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Sequentialization and Procedural Complexity in Automata Networks

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Abstract. In this article we consider finite automata networks (ANs) with two kinds of update schedules: the parallel one (all automata are updated all together) and the sequential ones (the automata are updated periodically one at a time according to a total order w). The cost of sequentialization of a given AN h is the number of additional automata required to simulate h by a sequential AN with the same alphabet. We construct, for any n and q , an AN h of size n and alphabet size q whose cost of sequentialization is at least $n/3$. We also show that, if $q \geq 4$, we can find one whose cost is at least $n/2 - \log_q(n)$. We prove that $n/2 + \log_q(n/2 + 1)$ is an upper bound for the cost of sequentialization of any AN h of size n and alphabet size q . Finally, we exhibit the exact relation between the cost of sequentialization of h and its procedural complexity with unlimited memory and prove that its cost of sequentialization is less than or equal to the pathwidth of its interaction graph.

Keywords: Automata networks, intrinsic simulation, parallel update schedule, sequential update schedules, procedural complexity.

1 Introduction

In this article, we study finite automata networks (ANs). They are models classically used for representing and analyzing natural dynamical systems like genetic or neural networks [8,5]. Moreover, they are also computational models on which we study computability and complexity properties which is the purpose of this paper. An AN h can be seen as a transformation of A^n with A a finite alphabet. Here, n is the number of automata, and the i -th component of h is the update function of the i -th automaton. We consider them with two types of update schedules. With the parallel one, automata are updated all together, at each time step. In other words, we just apply h . With the sequential ones, automata are updated sequentially, according to a total order w . They have been several works on the influence of the update schedules on the function computed by

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an AN [6,1]. Here, like in [7] we take the opposite approach. We have an AN h with a parallel update schedule and try to find an AN f with a sequential update schedule w which computes the same function. However, sometime it is impossible. For instance, the transformation of $\{0, 1\}^2$ which exchanges the two values $h : (x_1, x_2) \mapsto (x_2, x_1)$ cannot be sequentialized. The famous XOR swap algorithm, $x_1 \leftarrow x_1 \oplus x_2$, $x_2 \leftarrow x_1 \oplus x_2$, $x_1 \leftarrow x_1 \oplus x_2$ does not apply here because we can only update one time each automaton between two time steps. However, what we can do is to consider the AN f with one additional automaton and the sequential update schedule $w := (3, 2, 1)$ which executes the three instructions $x_3 \leftarrow x_1$, $x_1 \leftarrow x_2$, $x_2 \leftarrow x_3$. We see that f with the update schedule w computes the transformation h if we only consider the 2 first automata. The goal of this paper is to determine the cost of sequentialization of an AN h , namely, the minimum number of additional automata that an AN f which sequentializes h will have. This paper is the direct sequel of [3] in which the same problem was studied for an alphabet of size 2 and with an imposed order of sequentialization. Definition 7, Theorem 1 and Lemma 3 are straightforward generalization of results published in [3]. All other results are new.

In Section 2, we define ANs, interaction graphs, the notion of a sequentialization and we present most of the notations that we use. In Section 3, we define the cost of sequentialization $\kappa(h, u)$ of an AN h respecting an order u . It is the minimum number of additional automata required for any AN f with a sequential update schedule w respecting the order u to compute h . We also define $\kappa^{\min}(h)$ which is like $\kappa(h, u)$ except that the sequential update schedules we consider are not constraint anymore. In Section 4, we give an upper and lower bounds for $\kappa(h, u)$ for the couple (h, u) which maximizes it. In Section 5, we prove different lower bounds depending on the alphabet size for $\kappa^{\min}(h)$ when h maximizes $\kappa^{\min}(h)$. In Section 6 we give the relation between κ^{\min} and the procedural complexity as defined in [4]. Finally, In Section 7, we prove an upper bound for $\kappa^{\min}(h)$ depending on the pathwidth of the interaction graph of h .

2 Definitions and notations

For all $i \in \mathbb{N}$, the interval between 1 and i is denoted by $[i] := \{1, 2, \dots, i\}$. For all $i, j \in \mathbb{N}$, with $i \leq j$, the closed interval between i and j is denoted by $[i, j] := \{i, i+1, \dots, j\}$ and the open one by $]i, j[:= [i, j] \setminus \{i, j\}$. For any $q \geq 2$ and $n \in \mathbb{N}$, let $F(n, q)$ be the set of functions from $[0, q]^n$ to $[0, q]^n$ (also called transformations of $[0, q]^n$). For all $I = \{i_1, i_2, \dots, i_p\} \subseteq [n]$

with $i_1 < i_2 < \dots < i_p$, the projection of x on I is denoted either by $\text{pr}_I(x)$ or by x_I . In other words, $\text{pr}_I(x) = x_I = (x_{i_1}, x_{i_2}, \dots, x_{i_p})$. For all vectors $x := (x_1, \dots, x_p)$ and $y := (y_1, \dots, y_t)$, their concatenation is denoted by $xy := (x_1, \dots, x_p, y_1, \dots, y_t)$.

Definition 1 (Coordinate functions). *Let $f \in F(n, q)$. For every $i \in [n]$, the i -th coordinate functions of f is the function $f_i := \text{pr}_i \circ f$.*

This means that we have $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$. In this paper, we make particular use of the superscript of a function f .

Definition 2 (Updates of a transformation). *For all $i \in [n]$, $f^i \in F(n, q)$ is the function which updates the i -th coordinate (i.e. executes f_i). For all $I \subseteq [n]$, f^I is the function which updates the coordinates of all elements of I synchronously. For any word $w := (w_1, w_2, \dots, w_t)$ on the alphabet $[n]$, f^w is the function which updates sequentially the coordinates w_1, \dots, w_t in the order given by w .*

Formally, we have

$$\begin{aligned} \forall x \in A^n, & \quad f^i(x) := (x_1, \dots, x_{i-1}, f_i(x), x_{i+1}, \dots, x_n). \\ \forall x \in A^n, j \in [n], & \quad f^I(x)_j := \begin{cases} f_j(x) & \text{if } j \in I \\ x_j & \text{otherwise.} \end{cases} \\ \forall w = (w_1, w_2, \dots, w_t) \in [n]^t, & \quad f^w := f^{w_t} \circ \dots \circ f^{w_2} \circ f^{w_1}. \end{aligned}$$

We say that f_i is a trivial coordinate function if for all $x \in A^n$, $f_i(x) = x_i$. The relation $y = f^i(x)$ can be expressed by $x \xrightarrow{f^i} y$. The set of permutations of $[n]$ is denoted by $\Pi([n])$. Let $w := (w_1, w_2, \dots, w_t) \in \Pi([n])$. If $w_j = i$ then we say that i is updated at step $w(i) := j$.

Definition 3 (Sequentialization). *An AN $f \in F(m, q)$, with the sequential update schedule $w \in \Pi([m])$ sequentializes an AN $h \in F(n, q)$ with $m \geq n$ if $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$.*

Remark 1. All the results of this paper remain true if we use the more general definition: $\exists I \subseteq [m]$, with $|I| = n$ such that $\text{pr}_I \circ f^w = h \circ \text{pr}_I$.

Definition 4 (Interaction graph). *The interaction graph $\text{IG}(h)$ of an AN $h \in F(n, q)$ is the directed graph $([n], E)$ with $(i, j) \in E$ if and only if i has an influence on j . More formally, $\forall i, j \in [n]$, $(i, j) \in E$ if and only if $\exists x, y \in A^n$ such that $x_{[n] \setminus \{i\}} = y_{[n] \setminus \{i\}}$ and $h_j(x) \neq h_j(y)$.*

We denote by $\text{IG}^*(h)$ be the undirected version of $\text{IG}(h)$.

3 Cost of sequentialization

In this section, we define the main question tackled in this paper. For all $u \in \Pi([n])$, we say that $w \in \Pi([m])$ respects u , if all the coordinates of $[n]$ are updated in the same order in u and in w . In other words, $\forall i, j \in [n]$, if $u(i) < u(j)$ then $w(i) < w(j)$.

Definition 5 ($\kappa(h, u)$). *Let $h \in F(n, q)$ and $u \in \Pi([n])$. The cost of sequentialization of h respecting u , denoted by $\kappa(h, u)$, is the smallest k such that there exists $f \in F(n+k, q)$ and $w \in \Pi([n+k])$, such that (f, w) sequentializes h and w respects u .*

Definition 6 ($\kappa^{\min}(h)$). *Let $h \in F(n, q)$. The cost of sequentialization of h , denoted by $\kappa^{\min}(h)$, is the smallest k such that there is a $f \in F(n+k, q)$ and a $w \in \Pi([n+k])$, such that (f, w) sequentializes h .*

Clearly, $\kappa^{\min}(h) = \min(\{\kappa(h, u) \mid u \in \Pi([n])\})$. Given n and q , the maximal cost of sequentialization respectively with or without imposed order is denoted by $\kappa_{n,q} := \max(\{\kappa(h, u) \mid h \in F(n, q) \text{ and } u \in \Pi([n])\})$ and $\kappa_{n,q}^{\min} := \max(\{\kappa^{\min}(h) \mid h \in F(n, q)\})$, respectively. Example 1 shows that, for some (h, u) , the difference between $\kappa^{\min}(h)$ and $\kappa(h, u)$ is large.

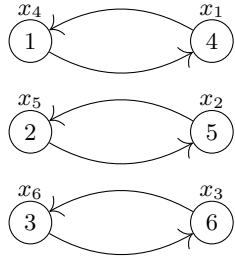


Fig. 1: Interaction graph of the AN h of Example 1.

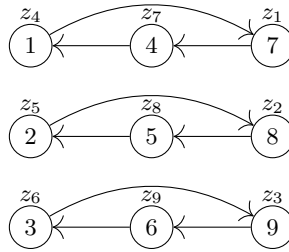


Fig. 2: Interaction graph of the AN f of Example 1.

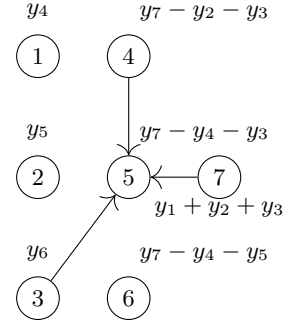


Fig. 3: Interaction graph of the AN g of Example 1 with only inner edges of the automaton 5 displayed.

Example 1. Let us consider the AN $h \in F(n, q)$ with $n = 6$ which computes the swaps of the values of 3 pairs of automata. In other words,

$$h : x \mapsto (x_4, x_5, x_6, x_1, x_2, x_3).$$

Figure 1 displays the interaction graph of h . Now, we consider the canonical sequential update schedule $u = (1, 2, \dots, 6)$ and we want to find an AN f and a update schedule w which sequentializes h respecting u . To do so, let us consider an AN $f \in F(9, 2)$ and $w \in \Pi([9])$. First, we define the order $w := (7, 8, 9, 1, 2, 3, 4, 5, 6)$ which updates the $n/2$ additional automata of f before it updates the n first ones. Then, we take f which copies the values of the first set of automata in the third, the second in the first and the third in the second. Formally, $f : z \mapsto z_{[4,6]}z_{[7,9]}z_{[3]}$. Figure 2 shows the interaction graph of f . Now, a simple expansion of f^w gives us

$$z = z_{[3]}z_{[4,6]}z_{[7,9]} \xrightarrow{f^{7,8,9}} z_{[3]}z_{[4,6]}z_{[3]} \xrightarrow{f^{1,2,3}} z_{[4,6]}z_{[4,6]}z_{[3]} \xrightarrow{f^{4,5,6}} h(z_{[6]})z_{[3]}.$$

Thus, we have $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$. As a result, (f, w) sequentializes h respecting u and $\kappa(h, u) \leq 3$. Moreover, Lemma 3 (Section 4), shows that there are no smaller (f, w) which would suit. Thus, we have $\kappa_{6,q} \geq \kappa(h, u) = n/2 = 3$. Next, we define $g \in F(7, 2)$ and $v \in \Pi([7])$ such that (g, v) (with only one more automaton than h) sequentializes h (but without respecting u). First, we define the order $v := (7, 1, 4, 2, 5, 3, 6)$ which, instead of updating $[n]$ in the order u , updates the pairs of automata $(1, 4)$, $(2, 5)$ and $(3, 6)$ one by one. Then, we take g such that for all $y \in \{0, 1\}^7$,

$$g : y \mapsto (y_4, y_5, y_6, y_7 - y_2 - y_3, y_7 - y_4 - y_3, y_7 - y_4 - y_5, y_1 + y_2 + y_3).$$

Figure 3 depicts the interaction graph of g with only the inner edges of the automaton 5 displayed. As above, a simple expansion of g^v gives us $g^v : y \mapsto h(y_{[6]})(y_1 + y_2 + y_3)$. Thus, $\text{pr}_{[n]} \circ g^v = h \circ \text{pr}_{[n]}$ and g has 1 more automata than h . As a result, (f, w) sequentializes h and $\kappa^{\min}(h) \leq 1$. A generalization of this example shows that for all even n and $q \geq 2$, $\exists h \in F(n, q), u \in \Pi([n])$ such that $\kappa(h, u) \geq \kappa^{\min}(h) + n/2 - 1$.

4 Confusion graph and $\kappa_{n,q}$

In [3], the NECC graph was defined. This graph is very useful to compute $\kappa(h, u)$. We rather call it the confusion graph in this paper.

Definition 7 (Confusion graph). *Let us consider $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. We call confusion graph $G_{h,u}$ the undirected graph whose vertices are all the configurations of $[0, q]^n$ and in which two configurations x and x' are neighbors if and only if $h(x) \neq h(x')$ and $\exists i \in [n], h^{\{u_1, \dots, u_i\}}(x) = h^{\{u_1, \dots, u_i\}}(x')$.*

In the sequel, we denote by $\chi(G)$ the *chromatic number* of the graph G , namely the minimum number of colors of a proper coloring of its vertices. In [3], the exact relation between the chromatic number of the confusion graph $G_{h,u}$ and $\kappa(h,u)$ was proven in the case where $q = 2$. We propose in Theorem 1 a straightforward generalization for any alphabet size.

Theorem 1. *Let us consider $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. Then we have $\kappa(h, u) = \lceil \log_q(\chi(G_{h,u})) \rceil$.*

In [3], the authors proved that for all n we can construct $h \in F(n, 2)$ whose cost of sequentialization respecting the order $u \in \Pi([n])$ is $\lfloor n/2 \rfloor$. Lemma 3 below is a straightforward generalization for any alphabet size.

Lemma 3. *For all $n \in \mathbb{N}$ and $q \geq 2$ we have $\kappa_{n,q} \geq \lfloor n/2 \rfloor$.*

Moreover, in [3], the authors showed that $\forall n \in \mathbb{N}, \kappa_{n,2} \leq 2n/3 + 2$. Theorem 2 below shows that we have in fact, $\kappa_{n,q} \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$ for any q . To prove it, we regroup all the configurations of the confusion graph $G_{h,u}$ which are equal in their second half ($x_{\{u_{n/2+1}, \dots, u_n\}} = x'_{\{u_{n/2+1}, \dots, u_n\}}$) and have the same image ($h(x) = h(x')$). We prove that a proper coloring of this graph is a proper coloring of the confusion graph. And then, we prove that the maximal degree of this factorized graph is at most $\lceil (n/2 + 1)q^{n/2} \rceil$. Since the chromatic number of a graph is at most its maximal degree (plus one), we deduce an upper bound for the chromatic number and then for $\kappa_{n,q}$.

Theorem 2. *For all $n \in \mathbb{N}$, $q \geq 2$ we have $\kappa_{n,q} \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$.*

5 Lower bounds for $\kappa_{n,q}^{\min}$

The goal of this section is to construct an AN with the biggest cost of sequentialization possible and thus deduce a lower bound for $\kappa_{n,q}^{\min}$. For any set I , the set of subsets of I of size k is denoted by $\binom{I}{k} := \{J \subseteq I \mid |J| = k\}$. For all $x \in A^n$ and $I \subseteq [n]$, let $x[I] := \{x' \in A^n \mid x'_{[n] \setminus I} = x_{[n] \setminus I}\}$ be the set of configurations of A^n which only differ from x in I . In Lemma 4, we prove that if we can find an encoding $b : \binom{[2k]}{k} \rightarrow A^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint, then there exists $h \in F(n, q)$ such that $\kappa^{\min}(h) \geq k$. To do so, we define the function h such that for all $x \in b(E)[E]$, $h_E(x) = x_{[2k] \setminus E}$ and $h_{[2k] \setminus E}(x) = x_E$. For any $u \in \Pi([n])$, we can define E as the k first coordinates updated by u in $[2k]$ and consider $x = b(E)$. The set $x[E]$ is a clique in the confusion graph $G_{h,u}$. Indeed, any

function which sequentializes h respecting u has, for any configuration in $x[E]$, to first erase the information in E and then to restore it in $[2k] \setminus E$. Since this clique is of size q^k , we have $\kappa_{h,u} \geq k$ for any u and $\kappa^{\min}(h) \geq k$.

Lemma 4. *Let $n, k \in \mathbb{N}$ and $q \geq 2$. If there is a function $b : \binom{[2k]}{k} \rightarrow [0, q]^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint then there exists a $h \in F(n, q)$ without trivial coordinate functions, with $\kappa^{\min}(h) \geq k$.*

Using Lemma 4 we could easily show that for any $q \geq 2$ and $n \in \mathbb{N}$, we have $\kappa_{n,q}^{\min} \geq \lfloor n/4 \rfloor$. Indeed, if we have $n = 4k$, we can use the second half of the configuration to encode the set E . In Theorem 3 we prove that for any alphabet, we can in fact encode any $E \in \binom{[2k]}{k}$ in a configuration x of size $3k$. To do so, we use the following technique: if $i \in \bar{E} := [2k] \setminus E$ then we have $x_i = 0$ if $i+1$ in E and 1 otherwise. Moreover, in $[2k+1, 3k]$, using the same technique, we indicate if each element of E is followed by another element of E or not. From this encoding and Lemma 4 we deduce a lower bound for κ^{\min} for any alphabet.

Theorem 3. *For all $n \in \mathbb{N}$ and $q \geq 4$, we have $\kappa_{n,q}^{\min} \geq \lfloor n/2 - \log_q(n) \rfloor$.*

Theorem 4 below states that, if we have an alphabet of size at least 4, we can encode any $E \in \binom{[2k]}{k}$ in a configuration of size $2k + \log_q(2k)$. To do so, we encode E in $[2k] \setminus E$ using the fact that in an alphabet of size 4 each coordinate can encode twice more information than with a bit. Then, we indicate in $[2k, 2k + \log_q(2k)]$ where the reading for decoding starts. From this encoding and Lemma 4 we deduce a lower bound for κ^{\min} .

Theorem 4. *For all $q \geq 4, n \in \mathbb{N}$, $\kappa_{n,q}^{\min} \geq \lfloor n/2 - \log_q(n) \rfloor$.*

6 Procedural complexity

Now, we study the relation between κ^{\min} and the procedural complexity as defined in [4]. The procedural complexity of h is the minimum number t of functions $g^{(1)}, \dots, g^{(t)}$ (each of which update at most one coordinate) that are required for $g^{(t)} \circ \dots \circ g^{(1)}$ to compute h . For all $q \geq 2$ and $n \geq 2$, let us denote by $F^*(n, q) \subseteq F(n, q)$ the set of functions which do not update more than one coordinate. In [4], the authors first studied the memoryless procedural complexity $\mathcal{L}(h)$. It is the necessary number of step to compute h with $g^{(1)}, \dots, g^{(t)}$ of same size than h . Then, they studied $\mathcal{L}(h|m)$ which is the procedural complexity using functions $g^{(1)}, \dots, g^{(t)}$ of a fixed size m . More formally, $\forall m \geq n$, $\mathcal{L}(h|m) :=$ smallest t such that

$\exists g^{(1)}, \dots, g^{(t)} \in F^*(m, q)$ such that $\text{pr}_{[n]} \circ g^{(t)} \circ \dots \circ g^{(1)} = h \circ \text{pr}_{[n]}$. Here, we also use $\mathcal{L}^*(h) := \min(\{\mathcal{L}(h|m) \mid n \leq m\})$ which is the procedural complexity with a size arbitrarily big. Let $\Omega(h)$ be the number of non-trivial coordinate functions of h . Theorem 5 shows that the procedural complexity of an AN h is equal to $\kappa^{\min}(h) + \Omega(h)$. Furthermore, it shows that the minimum procedural complexity is reached when we use $\kappa^{\min}(h)$ additional automata. It is directly deduced from Lemma 5 and Lemma 6.

Theorem 5. *Let $h \in F(n, q)$ and $k := \kappa^{\min}(h)$. We have $\mathcal{L}^*(h) = \mathcal{L}(h|n+k) = \Omega(h) + k$.*

In Lemma 5, we prove that $\mathcal{L}^*(h) \leq \Omega(h) + \kappa^{\min}(h)$. We use the fact that by definition of $k := \kappa^{\min}(h)$ there is $f \in F(n+k, q)$ and $w \in \Pi([n+k])$ such that the $n+k$ instructions $f^{w_1}, \dots, f^{w_{n+k}} \in F^*(n+k, q)$ compute h . With that, we already have $\mathcal{L}(h|n+k) \leq n+k$. Furthermore, for each i such that h_i is trivial, we can remove the function f^i of the list of instructions and still compute h . As a result, we have $\mathcal{L}(h|n+k) \leq n+k - (n - \Omega(h)) = \Omega(h) + k$, and by definition of $\mathcal{L}^*(h)$ we have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k)$.

Lemma 5. *Let $h \in F(n, q)$ and $k := \kappa^{\min}(h)$. We have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k) \leq \Omega(h) + k$.*

In Lemma 6, we prove that $\Omega(h) + k \leq \mathcal{L}^*(h)$ with $k := \kappa^{\min}(h)$. To do so, we take a set of functions $g^{(1)}, \dots, g^{(t)} \in F^*(m, q)$ which compute h . We consider an order $w \in \Pi([n])$ which updates all coordinate of $[n]$ in the same order that $g^{(1)}, \dots, g^{(t)}$ update them for the last time. Then we prove that h can be sequentialized respecting w with less than $\mathcal{L}^*(h) - \Omega(h)$ additional automata. Let $J = \{j_1, \dots, j_\ell\}$ be the set of steps such that either $g^{(j_i)}$ updates a coordinate of $]n, m]$, either it updates a coordinate of $[n]$ that will be updated again later. We have $\ell = \mathcal{L}^*(h) - \Omega(h)$. Then, we define $c : A^n \rightarrow A^k$ such that $c_i(x)$ equals $(g^{(j_i)} \circ \dots \circ g^{(1)}(x(0)^{m-n}))_a$ with a the coordinate updated by $g^{(j_i)}$. Then, we prove that c is a proper coloring of the confusion graph $G_{h,w}$ and that $\Omega(h) + k \leq \mathcal{L}^*(h)$.

Lemma 6. *Let $h \in F(n, q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h) + k \leq \mathcal{L}^*(h)$.*

In [4], Proposition 12 states that $\forall h \in F(n, q)$, we have $\mathcal{L}(h|n-1) \leq 2n-1$. In Corollary 1 bellow, we refine this bound using Theorem 2, Theorem 5 and the fact that $\forall h \in F(n, q)$, $\Omega(h) \leq n$.

Corollary 1. For all $h \in F(n, q)$, $\mathcal{L}(h|m) \leq m$ with $m := n + \lceil n/2 + \log_q(n/2 + 1) \rceil$.

In the following Corollary 2, we give a lower bound for the procedural complexity with unlimited memory. It is a direct corollary of Theorem 5, Lemma 4, Theorem 3, Theorem 4 in which we construct an AN h without trivial coordinate functions (and thus we have $\Omega(h) = n$).

Corollary 2. For all $n, q \geq 2$ there is $h \in F(n, q)$ such that $\mathcal{L}^*(h) \geq n + \lfloor n/3 \rfloor$. Furthermore, if $q \geq 4$ there is $h \in F(n, q)$ such that $\mathcal{L}^*(h) \geq n + \lfloor n/2 - \log_q(n) \rfloor$.

7 Bound for $\kappa^{\min}(h)$ using interaction graph

Let us now present a way to upper bound $\kappa^{\min}(h)$ for an AN h using the pathwidth of the interaction graph of h [2].

Definition 8 (Pathwidth). A path decomposition of an undirected graph $G = (V, E)$ is a sequence of subsets X_1, \dots, X_p of vertices such that

- $\forall (v, v') \in E, \exists X_i$ such that $v, v' \in X_i$.
- If $v \in X_i$ and $v \in X_j$ with $i < j$ then $\forall k \in [i, j], v \in X_k$

The size of a path decomposition is the size of the largest X_ℓ minus one. The pathwidth $\text{Pw}(G)$ is the minimum size of a path decomposition of G .

Theorem 6 shows that the pathwidth of the graph $IG^*(h)$ is an upper bound for $\kappa^{\min}(h)$. It can be deduced directly from Lemma 7 and Lemma 8.

Theorem 6. For any AN h , $\kappa^{\min}(h) \leq \text{Pw}(IG^*(h))$.

Lemma 7 shows that from a path decomposition of a graph G of size s , we can construct a partition c of its vertices in s sets, and an update schedule u with properties allowing an efficient sequentialization by Lemma 8. We define c (resp. u) using a greedy algorithm. We iterate the subsets X_1, \dots, X_n of the path decomposition and choose the value $c(i)$ (resp. $u(i)$) the first (resp. last) time we see i .

Lemma 7. Let $G = ([n], E)$ be an undirected graph and let $s = \text{Pw}(G)$. Then there are functions $c : [n] \rightarrow [s]$ and $u \in \Pi([n])$ with the following property. For all $i \in [n]$, we have either 1) for all k neighbor of i in G we have $u(i) \leq u(k)$ or 2) for all $j, k \in [n]$ with $c(i) = c(j)$, $u(i) < u(j)$ and k neighbor of j in G we have $u(i) \leq u(k)$.

Lemma 8 shows how to use c and u defined in Lemma 8 to sequentialize h respecting u . Each additional automaton j (denoted from 1 to s) computes the sum modulo q of the images $\{h_i(x) \mid i \in [n] \text{ and } c(i) = j\}$. Then, each automaton of coordinate j can compute $h_j(x)$, either because all neighbors of j in G have not been updated yet, or because it can compute all $h_j(x)$ such that $i \neq j$ and $c(i) = c(j)$.

Lemma 8. *Let $h \in F(n, q)$. Let $G = \text{IG}^*(h)$. If we have $c : [n] \rightarrow [s]$ and $u \in \Pi([n])$ such that G, c, u have the same properties as in Lemma 7, then we have $\kappa(h, u) \leq s$.*

8 Conclusion and future research

We have seen that $\lfloor n/2 - \log_q(n) \rfloor \leq \kappa_{n,q}^{\min} \leq \kappa_{n,q} \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$. Thus, for any fixed n , the limit of $\kappa_{n,q}^{\min}$ and $\kappa_{n,q}$ when q tends to infinity is $n/2$. It is an argument in favor of the conjecture made in [3] which states that for any n and q , $\kappa_{n,q} = \lfloor n/2 \rfloor$ and which is still open. It would be interesting to investigate a variant of the problem presented in this paper, where additional automata are forbidden but several updates of the same automaton are allowed. The task is then to know, for given n and q , the minimum time $t(q, n)$ such that $\forall h \in F(n, q)$, $\exists f \in F(n, q)$, $w \in [n]^{t'}$ with $t' \leq t(q, n)$ such that $f^w = h$. The value of $t(2, 2)$ is not defined because for the AN $h \in F(2, 2)$ such that $(0, 0) \xrightarrow{h} (0, 1) \xrightarrow{h} (1, 1) \xrightarrow{h} (1, 0) \xrightarrow{h} (0, 0)$ there are no such f . However, with computers, we established that $t(3, 2) = 22$. We can easily see that $\mathcal{L}_{n,q} := \max(\{\mathcal{L}(h) \mid h \in F(n, q)\})$ is a lower bound for $t(n, q)$, and in [4], it is stated that $2n - 1 \leq \mathcal{L}_{n,q} \leq 4n - 3$.

References

1. J. Aracena. On the robustness of update schedules in Boolean networks. *Biosystems*, 97:1–8, 2009.
2. H. L. Bodlaender and T. Kloks. Better algorithms for the pathwidth and treewidth of graphs. In *Proceedings of ICALP'91*, volume 510 of *LNCS*, pages 544–555. Springer, 1991.
3. F. Bridoux, P. Guillon, K. Perrot, S. Sené, and G. Theyssier. On the cost of simulating a parallel Boolean automata network by a block-sequential one. In *Proceedings of TAMC'07*, volume 10185 of *LNCS*, pages 112–128. Springer, 2017.
4. M. Gadouleau and S. Riis. Memoryless computation: New results, constructions, and extensions. *Theoretical Computer Science*, 562:129–145, 2015.
5. E. Goles and S. Martínez. *Neural and Automata Networks: Dynamical Behavior and Applications*. Kluwer Academic Publishers, 1990.
6. E. Goles and L. Salinas. Comparison between parallel and serial dynamics of Boolean networks. *Theoretical Computer Science*, 396:247–253, 2008.

7. M. Tchente. Sequential simulation of parallel iterations and applications. *Theoretical Computer Science*, 48:135–144, 1986.
8. R. Thomas. Boolean formalization of genetic control circuits. *Journal of Theoretical Biology*, 42:563–585, 1973.

A Proof of Theorem 1

We can deduce Theorem 1 directly from Lemma 1 and Lemma 2.

Theorem 1 (Theorem 1). *Let us consider $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. Then we have $\kappa(h, u) = \lceil \log_q(\chi(G_{h,u})) \rceil$.*

Lemma 1 shows that we can use any (f, w) which sequentializes h respecting u to construct a proper coloring of $G_{h,u}$. Indeed, we can color the vertices of the graph $G_{h,u}$ using the values of the additional automata of f after their update. Thus, this coloring does not use more than q^k colors with k the number of additional automata of f .

Lemma 1. *Let us consider $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. Then we have $\lceil \log_q(\chi(G_{h,u})) \rceil \leq \kappa(h, u)$.*

Proof. Without loss of generality, let us say that u is the canonical sequential update schedule $(1, 2, \dots, n)$. Let $k := \kappa(h, u)$, $m := n + k$, $f \in F(m, q)$ and $w \in \Pi([m])$ respecting u such that $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$. Let x, x' be two neighbors in the confusion graph $G_{h,u}$. Let $y := (0)^k$ (a word of size k containing only the letter 0). Let $z := xy$ and $z' := x'y$. Let us prove that $f^w(z)_{[n+1,m]} \neq f^w(z')_{[n+1,m]}$. For the sake of contradiction, let us say that $f^w(z)_{[n+1,m]} = f^w(z')_{[n+1,m]}$. Since x and x' are neighbors in $G_{h,u}$, we know that $h(x) \neq h(x')$ and $\exists i \in [n]$, $h^{[i]}(x) = h^{[i]}(x')$. Let us consider the biggest of these i . So we have $h^{[i+1]}(x) \neq h^{[i+1]}(x')$ and then $h_{i+1}(x) \neq h_{i+1}(x')$. Let $j = w(i+1)$. Let us prove that $f^{w_1, \dots, w_{j-1}}(z) = f^{w_1, \dots, w_{j-1}}(z')$. First, we have $f^{w_1, \dots, w_{j-1}}(z)_{[n]} = h^{[i]}(x) = h^{[i]}(x') = f^{w_1, \dots, w_{j-1}}(z')_{[n]}$. Furthermore, for all $a \in [n+1, m]$ which is not updated before the step j in w we have $f^{w_1, \dots, w_{j-1}}(z)_a = y_{a-n} = f^{w_1, \dots, w_{j-1}}(z')_a$. Finally, for all $a \in [n+1, m]$ updated before the step j in w we have $f^{w_1, \dots, w_{j-1}}(z)_a = f^{w_1, \dots, w_{j-1}}(z')_a$ because we assumed that $f^w(z)_{[n+1,m]} = f^w(z')_{[n+1,m]}$. As a result, $f^{w_1, \dots, w_{j-1}}(z) = f^{w_1, \dots, w_{j-1}}(z')$. However, $f_{w_j} \circ f^{w_1, \dots, w_{j-1}}(z) = h_{i+1}(x) \neq h_{i+1}(x') = f_{w_j} \circ f^{w_1, \dots, w_{j-1}}(z')$. This is a contradiction. Consequently, we have, $f^w(z)_{[n+1,m]} \neq f^w(z')_{[n+1,m]}$. More generally, if x and x' are neighbors in $G_{h,u}$ then $f^w(xy)_{[n+1, n+k]} \neq f^w(x'y)_{[n+1, n+k]}$. In other words, $c : x \mapsto f^w(xy)_{[n+1, n+k]}$ gives a proper coloring of the confusion graph $G_{h,u}$. As a result, the confusion graph needs at most q^k colors because $f^w(xy)_{[n+1, n+k]}$ is a word of size k on the alphabet q . Thus, $\chi(G_{h,u}) \leq q^k$ and $\lceil \log_q(\chi(G_{h,u})) \rceil \leq k \leq \kappa(h, u)$.

Conversely, Lemma 2 states that we can construct a couple (f, w) which sequentializes h respecting u from a proper coloring of $G_{h,u}$. If this

coloring uses less than q^k colors then f is of size at most $n + k$ and then the cost of sequentialization is at most k .

Lemma 2. *Let us consider the AN $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. Then we have $\kappa(h, u) \leq \lceil \log_q(\chi(G_{h,u})) \rceil$.*

Proof. Let $k := \lceil \log_q(\chi(G_{h,u})) \rceil$, $m := n + k$. Let $A := [0, q]$. Let $w \in \Pi([m])$ which first update the k last automata and then the n first automata in the same order than u . In other words, $w := (n + 1, \dots, n + k, u_1, \dots, u_n)$. Let $c : A^n \rightarrow A^k$ be a proper coloring of $G_{h,u}$. For all $i \in [n]$, let us define $p^{(i)} : A^m \rightarrow P(A^n)$ with $P(A^n) := \{E \mid E \subseteq A^n\}$ the set of subsets of A^n . First, $p^{(1)} : z \mapsto \{z_{[n]}\}$ and then $\forall i \in [2, n]$, $p^{(i)} : z \mapsto \{x \in A^n \mid h^{\{u_1, \dots, u_{i-1}\}}(x) = z_{[n]}\}$ and $c(x) = z_{[n+1, m]}$. Let $f \in F(n + k, q)$ such that:

- $\forall i \in [n + 1, m]$, $f_i = c_i \circ \text{pr}_{[n]}$.
- $\forall i \in [n]$, $f_{u_i} : z \mapsto z_{u_i}$ if $p^{(i)}(z) = \emptyset$ and $h_{u_i}(x)$ with $x \in p^{(i)}(z)$ otherwise.

Let us prove that $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$. Let $x \in A^n$ and $z \in A^m$ with $z_{[n]} = x$. By, induction let us prove that,

$$\forall i \in [0, n], f^{w_1, \dots, w_{k+i}}(z) = h^{\{u_1, \dots, u_i\}}(x)c(x).$$

First, for $i = 0$, we have,

$$f^{w_1, \dots, w_k}(z) = f^{n+1, \dots, n+k}(z) = x(c_1(x), c_2(x), \dots, c_k(x)) = xc(x).$$

Second, let $i \in [n]$ and let us suppose that,

$$f^{w_1, \dots, w_{k+(i-1)}}(z) = h^{\{u_1, \dots, u_{i-1}\}}(x)c(x).$$

We have $f_{w_{k+i}} \circ f^{w_1, \dots, w_{k+(i-1)}}(z) = h_{u_i}(x')$ with $x' \in p^{(i)}(f^{w_1, \dots, w_{k+(i-1)}}(z))$. We have $x \in p^{(i)}(f^{w_1, \dots, w_{k+(i-1)}}(z))$ because $(f^{w_1, \dots, w_{k+(i-1)}}(z))_{[n+1, m]} = c(x)$ and $(f^{w_1, \dots, w_{k+(i-1)}}(z))_{[n]} = h^{\{u_1, \dots, u_{i-1}\}}(x)$. Let us prove that $h_{u_i}(x') = h_{u_i}(x)$. For the sake of contradiction let us say that $h_{u_i}(x') \neq h_{u_i}(x)$. Thus, $h(x) \neq h(x')$. However, $x, x' \in p^{(i)}(f^{w_1, \dots, w_{k+(i-1)}}(z))$ thus $h^{\{u_1, \dots, u_{i-1}\}}(x) = h^{\{u_1, \dots, u_{i-1}\}}(x')$ and $c(x) = c(x')$. Consequently, x and x' are neighbors in the confusion graph but they have the same color. This is a contradiction. As a result, $h_{u_i}(x') = h_{u_i}(x)$. Thus, $\forall i \in [0, n]$, $f^{w_1, \dots, w_{k+i}}(z) = h^{\{u_1, \dots, u_i\}}(x)c(x)$. As a consequence, $f^w(z) = h(x)c(x)$ and $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$. And since f has k additional automata, we have $\kappa(h, u) \leq k = \lceil \log_q(\chi(G_{h,u})) \rceil$.

B Proof of Lemma 3

To prove Lemma 3, we can construct a couple (h, u) such that $G_{h,u}$ has a clique of size $q^{n/2}$. Since the chromatic number of a graph is at least the size of its biggest clique, we have $\chi(G_{h,u}) \geq q^{n/2}$. As a result, $\kappa_{h,u} = \log(\chi(G_{h,u})) \geq n/2$ and we get Lemma 3 from that.

Lemma 3 (Lemma 3). *For all $q \geq 2$ and $n \in \mathbb{N}$, we have $\kappa_{n,q} \geq \lfloor n/2 \rfloor$.*

Proof. Let $k := \lfloor n/2 \rfloor$. Let us consider $h \in F(n, q)$ such that:

- $\forall i \in [k], h_i : x \mapsto x_{i+k}$
- $\forall i \in [k+1, 2k], h_i : x \mapsto x_{i-k}$
- If n is odd let $h_n : x \mapsto x_n$.

We also consider the canonical sequential update schedule $u := (1, 2, \dots, n)$. Let us consider the set of all configurations X which have only 0 in their second half. In other words, $X := \{x \in A^n \mid x_{[k+1,n]} = (0)^{n-k}\}$ ($(0)^{n-k}$ being a word of size $n - k$ containing only the letter 0). Let $x, x' \in X$ such that $x \neq x'$. We have $x_{[k+1,n]} = (0)^{n-k} = x'_{[k+1,n]}$. Thus, $x_{[k]} \neq x'_{[k]}$ and $\exists i \in [k]$ such that $x_i \neq x'_i$ and $h_{i+k}(x) = x_i \neq x'_i = h_{i+k}(x')$. Thus, $h(x) \neq h(x')$. However, when we update the first half of the automata, x and x' both become the configuration $(0)^n$. Indeed, $\forall i \in [k], f_i(x) = x_{i+k} = 0$. Then, we have $h^{[k]}(x) = (0)^n = h^{[k]}(x')$. As a result, (x, x') are neighbors in $G_{h,u}$. As a consequence, every two distinct vertices of X are neighbors. Thus, X is a clique. Moreover, X is a clique of size q^k . Thus, $\chi(G_{h,u}) \geq q^k$ and $\kappa(h, u) \geq \lceil \log_q(\chi(G_{h,u})) \rceil \geq \lceil \log_q(q^k) \rceil = k = \lfloor n/2 \rfloor$. Hence, $\forall q \geq 2, \forall n \in \mathbb{N}, \kappa_{n,q} \geq \lfloor n/2 \rfloor$.

Remark 2. In [4], Theorem 5 shows that if $h \in F(n, q)$ is a permutation, then for any $u \in Pi([n])$ we have $\kappa(h, u) \leq n/2$ if n is even and $\lfloor n/2 \rfloor + 1$ otherwise. As a result, the problem is almost solved for the permutations.

C Proof of Theorem 2

Theorem 2 (Theorem 2). *For all $n \in \mathbb{N}, q \geq 2$ we have $\kappa_{n,q} \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$.*

Proof. Let $h \in F(n, q)$ and $A := [0, q[$. Without loss of generality, let us say that u is the canonical sequential update schedule $(1, 2, \dots, n)$. Let E be the set of edges of the confusion graph $G_{h,u}$. Let $X = \{X_1, \dots, X_p\}$ be a partition of A^n , such that x, x' are in the same set X_i if and only if the two following conditions are respected:

- They are equal on the second half of the coordinates which will be updated in u . In other words, $x_{\{u(n/2+1), \dots, u(n)\}} = x'_{\{u(n/2+1), \dots, u(n)\}}$ or, more simply, $x_{]n/2, n]} = x'_{]n/2, n]}$ because we said that $u = (1, 2, \dots, n)$.
- They have the same image by h . In other words, $h(x) = h(x')$.

For all $x \in A^n$, let us denote by $X(x)$ the set $X_i \in X$ which contains x . Let $x^{(1)} \in X_1, x^{(2)} \in X_2, \dots, x^{(p)} \in X_p$. Let us consider the undirected graph $G' = (X, E')$ where two sets X_i and $X_{i'}$ are neighbors in G' if and only if there are two configurations $x \in X_i$ and $x' \in X_{i'}$ neighbors in the confusion graph $G_{h,u}$. Without loss of generality, let us consider the neighbors N of X_1 in G' . If $X_j \in N$ then $\exists x \in X_1, x' \in X_j$ such that $\exists i \in [n], h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Let us split N in $n/2 + 1$ sets:

- Let us denote by $N_{[n/2]}$ the set of sets X_j such that $\exists i \in [n/2], x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Since $h^{[i]}(x') = h^{[i]}(x)$, we have $x'_{]i, n]} = x_{]i, n]}$ and $x'_{]n/2, n]} = x_{]n/2, n]} = x_{]n/2, n]}^{(1)}$ because $i \leq n/2$. In other words, $\forall X_j \in N_{[n/2]}$, we have $x' \in X_j$ such that $x'_{]n/2, n]} = x_{]n/2, n]}^{(1)}$. However, there is only $q^{n/2}$ such configurations x' . Thus, $|N_{[n/2]}| \leq q^{n/2}$.
- For all $i \in [n/2 + 1, n]$, let us denote by N_i , the set of sets X_j such that, $\exists x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Let $X_j \in N_i$ and let $x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$. Thus, we have $x_{]i, n]}^{(j)} = x'_{]i, n]} = x_{]i, n]} = x_{]i, n]}^{(1)}$ because $i > n/2$. Thus, the value of $x_{]n/2+1, n]}^{(j)}$ is fixed on the interval $[i, n]$ and can vary only on the interval $[n/2 + 1, i]$. As a result, the second half of $x^{(j)}$ can take $q^{i-n/2}$ values. Furthermore, $h_{[i]}(x^{(j)}) = h_{[i]}(x) = h_{[i]}(x') = h_{[i]}(x^{(1)})$. Thus, the value of $h(x^{(j)})$ is fixed on the interval $[i]$ and can vary only on the interval $[i, n]$. As a result, $h(x^{(j)})$ can take q^{n-i} different values. Now if two configurations x' and x'' have the same image by h and are equal one their second half then they are in the same set X_j . Thus, $|N_i| \leq q^{i-n/2} * q^{n-i} = q^{n/2}$.

We have $N = N_{[n/2]} \cup N_{n/2+1} \cup \dots \cup N_n$. Thus, $|N| \leq (n/2 + 1)q^{n/2}$. As a consequence, the degree of X^1 in G' is less than $(n/2 + 1)q^{n/2}$ (strictly less because X^1 is in N but is not neighbor of himself). As a result, $\chi(G') \leq d(G') + 1 \leq (n/2 + 1)q^{n/2}$ with $d(G')$ the degree of G' . We can see that any coloring of this graph G' gives a proper coloring of the confusion graph. Indeed, we can color all the configurations of a set X_i in $G_{h,u}$ as we color X_i in G' . If two configurations x and x' are neighbors in the confusion graph $G_{h,u}$, then $X(x)$ and $X(x')$ are neighbors in G' and will not have the

same color. Thus, $\chi(G_{h,u}) \leq \chi(G') \leq (n/2 + 1) * q^{n/2}$. As a consequence, according to Theorem 1, we have, $\kappa(h, u) \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$. Hence, $\forall n \in \mathbb{N}, \kappa_{n,q} \leq \lceil n/2 + \log_q(n/2 + 1) \rceil$.

D Proof of Lemma 4

Lemma 4 (Lemma 4). *Let $n, k \in \mathbb{N}$ and $q \geq 2$. If there is a function $b : \binom{[2k]}{k} \rightarrow [0, q]^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint then there exists $h \in F(n, q)$ without trivial coordinate function, with $\kappa^{min}(h) \geq k$.*

Proof. Let $B := \bigcup_{E \in \binom{[2k]}{k}} b(E)[E]$. Let $a : B \rightarrow [0, q]^n$ such that $\forall E \in \binom{[2k]}{k}, \forall x \in b(E)[E], a(x) = E$. Let $h \in F(n, q)$ such that: $\forall x \in B,$

- $h_{a(x)}(x) = x_{[2k] \setminus a(x)}$.
- $h_{[2k] \setminus a(x)}(x) = x_{a(x)}$.
- $\forall i \in [2k + 1, n], h_i(x) = 0$.

and $\forall x \in [0, q]^n \setminus B, h(x) = (0)^n$. We can see that h does not have any trivial coordinate function. Indeed, for all $i \in [2k + 1, n]$ we have $h_i : x \mapsto 0$ which is nontrivial. Furthermore, if we take $x, y \in b(E)[E]$ with $E = [k + 1, 2k]$, and $x_E = (0)^k$ and $y_E = (1)^k$, we see that

$$\forall i \in [k], h_i(x) = x_{n/2+i} = 1 \neq 0 = y_{n/2+i} = h_i(y).$$

However, $\forall i \in [k], i \notin E$ and thus $x_i = y_i$ because $x, y \in E$. Thus, either $h_i(x) \neq x_i$ or $h_i(y) = y_i$. Either way, h_i is nontrivial. Thus, for all $i \in [k], h_i$ is nontrivial. The same way, we can prove that there are no trivial coordinate functions whose index is in $[k + 1, 2k]$. As a result, h does not have any trivial coordinate function. Let us prove that $\forall u \in \Pi([n]), \kappa(h, u) \geq k$. Let us consider the sequential update schedule $u \in \Pi([n])$. Let $E \in \binom{[2k]}{k}$ be the set of the k first automata of $[2k]$ updated in u . Let $E' = [2k] \setminus E$. Furthermore, let i be the first step at which all automata of E are updated in u . In other words, we have $E \subseteq \{u_1, \dots, u_i\}$ and $E' \cap \{u_1, \dots, u_i\} = \emptyset$. Let $z = b(E)$. We will prove that $z[E]$ is a clique in the confusion graph $G_{h,u}$. Let $x, y \in z[E]$ with $x \neq y$. First let us prove that $h^{\{u_1, \dots, u_i\}}(x) = h^{\{u_1, \dots, u_i\}}(y)$. We have:

$$\begin{aligned} - h^{\{u_1, \dots, u_i\}}(x)_E &= h^{\{u_1, \dots, u_i\}}(x)_{a(x)} = x_{[2k] \setminus a(x)} = x_{E'} = z_{E'} = y_{E'} = \\ & y_{[2k] \setminus a(y)} = h^{\{u_1, \dots, u_i\}}(y)_{a(y)} = h^{\{u_1, \dots, u_i\}}(y)_E. \end{aligned}$$

- $h^{\{u_1, \dots, u_i\}}(x)_{E'} = (x)_{E'} = z_{E'} = y_{E'} = h^{\{u_1, \dots, u_i\}}(y)_{E'}$ because $E' \cap \{u_1, \dots, u_i\} = \emptyset$.
- $\forall j \in [2k+1, n]$, with $j \in \{u_1, \dots, u_i\}$ we have $h^{\{u_1, \dots, u_i\}}(x)_j = h_j(x) = 0 = h_j(y) = h^{\{u_1, \dots, u_i\}}(y)_j$.
- $\forall j \in [2k+1, n]$, with $j \notin \{u_1, \dots, u_i\}$ we have $h^{\{u_1, \dots, u_i\}}(x)_j = x_j = z_j = y_j = h^{\{u_1, \dots, u_i\}}(y)_j$.

As a result, $h^{\{u_1, \dots, u_i\}}(x) = h^{\{u_1, \dots, u_i\}}(y)$. Now, $x \neq y$ and $x, y \in z[E]$. Thus, $x_E \neq y_E$ and $h(x)_{E'} = x_E \neq y_E = h(y)_{E'}$. As a result, x and y are neighbors in $G_{h,u}$ and then $z[E]$ is a clique. Furthermore, $z[E]$ is of size q^k . Thus, $\chi(G_{h,u}) \geq q^k$. As a consequence, for any sequential update schedule u we have $\kappa(h, u) \geq k$ and then $\kappa^{\min}(h) \geq k$.

E Proof of Theorem 3

Theorem 3 (Theorem 3). *For all $q \geq 2$ and $n \in \mathbb{N}$, $\kappa_{n,q}^{\min} \geq \lfloor n/3 \rfloor$.*

Proof. Let $A := [0, q[$. In this proof, c^i refers to i times the composition of c . Let $n = 3k$. (if $n = 3k + 1$ or $n = 3k + 2$ we just add one or two useless automata and the demonstration is the same). Let $b : \binom{[2k]}{k} \rightarrow A^n$ such that $\forall E = \{e_1, e_2, \dots, e_k\} \in \binom{[2k]}{k}$, $\forall x \in b(E)[E]$ we have:

- $\forall e \in \overline{E} = [2k] \setminus E$, $x_e = 0$ if $e + 1 \in E$ and 1 otherwise.
- $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise.
- $\forall \ell \in [k-1]$, $x_{2k+\ell+1} = 0$ if $j+1 \pmod{2k} \in E$ and 1 otherwise with $j = e_\ell$.

Let $a : B \rightarrow \binom{[2k]}{k}$ be the function which decodes the subset encoded in a configuration such that $a = g \circ c^{2k} \circ h$ with:

- $h : x \mapsto (x, \{1\}, \{\}, 1)$ if $x_{2k+1} = 0$ and $(x, \{\}, \{1\}, 1)$ otherwise.
- c such that for all $x \in B$, I, \overline{I} subsets of $[n]$ and $e \in [2k]$:
 - If $e \in \overline{I}$:
 - * If $x_e = 0$ then $c(x, I, \overline{I}, e) = (x, I \cup \{e+1\}, \overline{I}, e+1)$.
 - * If $x_e = 1$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
 - $e \in I$:
 - * If $|I| = k$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
 - * Otherwise, let $\ell = |I|$ and $b = x_{2k+\ell+1}$.
 - If $b = 0$ then $c(x, I, \overline{I}, e) = (x, I \cup \{e+1\}, \overline{I}, e+1)$.
 - If $b = 1$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
- $g : (x, I, \overline{I}, q) \mapsto I$.

By induction, let us prove that:

$$\forall i \in [2k], \forall x \in b(E)[E], c^{i-1}(h(x)) = (x, E \cap [i], \overline{E} \cap [i], i).$$

First, for $i = 1$ we have $c^{i-1}(h(x)) = c^0(h(x)) = h(x)$. There are 2 cases:

- If $1 \in E$, then $x_{2k+1} = 0$ because $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise. Thus, $h(x) = (x, \{1\}, \{1\}, 1)$. Furthermore, $E \cap [1] = \{1\}$ and $\overline{E} \cap [1] = \{1\}$. As a result, we have $c^{i-1}(h(x)) = (x, E \cap [1], \overline{E} \cap [1], 1)$.
- If $1 \in \overline{E}$, then $x_{2k+1} = 1$ because $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise. Thus, $h(x) = (x, \{1\}, \{1\}, 1)$. Furthermore, $E \cap [1] = \{1\}$ and $\overline{E} \cap [1] = \{1\}$. As a result, we have $c^{i-1}(h(x)) = (x, E \cap [1], \overline{E} \cap [1], 1)$.

Next, let us suppose that for $i \in [2k[$, we have $c^{i-1}(h(x)) = (x, E \cap [i], \overline{E} \cap [i], i)$. Let $I = E \cap [i]$, $\overline{I} = \overline{E} \cap [i]$, $e = i$. Let $c^{i-1}(h(x)) = (x, I, \overline{I}, e)$.

- if $e \in \overline{E}$, then we have $e \in \overline{I}$. There are two cases:
 - If $e + 1 \in E$ then $x_e = 0$ because $\forall e \in \overline{E}$, $x_e = 0$ if $e + 1 \in E$ and 1 otherwise. Then $c^i(h(x)) = (x, I \cup \{e + 1\}, \overline{I}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
 - If $e + 1 \in \overline{E}$ then $x_e = 1$ because $\forall e \in \overline{E}$, $x_e = 0$ if $e + 1 \in E$ and 1 otherwise. Then $c^i(h(x)) = (x, I, \overline{I} \cup \{e + 1\}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
- if $e \in E$ then we have $e \in I$. There are two cases:
 - If $e + 1 \in E$. Then we have $|I| < k$ because $I = E \cap [i] \subseteq E$ and $|E| = k$ and $(e + 1) \in E \setminus I$. Let $\ell = |I|$. We have $e = e_\ell$. We have $x_{2k+\ell+1} = 0$ because $\forall j \in [k - 1]$, $x_{2k+\ell+1} = 0$ if $j + 1 \in E$ and 1 otherwise with $j = e_\ell$. Then $c^i(h(x)) = (x, I \cup \{e + 1\}, \overline{I}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
 - $e + 1 \in \overline{E}$. There are two cases:
 - * If $e = e_k$. Then $|I| = k$, thus $c^i(h(x)) = (x, I, \overline{I} \cup \{e + 1\}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
 - * If $e = e_i$ with $i < k$. Then we have $|I| < k$. Let $\ell = |I|$. We have $e = e_\ell$. We have $x_{2k+\ell+1} = 1$ because $\forall j \in [k - 1]$, $x_{2k+\ell+1} = 0$ if $j + 1 \in E$ and 1 otherwise with $j = e_\ell$. Then $c^i(h(x)) = (x, I, \overline{I} \cup \{e + 1\}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.

By induction, we have $\forall x \in A^n$, $\forall i \in [2k]$,

$$c^{i-1}(h(x)) = (x, E^0 \cap V(i), E^1 \cap V(i), \overline{E}^0 \cap V(i), \overline{E}^1 \cap V(i), m+i+1, r(m+i+1)).$$

In particular, we have $c^{2k}(h(x)) = (x, E, \overline{E}, q)$. As a consequence, $a(x) = E$. Thus, all the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint. Using Lemma 4, we conclude that $\kappa_{n,q}^{min} \geq \lfloor n/3 \rfloor$.

F Proof of Theorem 4

Theorem 4 (Theorem 4). For all $q \geq 4, n \in \mathbb{N}, \kappa_{n,q}^{\min} \geq \lfloor n/2 - \log_q(n) \rfloor$.

Proof. Let $A := [0, q[$. Let $n = 2k + \lceil \log_q(2k) \rceil$. Only in this proof, to simplify the use of modulo, we index the coordinates starting from 0 and not from 1. Furthermore, each addition or subtraction is done modulo $2k$, and we will consider that if $a < b$ then $[b, a] = [a, 2k[\cup [0, b]$. For all $I \subseteq [0, 2k[$, let $\Delta_I : E \mapsto |I \cap E| - |I \setminus E|$. Let us consider the two functions $M : \binom{[0, 2k[}{k} \rightarrow [-k, k]$ and $m : \binom{[0, 2k[}{k} \rightarrow [0, 2k[$ such that $\forall E \in \binom{[0, 2k[}{k}$,

- $M(E) := \max(\{\Delta_{[i]}(E) \mid i \in [0, 2k[\})$.
- $\Delta_{[m(E)]}(E) = M$.

For instance if we have $k := 4$, and $E := \{2, 4, 5, 6\}$ then,

i	0	1	2	3	4	5	6	7
$\in E$	No	No	Yes	No	Yes	Yes	Yes	No
$\Delta_{[0,i]}(E)$	-1	-2	-1	-2	-1	0	1	0

Furthermore, $m(E) = 6$ and $M(E) = \Delta_{[0,6]}(E) = 1$. For all $E \in \binom{[0; 2k[}{k}$, let us denote by $E^0, E^1, \overline{E}^0, \overline{E}^1$ the subsets of $[0, 2k[$ such that

- $E^0 = \{e \in E \mid e - 1 \in \overline{E}\}$.
- $E^1 = E \setminus E^0 = \{e_1, e_2, \dots, e_p\}$ with $e_1 - m(E) - 1 < e_2 - m(E) - 1 < \dots < e_p - m(E) - 1$.
- $\overline{E}^0 = \{e' \in \overline{E} \mid e' + 1 \in E\}$.
- $\overline{E}^1 = \overline{E} \setminus \overline{E}^0 = \{e'_1, e'_2, \dots, e'_p\}$ with $e'_1 - m(E) - 1 < e'_2 - m(E) - 1 < \dots < e'_p - m(E) - 1$.

In other words, we sort the elements of E^1 and \overline{E}^1 in the order $m+1, m+2, \dots, 2k-1, 0, 1, \dots, m$. If we take again our example where $k := 4$, and $E := \{2, 4, 5, 6\}$ we have $E^0 = \{2, 4\}$, $E^1 = \{e_1 = 5, e_2 = 6\}$, $\overline{E}^0 = \{1, 3\}$ and $\overline{E}^1 = \{e'_1 = 7, e'_2 = 0\}$. Indeed we have $e_1 - m(E) - 1 = 5 - 6 - 1 = 6 \leq e_2 - m(E) - 1 = 6 - 6 - 1 = 7$ and $e'_1 - m(E) - 1 = 7 - 6 - 1 = 0 \leq e'_2 - m(E) - 1 = 0 - 6 - 1 = 1$. Let $v : [0, 2k[\rightarrow A^{n-2k}$ be an injective function and let v^{-1} be the inverse function of v . Let $b : \binom{[0, 2k[}{k} \rightarrow A^n$ such that if $x = b(E)$ then we have,

- $x_E = (0)^k$.
- $\forall e \in \overline{E}^0, x_e = 0$ if $e + 2 \in E$ and 1 otherwise.
- $\forall e'_j \in \overline{E}^1, x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise.

- $x_{[2k,n[} = v(m(E))$.

Again, with the same example, for all $y \in b(E)[E]$ we have:

i	0	1	2	3	4	5	6	7
$\in E$	No	No	Yes	No	Yes	Yes	Yes	No
y_i	3	1	y_2	0	y_4	y_5	y_6	2

Indeed,

- $y_3 = 0$ because $3 \in \overline{E}^0$ and $3 + 2 \in E$.
- $y_1 = 1$ because $1 \in \overline{E}^0$ and $1 + 2 \notin E$.
- $y_7 = 2$ because $e'_1 = 7 \in \overline{E}^1$ and $e_1 + 1 = 5 + 1 \in E$.
- $y_0 = 3$ because $e'_2 = 0 \in \overline{E}^1$ and $e_2 + 1 = 6 + 1 \notin E$.

Let $B =: \{x \in b(E)[E] \mid E \in \binom{[2k]}{k}\}$ be the set of configuration which encodes a set E . Let us consider the function $a : B \rightarrow \binom{[2k]}{k}$ which decodes the set encoded by any configuration of B such that $a = g \circ c^{2k} \circ h$ with:

- $h : x \mapsto (x, \emptyset, \emptyset, \emptyset, \emptyset, m + 1, 0)$ with $m = v^{-1}(x_{[2k,n[})$.
- c such that for all $I^0, I^1, \overline{T}^0, \overline{T}^1$ subsets of $[0, 2k[$, $e \in [0, 2k[$ and $q \in [0, 3[$,
 - if $q = 0$:
 - * if $x_e = 0$ or 1 then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0, I^1, \overline{T}^0 \cup \{e\}, \overline{T}^1, e + 1, 1)$.
 - * if $x_e = 2$ or 3 then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0, I^1, \overline{T}^0, \overline{T}^1 \cup \{e\}, e + 1, 0)$.
 - if $q = 1$:
 - * if $x_{e-1} = 0$ then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0 \cup \{e\}, I^1, \overline{T}^0, \overline{T}^1, e + 1, 2)$.
 - * if $x_{e-1} = 1$ then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0 \cup \{e\}, I^1, \overline{T}^0, \overline{T}^1, e + 1, 0)$.
 - if $q = 2$, let $j = |I^1|$, $e' = \overline{T}^1_j$ (the j -th element of \overline{T}^1 when we sort them in the order $m + 1, m + 2, \dots, 2k - 1, 0, 1, \dots, m$).
 - * if $x_{e'} = 2$ then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0, I^1 \cup \{e\}, \overline{T}^0, \overline{T}^1, e + 1, 2)$.
 - * if $x_{e'} = 3$ then $c(x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) = (x, I^0, I^1 \cup \{e\}, \overline{T}^0, \overline{T}^1, e + 1, 0)$.
- $g : (x, I^0, I^1, \overline{T}^0, \overline{T}^1, e, q) \mapsto I^0 \cup I^1$.

With the same example, let $y \in b(E)[E]$ and let us compute $a(y)$.

$$\begin{aligned}
y &= (3, 1, y_2, 0, y_4, y_5, y_6, 2)v(m(E)) \\
&\xrightarrow{h} (y, \emptyset, \emptyset, \emptyset, \emptyset, 7, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \emptyset, \{7\}, 0, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \emptyset, \{7, 0\}, 1, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \{1\}, \{7, 0\}, 2, 1) \\
&\xrightarrow{c} (y, \{2\}, \emptyset, \{1\}, \{7, 0\}, 3, 0) \\
&\xrightarrow{c} (y, \{2\}, \emptyset, \{1, 3\}, \{7, 0\}, 4, 1) \\
&\xrightarrow{c} (y, \{2, 4\}, \emptyset, \{1, 3\}, \{7, 0\}, 5, 2) \\
&\xrightarrow{c} (y, \{2, 4\}, \{5\}\{1, 3\}, \{7, 0\}, 6, 2) \\
&\xrightarrow{c} (y, \{2, 4\}, \{5, 6\}\{1, 3\}, \{7, 0\}, 7, 0) \\
&\xrightarrow{g} \{2, 4, 5, 6\} = E.
\end{aligned}$$

Thus, we have $\forall y \in b(E)[E]$, $a(y) = E$. If we can prove that for all $E \in \binom{[0, 2k]}{k}$ and for all $y \in b(E)[E]$, we have $a(y) = E$, then we prove that the sets $b(E)[E]$ are disjoint. Furthermore, with Lemma 4, we can conclude that $\kappa_{2k+\log(k), q}^{min} \geq k$. In the remaining of the proof, we prove it formally

for all k and E . Let $k \in \mathbb{N}$, $E \in \binom{[0, 2k]}{k}$. Let $r_E : \ell \mapsto \begin{cases} 0 & \text{if } \ell \in \bar{E} \\ 1 & \text{if } \ell \in E^0 \\ 2 & \text{otherwise} \end{cases}$. By

induction, let us prove that for all $i \in [0, 2k]$,

$$c^i(h(x)) = (x, E^0 \cap V(i), E^1 \cap V(i), \bar{E}^0 \cap V(i), \bar{E}^1 \cap V(i), m+i+1, r(m+i+1)).$$

with $V(0) = \emptyset$, and $\forall i \in [2k], V(i) = [m+1, m+i]$. First, let us prove that $m+1 \notin E$. For the sake of contradiction, let us say that $m+1 \in E$. Then we have $\Delta_{[m+1]}(E) = \Delta_{[m]}(E) + \Delta_{[m+1, m+1]}(E) = M(E) + 1$. This is absurd because $M(E) = \max(\{\Delta_{[i]}(E) \mid i \in [2k]\})$. As a result $m+1 \notin E$. Thus, $r_E(m+0+1) = 0$. Furthermore,

$$\begin{aligned}
c^0(h(x)) &= h(x) = (x, \{\}, \{\}, \{\}, \{\}, m+0+1, 0) \\
&= (x, E^0 \cap V(0), E^1 \cap V(0), \bar{E}^0 \cap V(0), \bar{E}^1 \cap V(0), m+0+1, r_E(m+0+1)).
\end{aligned}$$

Next, let us suppose that the induction hypothesis hold for $i \in [0, 2k[$. Let $I = E \cap V(i)$, $\bar{I} = \bar{E} \cap V(i)$, $I^0 = E^0 \cap V(i)$, $I^1 = E^1 \cap V(i)$,

$\bar{T}^0 = \bar{E}^0 \cap V(i)$, $\bar{T}^1 = \bar{E}^1 \cap V(i)$, $e = m + i + 1$ and $q = r_E(e)$. Thus, $c^i(h(x)) = (x, I^0, I^1, \bar{T}^0, \bar{T}^1, e, q)$. There are four cases:

- $e = m + i + 1 \in \bar{E}^0$. As a consequence, we have $q = r_E(e) = 0$. Furthermore, we have $x_e = 0$ or 1 because $\forall e \in \bar{E}^0$, $x_e = 0$ if $e+2 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1, \bar{T}^0 \cup \{e\}, \bar{T}^1, e+1, 1)$. By definition of \bar{E}^0 , we have $e+1 \in E^0$ and then $r_E(e+1) = 1$. Thus, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \bar{E}^0 \cap V(i+1), \bar{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m + i + 1 \in \bar{E}^1$. As a consequence, we have $q = r_E(e) = 0$. Furthermore, we have $x_e = 2$ or 3 because $\forall e'_i \in \bar{E}^1$, $x_{e'_i} = 2$ if $e_i+1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1, \bar{T}^0, \bar{T}^1 \cup \{e\}, e+1, 1)$. By definition of \bar{E}^1 , we have $e+1 \in \bar{E}$ and then $r_E(e+1) = 0$. Thus, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \bar{E}^0 \cap V(i+1), \bar{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m + i + 1 \in E^0$. By induction hypothesis, we have $q = r_E(e) = 1$. Furthermore, by definition of E^0 , $e-1 \in \bar{E}^0$. There are two subcases:
 - $e+1 \in E$. We have $x_{e-1} = 0$ because $\forall (e-1) \in \bar{E}^0$, $x_{e-1} = 0$ if $(e-1)+2 = e+1 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0 \cup \{e\}, I^1, \bar{T}^0, \bar{T}^1, e+1, 2)$. And since $e+1 \in E$ and $e \notin \bar{E}$, then $e+1 \in E^1$ and $r_E(e+1) = 2$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \bar{E}^0 \cap V(i+1), \bar{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
 - $e+1 \in \bar{E}$. We have $x_{e-1} = 1$ because $\forall (e-1) \in \bar{E}^0$, $x_{e-1} = 0$ if $(e-1)+2 = e+1 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0 \cup \{e\}, I^1, \bar{T}^0, \bar{T}^1, e+1, 0)$. And since $e+1 \in \bar{E}$, $r_E(e) = 0$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \bar{E}^0 \cap V(i+1), \bar{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m + i + 1 \in E^1$. As a consequence, we have $q = r_E(e) = 2$. Let $j = |I^1|$. Let us prove that $|\bar{T}^1| < |I^1|$. First, we have $|I^0| = |\bar{T}^0|$. Indeed, for all $u \in \bar{T}^0$, we have also $u \in \bar{E}^0$ and then $u+1 \in E^0$. Furthermore, $e \in E^1$ and thus $e-1 = m+i \notin \bar{E}^1$. Thus, $u \in [m+1, m+i+1[$, $u+1 \in [m+1, m+i+1]$. Consequently, $u+1 \in V(i)$ and $u+1 \in I^0$. As a result, for all $u \in \bar{T}^0$, we have $u+1 \in I^0$. As a consequence, $|\bar{T}^0| \leq |I^0|$. Reversely, for all $v \in I^0$, $v \in E^0$ and then $v-1 \in \bar{E}^0$. Furthermore, $m+1 \in \bar{E}$. Thus, $v \in]m+1, m+i+1]$ and $v-1 \in [m+1, m+i+1]$. As a result, $v-1 \in V(i)$ and $v-1 \in \bar{T}^0$. Consequently, for all $v \in I^0$, we have $v-1 \in \bar{T}^0$. As a consequence, $|I^0| \leq |\bar{T}^0|$ and

then $|I^0| = |\overline{I}^0|$. Now, $\Delta_{[m+i+1]}(E) = \Delta_{[m]}(E) + \Delta_{[m+1, m+i]}(E) + |\{e\}| = M(E) + |I| - |\overline{I}| + 1 = M(E) + |I^0| + |I^1| - |\overline{I}^0| - |\overline{I}^1| + 1 = M(E) + |I^1| - |\overline{I}^1| + 1$. If $|I^1| \geq |\overline{I}^1|$ then $\Delta_{[m+i+1]}(E) > M(E)$ which is absurd. Thus, $|I^1| < |\overline{I}^1|$. Let $e' = |\overline{I}_j^1|$. We have $e = e_j$ and $e' = e'_j$. There are two cases:

- $e + 1 \in E$. Then $x_{e'} = 2$ because $\forall e'_j \in \overline{E}^1$, $x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e + 1, 2)$. Furthermore, $e + 1 \in E$ and $e \notin \overline{E}$ then $e + 1 \in E^1$ and $r_E(e + 1) = 2$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i + 1), E^1 \cap V(i + 1), \overline{E}^0 \cap V(i + 1), \overline{E}^1 \cap V(i + 1), m + i + 2, r_E(m + i + 2))$.
- $e + 1 \in \overline{E}$. Then $x_{e'} = 3$ because $\forall e'_j \in \overline{E}^1$, $x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e + 1, 0)$. Furthermore, $e + 1 \in E$ and $e \notin \overline{E}$ then $e + 1 \in E^1$ and $r_E(e + 1) = 0$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i + 1), E^1 \cap V(i + 1), \overline{E}^0 \cap V(i + 1), \overline{E}^1 \cap V(i + 1), m + i + 2, r_E(m + i + 2))$.

By induction, we can see that $\forall i \in [2k]$, $c^{i+1}(h(x)) = (x, E^0 \cap V(i + 1), E^1 \cap V(i + 1), \overline{E}^0 \cap V(i + 1), \overline{E}^1 \cap V(i + 1), m + i + 2, r_E(m + i + 2))$. As a result, $a(x) = g(c^{2k}(h(x))) = g(x, E^0, E^1, \overline{E}^0, \overline{E}^1, m + 2k, r_E(m + 2k + 1)) = E^0 \cup E^1 = E$. Since there is a function a such that $\forall E \in \binom{[0, 2k]}{k}$, $\forall x \in b(E)[E]$, $a(x) = E$, we know that all the sets $b(E)[E]$ are disjoint. Using Lemma 4, we conclude that $\kappa_{2k+\log(k), q}^{\min} \geq k$.

G Proof of Lemma 5

Lemma 5 (Lemma 5). *Let $h \in F(n, q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h) + k \leq \mathcal{L}^*(h)$.*

Proof. Let $h \in F(n, q)$, $k := \kappa^{\min}(h)$, $A := [0, q]$ and $m := n + k$. By definition of $\kappa^{\min}(h)$ there exists $f \in F(m, q)$ and $w \in \Pi([m])$ such that f^w simulates h . Thus, $\text{pr}_{[n]} \circ f^{w_n} \circ \dots \circ f^{w_1} = h \circ \text{pr}_{[n]}$. By definition, $\forall i \in [m]$, f^i does not update more than one coordinate. Then, $f^{w_1}, \dots, f^{w_m} \in F^*(m, q)$. Let us consider the set T of the coordinates of the trivial functions of h and let $w' \in \Pi([m] \setminus T)$ be an order respecting w which does update the coordinates of T . In other words, $\forall i, j \in \Pi([m] \setminus T)$, if $w(i) < w(j)$ then $w'(i) < w'(j)$. Let us prove that $f^{w'} = f^w$. Let h_i be a trivial coordinate function. Thus, $\forall x \in A^n$, $h_i(x) = x_i$. And for all $y \in A^k$ and $z := xy$, we have $(f^{w'}(z))_i = h_i(x) = x_i = z_i$. Furthermore,

since $w \in \Pi([m])$, the coordinate i is updated only one time in w in step $j := w(i)$. Thus, $f_{w_j} \circ f^{w_1, \dots, w_{j-1}}(z) = (f^w(z))_i = z_i$. Furthermore, since i is not updated before the step j , we have $(f^{w_1, \dots, w_{j-1}}(z))_i = z_i$. As a result, $f^{w_j} \circ f^{w_1, \dots, w_{j-1}} = f^{w_1, \dots, w_{j-1}}$, $f^w = f^{w_1, \dots, w_{j-1}, w_j, \dots, w_m}$. Using the same method for all j such that h_{w_j} is trivial we get $f^w = f^{w'}$. The order w' is of size $\Omega(h) + k$. As a result, we have $\mathcal{L}(h|n+k) \leq \Omega(h) + k$. And by definition of $\mathcal{L}^*(h)$ we have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k)$.

H Proof of Label 6

Lemma 6 (Label 6). *Let $h \in F(n, q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h) + k \leq \mathcal{L}^*(h)$.*

Proof. Let $\ell := \mathcal{L}^*(h)$, $m \leq n$ and $g^{(1)}, \dots, g^{(\ell)} \in F^*(m, q)$ such that $\text{pr}_{[n]} \circ g^{(\ell)} \circ \dots \circ g^{(1)} = h \circ \text{pr}_{[n]}$. We can assume that for all $i \in [\ell]$, the function $g^{(i)}$ updates one coordinate. Otherwise, g^i would be the identity function and we could remove it and have $\ell > \mathcal{L}^*(h)$ which is absurd. Let $u \in [m]^t$ such that, for all $i \in [\ell]$, u_i is the coordinate updated by $g^{(i)}$. Let $I = \{i_1, i_2, \dots, i_p\}$ with $i_1 < i_2 < \dots < i_p$ the set of steps where a coordinate of $[n]$ is updated for the last time in u . In other words, $\forall j \in [p]$, $u_{i_j} \in [n]$ and $\forall i \in [i_j + 1, \ell]$, $u_i \neq u_{i_j}$. We know that $\Omega(h) \leq p$ because, to compute h , each coordinate of a nontrivial function of h needs to be updated at least once. Indeed, if h_i is nontrivial, then $\exists x \in A^n$, $h_i(x) \neq x_i$. If i is not updated in u , then $\forall y \in A^{m-n}$, $\text{pr}_i \circ g^{(\ell)} \circ \dots \circ g^{(1)}(xy) = x_i \neq h_i(x)$ and h is not computed. Let $k := \ell - p \leq \mathcal{L}^*(h) - \Omega(h)$. Let v be an order which updates all the coordinates of $[n]$ not updated by $g^{(1)}, \dots, g^{(\ell)}$. Let $u' := (u_{i_1}, u_{i_2}, \dots, u_{i_p})$ be an order which updates the coordinates of $[n]$ updated by $g^{(1)}, \dots, g^{(\ell)}$ in the same order that u updates them for the last time. Let $w := uv \in \Pi([n])$ be a permutation of $[n]$. Let $J = \{j_1, j_2, \dots, j_k\} = [\ell] \setminus I$ with $j_1 < j_2 < \dots < j_k$. For all $i \in [n]$, let $\tilde{g}^{(i)} : A^m \rightarrow A$ be the function which return the value of the coordinate updated by $g^{(i)}$. In other words, $\tilde{g}^{(i)} = \text{pr}_{u_i} \circ g^{(i)}$. Let $y := (0)^{m-n}$ (a word of size $m - n$ containing only the letter 0). Let $c : A^n \rightarrow A^k$, such that, $\forall x \in A^n$, $\forall i \in [k]$, $c_i(x) = \tilde{g}^{(j_i)} \circ g^{(j_i-1)} \circ \dots \circ g^{(1)}(xy)$. Let us prove that c give a proper coloring of the confusion graph $G_{h,w}$. Let $x, x' \in A^n$, be neighbors in the confusion graph $G_{h,w}$. In other words, $h(x) \neq h(x')$ but $\exists i \in [n]$, $h^{\{w_1, w_2, \dots, w_i\}}(x) = h^{\{w_1, w_2, \dots, w_i\}}(x')$. For the sake of contradiction, let us say that $c(x) = c(x')$. Let $z := xy$ and $z' := x'y$. Let $e = \max(\{i \in [n] \mid h^{\{w_1, \dots, w_i\}}(x) = h^{\{w_1, \dots, w_i\}}(x')\})$. Let $b \in [p]$ be the last step of u in which the coordinate w_e is updated. Let

$r := g^{(b)} \circ \dots \circ g^{(1)}(z)$ and $r' := g^{(b)} \circ \dots \circ g^{(1)}(z')$. Let us prove that $r = r'$. For all $a \in [m]$ not yet updated in u at step b :

- if $a \in [n]$ then $r_a = x_a = x'_a = r'_a$ because $h^{\{w_1, w_2, \dots, w_e\}}(x) = h^{\{w_1, w_2, \dots, w_e\}}(x')$ and thus $x_{[n] \setminus \{w_1, \dots, w_e\}} = x'_{[n] \setminus \{w_1, \dots, w_e\}}$.
- if $a \in [n+1, m]$ then $r_a = z_a = y_{a-n} = z'_a = r'_a$.

For all $a \in [m]$ already updated in u at step b :

- if $a \in [n]$ and a is updated for the last time then $r_a = h_a(x) = h_a(x') = r'_a$ because $h^{\{w_1, w_2, \dots, w_i\}}(x) = h^{\{w_1, w_2, \dots, w_i\}}(x')$ and thus $h(x)_{[n] \setminus \{w_1, \dots, w_e\}} = h(x')_{[n] \setminus \{w_1, \dots, w_e\}}$.
- Otherwise, let $d < b$ be the last step in u before b such that a is updated. In other words, $u_d = a$ and $\forall i \in]d, b[, u_i \neq a$. We have $r_a = \tilde{g}^{(d)} \circ g^{(d-1)} \circ \dots \circ g^{(1)}(z) = c_d(x) = c_d(x') = \tilde{g}^{(d)} \circ g^{(d-1)} \circ \dots \circ g^{(1)}(z') = r'_a$.

Thus, we have $g^{(b)} \circ \dots \circ g^{(1)}(z) = g^{(b)} \circ \dots \circ g^{(1)}(z')$ and thus $g^{(\ell)} \circ \dots \circ g^{(1)}(z) = g^{(\ell)} \circ \dots \circ g^{(1)}(z')$. However, $pr_{[n]} \circ g^{(\ell)} \circ \dots \circ g^{(1)}(z) = h(x) \neq h(x') \neq g^{(\ell)} \circ \dots \circ g^{(1)}(z')$. This is absurd, so if two configurations x, x' are neighbors in the confusion graph then $c(x) \neq c(x')$. Thus, c gives a proper coloring of the confusion graph $G_{h,w}$ and it uses at most $q^k = q^{\ell-p} \leq q^{\mathcal{L}^*(h) - \Omega(h)}$ colors. As a result, $\kappa(h, w) \leq \mathcal{L}^*(h) - \Omega(h)$.

I Proof of Lemma 7

Lemma 7 (Lemma 7). *Let $G = ([n], E)$ be an undirected graph and let $s = \text{Pw}(G)$. Then there are functions $c : [n] \rightarrow [s]$ and $u \in \Pi([n])$ with the following property. For all $i \in [n]$, we have either 1) for all k neighbor of i in G we have $u(i) \leq u(k)$ or 2) for all $j, k \in [n]$ with $c(i) = c(j)$, $u(i) < u(j)$ and k neighbor of j in G we have $u(i) \leq u(k)$.*

Proof. Let $G = ([n], E)$, $s = \text{Pw}(G)$ and X_1, \dots, X_p a minimal path decomposition of G . In other words:

- $\forall i \in [n], \forall a, b \in [p]$ with $a < b$, if $i \in X_a$ and $i \in X_b$ then $\forall \ell \in [a, b]$, $i \in X_\ell$.
- $\forall (i, j) \in E, \exists a \in [p]$ such that $i \in X_a$ and $j \in X_a$.
- $\forall i \in [n], \exists a \in [p]$ such that $i \in X_a$.
- $\forall i \in [p], |X_i| \leq s + 1$.

For all $i \in [n]$, let $X(i) = \{X \in \{X_1, \dots, X_p\} \mid i \in X\}$. Let $b : i \mapsto \min(\{j \mid X_j \in X(i)\})$ and $e : i \mapsto \max(\{j \mid X_j \in X(i)\})$. We will assume

that $\forall \{a, b\} \subseteq [p]$, we do not have $X_a \subseteq X_b$ since otherwise we could remove X_a and still have a valid path decomposition of same size. As a result, for all $a \in [p]$, there exists $j \in X_a$ such that $e(j) = a$. Indeed, if that was not the case, we would have $X_a \subseteq X_{a+1}$. Let $u \in \Pi([n])$ be an order respecting e and $v \in \Pi([n])$ be an order respecting b . In other words, for all $\{i, j\} \subseteq [n]$, if $e(i) < e(j)$ then $u(i) < u(j)$ and if $b(i) < b(j)$ then $v(i) < v(j)$. For all j in $[n]$ taken in the order v , let us define $c(j)$ as such:

- If, in the set of images by c of $X_{b(j)}$ already defined, there are value of $[s]$ not used then let $c(j)$ be the minimal of them. More formally, if $\{c(k) \mid k \in X_{b(j)} \text{ and } v(k) < v(j)\} \neq [s]$ then let $c(j) := \min([s] \setminus \{c(k) \mid k \in X_{b(j)} \text{ and } v(k) < v(j)\})$.
- Otherwise, if there is $k \in X_{b(j)}$ such that $v(k) < v(j)$ and $e(k) = b(j)$, then let us consider the i which minimize $u(i)$. In other words, let us consider i such that $u(i) = \min(\{u(k) \mid k \in X_{b(j)}\})$ and let $c(j) := c(i)$. We remark that since $\forall k \in X_{b(j)}, b(j) \leq e(k)$, we have $e(i) = b(j)$ (and not $e(i) < b(j)$).
- Otherwise, let $c(j) := 0$. In this case, we have $b(j) = e(j)$ because $\forall k \in X_{b(j)} \setminus \{j\}, b(j) < e(k)$ and by hypothesis $\forall a \in [p]$, there exists $j \in X_a$ such that $e(j) = a$.

We remark that with this construction of c , $\forall a \in [p]$ there is at most one $\{i, j\} \subseteq X_a$ such that $c(i) = c(j)$ because $|X_a| \leq s + 1$. Let $i \in [n]$. First, let us consider the case where $c(i)$ is defined using the third case. Then, we have $b(i) = e(i)$ and thus $X(i) = \{X_{b(i)}\}$. By definition of a path decomposition, for all neighbor k of i in G , we have $k \in X_{b(i)}$. Furthermore, $\forall k \in X_{b(i)}, e(i) = b(i) < e(k)$. Thus, $\forall k \in X_{b(i)}, u(i) < u(k)$. As a result, i respects the condition 1) for all k neighbor of i in G we have $u(i) \leq u(k)$. Next, let us assume that $c(i)$ is not defined using the third case. Let $j \in [n]$ such that $c(i) = c(j)$ and $u(i) < u(j)$. Let us prove that for all k neighbor of j in G , $u(i) \leq u(k)$. First let us prove that we have $e(i) \leq b(j)$. For the sake of contradiction, let us say that $b(j) < e(i)$. There are 2 cases:

- $v(i) < v(j)$. Thus, $b(i) \leq b(j) < e(i)$. However,
 - Since, $b(i) \leq b(j) < e(i)$, we have $b(j) \in [b(i), e(i)]$ and thus $i \in X_{b(j)}$. Thus, there exists $k \in X_{b(j)}$ ($k := i$) such that $c(k) = c(j)$ and $v(k) < v(j)$. As a result, $c(j)$ cannot have been defined using the first case of the definition.
 - We have $b(j) < e(i)$. Furthermore, by hypothesis, we know that there is $k \in X_{b(j)}$, such that $e(k) = b(j)$. Thus, $e(k) < e(i)$ and then $u(k) < u(i)$. As a consequence, we have $c(i) = c(j)$ but

$u(i) \neq \min(\{u(k)|k \in X_{b(j)}\})$. As a result, $c(j)$ cannot have been defined using the second case of the definition.

- We have $b(j) < e(i) \leq e(j)$ and thus $b(j) \neq e(j)$. As a result, $c(j)$ cannot have been defined using the third case of the definition.
- $v(j) < v(i)$ Thus, $b(j) \leq b(i) < e(i)$. However,
 - Since, $b(j) \leq b(i) < e(i) \leq e(j)$, we have $b(i) \in [b(j), e(j)]$ and thus $j \in X_{b(i)}$. Thus, there is $k \in X_{b(i)}$ such that $c(k) = c(i)$ and $v(k) < v(i)$. As a result, $c(i)$ cannot have been defined using the first case of the definition.
 - We have $b(i) \leq b(j) < e(i) \leq e(j)$ and thus $b(i) < e(j)$. Furthermore, we know that there is $k \in X_{b(i)}$, such that $e(k) = b(j)$. Thus, $e(k) < e(j)$ and then $u(k) < u(j)$. As a consequence, we have $c(j) = c(i)$ but $u(j) \neq \min(\{u(k)|k \in X_{b(i)}\})$. As a result, $c(i)$ cannot have been defined using the second case of the definition.
 - By hypothesis, $c(i)$ is not defined using the third case of the definition.

All cases raise a contradiction. As a result, we have $e(i) \leq b(j)$. There are 2 cases:

- If $e(i) < b(j)$, then let us prove that for each k neighbor of j in G , $u(i) < u(k)$ (and then $u(i) \leq u(k)$). Let k be a neighbor of j in G . Thus, $\exists a \in [p], k \in X_a$ and $j \in X_a$ and thus $b(j) \leq b(k)$. As a consequence, $e(i) < b(j) \leq b(k) \leq e(k)$ and then $u(i) < u(k)$.
- If $e(i) = b(j)$, then let us prove that for each k neighbor of j in G , $u(i) \leq u(k)$. Let k be a neighbor of j in G . If $b(j) < b(k)$, then like in the previous case, we have $e(i) = b(j) < b(k) \leq e(k)$ and thus $u(i) < u(k)$. Otherwise, let us prove that $k \in X_{e(i)}$. We have $b(k) \leq b(j) = e(i)$ and since k is a neighbor of j then $e(i) = b(j) \leq e(k)$. Thus, $e(i) \in [b(k), e(k)]$ and thus $k \in X_{e(i)}$. We remark that i and j are in $X_{b(j)} = X_{e(i)}$ and $c(i) = c(j)$. Then, the value $c(j)$ corresponds to the second cases in the definition and we have $\forall k \in X_{b(j)}, u(i) \leq u(k)$ (the only case where $u(i) = u(k)$ being when $i = k$).

J Proof of Lemma 8

Lemma 8 (Lemma 8). *Let $h \in F(n, q)$. Let $G = \text{IG}^*(h)$. If we have $c : [n] \rightarrow [s]$ and $u \in \Pi([n])$ such that G, c, u have the same properties as in Lemma 7, then we have $\kappa(h, u) \leq s$.*

Proof. For all $j \in [n]$, let $v(j) := \{k \in [n] \mid (k, j) \in E\}$. For all $i \in [n]$, let $g_i : A^{|v(i)|} \rightarrow A$ such that $g_i \circ pr_{v(i)} = h_i$. In other words, $\forall x \in A^n$, $g_i(x_{v(i)}) = h_i(x)$. We know that such a function exists by definition of the interaction graph. Let I_1, I_2, \dots, I_s a partition of $[n]$ such that $\forall \ell \in [s]$, $I_\ell := \{i \in [n] \mid c(i) = \ell\}$. For all $j \in [n]$, let $I(j) = I_{c(j)}$. Let $w = (n+1, n+2, \dots, n+s, u(1), \dots, u(n))$. Without loss of generality, let us say that u is the canonical update schedule $(1, \dots, n)$. Let $f \in F(n+s, q)$ such that $\forall z = xy \in A^{n+s}$,

$$\begin{aligned} - \forall i \in [n], f_i(z) &= \begin{cases} h_i(x) & \text{if } \forall k \in v(i), u(i) \leq u(k) \\ y_{c(i)} - \sum_{j \in I(i) \text{ with } u(j) < u(i)} x_j - \sum_{j \in I(i) \text{ with } u(i) < u(j)} h_j(x) & \text{otherwise} \end{cases} \\ - \forall \ell \in [s], f_{n+\ell}(z) &= \sum_{j \in I_\ell} h_j(x). \end{aligned}$$

Let $y' = f^{w_1, \dots, w_s}(z)_{[n+1, n+s]} = (\sum_{j \in I_1} h_j(x), \dots, \sum_{j \in I_s} h_j(x))$. Let us prove by induction that, being assumed that $[0] = \emptyset$, we have $\forall i \in [0, n]$, $f^{w_1, \dots, w_{s+i}}(z)_{[n]} = h^{[i]}(x)$. First $f^{w_1, \dots, w_s}(z)_{[n]} = (xy')_{[n]} = x = h^\emptyset(x)$. Next, let $i \in [n]$, let us suppose that $f^{w_1, \dots, w_{s+i-1}}(z)_{[n]} = h^{[i-1]}(x)$. Let $z' = x'y' = f^{w_1, \dots, w_{s+i-1}}(z)$. There are two cases. If $\forall k \in v(i)$, $u(i) \leq u(k)$ then $f_i(z') = h_i(x')$. In this case we have, $x'_{v(i)} = x_{v(i)}$. Thus, $f_i(z') = h_i(x)$. Otherwise, we have $\forall j \in I(i)$ with $u(i) < u(j)$, $\forall k \in v(j)$, $u(i) < u(k)$. In other words, for each such k we have $x'_k = x_k$ and thus $x'_{v(j)} = x_{v(j)}$. Let $\ell = c(i)$. We have $f^i(z') = y'_\ell - \sum_{j \in I_\ell \text{ with } u(j) < u(i)} x'_j - \sum_{j \in I_\ell \text{ with } u(i) < u(j)} h_j(x')$

We know that:

$$\begin{aligned} - y'_\ell &= \sum_{j \in I_\ell} h_j(x). \\ - \forall j \in I_\ell \text{ with } u(j) < u(i), x'_j &= h_j(x) \\ - \forall j \in I_\ell \text{ with } u(i) \leq u(j), h_j(x') &= g_j(x'_{v(j)}) = g_j(x_{v(j)}) = h_j(x) \\ &\text{(because } \forall k \in v(j), (k, j) \in E, \text{ and then, by hypothesis of this lemma, } \\ &u(i) \leq u(k)). \end{aligned}$$

Thus,

$$\begin{aligned}
f_i(z') &= y'_\ell - \sum_{j \in I_\ell \text{ with } u(j) < u(i)} x'_j - \sum_{j \in I_\ell \text{ with } u(i) < u(j)} h_j(x') \\
&= \sum_{j \in I_\ell} h_j(x) - \sum_{j \in I_\ell \text{ with } u(j) < u(i)} h_j(x) - \sum_{j \in I_\ell \text{ with } u(i) < u(j)} h_j(x) \\
&= \sum_{j \in I_\ell \text{ with } u(j) = u(i+1)} h_j(x) \\
&= h_i(x)
\end{aligned}$$

As a result, in both case we have $f_i(z') = h_i(x)$. Moreover, $f^{w_1, \dots, w_{s+i}}(z)_{[n]} = f^{s+i}(z')_{[n]} = h^{[i+1]}(x)$ and by induction $\forall i \in [0, n]$, $f^{w_1, \dots, w_{s+i}}(z)_{[n]} = h^{[i]}(x)$. In particular, we have $f^w(z)_{[n]} = h(x)$ and then $\text{pr}_{[n]} \circ f^w = h \circ \text{pr}_{[n]}$. Thus, $\kappa(h, w) \leq s$. As a result, $\kappa^{\min}(h) \leq s$.