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# ON COURANT'S NODAL DOMAIN PROPERTY FOR LINEAR COMBINATIONS OF EIGENFUNCTIONS PART II

PIERRE BÉRARD AND BERNARD HELFFER

ABSTRACT. This paper is about the extension of Courant's nodal domain theorem to sums of eigenfunctions.

The second section in Arnold's last published paper "Topological properties of eigenoscillations in mathematical physics" (Proceedings of the Steklov Institute of Mathematics 2011, vol. 273, pp. 25-34) is entitled *Courant-Gelfand theorem*, and states that the zeros of any linear combination of the  $n$  first eigenfunctions of the Sturm-Liouville problem

$$-y''(s) + q(x)y(x) = \lambda y(x) \text{ in } ]0, 1[, \text{ with } y(0) = y(1) = 0,$$

divide the interval into at most  $n$  connected components. Arnold describes the idea of proof suggested by Gelfand, and concludes that "the lack of a published formal text with a rigorous proof ... is still distressing."

In the first part of our paper, we discuss this *Courant-Gelfand theorem*, which actually goes back to Sturm (1836), and give a complete proof following Gelfand's ideas. In the second part, we prove that the extension of Courant's nodal domain theorem to sums of eigenfunctions is false for the equilateral rhombus with Neumann boundary condition, and we explain how to construct counterexamples in higher dimension. This paper is a companion to arXiv:1705.03731, which contains several counterexamples to the extension of Courant's theorem to sums of eigenfunctions.

## 1. INTRODUCTION

1.1. **Notation.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain (open and connected), with a piecewise  $C^1$  boundary, or a compact Riemannian manifold with or without boundary. Let  $V$  be a given continuous real valued function on  $\overline{\Omega}$ .

Assume that the boundary  $\Gamma = \partial\Omega$  is partitioned as

$$(1.1) \quad \Gamma = \overline{\bigsqcup_{j=1}^k \Gamma_j},$$

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where the  $\Gamma_j$  are open subsets of  $\partial\Omega$ , and where  $\sqcup$  denotes the disjoint union.

We consider the eigenvalue problem

$$(1.2) \quad -\Delta u + Vu = \mu u \text{ in } \Omega,$$

with the boundary condition  $\mathbf{b} \in \{\mathfrak{d}, \mathfrak{n}\}$  on  $\Gamma$ , where  $\mathfrak{d}$  stands for the *Dirichlet* boundary condition, and  $\mathfrak{n}$  for the *Neumann* boundary condition. In Section 4, we will also consider mixed boundary conditions  $\mathbf{b} = \mathbf{b}_1 \cdots \mathbf{b}_k$ , with  $\mathbf{b}_j$  on  $\Gamma_j$ , for  $1 \leq j \leq k$ . For short, we will speak of the eigenvalue problem for  $(\Omega, \mathbf{b})$ .

Alternatively, we consider the variational eigenvalue problem for the quadratic form

$$(1.3) \quad u \mapsto \int_{\Omega} (|\nabla u(x)|^2 + V(x)u(x)^2) dx$$

with domain  $H_{\mathbf{b}_1 \cdots \mathbf{b}_k}^1(\Omega) \subset L^2(\Omega)$ , where

$$(1.4) \quad H_{\mathbf{b}_1 \cdots \mathbf{b}_k}^1(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Gamma_j} = 0 \text{ whenever } \mathbf{b}_j = \mathfrak{d}\}.$$

As usual, we list the eigenvalues of  $(\Omega, \mathbf{b})$  in nondecreasing order, with multiplicities, starting with the index 1,

$$(1.5) \quad \mu_1(\Omega, \mathbf{b}) < \mu_2(\Omega, \mathbf{b}) \leq \mu_3(\Omega, \mathbf{b}) \leq \cdots$$

We denote by  $\mathcal{E}(\mu_m(\Omega, \mathbf{b}))$  the eigenspace of  $-\Delta + V$  associated with the eigenvalue  $\mu_m(\Omega, \mathbf{b})$ .

Given an eigenvalue  $\mu$  of  $(\Omega, \mathbf{b})$ ,  $\text{mult}(\mu)$  denotes the multiplicity of  $\mu$ , i.e., the dimension of  $\mathcal{E}(\mu)$ , with the convention that  $\text{mult}(\mu) = 0$  if  $\mu$  is not an eigenvalue. We also introduce the index

$$(1.6) \quad \kappa(\mu) := \min\{m \mid \mu = \mu_m(\Omega, \mathbf{b})\}.$$

To make notation lighter, we skip mentioning  $\Omega$ , or the boundary condition  $\mathbf{b}$ , whenever the context is clear.

We denote by  $\mathcal{Z}(u)$  the *nodal set* of a continuous function  $u$ ,

$$(1.7) \quad \mathcal{Z}(u) = \overline{\{x \in \Omega \mid u(x) = 0\}},$$

and by  $\beta_0(u)$  the number of *nodal domains* of  $u$ ,

$$(1.8) \quad \beta_0(u) = \text{number of connected components of } \Omega \setminus \mathcal{Z}(u).$$

## 1.2. Courant's theorem and the extended Courant Property.

The following theorem was proved by R. Courant<sup>1</sup> in 1923, see [10, § VI.6]),

**Theorem 1.1.** *Let  $\mu_m(\Omega, \mathbf{b})$  be the  $m$ -th eigenvalue of  $(\Omega, \mathbf{b})$  according to the order (1.5). Then, for any eigenfunction  $u \in \mathcal{E}(\mu_m(\Omega, \mathbf{b}))$ ,*

$$(1.9) \quad \beta_0(u) \leq \kappa(\mu_m) \leq m.$$

<sup>1</sup>Richard COURANT (1888-1972).

The first footnote in [10, p. 454] states that this theorem *may be generalized as follows: Any linear combination of the first  $n$  eigenfunctions divides the domain, by means of its nodes, into no more than  $n$  subdomains. See the Göttingen dissertation of H. Herrmann, Beiträge zur Theorie der Eigenwerten und Eigenfunktionen, 1932.*

For later purposes, we introduce the following definition.

**Definition 1.2.** *We say that the Extended Courant property is true for the eigenvalue problem  $(\Omega, \mathbf{b})$ , or simply that the  $\text{ECP}(\Omega, \mathbf{b})$  is true, if, for any  $m \geq 1$ , and for any linear combination  $v = \sum_{\mu_j \leq \mu_m} u_{\mu_j}$ , with  $u_{\mu_j} \in \mathcal{E}(\mu_j(\Omega, \mathbf{b}))$ ,*

$$(1.10) \quad \beta_0(v) \leq \kappa(\mu_m) \leq m.$$

**Remarks 1.3.** *Some remarks are in order.*

- (1) *For planar domains with the Dirichlet boundary condition, Pleijel [22] proved that the upper bound in (1.9) is achieved for finitely many values of  $m$  only. Since then, this result has been extended to higher dimensions, and to the manifold case (with the same ingredients as in Pleijel's original proof: a Faber-Krahn inequality, and the Weyl asymptotic law for the eigenvalues). In an attempt to generalize his statement to the Neumann boundary condition, Pleijel (ibidem) mentions the ECP, and says that as far as he has been able to find, there is no proof of this assertion in the literature. The fact that Pleijel's property holds for domains with Neumann boundary condition has recently been settled by C. Léna [16].*
- (2) *As pointed out by Gladwell and Zhu [12], H. Herrmann's thesis [13] does not contain any statement, let alone any proof, of the ECP. They refer to the ECP as the Courant-Herrmann conjecture.*
- (3) *In the early 1970s, V. Arnold<sup>2</sup> pointed out that the ECP can be applied to partially answer Hilbert's 16th problem, see Section 2.*
- (4) *It turns out that, for  $d \geq 2$ , there exist domains  $\Omega$  for which the  $\text{ECP}(\Omega)$  is false. We refer to [7] for references, more historical remarks and some counterexamples.*

Our motivations to investigate the *Extended Courant property* came from reading the survey [14]. This induced us to read [4], Arnold's last paper, submitted for publication six month before his death. We were also motivated by the paper of Gladwell and Zhu [12].

The present paper is organized as follows. In Section 2, we discuss [4, Section 2] in which Arnold relates an interesting story about the

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<sup>2</sup>Vladimir Igorevich ARNOLD (1937-2010).

*generalized Courant theorem*. In Section 3, we come back to the 1-dimensional case, i.e., to Sturm's oscillation theorem for sums of eigenfunctions. We in particular give a proof of a weak form of Sturm's result using an idea of I. Gelfand<sup>3</sup>, described in [4]. In Section 4, we give a new counterexample to the ECP, namely the equilateral rhombus in  $\mathbb{R}^2$ , with Neumann boundary condition. Finally, in Section 5, we explain how one can construct counterexamples to the ECP in higher dimensions, or examples which satisfy a weak form of the ECP.

## 2. ARNOLD AND THE EXTENDED COURANT PROPERTY

**2.1. The ECP for the sphere and the projective space.** Let  $(\mathbb{S}^d, g_0)$  be the round  $d$ -dimensional sphere in  $\mathbb{R}^{d+1}$ . The eigenvalues of the Laplace-Beltrami operator, as point of the spectrum, are the numbers  $\hat{\mu}_k = k(k + d - 1)$ , with  $k \in \mathbb{N}$ . The associated eigenspace is  $\mathcal{H}_k(\mathbb{S}^d)$ , the spherical harmonics of degree  $k$ , which are the restrictions to the sphere of the harmonic homogeneous polynomials of degree  $k$  in  $(d + 1)$  variables, with

$$\dim(\mathcal{H}_k(\mathbb{S}^d)) = \binom{d+k}{d} - \binom{d+k-2}{d}$$

Applying Courant's theorem to a spherical harmonic of degree  $k$ ,  $u \in \mathcal{H}_k(\mathbb{S}^d)$ , we obtain

$$(2.1) \quad \beta_0(u) \leq \tau(\hat{\mu}_k) = \binom{d+k-1}{d} + \binom{d+k-2}{d} + 1.$$

Arnold [4, p. 27] refers to the ECP as the *generalized Courant theorem*. Since spherical harmonics of even degree can be seen as the eigenfunctions of the Laplace-Beltrami operator on the projective space  $(\mathbb{RP}^d, g_0)$ , he noticed that a homogeneous polynomial  $p$ , of even degree, in  $\mathbb{R}^{d+1}$ , seen as a function on  $\mathbb{RP}^d$ , can be written as a sum of even spherical harmonics, and hence that the ECP gives some information on the algebraic hypersurface  $p^{-1}(0)$ .

Citation from Arnold [4, p. 27].

... I immediately deduced from the generalized Courant theorem new results in Hilbert's famous (16th) problem. ... And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. ... Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples

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<sup>3</sup>Israel Moiseevich GELFAND (1913-2009).

were soon produced by Viro). Courant died in 1972 and could not have known about this counterexample<sup>4</sup>.

**Remark 2.1.** *An immediate consequence of Viro's examples [25], published in 1979, is that the  $\text{ECP}(\mathbb{RP}^3, g_0)$  is false and hence, that so does the  $\text{ECP}(\mathbb{S}^3, g_0)$ .*

**2.2. The two dimensional case.** In this case, (2.1) gives

$$(2.2) \quad \beta_0(u) \leq \tau(\hat{\mu}_k) = k^2 + 1, \quad \text{for all } u \in \mathcal{H}_k(\mathbb{S}^2).$$

It turns out, see [17], that this estimate can be improved to

$$(2.3) \quad \beta_0(u) \leq k(k-1) + 2, \quad \text{for all } u \in \mathcal{H}_k(\mathbb{S}^2).$$

**Remark 2.2.** *The fact that the  $\text{ECP}(\mathbb{RP}^2, g_0)$  is true, is stated by Arnold in [1, 5]. In [17, p. 305], Leydold proves the following more general result, and provides the only proof that  $\text{ECP}(\mathbb{RP}^2, g_0)$ , we are aware of.*

**Theorem 2.3** (Leydold). *Let  $p$  be a spherical polynomial of degree  $k$  on  $(\mathbb{S}^2, g_0)$ , i.e., the restriction to the sphere of a homogeneous polynomial of degree  $k$  in  $\mathbb{R}^3$ . Then,*

$$\beta_0(p) \leq k(k-1) + 2.$$

Note that the upper bound in the above theorem is the same as in (2.2). The proof, however, is not in the spirit of the proof of Courant's theorem, but uses real algebraic geometry, see the remarks following the proof of Theorem 1 in [17, p. 305].

**Remark 2.4.** *In the same papers, Arnold also states (without proof) that the  $\text{ECP}(\mathbb{S}^2, g)$  is false for a generic metric  $g$  on  $\mathbb{S}^2$ . A proof that  $\text{ECP}(\mathbb{S}^2, g_0)$  holds seems missing.*

**2.3. Arnold and Gelfand.** In [4, Section 2], Arnold mentions that he discussed the open question of the generalized Courant theorem with many colleagues. He in particular relates a discussion with Gelfand, which we transcribe below, using Arnold's words, in the form of an imaginary dialog.

(Gelfand) *I thought that, except for me, nobody paid attention to Courant's remarkable assertion. But I was so surprised that I delved into it and found a proof.*

(Arnold is quite surprised, but does not have time to mention the counterexamples before Gelfand continues.)

*However, I could prove this theorem of Courant only for oscillations of one-dimensional media, where  $m = 1$ .*

(Arnold) *Where could I read it?*

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<sup>4</sup>[1], the first paper of Arnold on this subject, we are aware of, dates from 1973.

(Gelfand) *I never write proofs. I just discover new interesting things. Finding proofs (and writing articles) is up to my students.*

In [14], Kuznetsov refers to the ECP as *Herrmann's theorem*, and relates that Gelfand's approach *so attracted Arnold that he included Herrmann's theorem for eigenfunctions of problem (★) together with Gelfand's hint into the 3rd Russian edition of his Ordinary Differential Equations.*

Citation from [2, Supplementary problems].

Problem 9. Prove that the zeros of a linear combination of the first  $n$  eigenfunctions of the Sturm-Liouville<sup>5</sup> problem

$$u_{xx} + q(x)u = \lambda u, u(0) = u(\ell) = 0, \quad q > 0 \quad (\star)$$

divide the interval  $[0, \ell]$  into at most  $n$  parts.

Hints (I.M. Gel'fand). Convert to fermions, i.e., to skew-symmetric solutions of the equation  $\sum u_{x_i x_i} + \sum q(x_i)u = \lambda u$  and use the fact that the first eigenfunction of this equation has no zeros inside the fundamental simplex  $0 < x_1 < \dots < x_n < \ell$ .

In Section 3.3, we shall come back to Gelfand's hints.

### 3. THE ECP IN DIMENSION 1

In dimension 1, the natural operator to work with is the Sturm-Liouville operator

$$(3.1) \quad u \mapsto L(u) := -\frac{d}{dt} \left( K \frac{du}{dt} \right) + Q u.$$

We consider both the boundary value problem in a bounded open interval, with either the Dirichlet or the Neumann boundary condition, and the closed eigenvalue problem on  $\mathbb{S}^1$  (no boundary). In both cases the ECP is true, see Theorem 3.1, Assertion 3, and Theorem 3.5, Assertion 3.

**3.1. The Sturm-Liouville problem in  $] \alpha, \beta [$ .** Let  $] \alpha, \beta [$  be a bounded open interval. Let  $K, G$  and  $Q$  be real valued  $C^\infty$  functions defined in a neighborhood of  $] \alpha, \beta [$ , with  $K, G$  strictly positive. We consider the Sturm-Liouville eigenvalue problems

$$(3.2) \quad \begin{cases} -\frac{d}{dt} \left( K \frac{du}{dt} \right) + Q u = \lambda G u \text{ in } ] \alpha, \beta [, \text{ and} \\ u(\alpha) = u(\beta) = 0, \text{ or} \\ \frac{du}{dt}(\alpha) = \frac{du}{dt}(\beta) = 0. \end{cases}$$

<sup>5</sup>Charles-François STURM (1803-1855). Joseph LIOUVILLE (1809-1882).

Note that, without loss of generality, we can always assume that  $Q \geq 0$ . Call  $\{\lambda_i^{\mathfrak{d}}, i \geq 1\}$  (resp.  $\{\lambda_i^{\mathfrak{n}}, i \geq 1\}$ ) the Dirichlet (resp. Neumann) eigenvalues of (3.2), and  $\{V_i^{\mathfrak{d}}, i \geq 1\}$  (resp.  $\{V_i^{\mathfrak{n}}, i \geq 1\}$ ) an associated orthonormal basis of eigenfunctions in  $L^2(] \alpha, \beta[, G dx)$ .

**Theorem 3.1** (Sturm 1836). *For  $\mathfrak{b} \in \{\mathfrak{d}, \mathfrak{n}\}$ , the Sturm-Liouville eigenvalue problem (3.2), has the following properties.*

- (1) *The eigenvalues  $\lambda_i^{\mathfrak{b}}$  are all simple.*
- (2) *For all  $j \geq 1$ , the eigenfunction  $V_j^{\mathfrak{b}}$  has exactly  $(j - 1)$  simple zeros in the interval  $] \alpha, \beta[$ .*
- (3) *For any pair of integers,  $1 \leq m \leq n$ , and for any set of real constants  $a_j, m \leq j \leq n$ , with  $a_m a_n \neq 0$ , the function*

$$(3.3) \quad Y = \sum_{j=m}^n a_j V_j^{\mathfrak{b}}$$

- (a) *has at most  $(n - 1)$  zeros, counted with multiplicities, in the interval  $] \alpha, \beta[$ ;*
- (b) *changes sign at least  $(m - 1)$  times in the interval  $] \alpha, \beta[$ .*

*Proof.* Assertions (1) and (2) are well-known. Assertion (3), is not so well-known; for more details, we refer to [8] and its bibliography.  $\square$

**Remark 3.2.** *In the framework of Fourier series, Assertion (3b) is often referred to as the “Sturm-Hurwitz<sup>6</sup>” theorem. Several papers of Arnold mention this theorem, for example [3]. Strangely enough, in the abstract of [4], Arnold writes that the “one-dimensional version of Courant’s [generalized] theorem is apparently valid”, and refers to Gelfand’s proof (see Subsection 3.3). Was Arnold not aware of the full strength of Sturm’s theorem?*

For later reference, we now indicate an intermediate result, and give a proof (due to Liouville [19], see also Rayleigh<sup>7</sup> [23, § 142]) that Assertion (3a) implies Assertion (3b).

**Lemma 3.3.** *Under the assumptions of the eigenvalue problem (3.2), the function  $Y$  cannot vanish at infinite order at a point  $\xi \in [ \alpha, \beta ]$ . In particular, the zeros of  $Y$  are isolated.*

*Proof.* For  $\ell \in \mathbb{N}$ , introduce the functions  $Y_\ell$  defined by

$$(3.4) \quad Y_\ell = \sum_{j=m}^n (-\lambda_j)^\ell a_j V_j,$$

where  $\lambda_j$  stands for  $\lambda_j^{\mathfrak{b}}$ , and  $V_j$  for  $V_j^{\mathfrak{b}}$ .

<sup>6</sup>Adolf HURWITZ (1859-1919)

<sup>7</sup>John William STRUTT, Lord RAYLEIGH (1842-1919).



Using (3.2), and the definition of  $Y_\ell$ , we infer the relation

$$(3.5) \quad G Y_{\ell+1} = K \frac{d^2 Y_\ell}{dt^2} + \frac{dK}{dt} \frac{dY_\ell}{dt} + Q Y_\ell.$$

Assume that  $Y$  vanishes at infinite order at some  $\xi$ . From (3.5), we deduce that for any  $\ell \geq 0$ ,  $Y_\ell$  vanishes at infinite order at  $\xi$ . Since  $a_n \neq 0$ , dividing by  $\lambda_n^\ell$ , and letting  $\ell$  tend to infinity, we see that  $V_n$  would vanish at  $\xi$  at infinite order also, a contradiction.  $\square$

*Proof that Assertion (3a) implies Assertion (3b).* Assume that the function  $Y$  in (3.3) changes sign precisely at the distinct points  $z_1 < \dots < z_k$  in  $] \alpha, \beta [$ . Following Liouville and Rayleigh, we introduce the  $(k+1) \times (k+1)$  determinant

$$(3.6) \quad D(z_1, \dots, z_k, x) = \begin{vmatrix} V_1(z_1) & \dots & V_1(z_k) & V_1(x) \\ \vdots & \vdots & \vdots & \vdots \\ V_{k+1}(z_1) & \dots & V_{k+1}(z_k) & V_{k+1}(x) \end{vmatrix}.$$

The function  $x \mapsto U(x) := D(z_1, \dots, z_k, x)$  is a linear combination of  $V_1, \dots, V_{k+1}$  which vanishes at the points  $z_1, \dots, z_k$ . According to Sturm's upper bound, Assertion (3a), the function  $U$  cannot vanish at any other point in  $] \alpha, \beta [$ , and furthermore vanishes at order one at each  $z_i$ . It follows that the function  $U Y$  does not change sign in  $] \alpha, \beta [$ .

We claim that  $k \geq m$ . Indeed, if  $k < m$ , using the orthogonality of the functions  $V_j$ , we would have

$$\int_\alpha^\beta U(x) Y(x) G(x) dx = 0,$$

and this would imply that  $Y \equiv 0$ , contradicting the assumption that  $a_m a_n \neq 0$ .  $\square$

**Remark 3.4.** *As a matter of fact, Liouville did not directly introduce the determinant (3.6). Given distinct values  $\{a_k, k \geq 1\} \subset ] \alpha, \beta [$ , he rather introduced a double sequence of functions  $W_{p,j}$  (not his notations), defined as follows. For  $p = 0$  and  $j \geq 1$ , define  $W_{0,j}(x) = V_j(x)$ . For  $p = 1$  and  $j \geq 2$ , define*

$$W_{1,j}(x) = W_{0,1}(a_1) W_{0,j}(x) - W_{0,1}(x) W_{0,j}(a_1).$$

*The functions  $W_{1,j}$  vanish at  $a_1$ , and the first one,  $W_{1,2}(x)$ , only vanishes at  $a_1$ , at order 1, a consequence of Theorem 3.1 Assertion (3a). For  $p \geq 2$  and  $j \geq p+1$ , define*

$$W_{p,j}(x) = W_{p-1,p}(a_p) W_{p-1,j}(x) - W_{p-1,p}(x) W_{p-1,j}(a_p).$$

*The functions  $W_{p,j}$ ,  $j \geq p+1$ , vanish at  $a_1, \dots, a_p$ , and the first one,  $W_{p,p+1}(x)$  only vanishes at these  $p$  points, at order 1 (same reason as above).*

*For  $k \geq 1$  the function  $W_{k,k+1}$  is actually proportional to the determinant in (3.6). Liouville introduced these functions in [19], to prove that*

the eigenfunctions of (3.1) form a complete set. The preceding proof that Assertion (3a) implies Assertion (3b), appears in [20].

Rayleigh introduced the determinant explicitly in [23, § 142].

**3.2. The Sturm-Liouville problem on  $\mathbb{S}^1$ .** With the same assumptions as above, except that  $K, G$  and  $Q$  are now defined on  $\mathbb{R}$ , and assumed to be  $2\pi$ -periodic, we consider the periodic Sturm-Liouville eigenvalue problem,

$$(3.7) \quad \begin{cases} -\frac{d}{dt} \left( K \frac{du}{dt} \right) + Q u = \lambda G u \text{ in } \mathbb{R}, \text{ and} \\ u \text{ is } 2\pi\text{-periodic.} \end{cases}$$

We denote the eigenvalues of (3.7) by  $\{\lambda_i^p, i \geq 1\}$ , and an orthonormal basis of eigenfunctions by  $\{V_i^p, i \geq 1\}$ . Equivalently, we view this eigenvalue problem on the circle  $\mathbb{S}^1$ .

**Theorem 3.5.** *The periodic Sturm-Liouville eigenvalue problem (3.7) has the following properties.*

(1) *The eigenvalues  $\lambda_i^p$  have multiplicity at most 2. More precisely, we have*

$$(3.8) \quad \lambda_1^p < \lambda_2^p \leq \lambda_3^p < \cdots \leq \lambda_{2k-1}^p < \lambda_{2k}^p \leq \lambda_{2k+1}^p < \lambda_{2k+2}^p \leq \cdots$$

(2) *The first eigenfunction,  $V_1^p$  does not vanish in  $\mathbb{S}^1$ , and, for all  $j \geq 1$ , the eigenfunctions  $V_{2j}^p$  and  $V_{2j+1}^p$  have exactly  $2j$  simple zeros in  $\mathbb{S}^1$ .*

(3) *For any pair of integers,  $1 \leq m \leq n$ , and any set of real constants  $a_j, m \leq j \leq n$ , with  $a_m a_n \neq 0$ , the function*

$$(3.9) \quad Y = \sum_{j=m}^n a_j V_j^p$$

(a) *has at most as many zeros, counted with multiplicities, as the function  $V_n^p$  in  $\mathbb{S}^1$ ;*

(b) *changes sign at least as many times as the function  $V_m^p$  in  $\mathbb{S}^1$ .*

**Remark 3.6.** *Note that the function  $Y$  cannot vanish at infinite order at one point, unless it is identically zero (by the same proof as for Lemma 3.3).*

*Proof.* For the first two assertions, we refer to [11, Theorem 3.1.2].

We did not find the third assertion in the literature. We give a proof à la Liouville [20]. In the proof, we write  $\lambda_j$  for  $\lambda_j^p$ , and  $V_j$  for  $V_j^p$ . For  $\ell \geq 0$ , introduce the functions

$$(3.10) \quad Y_\ell = \sum_{j=m}^n (\lambda_1 - \lambda_j)^\ell a_j V_j.$$

Write the equation (3.7) for  $V_1$  and for  $V_j, m \leq j \leq n$ .

$$(3.11) \quad \begin{cases} -\frac{d}{dt} \left( K \frac{dV_1}{dt} \right) + Q V_1 = \lambda_1 G V_1, \\ -\frac{d}{dt} \left( K \frac{dV_j}{dt} \right) + Q V_j = \lambda_j G V_j. \end{cases}$$

Multiply the first equation by  $V_j$ , the second by  $(-V_1)$ , add, and collect the terms to obtain,

$$(3.12) \quad \frac{d}{dt} \left( K V_1^2 \frac{d}{dt} \left( \frac{V_j}{V_1} \right) \right) = (\lambda_1 - \lambda_j) G V_1 V_j.$$

Sum these relations for  $j$  from  $m$  to  $n$ . Then,

$$(3.13) \quad \frac{d}{dt} \left( K V_1^2 \frac{d}{dt} \left( \frac{Y}{V_1} \right) \right) = G V_1 Y_1.$$

More generally, for all  $\ell \geq 0$ , we have the relation

$$(3.14) \quad \frac{d}{dt} \left( K V_1^2 \frac{d}{dt} \left( \frac{Y_\ell}{V_1} \right) \right) = G V_1 Y_{\ell+1}.$$

To prove Assertion (3a), we then use Rolle's lemma for  $2\pi$ -periodic functions.

**Lemma 3.7.** *Let  $f$  be a  $2\pi$ -periodic  $C^\infty$  function. If  $f$  has  $p$  zeros counted with multiplicity in  $\mathbb{S}^1$ , then  $\frac{df}{dt}$  has at least  $p$  zeros counted with multiplicities.*

Applying Rolle's lemma to (3.14), we see that if  $Y$  has  $p$  zeros counted with multiplicities, then for all  $\ell \geq 0$ ,  $Y_\ell$  has at least  $p$  zeros counted with multiplicities. We can always assume that  $\lambda_{n-1} < \lambda_n$ . Then,  $\frac{\lambda_j}{\lambda_n} < 1$ . For  $\ell$  large enough, we infer that the function  $Y_\ell$  has the same number of zeros as  $V_n$ , so that  $Y$  has at most as many zeros as  $V_n$ , counted with multiplicities.

To prove Assertion (3b), we use the method of Liouville indicated in Subsection 3.1. Recall that for  $j \geq 1$ ,  $V_{2j}$  and  $V_{2j+1}$  have exactly  $2k$  zeros of order 1. Let  $k$  be the number of sign changes of the function  $Y$ . We claim that  $k = 2k_1$ . Indeed, we may choose an origin, and arrange the zeros in increasing order, so that,

$$0 < z_1 < z_2 < \cdots < z_k < 2\pi.$$

Without loss of generality, we may assume that  $Y(0) > 0$ . We then have, for  $a$  small enough,  $Y(z_1 - a) > 0$  and hence  $Y(z_1 + a) < 0$ . We easily conclude that  $Y(z_k - a)(-1)^k < 0$ , and  $Y(z_k + a)(-1)^k > 0$ . In particular,  $Y(2\pi)(-1)^k > 0$ . since  $Y(0) = Y(2\pi)$ , we conclude that  $k$  is even.

We then use the function  $U(x)$  defined by (3.6), with  $k = 2k_1$ . From Assertion (3a), we deduce that the only zeros of  $U$  are the  $z_j$ 's, and that they are of order 1, so that  $U$  changes sign at precisely the  $z_j$ 's.

It follows that  $UY$  as a constant sign on  $S^1$ , which we may assume to be positive.

We claim that  $k = 2k_1$  is larger than or equal to the number of zeros of  $V_m$ , which is equal to  $2 \left\lfloor \frac{m}{2} \right\rfloor$ , where  $\lfloor \cdot \rfloor$  denotes the integer part. If not, we would have  $2k_1 \leq 2 \left\lfloor \frac{m}{2} \right\rfloor - 2$ , and hence  $2k + 1 \leq m - 1$ . From this inequality, and the orthogonality of the  $V_k$ 's, we deduce that  $\int_{S^1} UYG dx = 0$ , and hence that  $Y \equiv 0$  which contradicts the assumption that  $a_m \neq 0$ .  $\square$

### 3.3. Gelfand and the ECP for the Sturm-Liouville problem.

3.3.1. *Arnold explanations on Gelfand's hints.* We come back to Problem 9 in Arnold's book [2, Supplementary problems], see Subsection 2.3. The quotations in this subsection are taken from [4, Section 2], in which Arnold is more explicit on Gelfand's hints to solve this problem.

Gelfand's idea was to replace the analysis of the system of  $n$  eigenfunctions of the one-particle quantum-mechanical problem by the analysis of the first eigenfunction of the  $n$ -particle problem (considering as particles, fermions rather than bosons).

According to Arnold, Gelfand's method reduces to proving the following Assertions A and B, which we write with the notations to be used in the next subsections.

**Assertion A.** The (antisymmetric) eigenfunction  $[\mathfrak{S}_n]$  of the operator  $[\mathfrak{h}^{(n)}]$  is the first eigenfunction for this operator (on functions satisfying the Dirichlet condition in the fundamental domain  $[\Omega_n]$ ).

**Assertion B.** Choosing the locations  $[(c_2, \dots, c_n)]$  of the other electrons (except for the first one), one can obtain any linear combination of the first  $n$  eigenfunctions of the one-electron problem as a linear combination

$$[\mathfrak{S}_n(x, c_2, \dots, c_n)]$$

(up to multiplication by a nonzero constant).

However, Arnold continues

Unfortunately, the arguments above do not yet provide a *proof* for this generalized theorem: many facts are still to be proved. For example, assertions A and B are far from obvious and require proving. ...

Gelfand did not publish anything concerning this: he only told me that he hoped his students would correct this drawback of his theory. He pinned high hopes on V.B. Lidskii and A.G. Kostyuchenko.

Arnold also mentions that Lidskii<sup>8</sup> claimed to have a proof, but finally concludes

Although [Lidskii's] arguments look convincing, the lack of a published formal text with a proof of the Courant-Gelfand theorem is still distressing.

In the next subsections, we answer Arnold's wish, and provide a complete solution of Problem 9, following Gelfand's hints (after modification of Assertion B).

3.3.2. *Warm-up: Gelfand's idea applied to the harmonic oscillator.* Let  $\mathfrak{h}^{(1)} := -\frac{d^2}{dx^2} + x^2$  be the *harmonic oscillator* on the line. The eigenvalues of this operator are the  $\{E_k = 2k + 1, k \in \mathbb{N}\}$ , with associated eigenfunctions  $\{h_k(x) = H_k(x) \exp(-x^2/2), k \in \mathbb{N}\}$ , where  $H_k$  is the  $k$ -th Hermite polynomial, [15, Chap. 3].

Introduce the corresponding  $n$ -particle Hamiltonian

$$\mathfrak{h}^{(n)} := \sum_{i=1}^n \left( -\frac{\partial^2}{\partial x_i^2} + x_i^2 \right),$$

on  $L^2(\mathbb{R}^n)$  identified with  $\otimes^n(L^2(\mathbb{R}))$ .

According to Gelfand's hint, we should consider Fermions. This means that we consider  $\wedge^n(L^2(\mathbb{R}))$ , which we can identify with the subspace of  $L^2(\mathbb{R}^n)$  defined by

$$L_F^2(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid u(\sigma_{ij}(x)) = -u(x), \forall i, j\},$$

where  $\sigma_{ij}$  is the transposition of the indices  $i$  and  $j$ ,  $i \neq j$ .

More precisely, we should look at the ground state of  $\mathfrak{h}^{(n)}$  acting on  $L_F^2(\mathbb{R}^n)$ . Define the Fermionic function

$$(3.15) \quad \mathfrak{S}_n(x_1, \dots, x_n) = \begin{vmatrix} h_1(x_1) & h_1(x_2) & \dots & h_1(x_n) \\ h_2(x_1) & h_2(x_2) & \dots & h_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x_1) & h_n(x_2) & \dots & h_n(x_n) \end{vmatrix},$$

known as the Slater<sup>9</sup> determinant in quantum mechanics.

Using the properties of Hermite polynomials, an immediate computation gives

$$\mathfrak{S}_n(x_1, \dots, x_n) = C_n (\prod_{i < j} (x_i - x_j)) \exp(-|x|^2/2),$$

for some constant  $C_n$ , where  $|x|^2 = x_1^2 + \dots + x_n^2$ . From this explicit expression, we immediately conclude that:

<sup>8</sup>Viktor Borisovich LIDSKII (1924-2008).

<sup>9</sup>John Clark SLATER (1900-1976).

$\mathfrak{S}_n$  does not vanish in the fundamental domain  $\Omega_n$ ,

$$\Omega_n := \{-\infty < x_1 < x_2 < \cdots < x_n < +\infty\},$$

of the natural action of the permutation group  $\mathfrak{s}_n$  on  $\mathbb{R}^n$

It follows that  $\mathfrak{S}_n$  is the ground state of the Dirichlet realization of  $\mathfrak{h}^{(n)}$  in  $\Omega_n$ . It follows that, as a function on  $\mathbb{R}^n$ ,  $\mathfrak{S}_n$  is the ground state of  $\mathfrak{h}^{(n)}$  acting on  $L_F^2(\mathbb{R}^n)$ . It has exactly  $n!$  nodal domains, which is also the cardinal of  $\mathfrak{s}_n$ .

We have proved Gelfand's Assertion A of Subsection 3.3.1 in the particular case of the quantum harmonic oscillator.

It is standard in Quantum mechanics (except that the usual context for the one-particle Hamiltonian is a 3D-space) that the ground state energy is the sum of the  $n$  first eigenvalues of the one-particle Hamiltonian (a consequence of the Pauli exclusion principle). This is for example the main motivation for considering this sum when analyzing the celebrated Lieb-Thirring's inequality in connection with the analysis of the *stability of matter* (see for example [18]). When all the eigenvalues are simple (this is the case when the one-particle hamiltonian is 1D), the ground state energy is simple, and the corresponding ground state is the Slater determinant of the first  $n$  eigenfunctions  $h_j$ .

As we shall now see, except for the explicit computations, the above arguments are in fact quite general.

### 3.3.3. The Sturm-Liouville problem with Dirichlet boundary condition.

We now consider a Sturm-Liouville eigenvalue problem and restrict, for simplicity, like Arnold in his paper [4], to the Dirichlet realization of the operator

$$(3.16) \quad \mathfrak{h}^{(1)} := -\frac{d^2}{dx^2} + q(x),$$

in the interval  $I = ]0, 1[$ .

Let  $\{(\lambda_j, h_j), j \geq 1\}$  be the eigenpairs of  $\mathfrak{h}$ , with

$$(3.17) \quad \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

and  $\{h_j, j \geq 1\}$  an associated orthonormal basis of eigenfunctions.

We consider the Dirichlet realization  $\mathfrak{h}^{(n)}$  of the associated  $n$ -particle operator in  $I^n$ ,

$$(3.18) \quad \mathfrak{h}^{(n)} := -\sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} + q(x_j) \right).$$

Denote by  $\vec{k} = (k_1, \cdots, k_n)$  a vector with positive integer entries, and by  $\vec{x} = (x_1, \cdots, x_n)$  a vector in  $I^n$ . The eigenpairs of  $\mathfrak{h}^{(n)}$  are the

$(\Lambda_{\vec{k}}, H_{\vec{k}})$ , with

$$(3.19) \quad \begin{cases} \Lambda_{\vec{k}} = \lambda_{k_1} + \cdots + \lambda_{k_n}, \text{ and} \\ H_{\vec{k}}(\vec{x}) = h_{k_1}(x_1) \cdots h_{k_n}(x_n), \end{cases}$$

where  $H_{\vec{k}}$  is seen as a function in  $L^2(\mathbb{I}^n, dx)$  identified with  $\otimes L^2(\mathbb{I}, dx_j)$ .

The symmetric group  $\mathfrak{s}_n$  acts on  $\mathbb{I}^n$  by  $\sigma(\vec{x}) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ , if  $\vec{x} = (x_1, \dots, x_n)$ . It consequently acts on  $L^2(\mathbb{I}^n)$ , and on the functions  $H_{\vec{k}}$  as well. A fundamental domain of the action of  $\mathfrak{s}_n$  on  $\mathbb{I}^n$  is the  $n$ -simplex

$$(3.20) \quad \Omega_n^{\mathbb{I}} := \{0 < x_1 < x_2 < \cdots < x_n < 1\}.$$

According to Gelfand's suggestion, we look at the restriction  $\mathfrak{h}_F^{(n)}$  of  $\mathfrak{h}^{(n)}$  to the *Fermion states*, i.e., to the functions  $\Psi$  which are antisymmetric with respect to the action of  $\mathfrak{s}_n$ ,

$$(3.21) \quad \Psi(\sigma(\vec{x})) = \varepsilon(\sigma) \Psi(\vec{x}),$$

for all  $\sigma \in \mathfrak{s}_n$ , where  $\varepsilon(\sigma)$  is the signature of the permutation  $\sigma$ . Equivalently, we view  $\mathfrak{h}_F^{(n)}$  as the Dirichlet realization of the operator  $-\Delta + \sum_j q(x_j)$  in the simplex  $\Omega_n^{\mathbb{I}}$ .

In this framework, Gelfand's Assertion A in Subsection 3.3.1, becomes

**Proposition 3.8.** *The ground state energy of  $\mathfrak{h}_F^{(n)}$  is*

$$(3.22) \quad \Lambda^{(n)} := \Lambda_{(1,2,\dots,n)} = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

*Furthermore, its multiplicity is one, and an associated (antisymmetric) eigenfunction is the Slater determinant  $\mathfrak{S}_n$ .*

*Proof.* An eigenfunction  $\Psi$  of  $\mathfrak{h}_F^{(n)}$  is given by a (finite) linear combination  $\Psi = \sum \alpha_{\vec{k}} H_{\vec{k}}$  of eigenfunctions of  $\mathfrak{h}^{(n)}$ , such that the corresponding  $\Lambda_{\vec{k}}$  are equal, and such that  $\Psi$  is antisymmetric. If  $\vec{k} = (k_1, \dots, k_n)$  is such that  $k_i = k_j$  for some pair  $i \neq j$ , using the permutation which exchanges  $i$  and  $j$ , we see that the corresponding  $\alpha_{\vec{k}}$  vanishes. It follows that the eigenvalues of  $\mathfrak{h}_F^{(n)}$  are the  $\Lambda_{\vec{k}}$  such that the entries of  $\vec{k}$  are all different. Using the fact that the eigenvalues of  $\mathfrak{h}^{(1)}$  are simple, it follows that the ground state energy of  $\mathfrak{h}_F^{(n)}$  is  $\Lambda^{(n)}$ .

**Claim 3.9.** *For any  $n$ , the Slater determinant  $\mathfrak{S}_n$  is not identically zero.*

The claim is clear when  $n = 1$ , because  $h_1$  is not identically zero, and when  $n = 2$ , because the functions  $h_1$  and  $h_2$  are linearly independent. We can now use an induction argument. Assume that  $\mathfrak{S}_{n-1}$  is

not identically zero, and that  $\mathfrak{S}_n$  is identically zero. Developing the determinant

$$(3.23) \quad \mathfrak{S}_n(x_1, \dots, x_n) = \begin{vmatrix} h_1(x_1) & h_1(x_2) & \dots & h_1(x_n) \\ h_2(x_1) & h_2(x_2) & \dots & h_2(x_n) \\ \vdots & \vdots & & \vdots \\ h_n(x_1) & h_n(x_2) & \dots & h_n(x_n) \end{vmatrix}$$

with respect to the last column, we obtain

$$(3.24) \quad 0 = s_1(x_1, \dots, x_{n-1}) h_1(x_n) + \dots + s_n(x_1, \dots, x_{n-1}) h_n(x_n),$$

for all  $(x_1, \dots, x_n)$ , where the last coefficient is

$$(3.25) \quad s_n(x_1, \dots, x_{n-1}) = \mathfrak{S}_{n-1}(x_1, \dots, x_{n-1}).$$

According to our induction assumption, there exists some  $(x_1^0, \dots, x_{n-1}^0)$  such that  $s_n(x_1^0, \dots, x_{n-1}^0) \neq 0$ . Because the functions  $h_j, 1 \leq j \leq n$  are linearly independent, this yields a contradiction with (3.24) applied to  $(x_1^0, \dots, x_{n-1}^0, x_n)$  with varying  $x_n$ . The claim is proved.

It is clear that  $\mathfrak{S}_n$  vanishes on  $\partial\Omega_n^I$ . Its restriction  $\mathfrak{S}_{\Omega_n^I}$  to  $\Omega_n^I$  satisfies the Dirichlet condition on  $\partial\Omega_n^I$ , and is an eigenfunction of  $\mathfrak{h}_F^{(n)}$  corresponding to  $\Lambda^{(n)}$ . Suppose that  $\mathfrak{S}_{\Omega_n^I}$  is not the ground state. Then, it has a nodal domain  $\omega$  strictly included in  $\Omega_n$ . Define the function  $U$  which is equal to  $\mathfrak{S}_{\Omega_n^I}$  in  $\omega$ , and to 0 elsewhere in  $I^n$ . It is clearly in  $H_0^1(\Omega_n^I)$ . Using  $\mathfrak{s}_n$ , extend the function  $U$  to a Fermi state  $U_F$  on  $I^n$ . Its energy is  $\Lambda^{(n)}$  which is the bottom of the spectrum of  $\mathfrak{h}_F^{(n)}$ . It follows that  $U_F$  is an eigenfunction of  $\mathfrak{h}_F^{(n)}$ , and a fortiori of  $\mathfrak{h}^{(n)}$ . This would imply that  $\mathfrak{S}_n$  is identically zero, a contradiction with Claim 3.9.  $\square$

Given  $\vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ , denote by  $S_{\vec{b}}(x)$ , the linear combination

$$(3.26) \quad S_{\vec{b}}(x) = \sum_{j=1}^n b_j h_j(x).$$

To solve Arnold's Problem 9 in Subsection 2.3 amounts to proving the following assertion.

$$(3.27) \quad \left\{ \begin{array}{l} \text{For any } \vec{b} \in \mathbb{R}^n \setminus \{0\}, \text{ the function } S_{\vec{b}} \\ \text{has at most } (n-1) \text{ zeros in } ]\alpha, \beta[. \end{array} \right.$$

Given any point  $\vec{c} := (c_1, c_2, \dots, c_{n-1}) \in I^{n-1}$  (with  $0 < c_1 < \dots < c_{n-1} < 1$ ), the function

$$I \ni x \mapsto \mathfrak{S}_n(c_1, c_2, \dots, c_{n-1}, x)$$

can be written as

$$(3.28) \quad \mathfrak{S}_n(c_1, \dots, c_{n-1}, x) = S_{\vec{s}(\vec{c})}(x) := \sum_{j=1}^n s_j(\vec{c}) h_j(x),$$



where the coefficients  $s_j(\vec{c})$  are given by the  $(n-1) \times (n-1)$  Slater determinants,

$$(3.29) \quad s_j(\vec{c}) = s_j(c_1, \dots, c_{n-1}) = \begin{vmatrix} h_1(c_1) & \dots & h_1(c_{n-1}) \\ \vdots & \vdots & \vdots \\ h_{j-1}(c_1) & \dots & h_{j-1}(c_{n-1}) \\ h_{j+1}(c_1) & \dots & h_{j+1}(c_{n-1}) \\ \vdots & \vdots & \vdots \\ h_n(c_1) & \dots & h_n(c_{n-1}) \end{vmatrix}.$$

Clearly, the sum  $S_{\vec{s}(\vec{c})}(x)$  vanishes on the set  $\{c_1, \dots, c_{n-1}\}$ . The fact that the Slater determinant does not vanish in the simplex  $\Omega_n^I$  tells us

$$(3.30) \quad \text{Property } \mathcal{P}: \quad S_{\vec{s}(\vec{c})}(x) = 0 \text{ if and only if } x \in \{c_1, \dots, c_{n-1}\}.$$

Hence  $S_{\vec{s}(\vec{c})}(x)$  vanishes at exactly the  $(n-1)$  distinct points  $c_1, \dots, c_{n-1}$  in  $I$ , and has exactly  $n$  nodal domains.

As a matter of fact, one can say more,

$$(3.31) \quad S_{\vec{s}(\vec{c})} \text{ changes sign at each } c_j.$$

Indeed, we know that  $\mathfrak{S}_n|_{\Omega_n^I}$  does not vanish and, without loss of generality, we may assume that it is positive. If  $x \notin \{c_1, \dots, c_{n-1}\}$ , then there exists some  $j \in \{1, \dots, n\}$  such that  $c_{j-1} < x < c_j$ , with  $c_0 = 0$  and  $c_n = 1$ . Then,  $\mathfrak{S}_n(c_1, \dots, c_{j-1}, x, c_j, \dots, c_{n-1}) > 0$  and hence, by the definition of  $\mathfrak{S}_n$  as determinant,  $(-1)^{n-j} \mathfrak{S}_n(c_1, \dots, c_{n-1}, x) > 0$ .

Gelfand's Assertion B in Subsection 3.3.1 means that given  $\vec{b} \in \mathbb{R}^n \setminus \{0\}$ , one can find  $c_1 < c_2 < \dots < c_{n-1}$  such that  $\vec{b} \parallel \vec{s}(\vec{c})$  (note that we consider the last location instead of the first one).

*Arnold does not observe that this statement is obviously wrong!* Indeed, (3.30)–Property  $\mathcal{P}$  tells us that this is only possible if  $S_{\vec{b}}$  has at least  $(n-1)$  zeroes, and changes sign at each of these zeros.

This might seem to be a problem, but we can simply observe that if  $S_{\vec{b}}$  has less than  $(n-1)$  zeroes, then (3.27) is true. So that we replace Gelfand's Assertion B by the following lemma.

**Lemma 3.10.** *Let  $\vec{b} \neq 0$ , and assume that  $S_{\vec{b}}$  has at least  $(n-1)$  zeroes  $c_1 < c_2 < \dots < c_{n-1}$ , then there exists a constant  $C \neq 0$  such that*

$$S_{\vec{b}}(x) = C \mathfrak{S}_n(c_1, \dots, c_{n-1}, x).$$

Note that we immediately deduce from the lemma and Property  $\mathcal{P}$  that in this case  $S_{\vec{b}}$  has exactly  $(n-1)$  zeroes, and that we have proved (3.27) and solved Problem 9. Hence, it remains to prove the lemma.

*Proof.* We observe that the vectors  $\vec{c}$  and  $\vec{s}(\vec{c})$  are explicitly given by the assumption, and that  $\vec{s}(\vec{c})$  is obtained by writing the  $(n-1)$  equations  $S_{\vec{b}}(c_j) = 0$  for  $j = 1, \dots, n-1$ .

What we have to prove is that  $\vec{b}$  and  $\vec{s}(\vec{c})$  are collinear. For this, we observe that the two vectors are orthogonal to the space generated by the  $(n - 1)$  vectors  $\vec{h}(c_j)$  where  $\vec{h} := (h_1, \dots, h_n)$ . It suffices to prove that the  $(n - 1)$  vectors  $\vec{h}(c_j)$  are linearly independent in  $\mathbb{R}^n$ . But if they were not linearly independent, we would have  $\mathfrak{S}_n(c_1, \dots, c_{n-1}, x) \equiv 0$ , which is excluded.  $\square$

**Remark 3.11.** *Note that we have so far not discussed the multiplicities of zeros. As a matter of fact, Arnold’s Problem 9 proposes a weak form of Sturm’s theorem, compare the statement (3.27) with Theorem 3.1, Assertion (3a). It would be interesting to have a proof “à la Gelfand” of the strong version, counting zeros with multiplicities. Note, however, that Gelfand’s proof is sufficient to prove Theorem 3.1, Assertion (3b). Indeed, suppose by contradiction that the function  $S_{\vec{b}} = \sum_{\ell=m}^n b_{\ell} h_{\ell}$  has only  $k$  zeroes  $c_1, c_2, \dots, c_k$  where it changes sign, with  $k < m - 1$ . Then, using the Slater determinant relative to the  $k + 1$  first eigenfunctions, and equation (3.31), we can construct a function  $Y$  which has the same sign changes as  $S_{\vec{b}}$  by fixing the first  $k$  variables. Then, the scalar product of  $Y$  and  $S_{\vec{b}}$  is strictly positive in contradiction with the orthogonality of  $Y$  and  $S_{\vec{b}}$  which we deduce from the inequality  $k < m - 1$ .*

#### 4. THE EQUILATERAL RHOMBUS

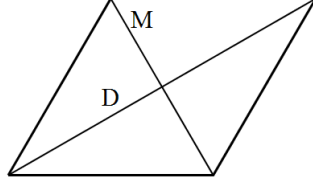
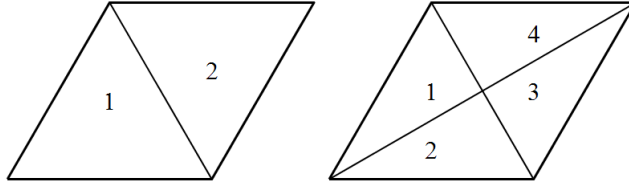
**4.1. Preparation.** Let  $\mathcal{R}h_e$  be the equilateral rhombus with sides of length 1, and vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(\frac{3}{2}, \frac{\sqrt{3}}{2})$  and  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ . Call  $D$  and  $M$  its diagonals (resp. the longer one and the shorter one), see Figure 4.1.

Call  $\mathcal{T}_e$  the equilateral triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ . Call  $\mathcal{T}_h$  the hemiequilateral triangle with vertices  $(0, 0)$ ,  $(\frac{\sqrt{3}}{2}, 0)$  and  $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ . See Figure 4.3.

The diagonal  $M$  divides the rhombus into two equilateral triangles  $\mathcal{T}_{e,1}$  and  $\mathcal{T}_{e,2}$ , isometric to  $\mathcal{T}_e$ . The diagonals  $D$  and  $M$  divide the rhombus into four hemiequilateral triangles  $\mathcal{T}_{h,j}$ ,  $1 \leq j \leq 4$ , isometric to  $\mathcal{T}_h$ , see Figure 4.2. In the sequel, we also use the generic notation  $\mathcal{T}_e$  (resp.  $\mathcal{T}_h$ ) for any of the equilateral triangles (resp. hemiequilateral triangles) into which the rhombus decomposes.

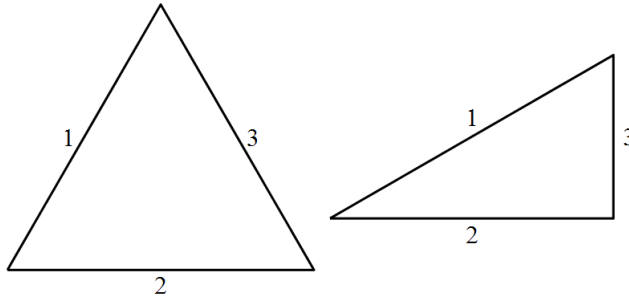
We also denote by  $D$  and  $M$  the mirror symmetries with respect to the lines supporting the diagonals of the rhombus, and by  $D^*$  and  $M^*$  the action of these symmetries on functions, for example  $D^*f = f \circ D$ .

The symmetries  $D$  and  $M$  act by isometries on the rhombus, and they commute. The eigenspaces of  $-\Delta$  for  $(\mathcal{R}h_e, \mathbf{b})$ , with  $\mathbf{b} \in \{\mathfrak{d}, \mathfrak{n}\}$ , decompose into summands corresponding to the action of these symmetries. The eigenfunctions in each summand correspond to eigenfunctions of  $-\Delta$  for the equilateral or hemiequilateral triangles into which the rhombus decomposes, with the boundary condition  $\mathbf{b}$  on the sides supported

FIGURE 4.1. The equilateral rhombus  $\mathcal{R}h_e$ , and its diagonalsFIGURE 4.2. Decomposition of the equilateral rhombus  $\mathcal{R}h_e$ 

by  $\partial\mathcal{R}h_e$ , and with mixed boundary conditions, either Dirichlet or Neumann, on the sides supported by the diagonals.

To be more explicit, we need naming the eigenvalues according to Subsection 1.1. For this purpose, we partition the boundaries of  $\mathcal{T}_e$  and  $\mathcal{T}_h$  into their three sides. For  $\mathcal{T}_h$ , we number the sides 1, 2, 3, in decreasing order of length, see Figure 4.3. For example,  $\mu_i(\mathcal{T}_h, \mathbf{n}\mathbf{d}\mathbf{n})$  denotes the  $i$ -th eigenvalue of  $-\Delta$  in  $\mathcal{T}_h$  with Neumann boundary condition on the longest (1) and shortest (3) sides, and Dirichlet boundary condition on the other side (2).

FIGURE 4.3. Labelling the sides of  $\mathcal{T}_e$  and  $\mathcal{T}_h$ 

More precisely, we decompose the space  $L^2(\mathcal{R}h_e)$  into orthogonal components,

$$(4.1) \quad L^2(\mathcal{R}h_e) = \mathcal{S}_{+,+} \oplus \mathcal{S}_{+,-} \oplus \mathcal{S}_{-,+} \oplus \mathcal{S}_{-,-},$$

where

$$(4.2) \quad \mathcal{S}_{\sigma,\tau} := \left\{ \phi \in L^2(\mathcal{R}h_e) \mid D^*\phi = \sigma\phi \text{ and } M^*\phi = \tau\phi \right\},$$

for  $\sigma, \tau \in \{+, -\}$ .

Since the isometries  $D^*$  and  $M^*$  commute with  $\Delta$ , this orthogonal decomposition descends to each eigenspace of  $-\Delta$  for  $(\mathcal{R}h_e, \mathbf{b})$ , with the boundary condition  $\mathbf{b} \in \{\mathfrak{d}, \mathfrak{n}\}$  on the boundary  $\partial\mathcal{R}h_e$ .

**4.2. A reflection principle.** In this subsection, we explain an elementary but useful “reflection principle” which we will use repeatedly in the sequel.

Consider the decomposition  $\mathcal{R}h_e = \mathcal{T}_{e,1} \sqcup \mathcal{T}_{e,2}$ . Note that  $M(\mathcal{T}_{e,1}) = \mathcal{T}_{e,2}$ . Choose a boundary condition  $\mathbf{a} \in \{\mathfrak{d}, \mathfrak{n}\}$  on  $\partial\mathcal{R}h_e$ . Given an eigenvalue  $\lambda$  of  $-\Delta$  for  $(\mathcal{R}h_e, \mathbf{a})$ , and  $\sigma \in \{+, -\}$ , consider the subspace  $\mathcal{E}(\lambda) \cap \mathcal{S}_{M,\sigma}$  of eigenfunctions  $\phi \in \mathcal{E}(\lambda)$  such that  $M^*\phi = \sigma\phi$ .

If  $0 \neq \phi \in \mathcal{E}(\lambda) \cap \mathcal{S}_{M,\sigma}$ , then  $\phi|_{\mathcal{T}_{e,1}}$  is an eigenfunction of  $-\Delta$  for  $(\mathcal{T}_{e,1}, \mathbf{aab})$ , with  $\mathbf{b} = \mathfrak{n}$  if  $\sigma = +$ , and  $\mathbf{b} = \mathfrak{d}$  if  $\sigma = -$ , associated with the same eigenvalue  $\lambda$ .

Conversely, let  $\psi$  be an eigenfunction of  $(\mathcal{T}_{e,1}, \mathbf{aab})$ , with eigenvalue  $\mu_m(\mathcal{T}_{e,1}, \mathbf{aab})$ , for some  $m \geq 1$ . Define the function  $\check{\psi}$  on  $\mathcal{R}h_e$  such that  $\check{\psi}|_{\mathcal{T}_{e,1}} = \psi$  and  $\check{\psi}|_{\mathcal{T}_{e,2}} = \sigma\psi \circ M$ . This means that  $\check{\psi}$  is obtained by extending  $\psi$  across  $M$  to  $\mathcal{T}_{e,2}$  by symmetry, in such a way that  $M^*\check{\psi} = \sigma\check{\psi}$ . It is easy to see that the function  $\check{\psi}$  is an eigenfunction of  $-\Delta$  for  $(\mathcal{R}h_e, \mathbf{n})$  (in particular it is smooth in a neighborhood of  $M$ ), with eigenvalue  $\mu_m(\mathcal{T}_{e,1}, \mathbf{aab})$ , so that  $\check{\psi} \in \mathcal{E}(\mu_m) \cap \mathcal{S}_{M,\sigma}$ .

The above considerations prove the first two assertions in the following proposition. The proof of the third and fourth assertions is similar, using the symmetries  $D$  and  $M$ , and the decomposition of  $\mathcal{R}h_e$  into the triangles  $\mathcal{T}_{h,j}$ ,  $1 \leq j \leq 4$ .

**Proposition 4.1** (Reflection principle). *For any  $\mathbf{a} \in \{\mathfrak{d}, \mathfrak{n}\}$  and any  $\lambda \in \text{sp}(\mathcal{R}h_e, \mathbf{a})$ ,*

- (i)  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{M,+} \neq \{0\}$  if and only if  $\lambda \in \text{sp}(\mathcal{T}_e, \mathbf{aan})$ , and the map  $\phi \mapsto \phi|_{\mathcal{T}_{e,1}}$  is a bijection from  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{M,+}$  onto  $\mathcal{E}(\lambda, (\mathcal{T}_e, \mathbf{aan}))$ ;
- (ii)  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{M,-} \neq \{0\}$  if and only if  $\lambda \in \text{sp}(\mathcal{T}_e, \mathbf{aad})$ , and the map  $\phi \mapsto \phi|_{\mathcal{T}_{e,1}}$  is a bijection from  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{M,-}$  onto  $\mathcal{E}(\lambda, (\mathcal{T}_e, \mathbf{aad}))$ .

More generally, define  $\epsilon(\mathfrak{n}) = +$  and  $\epsilon(\mathfrak{d}) = -$ . Then, for any  $\lambda \in \text{sp}(\mathcal{R}h_e, \mathbf{a})$ , and any  $\mathbf{b}, \mathbf{c} \in \{\mathfrak{d}, \mathfrak{n}\}$ ,

- (iii)  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{\epsilon(\mathbf{b}), \epsilon(\mathbf{c})} \neq \{0\}$  if and only if  $\lambda \in \text{sp}(\mathcal{T}_h, \mathbf{abc})$ , and the map  $\phi \mapsto \phi|_{\mathcal{T}_{h,1}}$  is a bijection from  $\mathcal{E}(\lambda, (\mathcal{R}h_e, \mathbf{a})) \cap \mathcal{S}_{\epsilon(\mathbf{b}), \epsilon(\mathbf{c})}$  onto  $\mathcal{E}(\lambda, (\mathcal{T}_h, \mathbf{abc}))$ .

Furthermore, the multiplicity of the number  $\lambda$  as eigenvalue of  $(\mathcal{R}h_e, \mathbf{a})$  is the sum, over  $\mathbf{b}, \mathbf{c} \in \{\mathfrak{d}, \mathfrak{n}\}$ , of the multiplicities of  $\lambda$  as eigenvalue of  $(\mathcal{T}_h, \mathbf{abc})$  (with the convention that the multiplicity is zero if  $\lambda$  is not an eigenvalue).

**4.3. Some useful results.** In this subsection, we recall some known results for the reader's convenience.

**4.3.1. Eigenvalue inequalities.** The following proposition is a particular case of a result of V. Lotoreichik and J. Rohleder [21, Proposition 2.3].

**Proposition 4.2.** *For the triangle  $\mathcal{T}_h$ , and for any  $i \geq 1$ , we have the inequalities,*

$$(4.3) \quad \begin{cases} \mu_i(\mathcal{T}_h, \mathbf{nnn}) < \mu_i(\mathcal{T}_h, \mathbf{n\partial n}) < \mu_i(\mathcal{T}_h, \mathbf{n\partial\partial}), \\ \mu_i(\mathcal{T}_h, \mathbf{nnn}) < \mu_i(\mathcal{T}_h, \mathbf{n\partial\partial}) < \mu_i(\mathcal{T}_h, \mathbf{n\partial n}), \end{cases}$$

and

$$(4.4) \quad \begin{cases} \mu_i(\mathcal{T}_h, \mathbf{\partial nn}) < \mu_i(\mathcal{T}_h, \mathbf{\partial\partial n}) < \mu_i(\mathcal{T}_h, \mathbf{\partial\partial\partial}), \\ \mu_i(\mathcal{T}_h, \mathbf{\partial nn}) < \mu_i(\mathcal{T}_h, \mathbf{\partial\partial\partial}) < \mu_i(\mathcal{T}_h, \mathbf{\partial\partial n}). \end{cases}$$

The following inequalities are due to B. Siudeja [24, Theorem 1.1].

**Proposition 4.3.** *The eigenvalues of  $\mathcal{T}_h$  with Neumann boundary condition are denoted by  $\nu_i$ . The eigenvalues of  $\mathcal{T}_h$  with mixed boundary conditions are denoted by  $\mu_i(\mathbf{abc})$ , with the sides listed in decreasing order. They satisfy the following inequalities.*

$$0 = \nu_1 < \mu_1(\mathbf{n\partial\partial}) < \mu_1(\mathbf{n\partial n}) = \nu_2 < \mu_1(\mathbf{\partial nn}) \cdots \\ \cdots < \mu_1(\mathbf{n\partial\partial}) < \mu_1(\mathbf{\partial\partial n}) < \mu_1(\mathbf{\partial\partial\partial}) < \delta_1.$$

**Remark 4.4.** *We do not know if there exist, for  $i \geq 2$ , any general inequalities between the eigenvalues  $\mu_i(\mathcal{T}_h, \mathbf{n\partial n})$  and  $\mu_i(\mathcal{T}_h, \mathbf{n\partial\partial})$  in (4.3), or between the eigenvalues  $\mu_i(\mathcal{T}_h, \mathbf{\partial\partial n})$  and  $\mu_i(\mathcal{T}_h, \mathbf{\partial\partial\partial})$  in (4.4).*

**4.3.2. Eigenvalues of some mixed boundary value problems for  $\mathcal{T}_h$ .** For later reference, we also describe the eigenvalues and eigenfunctions of four mixed eigenvalue problems for  $\mathcal{T}_h$ . This description follows easily for example from [7, Appendix A] and [6].

The eigenvalues of the equilateral triangle  $\mathcal{T}_e$ , with either the Dirichlet or the Neumann boundary condition on  $\partial\mathcal{T}_e$ , are the numbers

$$(4.5) \quad \hat{\lambda}(m, n) = \frac{16\pi^2}{9} (m^2 + mn + n^2),$$

with  $(m, n) \in \mathbb{N} \times \mathbb{N}$  for the Neumann boundary condition, and  $(m, n) \in \mathbb{N}^\bullet \times \mathbb{N}^\bullet$  for the Dirichlet boundary condition (here  $\mathbb{N}^\bullet = \mathbb{N} \setminus \{0\}$ ). The multiplicities are given by,

$$(4.6) \quad \text{mult}(\hat{\lambda}_0) = \# \left\{ (m, n) \in \mathcal{L} \mid \hat{\lambda}(m, n) = \hat{\lambda}_0 \right\},$$

with  $\mathcal{L} = \mathbb{N} \times \mathbb{N}$  for the Neumann boundary condition, and  $\mathcal{L} = \mathbb{N}^\bullet \times \mathbb{N}^\bullet$  for the Dirichlet boundary condition.

One can associate one or two real eigenfunctions with such a pair  $(m, n)$ . When  $m = n$ , there is only one associated eigenfunction, and it is  $D$ -invariant. When  $m \neq n$ , there are two associated eigenfunctions, one symmetric with respect to  $D$ , the other one anti-symmetric. As a consequence, one can explicitly describe the eigenvalues and eigenfunctions of the four eigenvalue problems  $(\mathcal{T}_h, \mathbf{nnn})$ ,  $(\mathcal{T}_h, \mathbf{n\partial n})$  (they arise from the Neumann problem for  $\mathcal{T}_e$ ), and  $(\mathcal{T}_h, \mathbf{\partial n\partial})$ ,  $(\mathcal{T}_h, \mathbf{\partial\partial\partial})$  (they arise from the Dirichlet problem for  $\mathcal{T}_e$ ).

The resulting eigenvalues are given in Table 4.1.

TABLE 4.1. Four mixed eigenvalue problems for  $\mathcal{T}_h$

Eigenvalue problem	Eigenvalues
$(\mathcal{T}_h, \mathbf{nnn})$	$\hat{\lambda}(m, n)$ , for $0 \leq m \leq n$
$(\mathcal{T}_h, \mathbf{n\partial n})$	$\hat{\lambda}(m, n)$ , for $0 \leq m < n$
$(\mathcal{T}_h, \mathbf{\partial n\partial})$	$\hat{\lambda}(m, n)$ , for $1 \leq m \leq n$
$(\mathcal{T}_h, \mathbf{\partial\partial\partial})$	$\hat{\lambda}(m, n)$ , for $1 \leq m < n$

**Remark 4.5.** *As far as we know, there are no such explicit formulas for the eigenvalues of the other mixed boundary value problems for  $\mathcal{T}_h$ .*

The following tables, display the first few eigenvalues, the corresponding pairs of integers, and the corresponding indexed eigenvalues for the given mixed boundary value problems for  $\mathcal{T}_h$ .

TABLE 4.2. First eigenvalues for  $(\mathcal{T}_h, \mathbf{nnn})$  and  $(\mathcal{T}_h, \mathbf{n\partial n})$

Eigenvalue	Pairs	$(\mathcal{T}_h, \mathbf{nnn})$	$(\mathcal{T}_h, \mathbf{n\partial n})$
0	(0, 0)	$\mu_1$	
$\frac{16\pi^2}{9}$	(0, 1), (1, 0)	$\mu_2$	$\mu_1$
$3 \times \frac{16\pi^2}{9}$	(1, 1)	$\mu_3$	
$4 \times \frac{16\pi^2}{9}$	(0, 2), (2, 0)	$\mu_4$	$\mu_2$
$7 \times \frac{16\pi^2}{9}$	(1, 2), (2, 1)	$\mu_5$	$\mu_3$
$9 \times \frac{16\pi^2}{9}$	(0, 3), (3, 0)	$\mu_6$	$\mu_4$

**Remark 4.6.** *For later reference, we point out that all the eigenvalues which appear in Tables 4.2 and 4.3 are simple.*

TABLE 4.3. First eigenvalues for  $(\mathcal{T}_h, \mathfrak{d}\mathfrak{n}\mathfrak{d})$  and  $(\mathcal{T}_h, \mathfrak{d}\mathfrak{d}\mathfrak{d})$ 

Eigenvalue	Pairs	$(\mathcal{T}_h, \mathfrak{d}\mathfrak{n}\mathfrak{d})$	$(\mathcal{T}_h, \mathfrak{d}\mathfrak{d}\mathfrak{d})$
$3 \times \frac{16\pi^2}{9}$	(1, 1)	$\mu_1$	
$7 \times \frac{16\pi^2}{9}$	(1, 2), (2, 1)	$\mu_2$	$\mu_1$
$12 \times \frac{16\pi^2}{9}$	(2, 2)	$\mu_3$	
$13 \times \frac{16\pi^2}{9}$	(1, 3), (3, 1)	$\mu_4$	$\mu_2$
$19 \times \frac{16\pi^2}{9}$	(2, 3), (3, 2)	$\mu_5$	$\mu_3$
$21 \times \frac{16\pi^2}{9}$	(1, 4), (4, 1)	$\mu_6$	$\mu_4$

**4.4. Rhombus with Neumann boundary condition.** In this subsection, we take the Neumann boundary condition on the boundary  $\partial\mathcal{R}h_e$  of the equilateral rhombus.

4.4.1. *The first Neumann eigenvalues of  $\mathcal{R}h_e$ .*

**Proposition 4.7.** *Let  $\nu_i$  denote the eigenvalues of  $(\mathcal{R}h_e, \mathfrak{n})$ . Then,*

$$(4.7) \quad 0 = \nu_1 < \nu_2 < \nu_3 = \nu_4 < \nu_5 \leq \dots$$

More precisely,

(i) *The second eigenvalue  $\nu_2$  is simple and satisfies*

$$(4.8) \quad \nu_2 = \mu_1(\mathcal{T}_e, \mathfrak{n}\mathfrak{n}\mathfrak{d}) = \mu_1(\mathcal{T}_h, \mathfrak{n}\mathfrak{n}\mathfrak{d}).$$

*If  $\psi_2 \in \mathcal{E}(\nu_2)$ , then it is invariant under the symmetry  $D$ , anti-invariant under the symmetry  $M$ , and  $\mathcal{Z}(\psi_2) = M$ .*

*Furthermore,  $\psi_2|_{\mathcal{T}_h}$  is a first eigenfunction of  $(\mathcal{T}_h, \mathfrak{n}\mathfrak{n}\mathfrak{d})$ , and  $\psi_2|_{\mathcal{T}_e}$  is a first eigenfunction of  $(\mathcal{T}_e, \mathfrak{n}\mathfrak{n}\mathfrak{d})$ .*

(ii) *For the eigenspace  $\mathcal{E}(\nu_3)$  we have*

$$(4.9) \quad \begin{cases} \dim(\mathcal{E}(\nu_3) \cap \mathcal{S}_{+,+}) = \dim(\mathcal{E}(\nu_3) \cap \mathcal{S}_{-,+}) = 1, \\ \mathcal{E}(\nu_3) \cap \mathcal{S}_{-,-} = \mathcal{E}(\nu_3) \cap \mathcal{S}_{+,-} = \{0\}. \end{cases}$$

*In particular, the eigenspace  $\mathcal{E}(\nu_3)$  is spanned by two linearly independent functions  $\psi_3$  and  $\psi_4$  which are  $M$  invariant, and whose restrictions to  $\mathcal{T}_e$  generate the eigenspace  $\mathcal{E}(\nu_2(\mathcal{T}_e))$ .*

*Proof.* According to the Reflection principle, Proposition 4.1, the first six eigenvalues of  $(\mathcal{R}h_e, \mathfrak{n})$  belong to the set

$$(4.10) \quad \{\mu_i(\mathcal{T}_h, \mathfrak{n}\mathfrak{a}\mathfrak{b}) \text{ for } 1 \leq i \leq 6 \text{ and } \mathfrak{a}, \mathfrak{b} \in \{\mathfrak{d}, \mathfrak{n}\}\}.$$

Among these numbers, the eigenvalues of  $(\mathcal{T}_h, \mathfrak{n}\mathfrak{n}\mathfrak{n})$  and  $(\mathcal{T}_h, \mathfrak{n}\mathfrak{d}\mathfrak{n})$  are known explicitly, and they are simple, see Tables 4.2.

Although the eigenvalues and eigenfunctions of  $(\mathcal{T}_h, \mathfrak{n}\mathfrak{n}\mathfrak{d})$  and  $(\mathcal{T}_h, \mathfrak{d}\mathfrak{n}\mathfrak{n})$  are, as far as we know, not explicitly known, they satisfy some inequalities: the obvious inequalities  $\mu_1 < \mu_2 \leq \dots$ , and the inequalities provided by Proposition 4.2 (see [21]), and Proposition 4.3 (see [24]).

Table 4.4 summarizes what we know about the eigenvalues which appear in (4.10). In blue the known values, in red the known inequalities (Propositions 4.2 and 4.3). The gray cells contain the eigenvalues, listed with multiplicities, for which we have no a priori information, except the trivial inequalities (black inequality signs).

**Remark 4.8.** *Note that we only display the first four eigenvalues in each line, because this turns out to be sufficient.*

**Remark 4.9.** *The reason why there are white empty cells in the 5th row is explained in Remark 4.4.*

$(\sigma, \tau)$	$(\mathcal{T}_h, \mathbf{nab})$	$\mu_1$		$\mu_2$		$\mu_3$		$\mu_4$
$(+, +)$	$(\mathcal{T}_h, \mathbf{nnn})$	0	<	$\frac{16\pi^2}{9}$	<	$3 \frac{16\pi^2}{9}$	<	$4 \frac{16\pi^2}{9}$
Prop. 4.2		$\wedge$		$\wedge$		$\wedge$		$\wedge$
$(+, -)$	$(\mathcal{T}_h, \mathbf{nn}\mathfrak{d})$		<		$\leq$		$\leq$	
Prop. 4.3		$\wedge$						
$(-, +)$	$(\mathcal{T}_h, \mathbf{n}\mathfrak{d}\mathbf{n})$	$\frac{16\pi^2}{9}$	<	$4 \frac{16\pi^2}{9}$	<	$7 \frac{16\pi^2}{9}$	<	$9 \frac{16\pi^2}{9}$
Prop. 4.2		$\wedge$		$\wedge$		$\wedge$		$\wedge$
$(-, -)$	$(\mathcal{T}_h, \mathbf{n}\mathfrak{d}\mathfrak{d})$		<		$\leq$		$\leq$	

TABLE 4.4.  $\mathcal{R}h_e$ , Neumann boundary condition

Upon inspection of Table 4.4, we conclude that the Neumann eigenvalues of  $\mathcal{R}h_e$  satisfy the following inequalities

$$(4.11) \quad 0 = \nu_1 < \nu_2 < \nu_3 = \nu_4 < \nu_5 \leq \nu_6 \leq \dots,$$

with

$$\nu_2 = \mu_1(\mathcal{T}_h, \mathbf{nn}\mathfrak{d}), \nu_3 = \mu_1(\mathcal{T}_h, \mathbf{n}\mathfrak{d}\mathbf{n}) = \mu_2(\mathcal{T}_h, \mathbf{nnn}).$$

We can a priori not draw any conclusion on  $\nu_5, \nu_6, \dots$

Table 4.4 actually provides further information,

$$(4.12) \quad \left\{ \begin{array}{l} \dim \mathcal{E}(\nu_1) \cap \mathcal{S}_{+,+} = 1, \\ \mathcal{E}(\nu_1) \cap \mathcal{S}_{\sigma,\tau} = \{0\} \text{ if } (\sigma, \tau) \neq (+, +), \\ \dim \mathcal{E}(\nu_2) \cap \mathcal{S}_{+,-} = 1, \\ \mathcal{E}(\nu_2) \cap \mathcal{S}_{\sigma,\tau} = \{0\} \text{ if } (\sigma, \tau) \neq (+, -), \\ \dim \mathcal{E}(\nu_3) \cap \mathcal{S}_{+,+} = \dim \mathcal{E}(\nu_3) \cap \mathcal{S}_{+,-} = 1, \\ \dim \mathcal{E}(\nu_3) \cap \mathcal{S}_{\sigma,\tau} = \{0\} \text{ if } (\sigma, \tau) = (-, +) \text{ or } (-, -). \end{array} \right.$$

The equality  $\nu_3 = \nu_4$  comes from the fact that the second Neumann eigenvalue of the equilateral triangle has multiplicity 2, with an eigen-space generated by one eigenfunction which is symmetric with respect to a side bisector, and another one which is anti-symmetric.



Note: For the reader's information, Table 4.5, displays numerical values for the eigenvalues: in the gray cells, the numerical values computed with MATLAB; in the other cells, the approximate values of the known eigenvalues.

$(\sigma, \tau)$	$(\mathcal{T}_h, \mathbf{nab})$	$\mu_1$		$\mu_2$		$\mu_3$		$\mu_4$
$(+, +)$	$(\mathcal{T}_h, \mathbf{nnn})$	0	<	17.55	<	52.64	<	70.18
		$\wedge$		$\wedge$		$\wedge$		$\wedge$
$(+, -)$	$(\mathcal{T}_h, \mathbf{nn\partial})$	7.16	<	37.49	$\leq$	90.06	$\leq$	120.87
		$\wedge$						
$(-, +)$	$(\mathcal{T}_h, \mathbf{n\partial n})$	17.55	<	70.18	<	122.82	<	157.91
		$\wedge$		$\wedge$		$\wedge$		$\wedge$
$(-, -)$	$(\mathcal{T}_h, \mathbf{n\partial\partial})$	47.63	<	110.36	$\leq$	189.52	$\leq$	224.68

TABLE 4.5.  $\mathcal{R}h_e$ , Neumann boundary condition

The proof of Proposition 4.7 is complete.  $\square$

**Remark 4.10.** *One can also deduce Proposition 4.7 from the proof of Corollary 1.3 in [24] which establishes that the first four Neumann eigenvalues of a rhombus  $\mathcal{R}h(\alpha)$  with smallest angle  $2\alpha > \frac{\pi}{3}$  are simple, and describes the nodal patterns of the corresponding eigenvalues. When  $2\alpha = \frac{\pi}{3}$  the eigenvalues  $\nu_3$  and  $\nu_4$  become equal, see also Remarks 4.1 and 4.2 in [24].*

4.4.2. *The Extended Courant Property for  $\mathcal{R}h_e$ .* As a corollary of Proposition 4.7, we obtain,

**Proposition 4.11.** *The equilateral rhombus with Neumann boundary condition provides a counterexample to the ECP. More precisely, there is a linear combination of eigenfunctions in  $\mathcal{E}(\nu_1) \oplus \mathcal{E}(\nu_3)$  with four nodal domains.*

*Proof.* Proposition 4.7, Assertion (ii) tells us that  $\mathcal{E}(\nu_3)$  contains an eigenfunction which comes from the second Neumann eigenfunction of  $\mathcal{T}_{e,1} = \mathcal{T}_e$  which is symmetric with respect to  $D$ . It then suffices to apply the same arguments as in [7, Section 3.1]. A second Neumann eigenfunction for  $\mathcal{T}_e$  is given by

$$(4.13) \quad \phi_2^n(x, y) = 2 \cos\left(\frac{2\pi x}{3}\right) \left( \cos\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right) \right) - 1.$$

Extend this function, by symmetry with respect to  $M$ , to a function  $\phi_2$  in  $\mathcal{R}h_e$ . The linear combination  $\phi_2 + 1$  vanishes on the line segments  $\{x = \frac{3}{4}\} \cap \mathcal{R}h_e$  and  $\{x + \sqrt{3}y = \frac{3}{2}\} \cap \mathcal{R}h_e$ , so that it has four nodal domains, see Figure 4.4, contradicting Courant's theorem.  $\square$

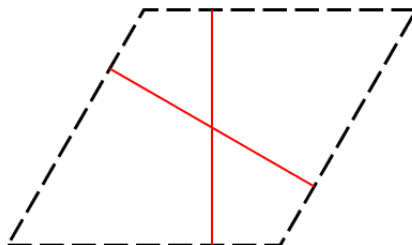


FIGURE 4.4. Nodal pattern of some linear combination in  $\mathcal{E}(\nu_1) \oplus \mathcal{E}(\nu_3)$ , with four nodal domains

Figure 4.5 displays the MATLAB rendering of the variation of the number of nodal domains (the eigenfunction produced by MATLAB is proportional to  $\phi_2$ , not equal, so that the bifurcation value is not 1 as in the proof of Proposition 4.11).

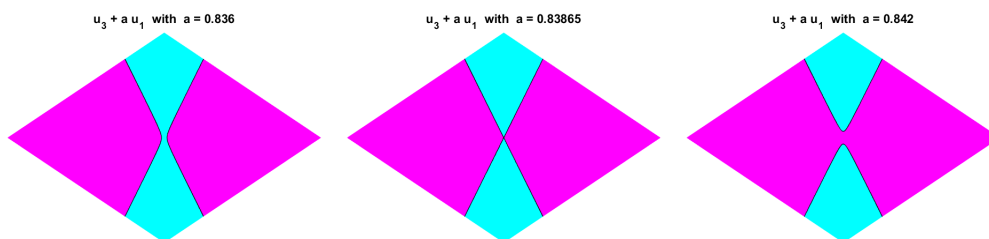
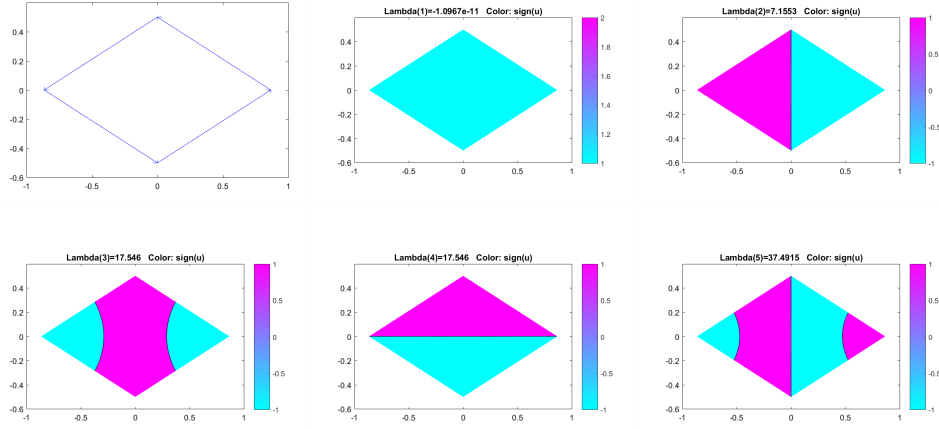
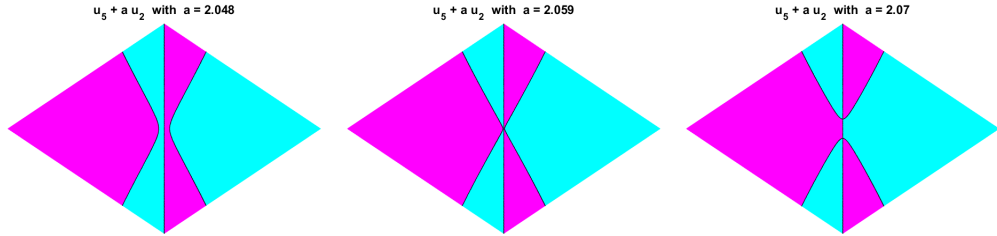


FIGURE 4.5. Nodal patterns of linear combinations in  $\mathcal{E}(\nu_1) \oplus \mathcal{E}(\nu_3)$  around the bifurcation

4.4.3. *A numerical result for  $\mathcal{R}h_e$  with Neumann boundary condition.* Numerical computations for the first Neumann eigenvalues of the equilateral rhombus give,

$$(4.14) \quad 0 = \nu_1 < \nu_2 < \nu_3 = \nu_4 < \nu_5 < \nu_6 < \nu_7 \dots ,$$

with the first five nodal patterns shown in Figure 4.6. Numerical computations of the eigenfunctions show that there are linear combinations in  $\mathcal{E}(\nu_2) \oplus \mathcal{E}(\nu_5)$  with six nodal domains, thus providing another (numerical) counterexample to the  $\text{ECP}(\mathcal{R}h_e, \mathbf{n})$ , see Figure 4.7. This counterexample can also be interpreted as a counterexample to the ECP for the equilateral triangle with mixed boundary conditions, Neumann on two sides, and Dirichlet on the third side.

FIGURE 4.6. Rhombus  $\mathcal{R}h_e$ , Neumann boundary conditionFIGURE 4.7. Nodal patterns of some linear combinations in  $\mathcal{E}(\nu_2) \oplus \mathcal{E}(\nu_5)$ 

**4.5. Rhombus with Dirichlet boundary condition.** The arguments of Subsection 4.4, applied to the Dirichlet eigenvalue problem  $(\mathcal{R}h_e, \mathfrak{d})$ , are not sufficient to prove that the ECP  $(\mathcal{R}h_e, \mathfrak{d})$  rigorously.

Numerical computations show that

$$(4.15) \quad 0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 = \delta_6 < \delta_7 \dots$$

with corresponding nodal patterns given by Figure 4.8.

Numerical computations also indicate that there are linear combinations in  $\mathcal{E}(\delta_2) \oplus \mathcal{E}(\delta_5)$  with 6 nodal domains, thus providing a counterexample to the ECP  $(\mathcal{R}h_e, \mathfrak{d})$ , see Figure 4.9. This counterexample, can also be interpreted as a counterexample to the ECP for the equilateral triangle with mixed boundary conditions, Dirichlet on two sides, and Neumann on the third one.

**Remark 4.12.** *Looking at the nodal patterns, and in analogy with what we did in the Neumann case, one could think of testing a linear combination in  $\mathcal{E}(\delta_1) \oplus \mathcal{E}(\delta_4)$ . This gives a linear combination with four nodal domains, which does not contradict Courant's theorem.*

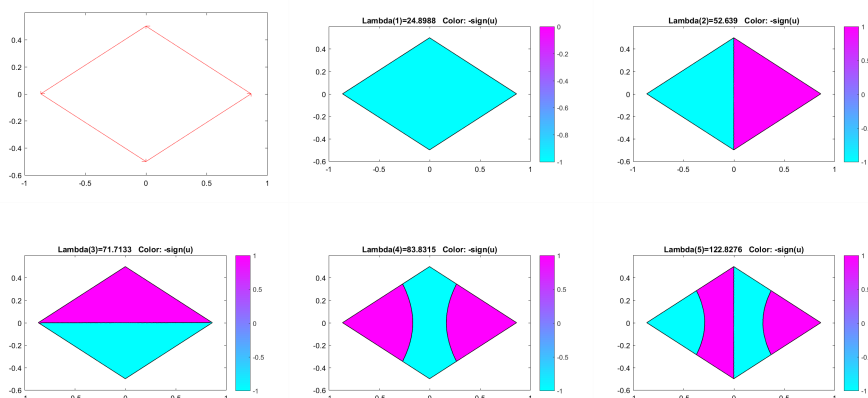


FIGURE 4.8. Rhombus  $\mathcal{R}h_e$ , Dirichlet boundary condition

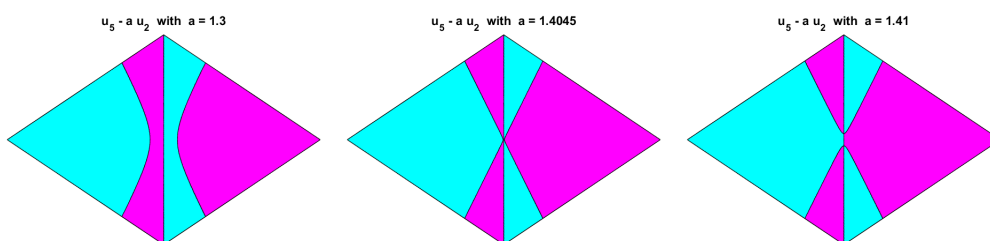


FIGURE 4.9. Counterexample to the ECP( $\mathcal{R}h_e$ ) with Dirichlet boundary condition

## 5. PRODUCT LIKE CONSTRUCTIONS

**1.** Our counterexamples in [7, 9], and in Section 4, are two dimensional. A natural question is to look for counterexamples to the ECP in higher dimensions. We already mentioned Viro’s examples (Section 2), which imply that the ECP( $\mathbb{R}P^3, g_0$ ) and ECP( $\mathbb{S}^3, g_0$ ) are false.

It is easy to construct counterexamples to the ECP in higher dimensions by using “collapsing products”. More precisely, let  $\Omega$  be any domain (or manifold) such that ECP( $\Omega$ ) is false. This means that there exists a linear combination  $v = \sum_{j=1}^m u_j$  of eigenfunctions of  $\Omega$ , associated with eigenvalues less than or equal to  $\mu_m(\Omega)$ , such that  $\beta_0(v)$  is larger than  $\kappa(\mu_m(\Omega))$ .

For example, take any closed connected Riemannian manifold  $(N, g)$ . Its first eigenvalue is  $\mu_1(N, g) = 0$ , with constant eigenfunction. Its second eigenvalue  $\mu_2(N, g)$  is positive. Take the product  $\Omega \times (N, \varepsilon^2 g)$ . For  $\varepsilon$  small enough, the first eigenvalue of  $(N, \varepsilon^2 g)$  is still 0, while the second eigenvalue is  $\varepsilon^{-2} \mu_2(N, g)$  and hence, can be chosen larger than  $\mu_m(\Omega)$ . It follows that the function  $v$ , viewed as a function on the product, is a linear combination of eigenfunctions of  $\Omega \times (N, \varepsilon^2 g)$  with  $\beta_0(v)$  larger than  $\kappa(\mu_m(\Omega \times (N, \varepsilon^2 g)))$ .

One can produce less trivial examples by taking warped products.

**2.** As far as we know, the projective space  $(\mathbb{RP}^2, g_0)$  is the sole known example such that the ECP is true. We relax the assumption and ask whether there are examples of manifolds such that the ECP is true for linear combinations of eigenfunctions with eigenvalues less than or equal to some given  $\lambda$ . The answer is positive: it suffices to consider a collapsing product (or warped product) whose first factor  $\Omega$  is either an interval or  $\mathbb{S}^1$ , equipped with a Sturm-Liouville operator.

An example of such a behaviour is given by circular sectors with angle tending to zero (a warped product which collapses to an interval).

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