Permeability of non-inertial porous matrices or under magnetic field

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ABSTRACT: We investigate the filtration law of an incompressible viscous Newtonian fluid through a rigid non-inertial porous medium or when submitted to a magnetic field. The filtration law is obtained by upscaling the flow at the pore scale. We use the method of multiple scale expansions which rigorously gives the macroscopic behaviour without any prerequisite on the form of the macroscopic equations. The filtration laws are shown to resemble Darcy's laws, but with a permeability tensor which depends on the angular velocity of the porous matrix or on the magnetic field, respectively. In both cases Hall – Onsager relations are verified.

1 INTRODUCTION

The steady-state slow flow of an incompressible liquid through a rigid inertial porous matrix and in the absence of magnetic field is described by the well known Darcy's law

$$\mathbf{v} = -\mathbf{K}\nabla p, \quad v_i = -K_{ij}\frac{\partial p}{\partial X_j}, \tag{1}$$

where $\mathbf{K}$ is the permeability tensor which is shown to be a positive symmetrical tensor.

In the paper we investigate the steady-state slow flow of an incompressible liquid through a rigid porous matrix in two different situations. Firstly, we consider a non-inertial porous matrix, with an angular rotation $\boldsymbol{\omega}$ with respect to a galilean frame and secondly we study the case of an electrically conducting liquid which is submitted to a magnetic field $\mathbf{H}$. The question into consideration is: what are the consequences of the existence of an angular rotation $\boldsymbol{\omega}$ or of a magnetic field $\mathbf{H}$ on Darcy's law (1)?

A first answer is given by the theory on non-equilibrium thermodynamics (de Groot & Mazur 1969). From this theory, it can be shown that the flow law (1) is valid, but that the permeability tensor now verifies the following Onsager relations

$$K_{ij}(\boldsymbol{\omega}) = K_{ij}(-\boldsymbol{\omega}), \quad K_{ij}(\mathbf{H}) = K_{ji}(-\mathbf{H}), \tag{2}$$

which can be seen as the filtration analog of Hall's effect which stands for the electric conductivity of a material submitted to a magnetic field. Note that Onsager relations arise in a straightforward manner from the microscopic laws of motion of the particles constituting the material (and the time reversal invariance of these laws) and the principles of statistical mechanics.

Phenomenological approaches are also available. In Vadasz (1993), the following dimensionless isotropic filtration law is introduced to describe flow through a rotating porous media

$$\mathbf{q} = -k(\nabla p + Ek^{-1}\mathbf{e}_\omega \times \mathbf{q}), \tag{3}$$

where $\mathbf{q}$ is the flow rate vector, k is the permeability, ∇p is the pressure gradient, $\mathbf{e}_\omega$ is a unit vector in the direction of the rotational velocity and Ek is the Ek-man number, i.e. the ratio of viscous term to Coriolis term in Navier-Stokes equations. For a flow in presence of a magnetic field, Rudraiah (1975) proposed the following macroscopic law

$$\mathbf{q} = k(-\nabla p + \mathbf{J} \times \mathbf{B}), \tag{4}$$

where $\mathbf{J}$ is the electric current density and $\mathbf{B}$ is the magnetic induction. Both phenomenological laws (3) and (4) are obtained by directly introducing the Coriolis force and the Lorentz force that act at the pore scale (see below the Stokes equation) in the macroscopic isotropic Darcy's law.

In the present study, we use a deterministic upscaling technique, by starting from the description at the pore scale to determine an equivalent description which is valid at a macroscopic scale – the Darcy scale. We use the method of multiple scale expansions. Heterogeneous system as for example porous media may be modelled by an equivalent macroscopic continuous system if the condition of separation of scales is

verified, (Bensoussan et al. 1978; Sanchez-Palencia 1980)

$$\varepsilon = \frac{l}{L} \ll 1, \tag{5}$$

where l and L are the characteristic lengths of the heterogeneities and of the macroscopic sample or excitation, respectively. The macroscopic equivalent model is obtained from the description at the heterogeneity scale by (Auriault 1991): (i) assuming the medium to be periodic, without loss of generality; (ii) writing the local description in a dimensionless form; (iii) evaluating the dimensionless numbers with respect to the scale ratio ε; (iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε; (v) solving the successive boundary-value problems that are obtained after introducing these expansions in the local dimensionless description. The macroscopic equivalent model is obtained from compatibility conditions which are the necessary conditions for the existence of solutions to the boundary-value problems. The main advantages of the method rely upon the possibility of: (a) avoiding prerequisites at the macroscopic scale; (b) modelling finite size macroscopic samples; (c) modelling macroscopically non-homogeneous media or phenomena; (d) modelling problems with several separations of scales; (e) modelling several simultaneous phenomena; (f) determining whether the system "medium + phenomena" is homogenisable or not; (g) providing the domains of validity of the macroscopic models.

In section 2, we investigate the flow through a non-inertial matrix, e.g. through a porous sample placed in a geotechnical centrifuge, (Auriault et al. 2000; Auriault et al. 2002). Deviations from Darcy's law are due to Coriolis forces which tend to modify the flow. Coriolis effects are measured by the inverse Ekman's number $Ek^{-1} = 2\rho\omega l^2/\mu$, where μ and ρ are the dynamic viscosity and the density of the liquid, respectively, ω is the angular velocity of the porous matrix and l is the pore characteristic size. As an example, consider a geotechnical centrifuge with an angular velocity $\omega \sim 300$ rpm, and filtrating water with $\mu \sim 10^{-3}$ Pa.s and $\rho \sim 10^3$ Kg/m^3. Then, Ek^{-1} number varies from 62 for $l \sim 10^{-3}$ m, for sand, to 6.2×10^{-5} for $l \sim 10^{-6}$ m, for clay. We investigate Ek^{-1} numbers of $O(1)$, i.e. $\varepsilon \ll Ek^{-1} \ll \varepsilon^{-1}$. Upscaling yields a macroscopic description in the form (1), with a pos-itive but non-symmetrical permeability tensor which verifies Onsager relation (2).

Section 3 is devoted to the flow of an electrically conducting liquid through a rigid conducting porous medium when submitted to a magnetic field, (Geindreau & Auriault 2001; Geindreau & Auriault 2002). The common characteristic of these MHD (magneto-hydrodynamic) flows in porous media is that they are all electromagnetically braked by the Lorentz force

$$\mathbf{F}_L = \sigma_p \mathbf{E} \times \mathbf{B} + \sigma_p (\mathbf{v} \times \mathbf{B}) \times \mathbf{B}, \tag{6}$$

where σ_p is the electrical conductivity of the fluid, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction applied and $\mathbf{v}$ is the fluid velocity. The influence of this braking force by comparison to the viscous friction is usually measured by the Hartmann number $Ha = (\sigma_p/\mu)^{1/2} Bl$. We investigate Ha numbers of $O(1)$, i.e. $\varepsilon \ll Ha \ll \varepsilon^{-1}$. The macroscopic description of the flow is again of the form (1). The permeability tensor is positive and verifies Onsager relation (2). However, contrary to the non-inertial case, the tensor $\mathbf{K}$ is symmetrical.

2 SEEPAGE LAW IN ROTATING POROUS MEDIA

In this section, we summarize the investigation conducted in Auriault et al. (2000, 2002). The porous medium period is denoted by Ω and is bounded by $\partial\Omega$, the fluid part of the unit cell is denoted by Ω_p, and the fluid-solid interface inside the unit cell is Γ. Relatively to the moving porous matrix frame, the momentum balance for the quasi-static flow of an incompressible viscous Newtonian liquid is

$$\mu \nabla^2 \mathbf{v} - \nabla p = \rho(\gamma_e + \gamma_c) \quad \text{in } \Omega_p, \tag{7}$$

where $\mathbf{v}$ is the velocity vector relative to the matrix frame, p is the pressure, ρ is the density and μ is the viscosity. Gravitational acceleration is included in the pressure term. Vectors γ_e and γ_c are the convective and the Coriolis accelerations, respectively, with respect to a galilean frame

$$\gamma_e = \gamma(O) + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{OM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OM}), \tag{8}$$

$$\gamma_c = 2\boldsymbol{\omega} \times \mathbf{v}, \tag{9}$$

where $\boldsymbol{\omega}$ is the angular velocity of the porous matrix, O is a fixed point of the porous matrix in the investigated period and M is a current point in Ω_p. Equation (7) is completed by the incompressibility condition and the adherence condition on Γ

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_p, \quad \mathbf{v} = 0 \quad \text{on } \Gamma. \tag{10}$$

2.1 *Dimensionless pore scale description*

We use the local length scale of a pore l as the characteristic length scale for the variations of the differential operators: we apply the so-called microscopic point of view (Auriault 1991). Other characteristic values

are denoted using the subscript c.To be precise, we consider a centrifuge of radius r at constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_\omega, \omega = \text{constant}$. The pore scale description introduces four dimensionless numbers: the ratio Q of pressure to viscous forces, the ratio R of the translational convective inertia to the viscous force, the ratio A of the rotational to the translational convective inertia and the ratio Ek (the Ekman number) of the viscous force to the Coriolis inertia. We have $A = O(\omega^2 l/\omega^2 r) = O(\varepsilon)$ and we assume $R = O(1)$. The estimation of R is the consequence of the hypothesis of separation of scales. Higher values would yield non-homogenisable problems, i.e., problems for which equivalent macroscopic description do not exist. As mentioned above we assume $Ek = O(1)$. For evaluating the latter dimensionless number Q, we introduce $\bar{p}_c$ which is the characteristic pressure increment in the sample. We obtain. $Q = \bar{p}_c l/\mu v_c = O(\varepsilon^{-1})$, The estimate for Q comes from a phenomenological argument, i.e. the viscous flow is locally driven by a macroscopic pressure gradient (Auriault 1991).

For simplicity we use the same names for the dimensionless variables as for the original variables. The formal dimensionless set that describes the flow is in the form

$$\mu \nabla^2 \mathbf{v} - \varepsilon^{-1} \nabla p = \rho(\gamma(O) + Ek^{-1} 2\boldsymbol{\omega} \times \mathbf{v} + \varepsilon \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OM})) \quad \text{in } \Omega_p, \tag{11}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_p, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma. \tag{12}$$

2.2 *Upscaling*

The next step is to introduce multiple scale coordinates (Sanchez-Palencia 1980; Bensoussan et al. 1978). The dimensionless macroscopic space variable $\mathbf{x} = \mathbf{X}/L$ is related to the dimensionless microscopic space variable $\mathbf{y} = \mathbf{X}/l$ by $\mathbf{x} = \varepsilon \mathbf{y}$. By following the multiple scale expansion technique, the velocity $\mathbf{v}$ and the pressure fluctuation p are looked for in the form of asymptotic expansions of powers of ε

$$\mathbf{v} = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon \mathbf{v}^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 \mathbf{v}^2(\mathbf{x}, \mathbf{y}) + \cdots \tag{13}$$

$$p = p^0(\mathbf{x}, \mathbf{y}) + \varepsilon p^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 p^2(\mathbf{x}, \mathbf{y}) + \cdots \tag{14}$$

Substituting these expansions in the set (11–12) gives, by identification of the like powers of ε, successive boundary value problems to be investigated. The lowest order approximation of the pressure verifies

$$\frac{\partial p^{(0)}}{\partial y_i} = 0, \quad p^{(0)} = p^{(0)}(\mathbf{x}). \tag{15}$$

The first order approximation of the velocity $\mathbf{v}^{(0)}$ and the second order approximation of the pressure $p^{(1)}$ are determined by the following set

$$\mu \frac{\partial^2 v_i^{(0)}}{\partial y_j \partial y_j} - G_i - \frac{\partial p^{(1)}}{\partial y_i} = 2\rho \varepsilon_{ijk} \omega_j v_k^{(0)} \quad \text{in } \Omega_p, \tag{16}$$

$$\frac{\partial v_i^{(0)}}{\partial y_i} = 0 \quad \text{in } \Omega_p, \quad v_i^{(0)} = 0 \quad \text{on } \Gamma, \tag{17}$$

where $\mathbf{v}^{(0)}$ and $p^{(1)}$ are Ω-periodic. Symbol ε_{ijk} is the permutation symbol. Vector $\mathbf{G}$ is the macroscopic driving force which is independent of $\mathbf{y}$ and which is defined by $\mathbf{G} = \nabla_x p^{(0)} + \rho\gamma(O)$, where the subscript x denotes the derivative with respect to the variable $\mathbf{x}$.

To investigate the above set and to obtain properties of the effective coefficients to be defined, we introduce the Hilbert space W of Ω-periodic, divergence free vectors, where the vectors vanish on Γ, with the scalar product

$$(\mathbf{u}, \mathbf{v})_W = \int_{\Omega p} \frac{\partial u_i}{\partial y_j} \frac{\partial v_i}{\partial y_j} \mathrm{d}y. \tag{18}$$

Now, let us multiply Equation (16) by $\mathbf{u} \in W$ and integrate over Ω_p. By using integration by parts, the divergence theorem, periodicity, and the boundary condition on Γ, one obtains:

$$\forall \mathbf{u} \in W, \int_{\Omega p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^{(0)}}{\partial y_j} \mathrm{d}y + \int_{\Omega p} 2\rho \varepsilon_{ijk} \omega_j v_k^{(0)} v_i \, \mathrm{d}y = -\int_{\Omega p} v_i G_i \, \mathrm{d}y. \tag{19}$$

Formulation (19) is strongly elliptic (John 1970), and there exists a unique $\mathbf{v}^{(0)}$ which is a linear vector function of $\mathbf{G}$

$$v_i^{(0)} = -k_{ij} G_j, \tag{20}$$

where the tensor field $\mathbf{k}$ depends on $\boldsymbol{\omega}$ and $\mathbf{y}$.

Finally, the volume balance (12) gives at the second order

$$\frac{\partial v_i^{(1)}}{\partial y_i} + \frac{\partial v_i^{(0)}}{\partial x_i} = 0 \quad \text{in } \Omega p. \tag{21}$$

By integrating over Ω_p, we obtain

$$\frac{\partial \langle v_i^{(0)} \rangle}{\partial x_i} = 0, \quad \langle v_i^{(0)} \rangle = -K_{ij} G_j,$$

$$K_{ij} = \frac{1}{\Omega} \int_{\Omega p} k_{ij} \, \mathrm{d}y, \tag{22}$$

which represents the macroscopic equivalent behaviour at the order $O(\varepsilon)$ approximation. The filtration tensor $\mathbf{K}$ depends on the angular velocity $\boldsymbol{\omega}$.

2.3 *Properties of the permeability tensor*

The properties of the permeability tensor $\mathbf{K}(\omega)$ result from the symmetries that are present in the variational formulation (19). By following Auriault et al. (2002), we obtain

$$K_{ij}G_jG_i = \frac{1}{\Omega_p}\int_{\Omega p} \mu \frac{\partial v_i^{(0)}}{\partial y_j}\frac{\partial v_i^{(0)}}{\partial y_j} \mathrm{d}y > 0, \tag{23}$$

which means that tensor $\mathbf{K}(\omega)$ is positive.

Investigating now the symmetry of $\mathbf{K}$, it can be shown that

$$\Omega(K_{pq} - K_{qp}) = 4\rho\varepsilon_{ijk}\omega_j \int_{\Omega p} k_{kp}k_{iq}\, \mathrm{d}y, \tag{24}$$

which generally does not cancel out when $\omega \neq 0$. Therefore, unlike the classical permeability tensor $\mathbf{K}(0)$, $\mathbf{K}(\omega)$ is not a symmetric tensor. The divergence operator in the balance $(22)_1$ kills the antisymmetric part of $\mathbf{K}$. However, this antisymmetric part is of importance for porous media that are heterogeneous at the macroscopic scale or in case of flux boundary conditions.

From (24) it is easy to show that tensor $\mathbf{K}$ verifies an Onsager relation

$$K_{pq}(\omega) = K_{qp}(-\omega), \tag{25}$$

which stands for the filtration analog of Hall's effect.

Finally, the permeability tensor of an isotropic porous medium of galilean permeability k and submitted to an angular velocity $\omega = \omega\mathbf{e}_3$ takes the form

$$\mathbf{K} = \begin{pmatrix} K_{11}(\omega) & K_{12}(\omega) & 0 \\ -K_{12}(\omega) & K_{11}(\omega) & 0 \\ 0 & 0 & k \end{pmatrix} \tag{26}$$

Note that tensor $\mathbf{K}$ is invariant under any rotation of $\mathbf{e}_3$ axis. Typical behaviours of $K_{11}(\omega)$ and $K_{12}(\omega)$ are shown in Figure 1.

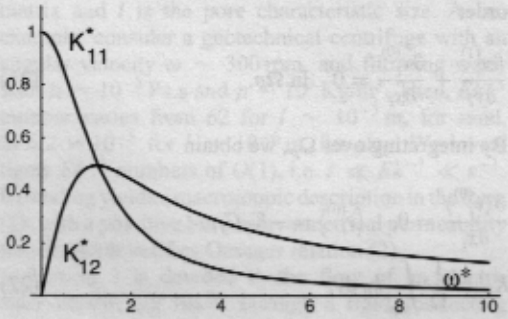

Figure 1. Dimensionless permeabilities $K_{11}^* = K_{11}(\omega)/k$ and $K_{12}^* = K_{12}(\omega)/k$ versus dimensionless frequency $\omega^* = \rho l^2\omega/4\mu$.

3 MAGNETOHYDRODYNAMIC FLOWS IN POROUS MEDIA

The porous medium is the one described in section 2, see also Geindreau & Auriault (2001, 2002) for more information. In the pores, the governing equations for the flow of an electrically conducting and incompressible Newtonian fluid are

$$-\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{J}\times\mathbf{B} = \mathbf{0} \quad \text{in } \Omega_p, \tag{27}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega p, \quad \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma, \tag{28}$$

where $\mathbf{v}$ is the velocity, p is the pressure, μ is the viscosity and $\mathbf{B}$ is the magnetic induction in the fluid. The current density $\mathbf{J}$ in the fluid is related to the fluid velocity $\mathbf{v}$ and the electric field $\mathbf{E}$ by Ohm's law, and to the magnetic field $\mathbf{H}$ by Maxwell's relation

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \nabla\times\mathbf{H} \quad \text{in } \Omega, \tag{29}$$

where $\sigma = \sigma_{p,s}$ is the electrical conductivity of the liquid and the solid, respectively and $\mathbf{v} = \mathbf{0}$ in the solid. The conducting solid skeleton Ω_s is assumed to be rigid. From (29) we have

$$\nabla\cdot\mathbf{J} = 0 \quad \text{in } \Omega. \tag{30}$$

The following relations are also valid

$$\nabla\times\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{31}$$

$$\nabla\cdot\mathbf{B} = \mathbf{0} \quad \text{with} \quad \mathbf{B} = \mu^*\mathbf{H} \quad \text{in } \Omega. \tag{32}$$

For simplicity, we have assumed that the conductivity σ and the magnetic permeability μ^* are isotropic. In particular we do not consider Hall effect at the microscale. Finally, the set of equations (27)–(32) is completed by the continuity on Γ of the normal components of the magnetic induction and the current density, and of the tangential components of the magnetic field and the electric field

$$(\mathbf{B}_p - \mathbf{B}_s)\cdot\mathbf{N} = 0 \quad \text{on } \Gamma, \tag{33}$$

$$(\mathbf{J}_p - \mathbf{J}_s)\cdot\mathbf{N} = (\sigma_p\mathbf{E}_p - \sigma_s\mathbf{E}_s)\cdot\mathbf{N} = 0 \quad \text{on } \Gamma, \tag{34}$$

$$(\mathbf{H}_p - \mathbf{H}_s)\times\mathbf{N} = \mathbf{0},\ (\mathbf{E}_p - \mathbf{E}_s)\times\mathbf{N} = \mathbf{0} \quad \text{on } \Gamma, \tag{35}$$

where $\mathbf{N}$ is the unit outward vector of Γ.

3.1 *Dimensionless pore scale description*

The pore scale description shows the following dimensionless numbers, where we use l as the characteristic

length

$$Q = \frac{|\nabla p|}{|\mu \nabla^2 \mathbf{v}|} = \frac{\bar{p}_c l}{\mu v_c},$$

$$Ha = \left(\frac{|\sigma p \mathbf{v} \times \mathbf{B}|}{|\mu \nabla^2 \mathbf{v}|} \right)^{1/2} = B_c l \left(\frac{\sigma_{pc}}{\mu} \right)^{1/2},$$

$$Rm = \frac{|\sigma_p \mathbf{v} \times \mathbf{B}|}{|\nabla \times \mathbf{H}|} = \mu^*_{pc} \sigma_{pc} v_c l,$$

$$K = \frac{|\mathbf{E}|}{|\mathbf{v} \times \mathbf{B}|} = \frac{E_c}{v_c B_c}, \quad S = \frac{\sigma_{pc}}{\sigma_{sc}}, \quad M = \frac{\mu^*_{pc}}{\mu^*_{sc}}.$$

As for the flow in rotating porous media, it can be shown that $Q = O(\varepsilon^{-1})$. The other dimensionless numbers depend on the problem under consideration. For example, the characteristic values in a metallic mushy zone (Moreau 1990; Lehmann et al. 1998) are $\sigma_{sc} \approx \sigma_{pc} \approx 10^6\,\Omega^{-1}\,\mathrm{m}^{-1}$, $\mu^*_{pc} \approx \mu^*_{sc}$ and $\mu^*_c \sigma_c \approx 1\,\mathrm{m}^{-2}\mathrm{s}$, $\mu \approx 10^{-3}$ Pa s, $B_c \approx 1$ T, $E_c \approx 10^{-2}\,\mathrm{V\,m}^{-1}$, $l \approx 100\,\mu\mathrm{m}$, $L \approx 10\,\mathrm{cm}$, $v_c \approx 10^{-3}\,\mathrm{ms}^{-1}$. Thus we obtain $\varepsilon \approx 10^{-3}$ and $Ha = O(1)$, $Rm = O(\varepsilon^2)$, $K = O(1)$, $S = O(1)$ and $M = O(1)$. For simplicity we use similar notations for dimensionless and dimensional quantities. The microscopic dimensionless set of equations that describes the MHD flow in a porous medium is the following, in which all quantities are now dimensionless quantities ($O(1)$):

$$\mu \nabla^2 \mathbf{v} - \varepsilon^{-1} \nabla p + \sigma_p (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \times \mathbf{B} = \mathbf{0} \quad \text{in } \Omega_p, \tag{36}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f, \tag{37}$$

$$\varepsilon^2 \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \nabla \times \mathbf{H} \quad \text{in } \Omega, \tag{38}$$

$$\nabla \cdot \mathbf{J} = \nabla \cdot [\sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})] = 0 \quad \text{in } \Omega, \tag{39}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{B} = \mu^* \mathbf{H} \quad \text{in } \Omega, \tag{40}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \tag{41}$$

$$\mathbf{v} = \mathbf{0}, \quad (\mathbf{B}_p - \mathbf{B}_s) \cdot \mathbf{N} = 0 \quad \text{on } \Gamma, \tag{42}$$

$$(\mathbf{J}_p - \mathbf{J}_s) \cdot \mathbf{N} = (\sigma_p \mathbf{E}_p - \sigma_s \mathbf{E}_s) \cdot \mathbf{N} = 0 \quad \text{on } \Gamma, \tag{43}$$

$$(\mathbf{H}_p - \mathbf{H}_s) \times \mathbf{N} = \mathbf{0} \quad \text{on } \Gamma, \tag{44}$$

$$(\mathbf{E}_p - \mathbf{E}_s) \times \mathbf{N} = \mathbf{0} \quad \text{on } \Gamma, \tag{45}$$

where $\mathbf{N}$ is a unit normal to Γ. In the above equations, the velocity $\mathbf{v}$ vanishes in Ω_s. Equation (38) shows that up to the second order of approximation, $\nabla \times \mathbf{H}$ is null and $\mathbf{H}$ is the gradient of a potential ψ which can be chosen such as it is continuous everywhere. We have in view a macroscopic description, at the scale L. Therefore we have the following estimation

$$H_c = O\left(\frac{\psi_c}{L} \right) = O\left(\frac{\varepsilon^{-1} \psi_c}{l} \right).$$

When using l to normalize, we obtain in the dimensionless form

$$\mathbf{H} = -\varepsilon^{-1} \nabla \psi + O(\varepsilon^2). \tag{46}$$

Similarly, equation (41) shows that the electric field $\mathbf{E}$ derives from a potential V. By following the same route as for $\mathbf{H}$, this is written in the following dimensionless form

$$\mathbf{E} = -\varepsilon^{-1} \nabla V. \tag{47}$$

The orders of magnitude of the Hartmann number Ha and the load factor K insure the coupling of the velocity field and of the electric flux at the pore scale. Therefore, we can anticipate that the macroscopic corresponding fluxes, that are averages of these local fluxes, are also coupled.

3.2 *Macroscopic behaviour*

The upscaling is conducted as in section 2. Details of analysis can be found in Geindreau and Auriault (2001, 2002). When returning to *dimensional* quantities, the equivalent macroscopic description is given within an approximation $O(\varepsilon)$ by

$$\nabla \times \langle \mathbf{H} \rangle = \mathbf{0}, \quad \langle \mathbf{H} \rangle = -\nabla \psi,$$
$$\nabla \cdot \langle \mathbf{B} \rangle = 0, \quad \langle \mathbf{B} \rangle = \boldsymbol{\mu}^{*\mathrm{eff}} \langle \mathbf{H} \rangle,$$
$$\nabla \times \langle \mathbf{E} \rangle = \mathbf{0}, \quad \langle \mathbf{E} \rangle = -\nabla V,$$
$$\nabla \cdot \langle \mathbf{J} \rangle = 0, \quad \langle \mathbf{J} \rangle = -\boldsymbol{\sigma}^{\mathrm{eff}} \nabla V - \boldsymbol{\eta}^{\mathrm{eff}} \nabla p,$$
$$\nabla \cdot \langle \mathbf{v} \rangle = 0, \quad \langle \mathbf{v} \rangle = -\mathbf{K} \nabla p - \boldsymbol{\chi}^{\mathrm{eff}} \nabla V,$$

where $\mathbf{K}$ is the permeability, $\boldsymbol{\sigma}^{\mathrm{eff}}$ is the effective electric conductivity, $\boldsymbol{\chi}^{\mathrm{eff}}$ is the electro-osmotic conductivity and $\boldsymbol{\eta}^{\mathrm{eff}}$ is the electric conductivity associated with the streaming potential effect. Under typical conditions as those encountered in magnetohydrodynamic flows, the mass flow and the electric current are described by two coupled equations which are both linear relations of the macroscopic gradient of pressure and of electric potential. Due to the small value of the magnetic Reynolds number in consideration, the macroscopic magnetic field is described by an independent classical magnetic field equation.

3.3 *Properties of the effective coefficients*

The properties of the effective coefficients are investigated from the equivalent variational formulations to the pore scale boundary value problems that are obtained during the upscaling process, see Geindreau & Auriault (2002).

As already mentioned in Hartmann (1937), the permeability $\mathbf{K}$ is strongly affected by the presence of a magnetic field. However, contrary to the case of

a rotating porous medium the permeability tensor is symmetric, as it is for seepage flows in absence of magnetic field. It is positive. The effective conductivity tensor σ^{eff} which depends on the magnetic field is also positive and symmetric.

$$K_{pq}(\langle \mathbf{B} \rangle) = K_{qp}(\langle \mathbf{B} \rangle),$$
$$\sigma^{\text{eff}}_{pq}(\langle \mathbf{B} \rangle) = \sigma^{\text{eff}}_{qp}(\langle \mathbf{B} \rangle). \tag{48}$$

Finally, it is possible to show that the different effective coefficients verify classical Onsager relations. In particular, the permeability $\mathbf{K}$ and the effective conductivity σ^{eff} verify the filtration analog of the Hall effect and the Hall effect, respectively (Landau & Lifshitz 1960)

$$K_{pq}(-\langle \mathbf{B} \rangle) = K_{qp}(\langle \mathbf{B} \rangle),$$
$$\sigma^{\text{eff}}_{pq}(-\langle \mathbf{B} \rangle) = \sigma^{\text{eff}}_{qp}(\langle \mathbf{B} \rangle). \tag{49}$$

Relations (48) and (49) show that $\mathbf{K}$ and σ^{eff} are even tensorial functions of the magnetic induction $\langle \mathbf{B} \rangle$. Moreover, the coupling between the macroscopic mass flux and electric current is characterized by,

$$\eta^{\text{eff}}_{pq} = -\chi^{\text{eff}}_{qp},$$
$$\boldsymbol{\chi}^{\text{eff}}(\langle \mathbf{B} \rangle) = -\chi^{\text{eff}}(-\langle \mathbf{B} \rangle),$$
$$\boldsymbol{\eta}^{\text{eff}}(\langle \mathbf{B} \rangle) = -\eta^{\text{eff}}(-\langle \mathbf{B} \rangle), \tag{50}$$

Tensors $\boldsymbol{\eta}^{\text{eff}}$ and $\boldsymbol{\chi}^{\text{eff}}$ are odd tensorial functions of $\langle \mathbf{B} \rangle$. Finally, the permeability tensor of an isotropic porous medium of permeability k in the absence of a magnetic field and submitted to a macroscopic magnetic induction $\langle \mathbf{B} \rangle = B\mathbf{e}_3$ takes the form

$$\mathbf{K} = \begin{pmatrix} K_{11}(B) & 0 & 0 \\ 0 & K_{11}(B) & 0 \\ 0 & 0 & k \end{pmatrix} \tag{51}$$

Typical behaviour of $K_{11}(B)$ is similar to behaviour of $K_{11}(\omega)$ in Figure 1.

4 CONCLUSIONS

The method of multiscale asymptotic expansions gives the macroscopic behaviours of a liquid flow through a rigid porous medium in rotation relatively to an inertial frame or submitted to a magnetic field, under the only condition of separation of scales. It is of interest to note that the macroscopic behaviours verify Onsager relations which are obtained from quite different analyses. Onsager relations arise from the time reversal invariance at the microscale and the upscaling is conducted using the principles of statistical mechanics whereas the present study is character-ized by irreversible behaviours at the microscale (the pore scale) and the upscaling process is deterministic. Note also that some of the above results cannot be deduced from Onsager's theory, e.g. the symmetries of the permeability and of the electric conductivity under a magnetic field or the decoupling of the magnetic field.

Finally, phenomenological approaches cannot give realistic results as it can be seen, e.g., in the case of parallel capillary pores for which (3) gives $k_{12} \neq 0$, whereas it should cancel out since perpendicular flow to pore axis is prevented.

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