\delta}(x - c_1 t) & \text{if } x \geq \xi_4 + c_1 t \end{cases}.$$

Furthermore, the interfaces satisfy, for all $t \geq 0$,

$$\begin{cases} x_0(t) < x_2(t) \\ \xi_1 + c_2^\delta t < x_2(t) \\ x_2(t) < x_3(t) \\ \xi_4 + c_1 t < \frac{\ln C_\delta}{\lambda_v(c_1)} + c_1 t \end{cases}.$$

4.3.5. *Verification of the differential inequalities.* Let us point out that by Theorem 2.3 and Theorem 2.4 and by construction of the pairs $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ and $\left(\underline{u_\delta}, \overline{v_\delta}\right)$, it suffices to verify the differential inequalities

$$P\left(\overline{u_\delta}, \underline{v_\delta}\right) \succeq F\left(\overline{u_\delta}, \underline{v_\delta}\right) \tag{4.10}$$

and

$$P\left(\underline{u_\delta}, \overline{v_\delta}\right) \preceq F\left(\underline{u_\delta}, \overline{v_\delta}\right) \tag{4.11}$$

where the functions are regular in order to establish that $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ and $\left(\underline{u_\delta}, \overline{v_\delta}\right)$ are indeed a super-solution and a sub-solution of (1.1) respectively. Also, the differential inequalities can also be verified before the translations are performed.

In what follows, for the sake of brevity, we voluntarily omit the mentions of the points (t,x), $x-c_1t$, $x-c_2t$ or $x-c_2^\delta t$ where the various functions are evaluated. In view of the construction, it should be unambiguous.

First, we consider (4.10). By Lemma 4.10 and Lemma 4.12, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(1,\underline{\theta_\delta}\left(x-c_2^\delta t\right)\right),$$

we find $\underline{\theta_\delta}\le\kappa\le\delta$ and

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}a\underline{\theta_\delta}\\ -c_2^\delta\underline{\theta_\delta}'-d\underline{\theta_\delta}''-r\underline{\theta_\delta}\left(1-\underline{\theta_\delta}-b\right)\end{pmatrix}\\&=\begin{pmatrix}a\underline{\theta_\delta}\\ -r\underline{\theta_\delta}\left(\delta-\underline{\theta_\delta}\right)\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

By definition of $\left(\overline{\varphi_\delta},\underline{\psi_\delta}\right)$, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(1,\underline{\psi_\delta}\left(x-c_2^\delta t\right)\right),$$

we find, using $\underline{\psi_\delta}\le 1+\delta$,

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}a\underline{\psi_\delta}\\ -c_2^\delta\underline{\psi_\delta}'-d\underline{\psi_\delta}''-r\underline{\psi_\delta}\left(1-\underline{\psi_\delta}-b\right)\end{pmatrix}\\&=\begin{pmatrix}a\underline{\psi_\delta}\\ -r\underline{\psi_\delta}\left(2\delta+b\left(\overline{\varphi_\delta}-1\right)\right)\end{pmatrix}\\&\succeq\begin{pmatrix}0\\ -2r\delta\underline{\psi_\delta}\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

Similarly, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(\overline{\varphi_\delta}\left(x-c_2^\delta t\right),\underline{\psi_\delta}\left(x-c_2^\delta t\right)\right),$$

we find

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}-c_2^\delta\overline{\varphi_\delta}'-\overline{\varphi_\delta}''-\overline{\varphi_\delta}\left(1-\overline{\varphi_\delta}-a\underline{\psi_\delta}\right)\\ -c_2^\delta\underline{\psi_\delta}'-d\underline{\psi_\delta}''-r\underline{\psi_\delta}\left(1-\underline{\psi_\delta}-b\overline{\varphi_\delta}\right)\end{pmatrix}\\&=\begin{pmatrix}\delta\overline{\varphi_\delta}\\ -2r\delta\underline{\psi_\delta}\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

By Lemma 4.5 and Lemma 4.13, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(\overline{w_\delta}(t,x),\underline{\psi_\delta}\left(x-c_2^\delta t\right)\right),$$

we find, using $\overline{w_\delta}\le\overline{\varphi_\delta}$ (Lemma 4.13(1)),

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}\partial_t\overline{w_\delta}-\partial_{xx}\overline{w_\delta}-\overline{w_\delta}\left(1-\overline{w_\delta}-a\underline{\psi_\delta}\right)\\ -c_2^\delta\underline{\psi_\delta}'-d\underline{\psi_\delta}''-r\underline{\psi_\delta}\left(1-\underline{\psi_\delta}-b\overline{w_\delta}\right)\end{pmatrix}\\&=\begin{pmatrix}\overline{w_\delta}\left(\overline{w_\delta}+a\underline{\psi_\delta}\right)\\ -r\underline{\psi_\delta}\left(2\delta+b\left(\overline{\varphi_\delta}-\overline{w_\delta}\right)\right)\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

By Lemma 4.9 and Lemma 4.13, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(\overline{w_\delta}\left(t,x\right),\underline{\pi_\delta}\left(x-c_1t\right)\right),$$

we find, using $\overline{w_\delta}\leq\frac{\delta}{b}$ (Lemma 4.13(3)),

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}\partial_t\overline{w_\delta}-\partial_{xx}\overline{w_\delta}-\overline{w_\delta}\left(1-\overline{w_\delta}-a\underline{\pi_\delta}\right)\\-c_1\underline{\pi_\delta}'-d\underline{\pi_\delta}''-r\underline{\pi_\delta}\left(1-\underline{\pi_\delta}-b\overline{w_\delta}\right)\end{pmatrix}\\&\succeq\begin{pmatrix}\overline{w_\delta}\left(\overline{w_\delta}+a\underline{\pi_\delta}\right)\\-r\underline{\pi_\delta}\left(\delta+\underline{\pi_\delta}-b\overline{w_\delta}\right)\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

By Lemma 4.8, Lemma 4.13 and construction of B, for all (t,x) such that

$$\left(\overline{u_\delta},\underline{v_\delta}\right)(t,x)=\left(\overline{w_\delta}\left(t,x\right),\underline{\beta_\delta}\left(x-c_1t\right)\right),$$

we find, By definition of $\underline{\beta_\delta}$, and that of η_β in Subsection 4.3.1,

$$\begin{aligned}P\left(\overline{u_\delta},\underline{v_\delta}\right)-F\left(\overline{u_\delta},\underline{v_\delta}\right)&=\begin{pmatrix}\partial_t\overline{w_\delta}-\partial_{xx}\overline{w_\delta}-\overline{w_\delta}\left(1-\overline{w_\delta}-a\underline{\beta_\delta}\right)\\-c_1\underline{\beta_\delta}'-d\underline{\beta_\delta}''-r\underline{\beta_\delta}\left(1-\underline{\beta_\delta}-b\overline{w_\delta}\right)\end{pmatrix}\\&\succeq\begin{pmatrix}\overline{w_\delta}\left(\overline{w_\delta}+a\underline{\beta_\delta}\right)\\rb\underline{\beta_\delta}\left(\overline{w_\delta}-B\mathrm{e}^{-\Lambda_\delta\left(c_2^\delta,c_1\right)(x-c_1t+\xi_\beta)}\right)\end{pmatrix}\\&\succeq(0,0).\end{aligned}$$

Finally, we consider the differential inequalities associated with $(\underline{u_\delta},\overline{v_\delta})$. By definition of $\underline{\chi_\delta}$, for all (t,x) such that

$$\left(\underline{u_\delta},\overline{v_\delta}\right)(t,x)=\left(\underline{\chi_\delta}\left(x-c_2t\right),1\right),$$

we find

$$\begin{aligned}P\left(\underline{u_\delta},\overline{v_\delta}\right)-F\left(\underline{u_\delta},\overline{v_\delta}\right)&=\begin{pmatrix}-c_2\underline{\chi_\delta}'-\underline{\chi_\delta}''-\underline{\chi_\delta}\left(1-a-\underline{\chi_\delta}\right)\\rb\underline{\chi_\delta}\end{pmatrix}\\&=\begin{pmatrix}0\\rb\underline{\chi_\delta}\end{pmatrix}\\&\preceq(0,0).\end{aligned}$$

By Lemma 4.7, Lemma 4.11 and by construction of $A=2C_\delta\mathrm{e}^{\lambda_v(c_1)x_w}$, for all (t,x) such that

$$\left(\underline{u_\delta},\overline{v_\delta}\right)(t,x)=\left(\underline{w_\delta}\left(t,x\right),1\right),$$

we find, using $C_\delta\mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\geq1$,

$$\begin{aligned}P\left(\underline{u_\delta},\overline{v_\delta}\right)-F\left(\underline{u_\delta},\overline{v_\delta}\right)&=\begin{pmatrix}\partial_t\underline{w_\delta}-\partial_{xx}\underline{w_\delta}-\underline{w_\delta}\left(1-\underline{w_\delta}-a\right)\\rb\underline{w_\delta}\end{pmatrix}\\&\preceq\begin{pmatrix}a\underline{w_\delta}\left(1-A\mathrm{e}^{-\lambda_v(c_1)(x-c_1t+x_w)}\right)\\0\end{pmatrix}\\&\preceq\begin{pmatrix}a\underline{w_\delta}\left(1-2C_\delta\mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\right)\\0\end{pmatrix}\\&\preceq(0,0).\end{aligned}$$

Similarly, for all (t,x) such that

$$\left(\underline{u_\delta},\overline{v_\delta}\right)(t,x)=\left(\underline{w_\delta}\left(t,x\right),C_\delta\mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\right),$$

we find

$$P\left(\underline{u_\delta},\overline{v_\delta}\right)-F\left(\underline{u_\delta},\overline{v_\delta}\right)=\begin{pmatrix}\partial_t\underline{w_\delta}-\partial_{xx}\underline{w_\delta}-\underline{w_\delta}\left(1-\underline{w_\delta}-aC_\delta \mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\right)\\ rb\underline{w_\delta}\end{pmatrix}$$
$$\preceq\begin{pmatrix}a\underline{w_\delta}\mathrm{e}^{-\lambda_v(c_1)(x-c_1t)}\left(C_\delta-A\mathrm{e}^{-\lambda_v(c_1)x_w}\right)\\ 0\end{pmatrix}$$
$$\preceq(0,0).$$

4.4. **Construction of the super-solutions for Theorem 1.1.** In this subsection, we prove Proposition 1.5.

Let $c_2>\max\left(c_{LLW},f^{-1}\left(2\sqrt{rd}\right)\right)$ and let $\delta\in(0,\delta^\star)$ such that $c^\delta_{LLW}<c_2$. Define

$$c_1^\delta=2\sqrt{r\left(1-2\delta\right)d}-\delta.$$

Recall from Subsection 4.1.12 that, given a fixed $\tilde{c}$, the function $c\mapsto\Lambda_\delta\left(c,\tilde{c}\right)$ is decreasing and bijectively maps $\left[2\sqrt{1-a_\delta},+\infty\right)$ onto

$$\left(\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4\left(a_\delta+1\right)}\right),\frac{1}{2}\left(\tilde{c}-\sqrt{\tilde{c}^2-4\left(\tilde{c}\sqrt{1-a_\delta}+2a_\delta-1\right)}\right)\right].$$

Thus the equation

$$\Lambda_\delta\left(c_2^\delta,c_1^\delta\right)=\Lambda\left(c_2,2\sqrt{rd}\right).$$

admits a unique solution c_2^δ if and only if

(4.12)
$$c_1^\delta-\sqrt{\left(c_1^\delta\right)^2-4\left(a_\delta+1\right)}<2\Lambda\left(c_2,2\sqrt{rd}\right)\leq c_1^\delta-\sqrt{\left(c_1^\delta\right)^2-4\left(c_1^\delta\sqrt{1-a_\delta}+2a_\delta-1\right)}.$$

Since $c_2\in(c_{LLW},+\infty)\subset\left(2\sqrt{1-a},+\infty\right)$, we have by the above discussion

$$2\sqrt{rd}-\sqrt{\left(2\sqrt{rd}\right)^2-4\left(a+1\right)}<2\Lambda\left(c_2,2\sqrt{rd}\right)<2\sqrt{rd}-\sqrt{\left(2\sqrt{rd}\right)^2-4\left(2\sqrt{rd}\sqrt{1-a}+2a-1\right)}.$$

By the facts that $c_1^\delta\to2\sqrt{rd}$ and $a_\delta\to a$ as $\delta\to0$, we deduce that we can in fact assume that δ is so small that (4.12) holds. Hence c_2^δ is well-defined.

Furthermore, by continuity, c_2^δ converges to c_2 as $\delta\to0$, and thus $c_2^\delta>c_{LLW}^\delta$. In summary, we can assume that δ is so small that c_1^δ and c_2^δ are well-defined, respectively close to c_1 and c_2, and satisfy the following:

$$c_{LLW}^\delta<c_2^\delta\quad\text{ and }\quad\Lambda_\delta\left(c_2^\delta,c_1^\delta\right)=\Lambda\left(c_2,2\sqrt{rd}\right)\tag{4.13}$$

4.4.1. *Super-solution.* The pair $\left(\overline{u_\delta},\underline{v_\delta}\right)$ is defined by

$$\overline{u_\delta}\left(t,x\right)=\begin{cases}\min\left(1,\overline{\varphi_{\delta,c_2^\delta}}\left(x-c_2^\delta t-\zeta_{1,\kappa}\right)\right) & \text{if } x<x_2\left(t\right)+\zeta_{1,\kappa}\\ \overline{w}_{\delta,c_2^\delta,c_1^\delta}\left(t,x-\zeta_{1,\kappa}-\zeta_2\right) & \text{if } x\geq x_2\left(t\right)+\zeta_{1,\kappa}\end{cases},$$

$$\underline{v_\delta}\left(t,x\right)=\begin{cases}\max\left(0,\theta_{\delta,c_2^\delta,A_\kappa}\left(x-c_2^\delta t\right)\right) & \text{if } x<\xi_{1,\kappa}+c_2^\delta t\\ \underline{\psi_{\delta,c_2^\delta}}\left(x-c_2^\delta t-\zeta_{1,\kappa}\right) & \text{if } x\in\left[\xi_{1,\kappa}+c_2^\delta t,x_3\left(t\right)+\zeta_{1,\kappa}\right)\\ \underline{\omega_{\delta,R_\delta}}\left(x-c_1^\delta t-\zeta_{1,\kappa}-\zeta_3\right) & \text{if } x\geq x_3\left(t\right)+\zeta_{1,\kappa}\end{cases},$$

where

- $\kappa\in(0,\delta]$ is fixed so small that, for all $t\geq0$, $\xi_{1,\kappa}+c_2^\delta t<x_2\left(t\right)+\zeta_{1,\kappa}$ (see Lemma 4.12);

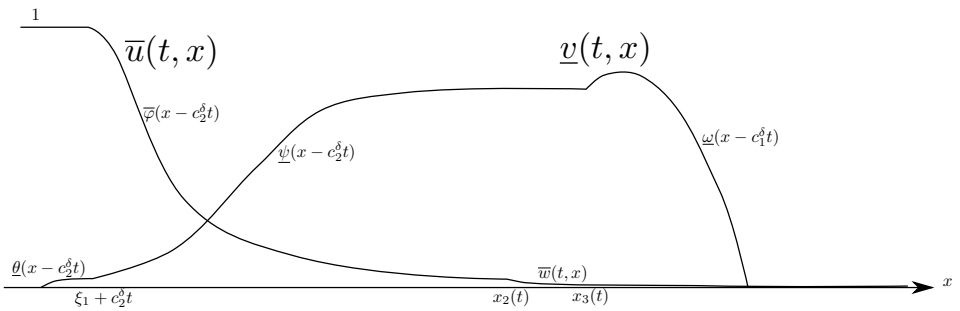


FIGURE 4.3. Super-solution $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ for Theorem 1.1

- ζ_3 is fixed so large that, for all $t \geq 0$, $x_2(t) < x_3(t)$ (see Lemma 4.14).

Thus, we have

$$\xi_{1,\kappa} + c_2^\delta t < x_2(t) + \zeta_{1,\kappa} < x_3(t) + \zeta_{1,\kappa} \quad \text{for all } t \geq 0.$$

4.4.2. *Cleansing.* Just as in the previous case, we normalize and simplify the notations so that $x_2(0) = 0$ and the super-solution is defined as follows:

$$\overline{u_\delta}(t,x) = \begin{cases} \min\left(1, \overline{\varphi_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < x_2(t) \\ \overline{w_\delta}(t,x) & \text{if } x \geq x_2(t) \end{cases},$$

$$\underline{v_\delta}(t,x) = \begin{cases} \max\left(0, \underline{\theta_\delta}\left(x - c_2^\delta t\right)\right) & \text{if } x < \xi_1 + c_2^\delta t \\ \underline{\psi_\delta}\left(x - c_2^\delta t\right) & \text{if } x \in \left[\xi_1 + c_2^\delta t, x_3(t)\right) \\ \underline{\omega_\delta}\left(x - c_1^\delta t\right) & \text{if } x \geq x_3(t) \end{cases}.$$

4.4.3. *Verification of the differential inequalities.* Just as in the previous case, we verify that $\left(\overline{u_\delta}, \underline{v_\delta}\right)$ is indeed a super-solution. The only new component to account for is $\underline{\omega_\delta}$, which can be handled easily in view of its definition.

4.5. **Construction of the sub-solutions for Theorem 1.2.** In this subsection, we prove Proposition 1.4.

Let $c_1 > 2\sqrt{rd}$ and $c_2 > c_{LLW}$ such that $c_1 \geq c_2$ and $c_1 < f(c_2)$. Let $c > c_2$ so close to c_2 that $c_1 < f(c)$ and let $\delta \in (0, \delta^\star)$ and

$$\tilde{c} \in \left(\max\left(c_1, f(c) - 4\sqrt{a}\right), f(c)\right).$$

4.5.1. *Sub-solution.* The pair $\left(\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}\right)$ is defined by

$$\underline{u_{\delta,\zeta,\kappa}}(t,x) = \begin{cases} \underline{\alpha_L}(x) & \text{if } x < x_{0,\kappa}(t) \\ \underline{\chi_c}(x - ct - \zeta_0) & \text{if } x \in [x_{0,\kappa}(t), x_1(t) + \zeta_0) \\ \frac{\underline{\chi_c}(\zeta - \zeta_0)}{\underline{z_{c,\tilde{c},\delta}}(0, X_z - \zeta_0)} \underline{z_{c,\tilde{c},\delta}}(t, x - \zeta_0 - \zeta) & \text{if } x \geq x_1(t) + \zeta_0 \end{cases},$$

$$\overline{v_{\delta,\zeta}}(t,x) = \min\left(1, \mathrm{e}^{-\lambda_v(\tilde{c})(x - y_{\delta,\zeta} - \tilde{c}t)}\right),$$

where

$$y_{\delta,\zeta} = \frac{\ln\delta - \ln(2a)}{\lambda_v(\tilde{c})} + \zeta_0 + \zeta$$

and $\kappa \in \left(0, \min\left(\frac{1-a}{2}, \frac{\delta}{2}\right)\right)$ and $\zeta > L$ are parameters.

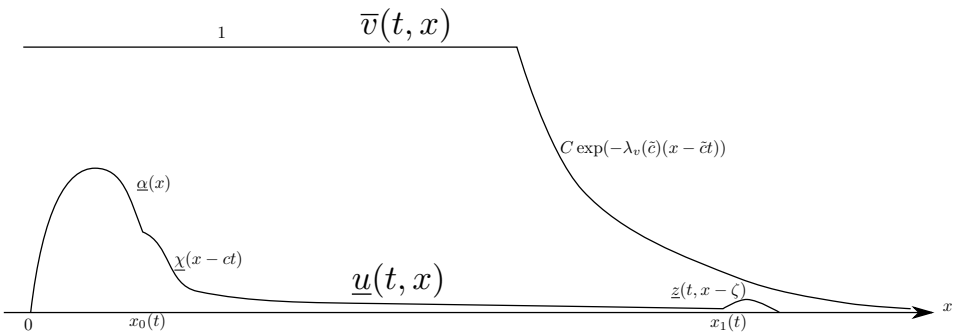


FIGURE 4.4. Sub-solution $\left(\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}\right)$ for Theorem 1.2

Note that $\overline{v_{\delta,\zeta}}(t,x) \leq \frac{\delta}{2a}$ for all $x \geq \zeta_0 + \zeta + \tilde{c}t$. By Lemma 4.18(3), we have $x_1(t) > \tilde{c}t + \zeta$ and thus $\overline{v_{\delta,\zeta}}(t,x) \leq \frac{\delta}{2a}$ for all $x \geq x_1(t) + \zeta_0$. Notice also that the support of $x \mapsto \underline{u_{\delta,\zeta,\kappa}}(0,x)$ is included in $[0, L + \zeta + 2R_z]$.

4.5.2. *Cleansing.* Again, we normalize and simplify:

$$\underline{u_{\delta,\zeta,\kappa}}(t,x) = \begin{cases} \underline{\alpha}(x) & \text{if } x < x_0(t) \\ \underline{\chi}(x-ct) & \text{if } x \in [x_0(t), x_1(t)) \\ \underline{z_\delta}(t, x-\zeta) & \text{if } x \geq x_1(t) \end{cases},$$

$$\overline{v_{\delta,\zeta}}(t,x) = \min\left(1, C_\delta \mathrm{e}^{-\lambda_v(\tilde{c})(x-\zeta-\tilde{c}t)}\right).$$

4.5.3. *Verification of the differential inequalities.* Again, we verify that $\left(\underline{u_{\delta,\zeta,\kappa}}, \overline{v_{\delta,\zeta}}\right)$ is a sub-solution. The only new components are $\underline{\alpha}$ and $\underline{z_\delta}$, the latter being handled with Lemma 4.6.

5. Discussion

As a preliminary remark, let us point out that analogous results can be obtained with the exact same method for the coexistence case $a < 1$, $b < 1$. In that case the solutions are characterized by a profile connecting $(0,0)$ to $(0,1)$ to $\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$.

5.1. **On the consequences of Theorem 1.1, Theorem 1.2 and Theorem 1.3.** Consider here the Cauchy problem associated with Theorem 1.1, namely the initial condition u_0 of the slower and stronger species has a support included in $(-\infty, 0]$ while the initial condition v_0 of the faster and weaker species has compact support. Treating $2\sqrt{rd}$ as a parameter, Theorem 1.1 says that, while the species v always spreads at speed $2\sqrt{rd}$ if it persists, the species u:

- lags behind v and spreads at speed c_{LLW} if $2\sqrt{rd} \geq f(c_{LLW})$;
- lags behind v and spreads at speed $f^{-1}(2\sqrt{rd}) > c_{LLW}$ if $2 < 2\sqrt{rd} < f(c_{LLW})$;
- drives v to extinction and spreads at speed 2 if $2\sqrt{rd} < 2$.

In general, it is unclear whether $c_{LLW} = 2\sqrt{1-a}$ or not. Hence the condition $2\sqrt{rd} \geq f(c_{LLW})$ might be difficult to check in practice. However, since

$$\max_{c \in \left[2\sqrt{1-a}, 2\right]} f(c) = f(2\sqrt{1-a}) = 2\left(\sqrt{1-a} + \sqrt{a}\right),$$

the condition $\sqrt{rd} > \sqrt{1-a} + \sqrt{a}$ always implies $2\sqrt{rd} > f(c_{LLW})$ and consequently always implies that u invades at speed c_{LLW}. In particular, the maximum of $a \mapsto \sqrt{1-a} + \sqrt{a}$ in $(0,1)$ being $\sqrt{2}$, if $rd > 2$, then u invades at speed c_{LLW} independently of the value of a and b. In ecological terms, if v is a sufficiently fast invader, then it decelerates optimally any stronger and slower competitor.

Applied to a pair $(u_0, 0)$, the nonexistence result reduces to a well-known property of the KPP equation satisfied by u in isolation: all solutions spread at least at speed 2.

In view of Figure 1.1, it is tempting to refer to the pair of speeds

$$(c_2^\star, c_1^\star) = \left(\max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd}\right)\right), 2\sqrt{rd}\right)$$

as a "minimal pair". But in our opinion, such a terminology would be misleading. Indeed, a very natural conjecture in view of the KPP literature is that the propagating terraces attract initial data with appropriate exponential decays ($\lambda_v(c_1)$ for v_0 and $\Lambda(c_2, c_1)$ for u_0). Assume this conjecture is true indeed, assume $2 < 2\sqrt{rd} < f(c_{LLW})$ and fix a compactly supported or Heavyside-like u_0. Then decreasing the decay of v_0 will accelerate the invasion of v but decelerate that of u (with the obvious convention that a compactly supported v_0 has an infinite decay).

More generally, this paper presents several results that are complementary to that of Lewis, Li and Weinberger, with several surprising consequences. It shows that c_{LLW} is not always the relevant speed when predicting the speed of the invasion of u in the territory of v. The initial spatial distribution of v has to be taken into account and in particular, it can be inappropriate to approximate a very large territory by an unbounded territory. Also, even if c_{LLW} is linearly determined and therefore only depends on a, the speed of u might still depend on rd.

Our acceleration result can be heuristically understood as a pulled property, in the sense that very small densities of u on the right of the territory of v are still sufficiently large to increase the speed of u on the left of the territory of v. Of course, it would be interesting to verify the existence of such pulled accelerations in real biological invasions. Indeed, at first glance, our result might very well be described by ecological modelers as a strong case against diffusion equations: dispersal operators preserving compact supports, like the nonlinear diffusion of the porous form, $\partial_t u - \Delta(u^m)$, will never lead to such a result. Actually, Guo and Wu [15] considered the analogous problem in a model with free boundaries –and therefore without long-range effects. Their results suggest the existence of two different spreading speeds. Subsequently, Du and Wu [7] considered the problem in higher dimension with radial symmetry. Under appropriate assumptions on the initial conditions, they showed that the spreading speed of the slower species is uniquely determined by the semi-wave system whose solutions were classified by Du, Wang and Zhou [6]. Therefore, in such a model, the second speed is never enhanced.

5.2. **On the boundary of the set of admissible pairs of speeds.** In the present paper, the question of existence at the boundary of the set of admissible pairs is not settled. It is in fact more subtle than expected.

Assuming only $2\sqrt{rd} > 2$, this boundary is naturally partitioned as $V \cup G \cup H \cup D$, where

$$V = \{c_{LLW}\} \times \left(\max\left(2\sqrt{rd}, f(c_{LLW})\right), +\infty\right),$$

$$G = \left\{ (c, f(c)) \mid c \in \left[c_{LLW}, \max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd} \right) \right) \right) \right\},$$

$$H = \left[\max\left(c_{LLW}, f^{-1}\left(2\sqrt{rd} \right) \right), 2\sqrt{rd} \right) \times \left\{ 2\sqrt{rd} \right\},$$

$$D = \left\{ (c, c) \mid c \geq 2\sqrt{rd} \right\},$$

and where G is possibly empty whereas V, H and D are always nonempty.

Points on $V \cup G$ should correspond to pairs (u_0, v_0) with u_0 supported in a left half-line and v_0 exponentially decaying. Using both Theorem 1.3 and Theorem 1.2 as well as a limiting argument and the comparison principle, it is possible to obtain the existence of such a terrace with a pair (u_0, v_0) of this form.

However, on H, which corresponds naively to pairs (u_0, v_0) with compactly supported v_0 and exponentially decaying u_0, such a construction seems to be impossible. A different, likely more delicate, argument is needed to deal with H. Still, we believe existence holds there.

On the contrary, on D, the question remains completely open. Indeed, on D, propagating terraces reduce to non-monotonic traveling waves connecting $(0, 0)$ to $(1, 0)$ with an intermediate bump of v. To the best of our knowledge, such traveling waves have never been studied. Even though it might be tempting to conjecture their nonexistence, we prefer to remain cautious here.

5.3. **On the proofs.** In the proof of Theorem 1.1, the approximated speed c_1^δ is necessary in the following sense: it is impossible to construct another $\underline{v}$ spreading this time exactly at speed $2\sqrt{rd}$. This is an immediate consequence of the Bramson shift for the KPP equation [2]: the level sets of the solution of the KPP equation satisfied by v in isolation with compactly supported initial data are asymptotically located at $2\sqrt{rd}t + s_{Bramson}(t)$, with $s_{Bramson}(t) = -\frac{3}{2}\log t + o(\log t)$. By comparison, it is then easily verified that for the solution (u, v) of our competitive system, there exists a time shift $s(t) \leq s_{Bramson}(t)$ such that the level sets of v in our problem are located at $2\sqrt{rd}t + s(t)$.

Similarly, in the proofs of Theorem 1.1 and of Theorem 1.3, we believe that the approximated speed c_2^δ are needed to account for a time shift $\tilde{s}(t) \neq 0$ describing the position of the level sets of u. The characterization of this shift is completely open; the only hint provided by our approach is that $\tilde{s}(t)$ is asymptotically nonnegative (contrarily to $s(t)$ and $s_{Bramson}(t)$).

Acknowledgments

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Appendix A. On competition–diffusion traveling waves connecting $(1, 0)$ to $(0, 1)$

In this appendix, the parameters (d, r, a, b) are not fixed anymore and can vary. We define

$$\Pi = (0, +\infty)^2 \times (0, 1) \times (1, +\infty).$$

For all $(d,r,a,b)\in\Pi$, $c_{LLW}^{d,r,a,b}$ denotes the associated spreading speed of the system (1.2). Subsequently, we define

$$E=\left\{(c,d,r,a,b)\in(0,+\infty)\times\Pi\mid c\geq c_{LLW}^{d,r,a,b}\right\}.$$

A.1. **Exact exponential decays.** For all $P=(c,d,r,a,b)\in E$, we define

$$\lambda_{1,P}^{-\infty}=\frac{\sqrt{c^2+4}-c}{2},$$
$$\lambda_{2,P}^{-\infty}=\frac{\sqrt{c^2+4rd\,(b-1)}-c}{2d},$$
$$\lambda_{1,P}^{+\infty}=\frac{c+\sqrt{c^2+4rd}}{2d},$$
$$\lambda_{2,P}^{+\infty}=\frac{c+\sqrt{c^2-4\,(1-a)}}{2},$$
$$\lambda_{3,P}^{+\infty}=\frac{c-\sqrt{c^2-4\,(1-a)}}{2}.$$

Lemma A.1. *Let $P=(c,d,r,a,b)\in E$ and (φ,ψ) be a profile of traveling wave solution of (1.1) with speed c. Define $R_P^{-\infty}:\lambda\mapsto\lambda^2+c\lambda-1$ and $R_P^{+\infty}:\lambda\mapsto d\lambda^2-c\lambda-r$.*

Then the asymptotic behaviors of (φ,ψ) are as follows.

(1) There exist $A>0$ and $B>0$ such that, as $\xi\to-\infty$:

(a) if $\lambda_{2,P}^{-\infty}>\lambda_{1,P}^{-\infty}$, then

$$\begin{cases}\varphi\left(\xi\right)=1-Ae^{\lambda_{1,P}^{-\infty}\xi}+h.o.t.\\ \psi\left(\xi\right)=Be^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\end{cases};$$

(b) if $\lambda_{2,P}^{-\infty}<\lambda_{1,P}^{-\infty}$, then $R_P^{-\infty}\left(\lambda_{2,P}^{-\infty}\right)<0$ and

$$\begin{cases}\varphi\left(\xi\right)=1+\frac{a}{R_P^{-\infty}\left(\lambda_{2,P}^{-\infty}\right)}Be^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\\ \psi\left(\xi\right)=Be^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\end{cases};$$

(c) if $\lambda_{2,P}^{-\infty}=\lambda_{1,P}^{-\infty}$, then $c+2\lambda_{2,P}^{-\infty}=\sqrt{c^2+4}>0$ and

$$\begin{cases}\varphi\left(\xi\right)=1-B\left|\xi\right|e^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\\ \psi\left(\xi\right)=\frac{c+2\lambda_{2,P}^{-\infty}}{a}Be^{\lambda_{2,P}^{-\infty}\xi}+h.o.t.\end{cases}.$$

(2) There exist $A\in\mathbb{R}$, $B\in\mathbb{R}$ and $C\geq0$ such that $B>0$ if $C=0$ and, as $\xi\to+\infty$:

(a) if $c>2\sqrt{1-a}$,

(i) if $\lambda_{1,P}^{+\infty}<\lambda_{3,P}^{+\infty}$, then $A>0$ and

$$\begin{cases}\varphi\left(\xi\right)=Be^{-\lambda_{2,P}^{+\infty}\xi}+Ce^{-\lambda_{3,P}^{+\infty}\xi}+h.o.t.\\ \psi\left(\xi\right)=1-Ae^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

(ii) if $\lambda_{1,P}^{+\infty}=\lambda_{3,P}^{+\infty}$, then $A>0$ if $C=0$ and

$$\begin{cases}\varphi\left(\xi\right)=\frac{2d\lambda_{1,P}^{+\infty}-c}{a}Ce^{-\lambda_{1,P}^{+\infty}\xi}+Be^{-\lambda_{2,P}^{+\infty}\xi}+h.o.t.\\ \psi\left(\xi\right)=1-\left(A+C\xi\right)e^{-\lambda_{1,P}^{+\infty}\xi}+h.o.t.\end{cases};$$

(iii) if $\lambda_{1,P}^{+\infty} \in \left(\lambda_{3,P}^{+\infty}, \lambda_{2,P}^{+\infty}\right)$, *then* $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$, $A > 0$ *if* $C = 0$ *and*

$$\begin{cases} \varphi(\xi) = Be^{-\lambda_{2,P}^{+\infty}\xi} + Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - Ae^{-\lambda_{1,P}^{+\infty}\xi} + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(iv) if $\lambda_{1,P}^{+\infty} = \lambda_{2,P}^{+\infty}$, *then* $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ *and*

$$\begin{cases} \varphi(\xi) = \frac{2d\lambda_{1,P}^{+\infty} - c}{a} Be^{-\lambda_{1,P}^{+\infty}\xi} + Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - B\xi e^{-\lambda_{1,P}^{+\infty}\xi} + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(v) if $\lambda_{1,P}^{+\infty} > \lambda_{2,P}^{+\infty}$, *then* $R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right) < 0$, $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ *and*

$$\begin{cases} \varphi(\xi) = Be^{-\lambda_{2,P}^{+\infty}\xi} + Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right)} Be^{-\lambda_{2,P}^{+\infty}\xi} + \frac{rb}{R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right)} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(b) if $c = 2\sqrt{1-a}$,

(i) if $\lambda_{1,P}^{+\infty} < \lambda_{2,P}^{+\infty}$, *then* $A > 0$ *and*

$$\begin{cases} \varphi(\xi) = (B + C\xi)\, e^{-\lambda_{2,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - Ae^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases}.$$

(ii) if $\lambda_{1,P}^{+\infty} = \lambda_{2,P}^{+\infty}$, *then* $2d\lambda_{1,P}^{+\infty} - c = \sqrt{c^2 + 4rd} > 0$ *and*

$$\begin{cases} \varphi(\xi) = \frac{2d\lambda_{1,P}^{+\infty} - c}{a} (B + C\xi)\, e^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - \left(B + \frac{1}{2}C\xi\right)\xi e^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(iii) if $\lambda_{1,P}^{+\infty} > \lambda_{2,P}^{+\infty}$, *then* $R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right) < 0$ *and*

$$\begin{cases} \varphi(\xi) = (B + C\xi)\, e^{-\lambda_{2,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}\left(\lambda_{2,P}^{+\infty}\right)} (B + C\xi)\, e^{-\lambda_{2,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

Proof. This result follows from a standard yet lengthy phase-plane analysis. The detailed proof can be found for instance in Kan-on [20] or in Morita–Tachibana [27]. □

Compiling these estimates, we obtain the following two corollaries.

Corollary A.2. *Let* $P = (c, d, r, a, b) \in E$ *and* (φ, ψ) *be a profile of traveling wave solution of the corresponding system with speed* c. *Then there exist* $i \in \{2, 3\}$, $C > 0$, $D > 0$ *and* $(i_+, j_+) \in \{0, 1\} \times \{0, 1, 2\}$ *such that, as* $\xi \to +\infty$,

$$\begin{cases} \varphi(\xi) = C\xi^{i_+} e^{-\lambda_{i,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - D\xi^{j_+} e^{-\min\left(\lambda_{1,P}^{+\infty}, \lambda_{i,P}^{+\infty}\right)\xi} + h.o.t. \end{cases}.$$

Corollary A.3. *Let* $P = (c, d, r, a, b) \in E$ *and* (φ, ψ) *be a profile of traveling wave solution of the corresponding system with speed* c. *Let* $i_- = 2 - \#\left\{\lambda_{1,P}^{-\infty}, \lambda_{2,P}^{-\infty}\right\}$.

Then there exist $A > 0$ and $B > 0$ such that, as $\xi \to -\infty$,

$$\begin{cases} \varphi(\xi) = 1 - A|\xi|^{i_-} e^{\min\left(\lambda_{1,P}^{-\infty}, \lambda_{2,P}^{-\infty}\right)\xi} + h.o.t. \\ \psi(\xi) = Be^{\lambda_{2,P}^{-\infty}\xi} + h.o.t. \end{cases}.$$

A.2. **Component-wise monotonicity of the profiles.** Thanks to Corollary A.2 and a sliding argument, we can show the component-wise monotonicity. We point out that Roques–Hosono–Bonnefon–Boivin [31] showed that the slow or fast decay problem is related to the pulled or pushed front problem.

Proposition A.4. *Let $P = (c, d, r, a, b) \in E$. Let $(\varphi, \psi) \in \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)$ be a profile of traveling wave solution of (1.1) with speed c connecting $(1,0)$ to $(0,1)$.*

Then (φ, ψ) is component-wise strictly monotonic, i.e.

$$(\varphi, \psi)(\xi_1) \succ (\varphi, \psi)(\xi_2) \quad \text{whenever} \quad \xi_1 < \xi_2.$$

Proof. The proof relies upon a sliding argument.

The sliding argument for monostable problems has three main steps: first, showing that if two profiles are correctly ordered at some point far on the left, then they remain correctly ordered everywhere on the left of this point; next, showing thanks to the first step and the exponential estimates at $+\infty$ that, up to some translation, the two profiles are globally ordered; finally, showing by optimizing the aforementioned translation that the two profiles actually coincide.

Notice that since the exponential estimates of Lemma A.1 can be differentiated, they imply the component-wise strict monotonicity of (φ, ψ) near $\pm\infty$. Thus we can define $R > 0$ such that (φ, ψ) is component-wise strictly monotonic in $\mathbb{R} \setminus [-R, R]$. In particular, we can assume that

$$(\varphi, \psi)(-R) \succ (\varphi, \psi)(\xi) \succ (\varphi, \psi)(R) \text{ for all } \xi \in (-R, R). \tag{A.1}$$

Step 1: We claim that there is $\tau_1 > 0$ such that for all $\tau \geq \tau_1$,

$$(\varphi, \psi)(\xi - \tau) \succ (\varphi, \psi)(\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{A.2}$$

In view of the monotonicity of (φ, ψ) in $\mathbb{R} \setminus (-R, R)$, and (A.1), the claim clearly holds once we take $\tau_1 = 2R$.

Step 2: Define $\tau^\star$ to be the infimum of all $\tau \in (0, 2R]$ such that (A.2) holds true. It remains to show that $\tau^\star = 0$. Suppose to the contrary that $\tau^\star > 0$. By construction,

$$(\varphi, \psi)(\xi - \tau^\star) \succeq (\varphi, \psi)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Moreover, by (A.1) and monotonicity of (φ, ψ) in $\mathbb{R} \setminus (-R, R)$, we see that for each $\tau \in \left[\frac{\tau^\star}{2}, 2\tau^\star\right]$,

$$(\varphi, \psi)(\xi - \tau) \succ (\varphi, \psi)(\xi) \text{ for all } \xi \in \mathbb{R} \setminus (-R + \tau, R),$$

and in particular for all $\xi \in \mathbb{R} \setminus (-R + \tau^\star/2, R)$. By the minimality of $\tau^\star > 0$, there exists $\xi^\star \in [-R + \tau^\star/2, R]$ such that equality holds for at least one of the components. The strong comparison principle yields

$$(\varphi, \psi)(\xi - \tau^\star) = (\varphi, \psi)(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

This implies (φ, ψ) is periodic with period $\tau^\star$, and contradicts $(\varphi, \psi)(-\infty) = (1, 0)$ and $(\varphi, \psi)(+\infty) = (0, 1)$. Hence $\tau^\star = 0$ and, subsequently, for all $\tau > 0$, we have

$$(\varphi, \psi)(\xi - \tau) \succ (\varphi, \psi)(\xi) \text{ for all } \xi \in \mathbb{R},$$

which exactly means that (φ, ψ) is component-wise strictly monotonic. □

A.3. **Ordering of the decays.** By a similar proof, we can characterize more precisely the decays. We point out that Roques–Hosono–Bonnefon–Boivin [31] showed that the slow or fast decay problem is related to the pulled or pushed front problem.

Lemma A.5. *Let $p=(d,r,a,b)\in\Pi$, $c\geq c^p_{LLW}$ and $\hat{c}\geq c$. Define $P=(c,p)\in E$ and $\hat{P}=(\hat{c},p)\in E$.*

Let $(\varphi,\psi)\in\mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)$ and $\left(\hat{\varphi},\hat{\psi}\right)\in\mathscr{C}^2\left(\mathbb{R},[0,1]^2\right)$ be two profiles of traveling wave solution of (1.1) with speed c and $\hat{c}$ respectively. Denote (i,C,D,i_+,j_+) and $\left(\hat{i},\hat{C},\hat{D},\hat{i_+},\hat{j_+}\right)$ the quantities given by Corollary A.2 when applied to (φ,ψ) and $\left(\hat{\varphi},\hat{\psi}\right)$ respectively.

Then at least one of the following estimates fails:

$$\hat{C}\xi^{\hat{i_+}}e^{-\lambda^{+\infty}_{\hat{i},\hat{P}}\xi}=o\left(C\xi^{i_+}e^{-\lambda^{+\infty}_{i,P}\xi}\right)\ \text{as}\ \xi\to+\infty,$$

$$\hat{D}\xi^{\hat{j_+}}e^{-\min\left(\lambda^{+\infty}_{1,\hat{P}},\lambda^{+\infty}_{\hat{i},\hat{P}}\right)\xi}=o\left(D\xi^{j_+}e^{-\min\left(\lambda^{+\infty}_{1,P},\lambda^{+\infty}_{i,P}\right)\xi}\right)\ \text{as}\ \xi\to+\infty.$$

Proof. The proof is by contradiction: we assume from now on that, on the contrary, the above two asymptotic estimates are satisfied. This means that, near $+\infty$, any translation of (φ,ψ) dominates $\left(\hat{\varphi},\hat{\psi}\right)$ (in the sense of the competitive ordering).

Here are the three steps of the sliding argument of this proof.

Step 1: choose $\xi_0\in\mathbb{R}$ sufficiently close to $-\infty$ and such that for all $\xi\leq\xi_0$,

$$\left(\hat{\varphi},\hat{\psi}\right)(\xi)\succeq\left(\frac{3}{4},\frac{1}{4}\right)\ \text{and}\ \hat{\varphi}(\xi)\geq\max\left(\frac{5-a}{8-4a},\frac{3+b}{4b}+\hat{\psi}(\xi)\right). \tag{A.3}$$

Notice that such a ξ_0 exists indeed, since $(\hat{\varphi},\hat{\psi})(-\infty)=(1,0)$, and $\max\left(\frac{5-a}{8-4a},\frac{3+b}{4b}\right)<1$ with $a<1$ and $b>1$. We claim that if there exists $\tau\in\mathbb{R}$ such that

$$(\varphi,\psi)(\xi_0-\tau)\succ\left(\hat{\varphi},\hat{\psi}\right)(\xi_0),$$

then

$$(\varphi,\psi)(\xi-\tau)\succ\left(\hat{\varphi},\hat{\psi}\right)(\xi)\ \text{for all}\ \xi\leq\xi_0.$$

Clearly, there exists $\varepsilon\in\left(0,\frac{1}{4}\right]$ such that

$$(\varphi,\psi)(\xi-\tau)\succeq\left(\hat{\varphi},\hat{\psi}\right)(\xi)+\varepsilon(-1,1)\ \text{for all}\ \xi\leq\xi_0.$$

Now, let $\varepsilon^\star\in\left[0,\frac{1}{4}\right]$ be the infimum of all these ε and assume by contradiction that $\varepsilon^\star>0$. In view of the limiting values at $-\infty$ and of the inequality at ξ_0, there exists $\xi^\star\in(-\infty,\xi_0)$ such that

$$(\varphi,\psi)(\xi^\star-\tau)\succeq\left(\hat{\varphi},\hat{\psi}\right)(\xi^\star)+\varepsilon^\star(-1,1)$$

with, most importantly, equality for at least one of the components. Let us verify that $(\varphi_{\varepsilon^\star},\psi_{\varepsilon^\star})=\left(\hat{\varphi},\hat{\psi}\right)+\varepsilon^\star(-1,1)$ is a sub-solution. Since (φ,ψ) satisfies by definition

$$\begin{cases}-\hat{\varphi}''-c\hat{\varphi}'=(\hat{c}-c)\hat{\varphi}'+\hat{\varphi}\left(1-\hat{\varphi}-a\hat{\psi}\right)\\ -d\hat{\psi}''-c\hat{\psi}'=(\hat{c}-c)\hat{\psi}'+r\hat{\psi}\left(1-\hat{\psi}-b\hat{\varphi}\right)\end{cases},$$

we find (note that, by Proposition A.4, $(\hat{\varphi}', \hat{\psi}') \preceq (0,0)$)

$$\begin{cases} -\varphi''_{\varepsilon^\star} - c\varphi'_{\varepsilon^\star} - \varphi_{\varepsilon^\star}\left(1 - \varphi_{\varepsilon^\star} - a\psi_{\varepsilon^\star}\right) < \varepsilon^\star\left(1 - (2-a)\,\varphi_{\varepsilon^\star} - (1-a)\,\varepsilon^\star - a\psi_{\varepsilon^\star}\right) \\ -d\psi''_{\varepsilon^\star} - c\psi'_{\varepsilon^\star} - r\psi_{\varepsilon^\star}\left(1 - \psi_{\varepsilon^\star} - b\varphi_{\varepsilon^\star}\right) > -r\varepsilon^\star\left(1 - (2-b)\,\psi_{\varepsilon^\star} - (b-1)\,\varepsilon^\star - b\varphi_{\varepsilon^\star}\right) \end{cases}.$$

From

$$\begin{pmatrix} 1 - (2-a)\,\varphi_{\varepsilon^\star} - (1-a)\,\varepsilon^\star - a\psi_{\varepsilon^\star} \\ -\left(1 - (2-b)\,\psi_{\varepsilon^\star} - (b-1)\,\varepsilon^\star - b\varphi_{\varepsilon^\star}\right) \end{pmatrix} \preceq \begin{pmatrix} 1 - (2-a)\,\hat{\varphi} + \frac{1-a}{4} - a\hat{\psi} \\ -\left(1 - (2-b)\,\hat{\psi} + \frac{b-1}{4} - b\hat{\varphi}\right) \end{pmatrix},$$

we deduce by (A.3) that

$$\begin{pmatrix} 1 - (2-a)\,\varphi_{\varepsilon^\star} - (1-a)\,\varepsilon^\star - a\psi_{\varepsilon^\star} \\ -\left(1 - (2-b)\,\psi_{\varepsilon^\star} - (b-1)\,\varepsilon^\star - b\varphi_{\varepsilon^\star}\right) \end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for all } \xi \leq \xi_0.$$

We are now in position to apply the strong comparison principle of Theorem 2.5 and deduce from the existence of $\xi^\star$ a contradiction. Hence $\varepsilon^\star = 0$, that is

$$(\varphi, \psi)\,(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \leq \xi_0.$$

Finally, by strong comparison principle the strict inequality must hold for any $\xi \leq \xi_0$.

Step 2: in this step, we show the existence of τ_1 such that, for all $\tau \geq \tau_1$,

$$(\varphi, \psi)\,(\xi - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

To this end, we fix ξ_0 as in (A.3) and choose $\tau_0 > 0$ large so that

$$(\varphi, \psi)\,(\xi_0 - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi_0) \text{ for all } \tau \geq \tau_0.$$

By Step 1, we deduce that for all $\tau \geq \tau_0$,

$$(\varphi, \psi)\,(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \leq \xi_0.$$

Next, we use the asymptotic behavior of (φ, ψ) and $(\hat{\varphi}, \hat{\psi})$ at $+\infty$ to choose $\tau_1 \geq \tau_0$ such that for all $\tau \geq \tau_1$,

$$(\varphi, \psi)\,(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \geq \xi_0.$$

The above two inequalities complete Step 2.

Step 3: define $\tau^\star$ as the infimum of all τ such that the preceding inequality holds true. By construction,

$$(\varphi, \psi)\,(\xi - \tau^\star) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

It suffices to show the existence of $\xi^\star \in \mathbb{R}$ such that $(\varphi, \psi)\,(\xi^\star - \tau^\star) \succeq (\varphi, \psi)\,(\xi^\star)$ with equality for at least one component. Granted, then the strong comparison principle yields

$$(\varphi, \psi)\,(\xi - \tau^\star) = \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R},$$

and the proof is ended. Suppose by contradiction that such a $\xi^\star$ does not exist, that is

$$(\varphi, \psi)\,(\xi - \tau^\star) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Now, the asymptotic behavior assumed at the beginning of this proof implies

$$\lim_{\xi \to +\infty} \frac{\varphi\,(\xi - \tau^\star)}{\hat{\varphi}\,(\xi)} = +\infty, \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{1 - \psi\,(\xi - \tau^\star)}{1 - \hat{\psi}\,(\xi)} = +\infty.$$

Hence, there exists $\xi_1 > 0$ large and $\delta > 0$ small such that for all $\tau \in (\tau^\star - \delta, \tau^\star)$,

$$(\varphi, \psi)(\xi - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \geq \xi_1.$$

By taking $\delta > 0$ small, we have also that, for all $\tau \in (\tau^\star - \delta, \tau^\star)$,

$$(\varphi, \psi)(\xi - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \quad \text{for all } \xi \in [\xi_0, \xi_1].$$

Finally, the result in Step 1 implies that for all $\tau \in (\tau^\star - \delta, \tau^\star)$,

$$(\varphi, \psi)(\xi - \tau) \succ \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

This contradicts the minimality of $\tau^\star$. □

From the preceding lemma, Lemma A.1 and the respective monotonicities of $c \mapsto \lambda_{1,P}^{+\infty}$, $c \mapsto \lambda_{2,P}^{+\infty}$ and $c \mapsto \lambda_{3,P}^{+\infty}$, we deduce the following corollary which is a refinement of Lemma A.1. Basically, it discards the possibility of solutions having a fast decay and a super-critical speed.

Corollary A.6. *Let $P = (c, d, r, a, b) \in E$ with $c > c_{LLW}^{d,r,a,b}$ and (φ, ψ) be a profile of traveling wave solution of (1.1) with speed c.*

Then there exist $A > 0$ and $C > 0$ such that, as $\xi \to +\infty$:

(1) if $\lambda_{1,P}^{+\infty} < \lambda_{3,P}^{+\infty}$, then

$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - Ae^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(2) if $\lambda_{1,P}^{+\infty} = \lambda_{3,P}^{+\infty}$, then

$$\begin{cases} \varphi(\xi) = \frac{2d\lambda_{1,P}^{+\infty} - c}{a} Ce^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 - C\xi e^{-\lambda_{1,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(3) if $\lambda_{1,P}^{+\infty} \in \left(\lambda_{3,P}^{+\infty}, \lambda_{2,P}^{+\infty}\right)$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and

$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}(\lambda_{3,P}^{+\infty})} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(4) if $\lambda_{1,P}^{+\infty} = \lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and

$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}(\lambda_{3,P}^{+\infty})} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

(5) if $\lambda_{1,P}^{+\infty} > \lambda_{2,P}^{+\infty}$, then $R_P^{+\infty}\left(\lambda_{3,P}^{+\infty}\right) < 0$ and

$$\begin{cases} \varphi(\xi) = Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \\ \psi(\xi) = 1 + \frac{rb}{R_P^{+\infty}(\lambda_{3,P}^{+\infty})} Ce^{-\lambda_{3,P}^{+\infty}\xi} + h.o.t. \end{cases};$$

Remark. We emphasize that there exists a unique translation of the profile such that the normalization $C = 1$ holds. The remaining degree of freedom in the first case above (A can still take any positive value *a priori*) is the main difficulty regarding uniqueness.

A.4. **Uniqueness and continuity.** We are now in position to establish the following uniqueness result.

Proposition A.7. *Let $P = (c, d, r, a, b) \in E$ such that $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$.*

Let $(\varphi, \psi) \in \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)$ and $\left(\hat{\varphi}, \hat{\psi}\right) \in \mathscr{C}^2\left(\mathbb{R}, [0,1]^2\right)$ be two profiles of traveling wave solution of (1.1) with speed c.

Then (φ, ψ) and $\left(\hat{\varphi}, \hat{\psi}\right)$ coincide up to translation.

Proof. The proof relies upon a sliding argument again.

In view of Corollary A.6, if $c > c_{LLW}^{d,r,a,b}$, the assumption $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$ immediately yields that the two profiles can be normalized so that they have the same decay at $+\infty$. Similarly, in view of Lemma A.1, if $c = c_{LLW}^{d,r,a,b}$, then the two profiles can be normalized so that their decays either coincide or are well-ordered. In all cases, we can fix *a priori* the roles of the two profiles so that (φ, ψ) dominates $\left(\hat{\varphi}, \hat{\psi}\right)$ near $+\infty$. By following the first two steps of the proof of Lemma A.5, we can assume without loss of generality the existence of $\tau_0 \in \mathbb{R}$ such that, for all $\tau \geq \tau_0$,

$$(\varphi, \psi)(\xi - \tau) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Next, define $\tau^\star \in \mathbb{R}$ as the infimum of all τ such that the preceding inequality holds true. It again suffices to show that there exists $\xi^\star \in \mathbb{R}$ where equality holds for one of the components. Assume on the contrary that no such $\xi^\star$ exists. Thus the preceding inequality is strict for both components for all $\xi \in \mathbb{R}$. Now, note that

$$\lim_{\xi \to +\infty} \frac{\varphi(\xi - \tau^\star)}{\hat{\varphi}(\xi)} \geq 1 \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{1 - \psi(\xi - \tau^\star)}{1 - \hat{\psi}(\xi)} \geq 1.$$

Next, we claim that

$$\lim_{\xi \to +\infty} \frac{\varphi(\xi - \tau^\star)}{\hat{\varphi}(\xi)} = 1 \quad \text{or} \quad \lim_{\xi \to +\infty} \frac{1 - \psi(\xi - \tau^\star)}{1 - \hat{\psi}(\xi)} = 1.$$

Otherwise we may further reduce $\tau^\star$, just as in the proof of Lemma A.5. Notice that this equality directly yields $\tau^\star = 0$, that is

$$(\varphi, \psi)(\xi) \succeq \left(\hat{\varphi}, \hat{\psi}\right)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Next, from the fact that $\lambda_{1,P}^{+\infty} \geq \lambda_{3,P}^{+\infty}$ and, depending on c, Corollary A.6 or Lemma A.1, both of the above limits are equal to 1.

Since the decay rate at $+\infty$ of both profiles coincide, we can reverse the profiles and repeat the proof. This leads to

$$\left(\hat{\varphi}, \hat{\psi}\right)(\xi) \succeq (\varphi, \psi)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Hence the two profiles actually coincide, which directly contradicts the assumption of nonexistence of $\xi^\star$.

In the end, $\xi^\star$ exists indeed and, by virtue of the strong comparison principle, the two normalized profiles coincide. In other words, the two profiles coincide up to translation. □

Corollary A.8. *Let $(d, r, a, b) \in \Pi$ such that $d \leq 2 + \frac{r}{1-a}$. Then each speed $c \geq c_{LLW}^{d,r,a,b}$ is associated with a unique profile (up to translation).*

Proof. It suffices to prove that, for all $c \geq c_{LLW}^{d,r,a,b}$, $\lambda^{+\infty}_{1,(c,d,r,a,b)} \geq \lambda^{+\infty}_{3,(c,d,r,a,b)}$.

Noticing that this inequality is equivalent to $R^{+\infty}_{(c,d,r,a,b)}\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right) \leq 0$, we find that we just have to prove that, for all $c \geq c_{LLW}^{d,r,a,b}$,

$$d \leq \frac{c\lambda^{+\infty}_{3,(c,d,r,a,b)} + r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2}$$

and, using the polynomial equation satisfied by $\lambda^{+\infty}_{3,(c,d,r,a,b)} = \frac{c-\sqrt{c^2-4(1-a)}}{2}$, this reads

$$d \leq \frac{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2 + 1 - a + r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2} = 1 + \frac{1-a+r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2}$$

It only remains to show that

$$\inf_{c \geq c_{LLW}^{d,r,a,b}} \frac{1-a+r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2} \geq 1 + \frac{r}{1-a}.$$

The above inequality follows actually quite easily:

$$\inf_{c \geq c_{LLW}^{d,r,a,b}} \frac{1-a+r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2} \geq \inf_{c \geq 2\sqrt{1-a}} \frac{1-a+r}{\left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2} = \frac{1-a+r}{\sup\limits_{c \geq 2\sqrt{1-a}} \left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2}$$

and, by monotonicity,

$$\sup_{c \geq 2\sqrt{1-a}} \left(\lambda^{+\infty}_{3,(c,d,r,a,b)}\right)^2 = \left(\lambda^{+\infty}_{3,\left(2\sqrt{1-a},d,r,a,b\right)}\right)^2 = 1-a.$$

□

Finally, as a consequence of the uniqueness, we also have the continuity of the profiles with respect to the parameters.

Proposition A.9. *Let*

$$E_u = \left\{ P \in E \mid \lambda^{+\infty}_{1,P} \geq \lambda^{+\infty}_{3,P} \right\}.$$

For all $P \in E_u$, let $\left(\Phi^P, \Psi^P\right)$ be the unique profile of traveling wave solution of (1.1) with speed c satisfying $\Psi^P(0) = \frac{1}{2}$.

Then $P \mapsto \left(\Phi^P, \Psi^P\right)$ is in $\mathscr{C}\left(intE_u, \mathscr{C}_b\left(\mathbb{R}, \mathbb{R}^2\right)\right)$.

Proof. Let $P_\infty \in \overline{\text{int}E_u}$ and $(P_n)_{n\in\mathbb{N}} \in (\text{int}E_u)^{\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} P_n = P_\infty$. By standard elliptic estimates (see Gilbarg–Trudinger [13]), the sequence $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges, up to a diagonal extraction, in $\mathscr{C}^2_{loc}$. *A fortiori* it converges pointwise in $\mathbb{R}$. The limit $(\Phi_\infty, \Psi_\infty)$ is continuous, monotonic, and satisfies $\Psi_\infty(0) = \frac{1}{2}$. Using standard elliptic estimates to study the asymptotic behaviors, we find easily

$$\lim_{-\infty}(\Phi_\infty, \Psi_\infty) \in \{(1,0),(0,0)\} \text{ and } \lim_{+\infty}(\Phi_\infty, \Psi_\infty) = (0,1).$$

If Φ_∞ is null in $\mathbb{R}$, then $\xi \mapsto \Psi_\infty(-\xi)$ is a KPP traveling wave with negative speed, which is impossible. Therefore the limit at $-\infty$ of $(\Phi_\infty, \Psi_\infty)$ is $(1,0)$. This shows that the sequence of monotonic functions $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges pointwise in

$[-\infty, +\infty]$, whence by a variant of the Dini theorem it converges uniformly in $\mathbb{R}$. In view of the preceding uniqueness result, the limit is exactly $\left(\Phi^{P_\infty}, \Psi^{P_\infty}\right)$. Finally, a classical uniqueness and compactness argument shows that the previous diagonal extraction was not necessary and the sequence $\left(\left(\Phi^{P_n}, \Psi^{P_n}\right)\right)_{n\in\mathbb{N}}$ converges indeed in $\mathscr{C}_b\left(\mathbb{R}, \mathbb{R}^2\right)$ to $\left(\Phi^{P_\infty}, \Psi^{P_\infty}\right)$. □

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