Interacting partially directed self-avoiding walk: a probabilistic perspective

Philippe Carmona, Gia Bao Nguyen, Nicolas Pétrélis, Torri Niccolo

Interacting partially directed self-avoiding walk: a probabilistic perspective

Philippe Carmona
Gia Bao Nguyen
Nicolas Pétrélis
Niccolò Torri

Laboratoire de Mathématiques Jean Leray UMR 6629
Université de Nantes, 2 Rue de la Houssinière
BP 92208, F-44322 Nantes Cedex 03, France
e-mail: philippe.carmona@univ-nantes.fr
e-mail: nicolas.petrelis@univ-nantes.fr

Department of Mathematics
KTH Royal Institute of Technology
Lindstedtsvägen 25, SE-100 44 Stockholm, Sweden
e-mail: nguyengb@kth.se

CNRS & Université Pierre et Marie Curie,
Laboratoire de Probabilités et Modèles Aléatoires,
Campus de Jussieu, 4 place Jussieu 75252 Paris Cedex 5
e-mail: niccolo.torri@upmc.fr
†

Abstract: * We review some recent results obtained in the framework of the 2-dimensional Interacting Self-Avoiding Walk (ISAW). After a brief presentation of the rigorous results that have been obtained so far for ISAW we focus on the *Interacting Partially Directed Self-Avoiding Walk* (IPDSAW), a model introduced in Zwanzig and Lauritzen (1968) to decrease the mathematical complexity of ISAW.

In the first part of the paper, we discuss how a new probabilistic approach based on a random walk representation (see Nguyen and Pétrélis (2013)) allowed for a sharp determination of the asymptotics of the free energy close to criticality (see Carmona, Nguyen and Pétrélis (2016)). Some scaling limits of IPDSAW were conjectured in the physics literature (see e.g. Brak et al. (1993)). We discuss here the fact that all limits are now proven rigorously, i.e., for the extended regime in Carmona and Pétrélis (2016), for the collapsed regime in Carmona, Nguyen and Pétrélis (2016) and at criticality in Carmona and Pétrélis (2017a).

The second part of the paper starts with the description of four open questions related to physically relevant extensions of IPDSAW. Among such extensions is the Interacting Prudent Self-Avoiding Walk (IPSAW) whose configurations are those of the 2-dimensional prudent walk. We discuss the main results obtained in Pétrélis and Torri (2016+) about IPSAW and in particular the fact that its collapse transition is proven to exist rigorously.

MSC 2010 subject classifications: Primary 60K35; Secondary 82B26, 82B41.
Keywords and phrases: Polymer collapse, phase transition, Wulff shape, local limit theorem, scaling limit.

* Invited paper to appear in a special volume of the Journal of Physics A, on the occasion of the 75th birthday of Stu Whittington.

Contents

†*Acknowledgements.* P. Carmona, N. Pétrélis and N. Torri thank the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.
*February 27, 2018

1. Introduction

The collapse transition is a well known example of phase transition. It takes place for instance when an homopolymer is dipped in a poor solvent. As the solvent temperature decreases, it reaches a threshold (the θ-point) below which the geometry of a typical polymer configuration changes drastically so that it looks pretty much like a compact ball.

A good mathematical model to investigate this phenomenon is the Interacting Self-Avoiding Walk (see Orr (1947) or Saleur (1986)). In size $L \in \mathbb{N}$, the configurations of ISAW are given by the L-step self-avoiding walk trajectories on $\mathbb{Z}^d$. A Gibbsian weight is assigned to each such configuration as $\beta \in [0, \infty)$ (the interaction intensity) times the number of *self-touchings*, i.e., pairs of sites of the walk adjacent on the lattice though not consecutive along the walk. Among lattice polymer models, the ISAW plays a central role because it fulfills the excluded volume effect, a feature that real world polymers indeed satisfy. However, few mathematical results are available so far, mostly because the mathematical understanding of self-avoiding walks remains fairly incomplete. At the moment, the existence of the free energy is established for small interaction parameter β (first in Ueltschi (2002) for random walk with infinite range step distribution and more recently in Hammond and Helmuth for a larger class of a priori laws on the walk including the simple random walk) but remains open elsewhere. In dimension $d \geq 5$ and for small β, the mean square displacement of ISAW is proven to be of order L (see Ueltschi (2002)) by using lace expansion techniques. There is so far, for $d \geq 2$, no rigorous proof of the existence of a phase transition for ISAW.

The mathematical complexity of ISAW has motivated the introduction of alternative models for self-interacting random walk. The challenge consists in designing models that, on one hand, are sophisticated enough to capture the most important physical features of the collapse phenomenon and, on the other hand, are tractable enough to allow for a deep mathematical investigation. In the physics literature, a lot of attention has been dedicated to exactly solvable models. For instance in Duplantier and Saleur (1987) a two-dimensional polymer model is investigated on the honeycomb lattice. A random environment is introduced by deleting some faces of the lattice in a percolation-type fashion. The edges of the missing faces are prohibited so that, by annealing on the environnement, the resulting model displays attractions between edges. The collapse transition of the model occurs when the deleted faces start to percolate and thanks to this analogy the critical exponents could be computed. Recent works support the idea that such exactly solvable models share common features with ISAW itself at criticality. In this spirit, numerical evidences are displayed in Gherardi (2013) to illustrate the correspondence between the two-dimensional ISAW at criticality and SLE_6 and both theoretical and numerical results are displayed

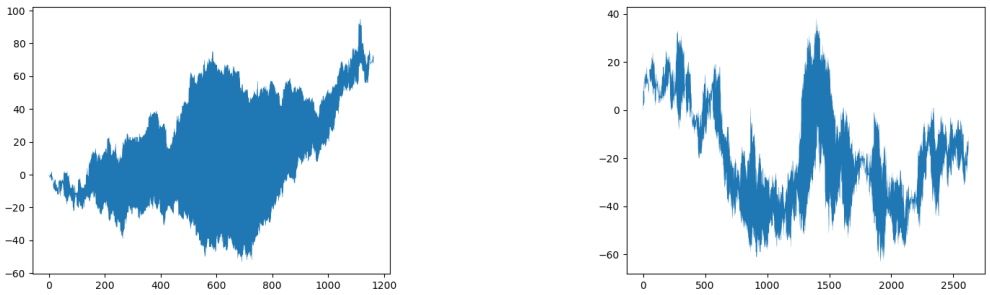


Fig 1: Two exact simulations of IPDSAW at critical temperature $\beta = \beta_c$ and with length $L = 60000$

in Vernier, Jacobsen and Saleur (2015) to try and determine the θ-point of ISAW (we actually refer to the introduction in Vernier, Jacobsen and Saleur (2015) for a concise and very clear state of the art on such exactly solvable models). Let us now focus on the mathematics literature where two other variants of ISAW received most of the attention.

The first of these variants is the Interacting Weakly-Self-Avoiding Walk (IWSAW), introduced in van der Hofstad and Klenke (2001). In size $L \in \mathbb{N}$, the set of allowed configurations for IWSAW is much larger than that of ISAW since it contains every L-step simple random walk trajectory on Z^d. However, the Hamiltonian of an IWSAW trajectory contains an additional term that penalizes the auto-contact, i.e., decreases the Gibbs weight by $-\gamma$ for every self-intersection of the trajectory. The phase diagram of IWSAW is conjectured to be divided into three phases, i.e., localized, collapsed and extended. In van der Hofstad and Klenke (2001), a critical curve $\beta = 2d\gamma$ is proven to separate the localized phase ($\beta > 2d\gamma$) inside which a typical trajectory remains in a box of finite size from the rest of the quadrant. Another critical curve $\gamma \mapsto \beta_c(\gamma)$ is conjectured to exist inside $\{(\gamma, \beta) \in [0, \infty)^2 \colon \beta < 2d\gamma\}$ that separates a collapsed phase where the end to end distance of a typical trajectory should be $L^{1/d}$ from an extended phase where this distance should be of the same order as that of the self-avoiding walk. In the limit $\gamma \to \infty$, it is expected that $\beta_c(\gamma)$ converges to the θ-point of ISAW. Recently, a continuous time version of IWSAW was investigated in Bauerschmidt, Slade and Wallace. In dimension 4, an area of the quadrant is isolated (corresponding to small γ and β) and proven to be part of the extended phase.

The second variant is the Interacting Partially-Directed Self-Avoiding walk (IPDSAW) and was first introduced in Zwanzig and Lauritzen (1968). This is a 2-dimensional model where the set of allowed configurations is narrowed (compared to that of ISAW) but the Hamiltonian remains unchanged. Until recently (see Section 5) the IPDSAW was the only 2-dimensional polymer model for which the collapse transition was rigorously established. It was first studied with transfer matrix methods (see Binder et al. (1990)) and then with combinatorial tools in Brak, Guttman and Whittington (1992) to compute the critical point β_c that partitions the phase diagram into an extended phase $\mathscr{E} := [0, \beta_c)$ and a collapsed phase $\mathscr{C} := [\beta_c, \infty)$.

A new probabilistic approach of IPDSAW has been introduced in Nguyen and Pétrélis (2013) which turned out to strongly simplify its investigation. In the present paper we review the results obtained using this new framework concerning the analytic properties of free energy in Carmona, Nguyen and

Pétrélis (2016) and Pétrélis and Torri (2016+) and also concerning the path properties of IPDSAW in Carmona and Pétrélis (2016) and Carmona and Pétrélis (2017b). For every result that we state here, we provide a sketch of its rigorous proof.

1.1. The model IPDSAW

Mathematical description of the model

The IPDSAW can be defined in two equivalent manners. In the original definition (see Zwanzig and Lauritzen (1968)), the polymer configurations are modeled by the trajectories of a two dimensional self-avoiding walk, taking unitary steps up, down and to the right, whereas in the alternative definition, the configurations are modeled by families of vertical stretches. In Sections 2 and 3, we will use the second definition since it fits with the probabilistic approach that we wish to display. However, in Section 5 we will come back to the original definition to present the IPSAW, a non-directed extension of IPDSAW that has been investigated in Pétrélis and Torri (2016+).

In size L, the allowed configurations of the polymer can be represented as families of oriented vertical stretches, i.e, $\Omega_L := \bigcup_{N=1}^{L} \mathscr{L}_{N,L}$, with

$$\mathscr{L}_{N,L} = \left\{ l \in \mathbb{Z}^N : \sum_{n=1}^{N} |l_n| + N = L \right\}. \tag{1.1}$$

With such configurations, the modulus of a given stretch corresponds to the number of monomers constituting this stretch and two consecutive vertical stretches are separated by one horizontal monomer (see Fig. 2). The repulsion exerted by the solvent on the monomers is taken into account by assigning to every configuration $l \in \Omega_L$ an energetic reward $\beta \in [0, \infty)$ every times it performs a *self-touching* that is every time it places two non-consecutive monomers at distance 1 from each other. By summing those microscopic interactions, we obtain for $N \in \{1, \dots, L\}$ the Hamiltonian of a given configuration $l \in \mathscr{L}_{N,L}$ as

$$H_L(l_1, \dots, l_N) = \sum_{n=1}^{N-1} (l_n \mathbin{\widetilde{\wedge}} l_{n+1}), \tag{1.2}$$

where

$$x \mathbin{\widetilde{\wedge}} y = \begin{cases} |x| \wedge |y| & \text{if } xy < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The preceding Hamiltonian is an exponential Gibbs weight that allows us to define the polymer measure on Ω_L as

$$P_{L,\beta}(l) = \frac{e^{\beta H_L(l)}}{Z_{L,\beta}}, \quad l \in \Omega_L, \tag{1.4}$$

where $Z_{L,\beta}$ is the partition function of the model. Finally, the free energy

$$f(\beta) := \lim_{L \to \infty} \frac{1}{L} \log Z_{L,\beta} \tag{1.5}$$

provides us with the exponential growth rate of the partition function.

1.2. Challenges

We can distinguish between two main types of questions that one tries to address when investigating lPDSAW:

1. *Determine the asymptotic development of the free energy close to criticality.* We will see below that the free energy of IPDSAW is trivial in its collapsed phase, i.e., $f(\beta) = \beta$ when $\beta \geq \beta_c$. Therefore, one wants to exhibit $\gamma, \alpha > 0$ such that
$$\widetilde{f}(\beta_c - \varepsilon) = \alpha \varepsilon^\gamma (1 + o(1)) \quad \text{as} \quad \varepsilon \to 0+, \tag{1.6}$$
with $\widetilde{f}(\beta) := f(\beta) - \beta$ the *excess free energy* of the system. One also expects that α may be expressed as the free energy of a counterpart model built with Brownian trajectories.
2. *Display the scaling limit of IPDSAW in each regime.* Compute the growth speed of the horizontal and vertical extensions of a typical IPDSAW trajectory in the extended phase ($\beta < \beta_c$), inside the collapsed phase ($\beta > \beta_c$) and at criticality ($\beta = \beta_c$). With those typical growth speeds in hand, determine the limiting shape of an appropriately rescaled typical trajectory of IPDSAW.

Section 2 below is dedicated to issues of type 1. With Section 3 we settle entirely the issues of type 2. In Section 4 we list some open problems related to ISAW and with Section 5 we give a first answer to one of them.

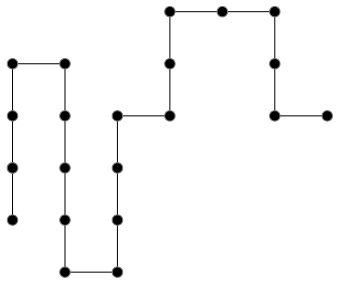


Fig 2: Example of a trajectory $l \in \mathscr{L}_{N,L}$ with $N = 6$ vertical stretches, a total length $L = 20$ and an Hamiltonian $H_L(l) = 6$.

2. Asymptotics of the free energy close to criticality

2.1. A probabilistic representation of the partition function

In Nguyen and Pétrélis (2013), a Random Walk representation of IPDSAW has been introduced (see Section 3 below for more details). With this new technique, a probabilistic expression of the partition function has been derived, i.e.,
$$\widetilde{Z}_{L,\beta} := c_\beta^{-1} Z_{L,\beta} e^{-\beta L} = \sum_{N=1}^{L} \Gamma_\beta^N \mathbf{P}_\beta(\mathscr{V}_{N,L-N}) \quad \text{with} \quad \Gamma_\beta := \frac{c_\beta}{e^\beta}, \tag{2.1}$$
where $\mathbf{P}_\beta$ is the law of a random walk $V := (V_i)_0^\infty$ starting from the origin ($V_0 = 0$) and with Laplace symmetric increments, i.e., $(V_{i+1} - V_i)_{i \geq 0}$ is an i.i.d. sequence of random variables satisfying
$$\mathbf{P}_\beta(V_1 = k) = \frac{e^{-\frac{\beta}{2}|k|}}{c_\beta} \quad \forall k \in \mathbb{Z} \quad \text{with} \quad c_\beta := \tfrac{1+e^{-\beta/2}}{1-e^{-\beta/2}}, \tag{2.2}$$
and where for every $N \in \{1, \dots, L\}$ the set $\mathscr{V}_{N,L-N}$ gathers the $N+1$ step trajectories of the random walk sweeping a geometric area $L-N$ and finishing at 0, i.e.,
$$\mathscr{V}_{N,L-N} := \{V \in \mathbb{Z}^{N+1} : G_N(V) = L - N, V_{N+1} = 0\} \quad \text{with} \quad G_N(V) = \textstyle\sum_{i=0}^N |V_i|. \tag{2.3}$$

The excess free energy $\widetilde{f}(\beta)$ corresponds to the exponential growth rate of $\widetilde{Z}_{L,\beta}$ and therefore can be deduced from the convergence radius of the grand canonical free energy. Using (2.1) we obtain

$$\sum_{L\geq 1}\widetilde{Z}_{L,\beta}e^{-\delta L}=\sum_{N=1}^{\infty}\big(\Gamma_\beta e^{-\delta}\big)^N\mathbf{E}_\beta\big(e^{-\delta\, G_N(V)}1_{\{V_{N+1}=0\}}\big). \tag{2.4}$$

and since $\beta\mapsto\Gamma_\beta$ is decreasing on $[0,\infty)$ and satisfies $\Gamma_0>1$ and $\lim_{\beta\to\infty}\Gamma_\beta=0$ we deduce from (2.4) that the critical point β_c of IPDSAW is the unique solution of $\Gamma_\beta=1$. With Theorem 2.1 below, we derive from (2.4) a simple formulation for the free energy. To that aim we set

$$h_\beta(\delta):=\lim_{N\to\infty}\frac{1}{N}\log\mathbf{E}_\beta\big(e^{-\delta\, G_N(V)}\big),\quad \delta\geq 0. \tag{2.5}$$

Theorem 2.1 (Carmona, Nguyen and Pétrélis (2016), Theorem A). *The excess free energy satisfies*

- $\widetilde{f}(\beta)$ *is the unique* δ*-solution of* $\log(\Gamma_\beta)-\delta+h_\beta(\delta)=0$ *for* $\beta<\beta_c$
- $\widetilde{f}(\beta)=0$ *for* $\beta\geq\beta_c$

Theorem 2.1 draws a tight link between the asymptotics of $\beta\mapsto\widetilde{f}(\beta)$ at β_c^- and the asymptotics of $\gamma\mapsto h_\beta(\gamma)$ at 0^+. This is the key to prove Theorem 2.2 which gives a complete answer to the first challenge raised in Section 1.2 (recall (1.6)).

Theorem 2.2 (Carmona, Nguyen and Pétrélis (2016), Theorem B). *The collapse transition of IPDSAW is second order with critical exponent* 3/2*. Moreover, the first order Taylor development of the excess free energy at* β_c^- *is given by*

$$\lim_{\varepsilon\to 0^+}\frac{\widetilde{f}(\beta_c-\varepsilon)}{\varepsilon^{3/2}}=\Big(\frac{c}{d}\Big)^{3/2}, \tag{2.6}$$

with $\sigma_\beta^2=\mathbf{E}_\beta(V_1^2)$ *and* $c=1+\frac{e^{-\beta_c/2}}{1-e^{-\beta_c}}$*, and with*

$$d=-\lim_{T\to\infty}\frac{1}{T}\log\mathbf{E}\big(e^{-\sigma_{\beta_c}\int_0^T|B(t)|dt}\big)=2^{-1/3}|a_1'|\sigma_{\beta_c}^{2/3}, \tag{2.7}$$

with a_1' *the smallest zero (in absolute value) of the first derivative of Airy function.*

Remark 2.3. *The computation of* $\mathbf{E}(e^{-s\int_0^1|B_s|ds})$ *for* $s>0$ *is due to Kac (1946) (see e.g. Janson (2007)).*

Let us explain in few words how Theorem 2.2 can be deduced from Theorem 2.1. One easily understand that the asymptotic development of $\widetilde{f}(\beta)$ at $(\beta_c)^-$ is strongly related to the fact that there exists a constant $c>0$ such that

$$h_\beta(\delta)=-c\,\delta^{2/3}+o(\delta^{2/3}),\quad \text{as } \delta\to 0^+. \tag{2.8}$$

These asymptotics are obtained by applying a coarse graining argument: we partition the V random walk trajectory into independent blocks, of size $T\delta^{-2/3}$ with $T\in\mathbb{N}$ chosen arbitrarily and δ small enough. Thus, a N-step V trajectory is decomposed into $N/(T\delta^{-2/3})$ blocks that are subsequently used to prove that as $\delta\searrow 0$, we have

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbf{E}_\beta(e^{-\delta G_N})\sim\lim_{T\to\infty}\frac{\delta^{2/3}}{T}\log\mathbf{E}_\beta(e^{-\delta G_{T\delta^{-2/3}}}). \tag{2.9}$$

Donsker's invariance principle ensures (assuming for simplicity $\mathbf{E}_\beta(V_1^2)=1$) (cf (Durrett, 2010, p. 405)) that

$$k^{-3/2}\sum_{i=1}^{Tk}|V_i|\xrightarrow{\text{Law}}\int_0^T|B(t)|dt\quad\text{as } k\to\infty, \tag{2.10}$$

where B is a standard Brownian motion. Thus, we choose $k = \delta^{-2/3}$ in (2.10) and since $|e^{-\delta G_{T\delta^{-2/3}}}| \leq 1$, we conclude that

$$\mathbf{E}_\beta\big(e^{-\delta G_{T\delta^{-2/3}}}\big) \to \mathbf{E}\big(e^{-\sigma_{\beta_c}\int_0^T |B(t)|dt}\big) \quad \text{as } \delta \to 0. \tag{2.11}$$

This last convergence combined with (2.9) implies that $h_\beta(\delta) \sim -c\,\delta^{2/3}$ where c can be expressed using the Laplace transform of the Brownian area, i.e.,

$$c = -\lim_{T\to\infty} \frac{1}{T} \log \mathbf{E}\big(e^{-\sigma_{\beta_c}\int_0^T |B(t)|dt}\big) > 0. \tag{2.12}$$

3. Geometric characterization of IPDSAW

3.1. Random walk representation

As mentioned above, the probabilistic expression of the partition function displayed in (2.1) is obtained after mapping appropriately the trajectories of IPDSAW onto random walk trajectories. Let us be more specific by recalling (1.1) and (2.3) and, for every $N \in \{1,\dots,L\}$, by settling a one-to-one correspondance T_N that maps $\mathscr{V}_{N,L-N}$ onto $\mathscr{L}_{N,L}$, i.e.,

$$T_N(V)_i = (-1)^{i-1} V_i \quad \text{for all} \quad i \in \{1,\dots N\}. \tag{3.1}$$

We note that $\forall x, y \in \mathbb{Z}$ one can write $x \mathbin{\widetilde{\wedge}} y = \frac{1}{2}(|x|+|y|-|x+y|)$ and therefore the partition function defined initially in (1.4) becomes

$$\begin{aligned} Z_{L,\beta} &= \sum_{N=1}^{L} \sum_{\substack{l\in\mathscr{L}_{N,L}\\ l_0=l_{N+1}=0}} \exp\Big(\beta\sum_{n=1}^{N}|l_n| - \tfrac{\beta}{2}\sum_{n=0}^{N}|l_n+l_{n+1}|\Big) \\ &= c_\beta\, e^{\beta L} \sum_{N=1}^{L} \big(\tfrac{c_\beta}{e^\beta}\big)^N \sum_{\substack{l\in\mathscr{L}_{N,L}\\ l_0=l_{N+1}=0}} \prod_{n=0}^{N} \frac{\exp\big(-\frac{\beta}{2}|l_n+l_{n+1}|\big)}{c_\beta}. \end{aligned} \tag{3.2}$$

At this stage, we note that for $l \in \mathscr{L}_{N,L}$ the increments of $(V_i)_{i=0}^{N+1} = (T_N)^{-1}(l)$ in (3.1) satisfy $V_i - V_{i-1} := (-1)^{i-1}(l_{i-1}+l_i)$. Therefore,

$$\widetilde{Z}_{L,\beta} = \sum_{N=1}^{L} \Gamma_\beta^N \sum_{\substack{l\in\mathscr{L}_{N,L}\\ l_0=l_{N+1}=0}} \mathbf{P}_\beta(V = (T_N)^{-1}(l))$$

which implies (2.1).

Another useful consequence of formula (2.1) is that it gives a method to sample IPDSAW trajectories with the help of random walk paths. To be more specific, let us denote by N_l the horizontal extension of a given $l \in \Omega_L$, i.e., $l \in \mathscr{L}_{N_l,L}$. Since in (2.1), the term indexed by N in the summation corresponds to the contribution of $\mathscr{L}_{N,L}$ to the partition function we can state that

$$P_{\beta,L}(N_l = k) = \frac{\Gamma_\beta^k \mathbf{P}_\beta(\mathscr{V}_{k,L-k})}{\sum_{k=1}^{L}\Gamma_\beta^k \mathbf{P}_\beta(\mathscr{V}_{k,L-k})}, \quad k \in \{1,\dots,L\}, \tag{3.3}$$

and that for every $N \in \{1,\dots,L\}$,

$$P_{L,\beta}(l \in \cdot \,|\, N_l = N) = \mathbf{P}_\beta(T_N(V) \in \cdot \,|\, V \in \mathscr{V}_{N,L-N}). \tag{3.4}$$

As a consequence, one can first sample an extension N under $P_{L,\beta}$ with (3.3) and then sample a V trajectory under $\mathbf{P}_\beta$ conditioned on $\mathscr{V}_{N,L-N}$ and finally apply T_N to V to obtain an IPDSAW trajectory. This method can be implemented to simulate long critical IPDSAW trajectory (see Fig. 1).

3.2. Scaling limit of IPDSAW in each regime

To describe geometrically an IPDSAW configuration $l \in \mathscr{L}_{N,L} \subset \Omega_L$, one may consider its *upper envelope* $\mathscr{E}_l^+$ (respectively *lower envelope* $\mathscr{E}_l^-$), namely the random process that links consecutively the top (resp. the bottom) of every vertical stretch constituting l, i.e., $\mathscr{E}_{l,0}^+ = \mathscr{E}_{l,0}^- = 0$ and $\mathscr{E}_{l,N+1}^+ = \mathscr{E}_{l,N+1}^- = l_1 + \cdots + l_N$ and

$$\begin{aligned} \mathscr{E}_{l,i}^+ &= \max\{l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i\}, \quad i \in \{1, \ldots, N\}, \\ \mathscr{E}_{l,i}^- &= \min\{l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i\}, \quad i \in \{1, \ldots, N\}. \end{aligned} \tag{3.5}$$

Since a given configuration l sampled from $P_{L,\beta}$ fills entirely the subset of $\mathbb{N} \times \mathbb{Z}$ trapped in-between those two envelopes, the scaling limit of IPDSAW (as its length L diverges) is obtained by determining the limiting law of $(\mathscr{E}_l^-, \mathscr{E}_l^+)$ rescaled in time and space appropriately.

Another geometric description of $l \in \Omega_L$ can be made by considering two auxiliary processes, i.e., the *profile* $|l| := (|l_i|)_{i=0}^{N+1}$ (with $l_0 = l_{N+1} = 0$ by convention) and the center-of mass walk $M_l := (M_{l,i})_{i=0}^{N+1}$ that links the middle of each stretch consecutively, i.e., $M_{l,0} = 0$ and $M_{l,N+1} = l_1 + \cdots + l_N$ and

$$M_{l,i} = l_1 + \cdots + l_{i-1} + \frac{l_i}{2}, \quad i \in \{1, \ldots, N\}. \tag{3.6}$$

Working with $(\mathscr{E}_l^-, \mathscr{E}_L^+)$ or with $(|l|, M_l)$ turns out to be equivalent since $\mathscr{E}_L^+ = M_l + \frac{|l|}{2}$ and $\mathscr{E}_L^- = M_l - \frac{|l|}{2}$. For simplicity, our results will be displayed with $(|l|, M_l)$ because asymptotically the profile and the center-of-mass always decorrelate.

We define a scaling operator $T_{\alpha,\beta}$ which rescales simultaneously the profile and the center-of-mass walk by L^α in time and by L^β in space, i.e.,

$$T_{\alpha,\beta}(l) = \frac{1}{L^\beta}\Big(M_{l,\lfloor tL^\alpha \rfloor \wedge N_l}, |l_{\lfloor tL^\alpha \rfloor \wedge N_l}|\Big)_{t \in [0,\infty]}. \tag{3.7}$$

Before stating Theorem 3.1 below, we recall that $\sigma_\beta^2 = \mathbf{E}_\beta(V_1^2)$ (see 2.2). Let us also say that in Theorem 3.1 the convergences occur in distribution for cadlag functions on $[0, \infty)$ endowed with the distance of uniform convergence on every compact subset of $[0, \infty)$. For simplicity, all processes in the statement of Theorem 3.1 have a finite time horizon but we implicitly consider that they remain constant afterwards and therefore are defined on $[0, \infty)$. Theorem 3.1 gathers results from (Carmona, Nguyen and Pétrélis, 2016, Theorem D) and (Carmona and Pétrélis, 2016, Theorem 2.8) and (Carmona and Pétrélis, 2017a, Theorem C).

Theorem 3.1. *For $L \in \mathbb{N}$, we consider an IPDSAW trajectory l sampled from $P_{L,\beta}$. Then,*

(1) if $\beta < \beta_c$,

$$\lim_{L\to\infty} T_{1,\frac{1}{2}}(l) = \alpha_\beta \left(B_s, 0\right)_{s \in [0, e_\beta)}, \tag{3.8}$$

with $e_\beta \in (0,1)$ and $\alpha_\beta > 0$ two explicit constants.

(2) if $\beta = \beta_c$,

$$\lim_{L\to\infty} T_{\frac{2}{3},\frac{1}{3}}(l) = \left(D_s, |B_s|\right)_{s \in [0, a_1]}, \tag{3.9}$$

with B and D two independent linear Brownian motions of variance $\frac{1}{4}\sigma_\beta^2$ and σ_β^2 respectively, with a_1 the time at which the geometric area swept by B reaches 1, i.e., $\int_0^{a_1} |B_u|\, du = 1$ and with B conditioned on the event $B_{a_1} = 0$.

(3) If $\beta > \beta_c$,

$$\lim_{L\to\infty} T_{\frac{1}{2},\frac{1}{2}}(l) = \left(0, \gamma_\beta(s)\right)_{s \in [0, a_\beta]}, \tag{3.10}$$

with a_β an explicit constant and γ_β a deterministic Wulff shape given by

$$\gamma_\beta(s) = a_\beta \int_0^s L'\big[(\tfrac{1}{2} - \tfrac{x}{a_\beta})\widetilde{h}_0\big(\tfrac{1}{a_\beta^2}, 0\big)\big]dx, \quad s \in [0, a_\beta] \tag{3.11}$$

with

$$a_\beta = \arg\max\big\{a \log \Gamma(\beta) - \tfrac{1}{a}\widetilde{h}_0\big(\tfrac{1}{a^2}, 0\big) + aL_\Lambda\big(\widetilde{H}\big(\tfrac{1}{a^2}, 0\big)\big),\ a \in (0, \infty)\big\}, \tag{3.12}$$

where

$$L_\Lambda(H) := \int_0^1 \log \mathbf{E}_\beta[e^{(xh_0+h_1)V_1}]dx, \quad H \in \mathscr{D}$$

with $L(h) := \log \mathbf{E}_\beta[e^{hV_1}]$ for $h \in (-\frac{\beta}{2}, \frac{\beta}{2})$ and $\mathscr{D} := \big\{H = (h_0, h_1)\colon \{h_0, h_0 + h_1\} \subset (-\frac{\beta}{2}, \frac{\beta}{2})\big\}$ and with $\widetilde{H} = (\widetilde{h}_0, \widetilde{h}_1)$ the inverse function of $\nabla L_\Lambda(H)$ that is a $\mathscr{C}^1$ diffeomorphism from $\mathscr{D}$ to $\mathbb{R}^2$.

With Theorem 3.1 we observe that the critical regime is characterized by the fact that the profile and center-of mass walk of a typical IPDSAW configuration display fluctuations of the same order (i.e., $L^{1/3}$). This is indeed not the case in the extended regime ($\beta < \beta_c$) and inside the collapsed regime ($\beta > \beta_c$) for different reasons.

When $\beta < \beta_c$ the self-interaction intensity is weak and therefore the qualitative behavior of a typical IPDSAW trajectory is not different from that of the random walk under its uniform measure (i.e., $\beta = 0$). To be more specific, the horizontal extension of a typical trajectory is of order L and the vertical stretches are typically of finite size. We will even see in the proof below that the vertical stretches have an exponential tail. As a consequence the profile vanishes when rescaling it in space by any function growing say faster than $\log L$ whereas the center-of-mass walk asymptotically decorrelates from the profile and displays Brownian fluctuations.

When $\beta > \beta_c$, in turn, a typical IPDSAW trajectory performs $L(1 + o(1))$ self-touchings (saturation) and therefore must be made of few large vertical stretches with alternating signs. As a consequence the horizontal extension and the vertical stretches of a typical configuration are both of order $\sqrt{L}$. This strong geometric constraint forces the profile rescaled in time and space by $\sqrt{L}$ to converge towards a deterministic Wulff shape (a sketch of the proof is displayed below). The rescaled center-of mass walk vanishes in the limit (3.10). The reason is that, the center-of-mass walk asymptotically decorrelates from the profile and therefore follows the law of a symmetric random walk of length $\sqrt{L}$ with vertical fluctuation of order $L^{1/4}$.

Remark 3.2. Appart from the extended case, the proof of Theorem 3.1 heavily relies on formulas (3.3–3.4) which allows us to work with random walk trajectories under a particular conditioning and subsequently to re-express the results in terms of IPDSAW via the applications T_N with $N \le L$ (recall 3.1).

Let us now give the main steps of the proof of Theorem 3.1 in each regime, starting with the collapsed phase.

Collapsed regime $\beta > \beta_c$.

Rewrite (2.1) as

$$\widetilde{Z}_{L,\beta} = \sum_{N=1}^{L} \exp\big(N\big[\log \Gamma_\beta + \tfrac{1}{N}\log \mathbf{P}_\beta(\mathscr{V}_{N,L-N})\big]\big). \tag{3.13}$$

There are two growth rates of N (as a function of L) for which $\mathbf{P}_\beta(\mathscr{V}_{N,L-N})$ has a non trivial exponential decay rate (as a function of N), namely, $N \sim L$ and $N \sim \sqrt{L}$. Therefore, and since $\Gamma_\beta < 1$, the sum in (2.1) is dominated by those terms indexed by $a\sqrt{L}$ with $a \in (0, \infty)$. Consequently, we set

$$g_\beta(u) := \lim_{N\to\infty} \mathbf{P}_\beta(G_N(V) = uN^2, V_N = 0), \quad u \in (0, \infty), \tag{3.14}$$

and $\widetilde{Z}_{L,\beta}$ is well approximated by

$$\sum_{a\in\frac{\mathbb{N}}{\sqrt{L}}} \exp\big(\sqrt{L}\big[a\log\Gamma_\beta + a g_\beta(\tfrac{1}{a^2})\big]\big)$$

so that a_β indeed equals $\arg\max\big\{a\log\Gamma_\beta + a g_\beta(\frac{1}{a^2}),\, a\in(0,\infty)\big\}$.

At this stage, proving (3.12) simply requires to provide an analytic expression of g_β. To that aim, for $u>0$ we observe that $\{G_N(V)=uN^2, V_N=0\}$ is a large deviation event. Its decay rate can indeed be expressed with J the rate function of Mogulskii Theorem applied to the rescaled process $\widetilde{V}_N := \big(\frac{1}{N}V_{\lfloor sN\rfloor}\big)_{s\in[0,1]}$ viewed as a random element in $\mathfrak{B}_{[0,1]}$ the set of cadlag real functions on $[0,1]$, endowed with the L^∞ norm. Thus $J:\mathfrak{B}_{[0,1]}\to[0,\infty]$ is defined as

$$J(\gamma)=\begin{cases}\displaystyle\int_0^1 L^*(\gamma'(t))dt & \text{if}\quad \gamma\in\mathscr{AC},\\ +\infty & \text{otherwise,}\end{cases} \tag{3.15}$$

where $\mathscr{AC}$ is the set of absolutely continuous functions and where L^* is the Legendre transform of L. Rewritting $\{G_N(V)=uN^2, V_N=0\}$ as $\{G(\widetilde{V}_N)=u, \widetilde{V}_N(1)=0\}$ (with $G(\gamma)=\int_0^1|\gamma(s)|ds$) and applying Mogulskii Theorem in (3.14) we obtain

$$g_\beta(u)=\inf\big\{J(\gamma),\,\gamma\in\mathfrak{B}_{[0,1]},\, G(\gamma)=u,\gamma(1)=0\big\} \tag{3.16}$$

from which we derive the closed formula $g_\beta(u)=-u\widetilde{h}_0\big(u,0\big)+L_\Lambda\big(\widetilde{H}\big(u,0\big)\big)$. The proof of (3.12) is therefore complete and it remains to prove (3.17) by observing that the infimum in (3.16) for $u=1/a_\beta^2$ is attained for $-\gamma^*_\beta$ and γ^*_β defined as

$$\gamma^*_\beta(s)=\int_0^s L'\big[(\tfrac12-x)\widetilde{h}_0\big(\tfrac{1}{a_\beta^2},0\big)\big]dx,\quad s\in[0,1], \tag{3.17}$$

so that γ_β simply satisfies $\gamma_\beta(s)=a_\beta\gamma^*_\beta(s/a_\beta)$ for $s\in[0,a_\beta]$.

Critical regime $\beta=\beta_c$

The critical regime is the most delicate since the fluctuations of $|l|$ and M_l are of the same magnitudes and must therefore be analyzed simultaneously.

A few more notations are required here. With V a random walk trajectory and with $j,k\in\mathbb{N}$ we associate $K_j=j+G_j(V)$ (recall 2.3) and $\xi_k:=\inf\{j\ge1\colon K_j\ge k\}$. We also associate with V an auxiliary process $M:=(M_i)_{i\in\mathbb{N}}$ build with the increments of V as follows: $M_0=0$ and for $j\in\mathbb{N}$

$$M_j:=\sum_{i=1}^{j-1}(-1)^{i-1}V_i+(-1)^{j-1}\frac{V_j}{2}=\frac12\sum_{i=1}^{j}(-1)^{i-1}(V_i-V_{i-1}). \tag{3.18}$$

Since $\Gamma_{\beta_c}=1$ the key tool here is the random walk representation (3.3–3.4) which guarantees that

$$P_{L,\beta_c}(l\in\cdot)=\mathbf{P}_\beta(T_{\xi_L}(V)\,|\,K_{\xi_L}=L, V_{\xi_L+1}=0). \tag{3.19}$$

A consequence of (3.19) is that $T_{\frac12,\frac12}(l)$ with l sampled from P_{L,β_c} has the same law as $(|\widetilde{V}|,\widetilde{M}) := (|\widetilde{V}_s|,\widetilde{M}_s)_{s\in[0,\infty)}$ defined as

$$(\widetilde{V}_s,\widetilde{M}_s)=\frac{1}{L^{1/3}}(V_{\lfloor sL^{2/3}\rfloor\wedge\xi_L}, M_{\lfloor sL^{2/3}\rfloor\wedge\xi_L}) \tag{3.20}$$

where V is sampled from $\mathbf{P}_{\beta_c}$ and conditioned on $\{K_{\xi_L} = L, V_{\xi_L+1} = 0\}$. Thus, Theorem 3.1 (b) can be proven by considering $(|\widetilde{V}|, \widetilde{M})$.

Outline of the proof. The strategy used in Carmona and Pétrélis (2017a) consists in decomposing every V trajectory into excursions $(\mathfrak{E}_r)_{r\in\mathbb{N}}$ away from the origin. The fact that the increments of V follow a symmetric discrete Laplace distribution yields that those excursions (in modulus) $(|\mathfrak{E}|_r)_{r\in\mathbb{N}}$ are independent and have the same distribution (except for the very first one). The conditioning $\{K_{\xi_L} = L, V_{\xi_L+1} = 0\}$ under which V is considered gives a particular importance to the geometric areas $(X_r)_{r\in\mathbb{N}}$ swept by the excursions. These areas are i.i.d. and heavy tailed random variables so that it suffices to consider finitely many excursions (those sweeping the largest area) to recover a fraction of the path arbitrary close to 1. For this reason, for $k \in \mathbb{N}$, we will truncate $(|\widetilde{V}|, \widetilde{M})$ outside the excursions sweeping an area larger than L/k. Since finitely many excursions of V (the largest ones) are required to reconstruct the truncated process $(|\widetilde{V}_{L,k}|, \widetilde{M}_{L,k})$, it should be sufficient to prove a convergence in distribution "excursion by excursion" to recover the convergence of the whole truncated process. Then, it remains to control the fluctuations of V and M on the small excursions of V in order to check that their contributions to the limiting process vanish as $k \to \infty$.

Let us be more specific and define the stopping times $(\tau_r)_{r\in\mathbb{N}}$ by the prescription $\tau_0 = 0$ and

$$\tau_{r+1} = \inf\{i > \tau_r : V_{i-1} \neq 0 \text{ and } V_{i-1}V_i \leq 0\}. \tag{3.21}$$

For every $r \in \mathbb{N}$ we denote by $|\mathfrak{E}|_r$ the r-th excursion of V in modulus, i.e.,

$$|\mathfrak{E}|_r = (i, |V_i|)_{i\in\{\tau_{r-1},\dots,\tau_r-1\}}, \tag{3.22}$$

and it turns out (see (Carmona and Pétrélis, 2017a, Proposition 3.1)) that provided we transform slightly the law of V_0, the sequence $(|\mathfrak{E}|_r)_{r\geq 1}$ is i.i.d. We introduce for every $r \in \mathbb{N}$ the sum X_r of the length and of the geometric area swept by $|\mathfrak{E}|_r$, i.e.,

$$X_r := \tau_r - \tau_{r-1} + \left|V_{\tau_{r-1}}\right| + \cdots + \left|V_{\tau_r-1}\right|. \tag{3.23}$$

With a slight abuse of notation, we will call X_r the geometric area swept by the r-th excursion and we define $\mathfrak{X}$ a random set of points on $\mathbb{N}_0$ as

$$\mathfrak{X} = \{0\} \cup \{X_1 + \cdots + X_n, n \in \mathbb{N}_0\}. \tag{3.24}$$

For simplicity, we transform the conditioning under which $(|\widetilde{V}|, \widetilde{M})$ is considered into $\{L \in \mathfrak{X}\}$. This does not change the scaling limit of $(|\widetilde{V}|, \widetilde{M})$ and lightens the presentation of the proof. Under the conditioning $\{L \in \mathfrak{X}\}$ we denote by ν_L the number of excursions completed by V when its geometric area reaches L.

Remark 3.3. A crucial result at this stage is that X_1 is heavy tailed. Deriving a local limit theorem for the geometric area swept by a random walk excursion (say with centered increments that have finite second moments) was an open issue until recently. The reason is that computing the characteristic function of such geometric area is difficult and therefore Gnedenko's type arguments can not be applied straightforwardly. In (Denisov, Kolb and Wachtel, 2015, Theorem 1.1), such a local limit theorem has been derived giving us $\lim_{n\to\infty} n^{4/3}\mathbf{P}_{\beta_c}(X_1 = n) = c_1$ and $\lim_{L\to\infty} L^{2/3}\mathbf{P}_{\beta_c}(L \in \mathfrak{X}) = c_2$ with $c_1, c_2 > 0$. Thus, by recalling 2.1, we obtain sharp asymptotics for the critical partition function, i.e.,

$$Z_{L,\beta_c} = e^{\beta_c L}\frac{c_3}{L^{2/3}}(1 + o(1)), \quad \text{with } c_3 > 0 \text{ explicit.}$$

Truncation of the profile and center-of-mass walk. As mentioned in the outline, since the variables $(X_r)_{r\geq 1}$ are heavy tailed, we truncate $(|\widetilde{V}|, \widetilde{M})$ outside the excursions sweeping an area larger than L/k. We recall (3.18) and (3.21) and for every $r \in \mathbb{N}$, we let $M^{\text{exc}}(r)$ be the contribution of the r-th excursion to the center-of-mass walk, i.e.,

$$M^{\text{exc}}(r) = \sum_{i=\tau_{r-1}}^{\tau_r-1} (-1)^{i-1} V_i. \tag{3.25}$$

For $x \in \mathbb{N}$, we truncate V outside the excursions of geometric area larger than x to obtain $(V_x^+(i))_{i\in\mathbb{N}\cup\{0\}}$. Similarly, with the help of (3.25) we define the discrete process $(M_x^+(i))_{i\in\mathbb{N}\cup\{0\}}$ which remains constant outside the excursions of geometric area larger than x and follows the center-of-mass walk elsewhere, i.e., for every $t \in \mathbb{N}$ and $i \in \{\tau_{t-1}, \dots, \tau_t - 1\}$

$$\begin{aligned} M_x^+(i) &:= \sum_{r=1}^{t-1} M^{\text{exc}}(r)\, 1_{\{X_r \geq x\}} + \Big[\sum_{j=\tau_{t-1}}^{i-1} (-1)^{j-1} V_j + (-1)^{i-1} \frac{V_i}{2} \Big] 1_{\{X_t \geq x\}}, \\ V_x^+(i) &:= V_i\, 1_{\{X_t \geq x\}}. \end{aligned} \tag{3.26}$$

For $k \in \mathbb{N}$, the truncated processes $\widetilde{V}_{L,k}$ and $\widetilde{M}_{L,k}$ are obtained from $V_{L/k}^+$ and $M_{L/k}^+$ as in (3.20).

Truncation of Brownian motion. We recall the definition of B and D in the statement of Theorem 3.1 (b). As in the discrete case, we truncate B and D outside the excursions of B sweeping a geometric area larger than $1/k$ to obtain B_k^+ and D_k^+ , i.e.,

$$\begin{aligned} D_k^+(s) &= \int_0^s 1_{\Gamma_k}(u)\, dD_u, \\ B_k^+(s) &= B_s\, 1_{\Gamma_k}(A_s), \end{aligned} \tag{3.27}$$

where $A_s := \int_0^s |B_s| ds$ is the geometric area swept by B up to time s and where $\Gamma_k := \left\{u > 0 : A_{d_u} - A_{g_u} \geq \frac{1}{k}\right\}$ with $d_u = d_u(B) := \inf\{t > u : B_t = 0\}$, $g_u = \sup\{t < u : B_t = 0\}$ so that $d_u - g_u$ (resp. $A_{d_u} - A_{g_u}$) is the length (resp. the geometric area) of the excursion straddling u.

The proof of Theorem (b) now consists in proving (3.28–3.30) below. To begin with we must control the fluctuations of V and M outside the large excursions of V, i.e., prove that for every $\varepsilon > 0$

$$\lim_{k\to\infty} \limsup_{L\to\infty} \mathbf{P}_{\beta_c}\Big(\sup_{i\leq \xi_L} \big|V_i - V_{L/k}^+(i)\big| + \big|M_i - M_{L/k}^+(i)\big| \geq \varepsilon L^{1/3} \,\Big|\, L \in \mathfrak{X} \Big) = 0, \tag{3.28}$$

and similarly for the fluctuations of B and D outside the large excursions of B, i.e.,

$$\lim_{k\to\infty} \mathbb{P}\Big(\sup_{s\leq a_1} \big|B_s - B_k^+(s)\big| + \big|D_s - D_k^+(s)\big| \geq \varepsilon \Big) = 0. \tag{3.29}$$

Then, we must show that for every $k \in \mathbb{N}$ the truncated discrete process $(\widetilde{V}_{L,k}, \widetilde{M}_{L,k})$ converge in distribution towards its continuous counterparts (B_k^+, D_k^+), i.e., for every $k \in \mathbb{N}$,

$$\big(\widetilde{V}_{L,k}, \widetilde{M}_{L,k}\big) \xrightarrow[L\to\infty]{d} (B_k^+(s), D_k^+(s))_{s\in[0,a_1]}. \tag{3.30}$$

The proof of (3.28–3.29) is displayed in (Carmona and Pétrélis, 2017a, Sections 5.2 and 5.3). The difficult part consists in controlling the fluctuations of the discrete center-of-mass walk M outside the large excursions of V $\big($i.e., of $M - M_{L/k}^+\big)$. The reason is that at the end of every excursion, i.e. at τ_k for $k \geq 1$, the V trajectory is located very close to the interface, and therefore controlling the fluctuations of $V - V_{L/k}^+$ requires a good control of the maximum of V on each of its small excursions. However, this is not the case for the center-of-mass walk, since M_{τ_k} has no reason to be near the origin. For this reason we must not only control the fluctuations of M inside every small excursions of V but also control the fluctuations of the discrete process of increments $\big((M_{\tau_r} - M_{\tau_{r-1}})\, 1_{\{X_r \leq L/k\}},\ r \leq \nu_L\big)$.

The proof of (3.30) is a reconstruction procedure displayed in (Carmona and Pétrélis, 2017a, Section 5.1). For every $k \in \mathbb{N}$, it consists in constructing on the same probability space a sequence of processes $(Y_L, Z_L)_{L\geq 1}$ and two independent Brownian motions B and D (with B conditioned on $B_{a_1} = 0$) so that for every $L \in \mathbb{N}$ the law of (Y_L, Z_L) equals that of $(\widetilde{V}_{L,k}, \widetilde{M}_{L,k})$ (with V sampled from $\mathbf{P}_{\beta_c}(\cdot \mid L \in \mathfrak{X})$) and (Y_L, Z_L) converges almost surely towards (B_k^+, D_k^+) as $L \to \infty$. To perform this reconstruction we need

two auxiliary convergence results. First, we recall (3.24) and for every $L \in \mathbb{N}$ we sample $\mathfrak{X}$ from $\mathbf{P}_\beta(\cdot \mid L \in \mathfrak{X})$. Then, the tail estimates of X_1 (see Remark 3.3) yields that $\frac{1}{L}\mathfrak{X} \cap [0, L]$ converges in law (in the space of closed subsets of $[0,1]$ endowed with the Hausdorff distance) towards $C_{1/3} \cap [0,1]$ conditioned on $1 \in C_{1/3}$ where $C_{1/3}$ is the 1/3-stable regenerative set. Second, we use (Carmona and Pétrélis, 2017b, Theorem A) which yields that, when considering an excursion of the V random walk conditioned on sweeping a prescribed geometric area L, the excursion itself and its associated center-of-mass walk, both rescaled in time by $L^{2/3}$ and in space by $L^{1/3}$ converge in distribution towards $(e_s, D_s)_{s\in[0,a_1]}$ where e is a Brownian excursion normalized by its area and D an independent Brownian motion.

Extended regime $\beta < \beta_c$

The extended regime is somehow the simplest to analyze and was considered in (Carmona and Pétrélis, 2016, Section 6). The technique consists in partitioning every IPDSAW configuration $l \in \Omega_L$ into $p(l) \in \mathbb{N}$ elementary patterns that do not interact with each other. A pattern is a path whose first zero length vertical stretch occurs only at the end of the path. Thus, for $l \in \Omega_L$ we set $T_0 = 0$ and $T_k(l) = \inf\{j \geq 1 + T_{k-1} : l_j = 0\}$ for $k \in \{1, \dots, p(l)\}$, so that the total length of the k-th pattern in l is

$$\sigma_k := T_k - T_{k-1} + \textstyle\sum_{i=T_{k-1}+1}^{T_k} |l_i|. \tag{3.31}$$

For $L \in \mathbb{N}$ we denote by $\widehat{Z}_{L,\beta}$ the contribution to the excess partition function $\widetilde{Z}_{L,\beta}$ (recall 2.1) of those trajectories that are made of exactly one pattern, i.e., satisfying $\sigma_1(l) = L$. For $\alpha \geq 0$ the moment generating function associated with $(\widehat{Z}_{L,\beta})_{L\in\mathbb{N}}$ is

$$\phi(\alpha) := \sum_{L\geq 1} \widehat{Z}_{L,\beta} e^{-\alpha L} \in]0, +\infty] \tag{3.32}$$

and the convergence abcissa $\hat{f}(\beta) := \inf\{\alpha : \phi(\alpha) < +\infty\}$. A key observation at this stage is the link between ϕ and $\widetilde{f}(\beta)$. It can be proven, using for instance the random walk representation, that in the extended phase we have $0 < \hat{f}(\beta) < \widetilde{f}(\beta)$ and moreover that $\phi(\widetilde{f}(\beta)) = 1$. This allows us to define the probability K on $\mathbb{N}$ as

$$K(n) = \widehat{Z}_{n,\beta}\, e^{-\widetilde{f}(\beta) n}, \quad n \in \mathbb{N}, \tag{3.33}$$

and to prove that K has an exponential tail. At this stage, a classical algebraic manipulation of the partition function $\widetilde{Z}_{L,\beta}$ (see (Giacomin, 2007, Section 1.2.1)) allows us to show that when l is sampled from $P_{L,\beta}$, the random variables $\{\sigma_i(l), i \in \{1, \dots, p(l)\}\}$ are i.i.d. (and conditioned on $\sigma_1 + \cdots + \sigma_{p(l)} = L$) with law K. As a consequence, a typical IPDSAW trajectory in the extended phase is made of $O(L)$ patterns of finite size. This yields that the vertical stretches of such configurations have length $O(1)$ and therefore explains why the rescaled profile in Theorem 3.1 (a) converges to 0. The Brownian limit of the center-of-mass walk in turn, is easily deduced from the fact that, because patterns do not interact energetically, their vertical displacements are i.i.d. random variables, centered for obvious symmetry reasons and with a finite second moment because the vertical displacement of a pattern is bounded by its total size.

3.3. More on the collapsed phase: uniqueness of the macroscopic bead.

Another meaningful manner of describing the geometry of IPDSAW consists in dividing its trajectories into beads. More precisely, a bead is made of vertical stretches of strictly positive length and arranged in such a way that two consecutive stretches have opposite directions (north and south) and are separated by one horizontal step (see Fig. 3). A bead ends when the polymer gives the same direction to two consecutive vertical stretches or when a zero length stretch appears.

Let us define beads rigorously. We pick $l \in \mathscr{L}_{N,L}$ we set $x_0 = 0$ and

$$x_j = \inf\{i \geq x_{j-1} + 1 :\ l_i \barwedge l_{i+1} = 0\} \quad \text{for} \quad j \geq 1.$$

We let $n(l)$ be the index of the last x_j that is well defined, so that l is partitioned into $\cup_{j=1}^{n(l)} B_j$ where the j-th bead B_j is defined as

$$B_j := \{l_{x_{j-1}+1}, \dots, l_{x_j}\}, \quad j \le n(l). \tag{3.34}$$

Determining the number of beads in a typical configuration is an interesting question raised for instance in Brak et al. (1993). The answer is fairly easy in the extended regime by using the patterns introduced in (3.31). Such a pattern contains at least one bead and since a typical trajectory consists of $O(L)$ patterns the same remains true for the beads. At criticality, the typical number of beads is $L^{1/3}$. This can be understood with the help of the random walk representation (recall (3.19)). Every bead of a given $l \in \mathscr{L}_{N,L}$ is indeed associated with an excursion of the associated random walk trajectory $V = (T_N)^{-1}(l)$. Thus the number of beads in a critical IPDSAW trajectory is also the number of excursions in a V trajectory sweeping a geometric area L and therefore with length $L^{2/3}$ (recall Theorem 3.1 (b)). At this stage, the fact that a V trajectory of length $L^{2/3}$ makes order $L^{1/3}$ excursions is sufficient to conclude. In the collapsed regime, the typical number of beads was expected to be small. This has been made rigorous with the following result which states that a typical IPDSAW configuration is almost completely trapped inside a *unique macroscopic bead*.

For $l \in \mathscr{L}_{N,L}$ and $j \le n(l)$ we let $I_j := |l_{x_{j-1}+1}| + \cdots + |l_{x_j}| + x_j - x_{j-1}$ be the size of the j-th bead of l (its number of monomers). We also set $j_{\max} = \arg\max\{|I_j|, j \le n_L(l)\}$ so that the size of the largest bead of l is $I_{j_{\max}}$.

Theorem 3.4 (Carmona, Nguyen and Pétrélis (2016), Theorem C). *For $\beta > \beta_c$, there exists a $c > 0$ such that*

$$\lim_{L\to\infty} P_{L,\beta}\big(I_{j_{max}} \ge L - c\,(\log L)^4\big) = 1. \tag{3.35}$$

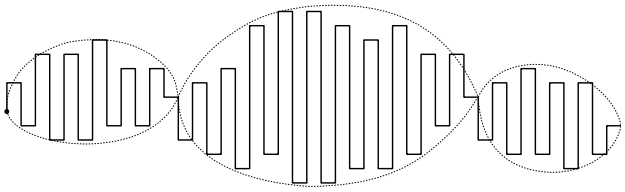


Fig 3: Example of a trajectory with 3 beads.

Theorem 3.4 can be deduced from Propositions 3.5 and 3.6 below. Proposition 3.5 gives bounds on the partition function restricted to those trajectories describing one bead only. We will discuss the proof of Proposition 3.5 at the end of the present section. Proposition 3.6 shows that the horizontal extension of a typical configuration inside the collapse phase is bounded above by $a\sqrt{L}$ for some $a > 0$. This has been discussed already in (3.13– 3.14).

The subset of Ω_L containing the one bead trajectories is $\Omega_L^\circ := \bigcup_{N=1}^{L} \mathscr{L}_{N,L}^\circ$, with

$$\mathscr{L}_{N,L}^\circ = \{l \in \mathscr{L}_{N,L} : \, l_i \,\widetilde{\wedge}\, l_{i+1} \neq 0 \ \forall i \in \{1, \dots, N-1\}\}, \tag{3.36}$$

and its contribution to the partition function is $Z_{L,\beta}^\circ := \sum_{l\in\Omega_L^\circ} e^{\beta H_L(l)}$.

Proposition 3.5 (Carmona, Nguyen and Pétrélis (2016), Proposition 4.2). *For $\beta > \beta_c$, there exist $c, c_1, c_2 > 0$ and $\kappa > 1/2$ such that*

$$\frac{c_1}{L^\kappa}\, e^{\beta L - c\sqrt{L}} \le Z_{L,\beta}^\circ \le \frac{c_2}{\sqrt{L}}\, e^{\beta L - c\sqrt{L}}, \quad L \in \mathbb{N}. \tag{3.37}$$

Proposition 3.6 (Carmona, Nguyen and Pétrélis (2016), Lemma 4.1). *For $\beta > \beta_c$, there exist $a, a_1, a_2 > 0$ such that*

$$P_{L,\beta}(N_L(l) \geq a_1\sqrt{L}) \leq a_2\, e^{-a\sqrt{L}}, \quad L \in \mathbb{N}. \tag{3.38}$$

Propositions 3.5 and 3.6 are sufficient to prove Theorem 3.4. The first step consists in showing that there exists an $s > 0$ such that a typical configuration has exactly one bead larger than $s(\log L)^2$. To that aim, we use Proposition 3.5 combined with the inequality $\sqrt{x} + \sqrt{y} - \sqrt{x+y} \geq \frac{1}{2}\sqrt{x \wedge y}$ to assert that a typical trajectory has at most one bead larger than $s(\log L)^2$. Then, we note that each bead contains at least one horizontal step. Therefore a configuration that has no bead larger than $s(\log L)^2$ has at least $L/s(\log L)^2$ horizontal steps and can not be typical because it would contradict Proposition 3.6.

The second step consists in bounding above the number of monomers that do not belong to the unique big bead. We denote by x_1 (resp. x_2) the number of monomers before (resp. after) this big bead and we assume for instance that $x_1 \geq c(\log L)^4$. There are at least $x_1/s(\log L)^2$ beads between 0 and x_1 and consequently at least $x_1/s(\log L)^2$ horizontal steps. The fact that x_1 is the end of a bead allows us to split the path at x_1 and to focus on the trajectory between 0 and x_1. Applying Proposition 3.6 with $L = x_1$ yields that, typically, less than $a_1\sqrt{x_1}$ of the first x_1 steps are horizontal. This provides a contradiction because, by choosing c large enough, the inequality $x_1 \geq c(\log L)^4$ yields that $a_1\sqrt{x_1} = o(x_1/s(\log L)^2)$. The proof of Theorem 3.4 is complete.

Let us give some insights concerning the proof of Proposition 3.5. The key tool here is the link between beads of IPDSAW and excursions of the V random walk. To be more specific, for $k \leq n$, we let $\mathcal{V}^+_{n,k}$ be the subset containing those positive excursions of the V random walk, returning to the origin after n steps, and sweeping an area k, i.e.,

$$\mathcal{V}^+_{n,k} := \{V :\, V_n = 0,\, G_n = k,\, V_i > 0\ \forall i \in \{1, \dots, n-1\}\}. \tag{3.39}$$

By mimicking (2.1) and by noticing that the T_N-transformation is a one-to-one correspondence between $\mathcal{V}^+_{N+1,L-N}$ and $\mathcal{L}^\circ_{N,L}$ we obtain that

$$\widetilde{Z}^\circ_{L,\beta} := \frac{1}{2c_\beta}\, e^{-\beta L} Z^\circ_{L,\beta} = \sum_{N=1}^{L} \Gamma^N_\beta \mathbf{P}_\beta(\mathcal{V}^+_{N+1,L-N}). \tag{3.40}$$

As discussed in (3.13–3.14) the sum in (3.40) is dominated by those terms indexed by $N \sim a\sqrt{L}$. Therefore, proving Proposition 3.5 requires the derivation of some sharp bounds on $\mathbf{P}_\beta(\mathcal{V}^+_{n,an^2})$ for $a > 0$ and $n \in \mathbb{N}$. By using tilting techniques from Dobrushin and Hryniv (1996) one obtains local limit theorems for $\mathbf{P}_\beta(A_n = an^2, V_n = 0)$ that is with a bridge instead of an excursion and with the algebraic area $A_n(V) := \sum_{i=1}^n V_i$ instead of the geometric area. This provides the upper bound in Proposition 3.5. To derive the lower bound in Proposition 3.5, it remains to bound from below the probability that a V trajectory remains positive when conditioned on $A_n = an^2, V_n = 0$ and this is the object of (Carmona, Nguyen and Pétrélis, 2016, Proposition 2.5).

4. Open problems

Open issues related to IPDSAW are numerous. Without pretending to be exhaustive, let us display four research directions that are both relevant from a physical standpoint and challenging mathematically.

1. **Disordered IPDSAW.** Taking into account an inhomogeneous solvent and/or a copolymer (instead of an homopolymer) when studying the collapse transition phenomenon would be an important improvement. With a copolymer, this could be achieved by introducing a random component in the self-touching intensity involving monomers i and j. One could replace β by $\beta + s\xi_{i,j}$ with $\{\xi_{i,j}, (i,j) \in \mathbb{N}^2\}$ an i.i.d. field of random variables and $s > 0$ a tuning parameter. In this framework, it would be particularly interesting to investigate the relevance of disorder ξ (Harris criterion), that is to figure out whether an arbitrary small $s > 0$ rounds the phase transition or not.

2. **Towards 2-dimensional ISAW.** One could consider an enhanced version of IPDSAW, i.e., a 2-dimensional model whose allowed configurations are not directed anymore. Of course the ISAW itself would be an ideal choice but we have seen that it is very hard to analyse at a rigorous level. Thus, one could consider a model that interpolates between IPDSAW and ISAW, in the sense that its allowed configurations are not directed anymore and have a connective constant strictly between that of IPDSAW trajectories and that of ISAW trajectories. This is the case for the Interacting Prudent Self-Avoiding Walk that has been investigated recently in Pétrélis and Torri (2016+) and will be discussed further in Section 5 below.
3. **Higher dimensions.** Obtaining rigorous mathematical results about the collapse transition of a three-dimensional extension of IPDSAW would be of great interest. One option would be to consider the above mentioned Interacting Prudent Self-Avoiding Walk in dimension 3.
4. **Collapse with adsorption.** The repulsion of monomers exerted by a poor solvent and the adsorption of those monomers along a hard wall are among the most basic interactions between an homopolymer and the medium around it (see Flory (1953)). Therefore, building a mathematical model taking both interactions into account is physically appealing. This has been done in dimension 2 for instance in Foster (1990) or Foster and Yeomans (1991) where the IPDSAW is perturbed by a pinning interaction at the x-axis that plays the role of an impenetrable horizontal interface. There are precise conjectures concerning the phase diagram of this model (see Foster and Yeomans (1991)). It is expected to be partitioned into 3 phases (collapsed, extended and pinned) separated by 3 critical curves and meeting at one tricritical point. So far, the boundary of the collapsed phase has been computed (see Foster (1990)) but the rest of the phase diagram lacks rigorous mathematical proofs.

5. A non directed model of ISAW: the IPSAW

In the spirit of the second class of open problems mentioned in Section 4, the Interacting Prudent Self-Avoiding Walk (IPSAW) is studied in Pétrélis and Torri (2016+). In size $L \in \mathbb{N}$, the set of configurations consists of the L-step prudent paths introduced in Turban and Debierre (1987), i.e.,

$$\Omega_L^{\mathtt{PSAW}} = \big\{ w := (w_i)_{i=0}^L \in (\mathbb{Z}^2)^{L+1} \colon\ w_0 = 0,\ w_{i+1} - w_i \in \{\leftarrow, \rightarrow, \downarrow, \uparrow\},\ 0 \le i \le L-1, \\ w \text{ satisfies the prudent condition} \big\}, \tag{5.1}$$

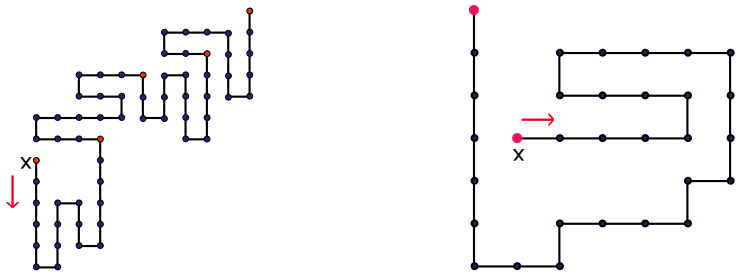
where the *prudent condition* for a path w means that it does not take any step in the direction of a lattice site already visited (see Figure 4). We define also $\Omega_L^{\mathtt{NE}}$ as the subset of $\Omega_L^{\mathtt{PSAW}}$ containing those trajectories with a general north-east orientation (see Figure 4), that is, all the prudent trajectories that do not take any step in the direction of the set $(-\infty, 0]^2$. We observe that partially-directed self-avoiding paths (recall 1.1) are in particular north-east prudent paths and that prudent paths are in particular self-avoiding paths, so that

$$\Omega_L \subset \Omega_L^{\mathtt{NE}} \subset \Omega_L^{\mathtt{PSAW}} \subset \Omega_L^{\mathtt{SAW}}, \quad L \in \mathbb{N}, \tag{5.2}$$

where $\Omega_L^{\mathtt{SAW}}$ denotes the set of L-step self-avoiding paths in $\mathbb{Z}^2$ taking unitary steps.

To help define the Hamiltonian, we associate with every path $w \in \Omega_L^{\mathtt{SAW}}$ the sequence of those points in the middle of each step, i.e., $u_i = w_{i-1} + \frac{w_i - w_{i-1}}{2}$ $(1 \le i \le L)$. The self-touchings performed by w correspond to the non-consecutive pairs (u_i, u_j) at distance one, i.e, $\|u_i - u_j\| = 1$, see Figure 5. Then, the Hamiltonian of every $w \in \Omega_L^{\mathtt{SAW}}$ is defined as

$$\mathrm{H}_L(w) := \sum_{\substack{i,j=0 \\ i<j}}^{L} 1_{\{\|u_i - u_j\| = 1\}}. \tag{5.3}$$



(a) NE-IPSAW (b) IPSAW

Fig 4: Examples of NE-PSAW (a) and PSAW (b) path. Each path starts at x and its orientation is given by the arrow.

The coupling parameter $\beta \geq 0$ stands for the self-interaction intensity and therefore, the partition functions of IPSAW and of the North-East model are respectively given by

$$\mathrm{Z}^{\mathtt{IPSAW}}_{\beta,L} := \sum_{w\in\Omega^{\mathtt{PSAW}}_L} e^{\beta\,\mathrm{H}_L(w)} \quad \text{and} \quad \mathrm{Z}^{\mathtt{NE}}_{\beta,L} := \sum_{w\in\Omega^{\mathtt{NE}}_L} e^{\beta\,\mathrm{H}_L(w)}, \tag{5.4}$$

and there exponential growth rate (free energies) by

$$\mathrm{F}^{\mathtt{IPSAW}}(\beta) := \lim_{L\to\infty} \frac{1}{L} \log \mathrm{Z}_{\beta,L} \quad \text{and} \quad \mathrm{F}^{\mathtt{NE}}(\beta) := \lim_{L\to\infty} \frac{1}{L} \log \mathrm{Z}^{\mathtt{NE}}_{\beta,L}. \tag{5.5}$$

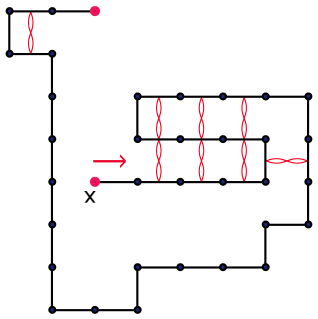


Fig 5: A prudent path w in which we highlight the self-touching. The path starts at x and its orientation is given by the arrow. In this example we have $L = 34$ and $\mathrm{H}_L(w) = 8$.

Remark 5.1. Defining the self-touchings of an IPDSAW configurations with pairs of edges or with pairs of sites is equivalent (see Fig. 2). This is not the case for non-directed models. The choice "by edge" made in (5.3) for IPSAW turns out to be more tractable than the choice "by site", but we expect that both choices lead to similar qualitative behaviors.

Remark 5.2. The connective constant of North-East prudent paths $\mathrm{F}^{\mathtt{NE}}(0)$ was computed in Bousquet-Mélou (2010) (under the name *two-sided*) and turns out to be strictly larger than that of partially-directed self-avoiding paths $f(0)$ (recall 1.4). This gives an incentive for studying IPSAW since it takes into account much more trajectories than IPDSAW and therefore, the proof of the existence of the collapse transition for IPSAW (see Theorem 5.5) is a real step forward. However, one should also acknowledge that IPSAW is still far from ISAW itself. It appears clearly with Theorems 5.5 and 5.6 since the collapse transitions of IPSAW and of ISAW are of different nature (see the discussion before Theorem 5.6 below).

Remark 5.3. The scaling limit of the 2-dimensional prudent walk has been derived in Beffara, Friedli and Velenik (2010) for its kinetic version and in Pétrélis, Sun and Torri (2017) with the uniform law. The prudent walk has also been used in Beaton and Iliev (2015) to build and investigate a non-directed model of polymer adsorption.

5.1. Existence of a collapse transition of IPSAW

The prudent model is, to our knowledge, the only *non-directed* model of a 2-dimensional interacting self-avoiding walk for which the existence of a collapse transition has been proven rigorously. This is the main result in Pétrélis and Torri (2016+) along with the equality between both free energies in (5.5) which (at $\beta = 0$) answers an open question raised in Bousquet-Mélou (2010).

Theorem 5.4 (Pétrélis and Torri (2016+), Theorem 2.1)**.** *For $\beta \geq 0$,*

$$\mathrm{F}^{\mathtt{IPSAW}}(\beta) = \mathrm{F}^{\mathtt{NE}}(\beta). \tag{5.6}$$

Theorem 5.5 (Pétrélis and Torri (2016+), Theorem 2.2)**.** *There exists a $\beta_c^{\mathtt{IPSAW}} \in (0, \infty)$ such that*

$$\begin{aligned} &\mathrm{F}^{\mathtt{IPSAW}}(\beta) > \beta \quad \textit{for every} \quad \beta < \beta_c^{\mathtt{IPSAW}}, \\ &\mathrm{F}^{\mathtt{IPSAW}}(\beta) = \beta \quad \textit{for every} \quad \beta \geq \beta_c^{\mathtt{IPSAW}}. \end{aligned} \tag{5.7}$$

Thus, the phase diagram $[0, \infty)$ is partitioned into a collapsed phase*, $\mathscr{C} := [\beta_c^{\mathtt{IPSAW}}, \infty)$ inside which the free energy (5.5) is linear and an* extended phase*, $\mathscr{E} = [0, \beta_c^{\mathtt{IPSAW}})$.*

The proof of Theorem 5.4 is purely combinatorial. It consists in building a sequence of path transformations $(M_L)_{L\in\mathbb{N}}$ such that for every $L \in \mathbb{N}$, M_L maps $\Omega_L^{\mathtt{PSAW}}$ onto $\Omega_L^{\mathtt{NE}}$ and satisfies the following properties:

- for every $w \in \Omega_L^{\mathtt{PSAW}}$, the difference between the Hamiltonians of w and of $M_L(w)$ is $o(L)$,
- the number of ancestors of a given path in $\Omega_L^{\mathtt{NE}}$ by M_L can be shown to be $e^{o(L)}$.

The prudent condition guaranties that every $w \in \Omega_L^{\mathtt{PSAW}}$ can be decomposed in a unique manner into at most $\sqrt{L}$ two-sided subpaths that are either North-East, South-East, North-West or South-West. The procedure encoded in M_L consists in detaching one by one the two-sided blocks composing w. This can be achieved by loosing at most $o(L)$ self-touchings. Then, using rotations and symmetries we concatenate the two-sided blocks so as to recover an L-step North-East path.

Thanks to Theorem 5.4, it is sufficient to prove Theorem 5.5 with the free energy of the North-East model. The prudent condition yields that every North-East path can be decomposed in a unique way into partially directed subpath (referred to as oriented blocks). Therefore, deriving a sharp upper bound on the North-East partition function $Z_{L,\beta}^{\mathtt{NE}}$ requires to bound from above

- the free energy of an oriented block of a given length,
- the self-touchings between different oriented blocks,
- the entropy carried by the fact that the number of oriented blocks and their respective lengths may fluctuate.

Controlling the free energy of an oriented block is achieved with the random walk representation since oriented blocks are IPDSAW trajectories. Dealing with the self-touchings occurring between blocks requires to observe that the i-th oriented block of a North-East path may only interact with the $(i-2)$-th and $(i-1)$-th blocks. Moreover, self-touchings may only appear between the first stretch of the i-th block and the inter-stretches of the $(i-1)$-th block or between the first stretch of the i-th block and the last stretch of the $(i-2)$-th block (see Figure 6). This allows us to derive an explicit upper bound on the total

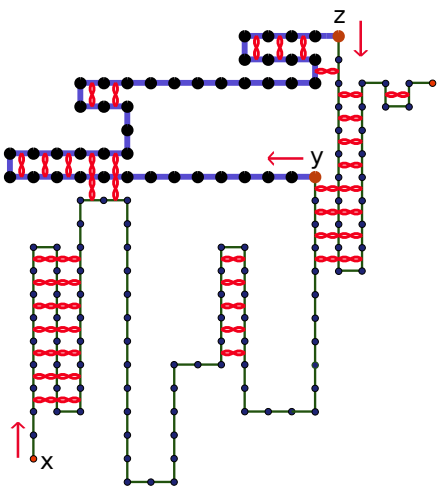


Fig 6: A NE-prudent path made of three blocks. The first block starts at x, the second block starts at y and the third block starts at z. Their orientation is given by the arrow. Interactions in each block and between different blocks are highlighted in red. We observe that, the first stretch of the third block interacts with both the inter-stretches of the second block and the last stretch of the first block.

number of self-touchings that may appear between the different oriented blocks of a North-East configuration, see Figure 7. Therefore, it remains to control the entropy related to the fact that the lengths of oriented blocks may fluctuate. This is again taken care of with the random walk representation. To be more precise, at the end of the proof we derive a random walk representation for the whole North-East path and this is sufficient to conclude that for β large enough, the free energy is not larger than β.

5.2. Conclusion

From (5.2) it is straightforward that $F^{\mathtt{IPSAW}}(\beta) \geq f(\beta)$ for every $\beta \geq 0$ (recall (1.5)). Thus, Theorem 5.5 implies that the critical point of IPSAW is not smaller than that of IPDSAW, i.e.,

$$\beta_c \leq \beta_c^{\mathtt{IPSAW}}. \tag{5.8}$$

It would be interesting to understand whether (5.8) is an equality or not. Even more challenging would be the computation of $\beta_c^{\mathtt{IPSAW}}$.

Let us stress also that even if the existence of a collapsed transition for IPSAW is proven, we do not have any results concerning its scaling limit in each regime (extended, critical and collapsed) as we did for IPDSAW in Section 3.2. In this spirit, at $\beta = 0$, the scaling limit of the prudent walk itself has been derived in Pétrélis, Sun and Torri (2017). We conjecture that in the extended phase the scaling limit should have a similar structure, that is, a straight line. More interesting is the inside of the collapsed phase, in which the limit shape is less clear. In analogy with the results obtained in Section 3.2 we only expect it to be deterministic.

We conclude with a few words about the 2-dimensional Interacting Self-Avoiding Walk (ISAW) defined exactly like the IPSAW in (5.3) but with a larger set of allowed configurations, that is

$$\Omega_L^{\mathtt{SAW}} := \big\{ w := (w_i)_{i=0}^L \in (\mathbb{Z}^2)^{L+1} \colon\ w_0 = 0,\ w_{i+1} - w_i \in \{\leftarrow, \rightarrow, \downarrow, \uparrow\},\ 0 \leq i \leq L-1, \\ w \text{ satisfies the self-avoiding condition} \big\}. \tag{5.9}$$

We denote by $Z_{L,\beta}^{\mathtt{ISAW}}$ the partition function of ISAW and we define its free energy as

$$F^{\mathtt{ISAW}}(\beta) := \liminf_{L\to\infty} \frac{1}{L} \log Z_{L,\beta}^{\mathtt{ISAW}}, \tag{5.10}$$

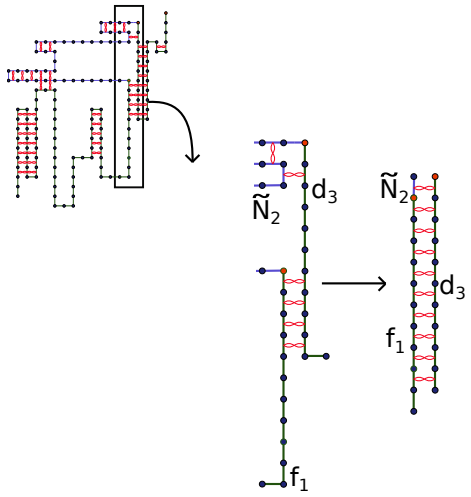


Fig 7: On the left, a NE-PSAW path made of three blocks. In the picture we zoom in on the interactions between the third block and the rest of path. We recall that the third block can only interact with its two preceding blocks, i.e., the first and the second one. We call f_1 the last vertical stretch of the first block and d_3 the first vertical stretch of the third block. The interactions between the first and the third blocks involve f_1 and d_3 while the interactions between the second and the third blocks involve d_3 and $\widetilde{N}_2$ (the number of inter-stretches in the second block that may truly interact with d_3, on the picture $\widetilde{N}_2 = 1$). Such interactions are bounded above by $(\widetilde{N}_2 + f_1)\widetilde{\wedge} d_3$.

Theorem 5.6 below shows that the conjectured collapse transition displayed by ISAW at some $\beta_c^{\texttt{ISAW}}$ does not correspond to a self-touching saturation as is the case for IPDSAW and IPSAW. The reason is that, even very dense ISAW trajectories can integrate small holes which are not compatible with the prudent condition. Introducing a small density of holes (εL) in a dense ISAW configuration of length L yields a loss of self-touchings of order εL, however this is overcome by the entropy gain associated with the choice of the locations of those holes (of order $-\varepsilon \log(\varepsilon) L$).

Theorem 5.6 (Pétrélis and Torri (2016+), Theorem 2.3)**.**

$$\mathrm{F}^{\texttt{ISAW}}(\beta) > \beta, \quad \textit{for every} \quad \beta \in [0, \infty). \tag{5.11}$$

References

Bauerschmidt, R., Slade, G. and Wallace, B. C. Four-dimensional weakly self-avoiding walk with contact self-attraction. https://arxiv.org/pdf/1610.08573.pdf (math.PR), to appear in J. Stat. Phys.

Beaton, N. R. and Iliev, G. K. (2015). Two-sided prudent walks: a solvable non-directed model of polymer adsorption. *J. Stat. Mech. Theory Exp.* **9** P09014, 23. MR3412072

Beffara, V., Friedli, S. and Velenik, Y. (2010). Scaling Limit of the Prudent Walk. *Electronic Communications in Probability* **15** 44–58.

Binder, P. M., Owczarek, A. L., Veal, A. R. and Yeomans, J. M. (1990). Collapse transition in a simple polymer model: exact results. *Journal of Physics A: Mathematical and General* **23** L975.

Bousquet-Mélou, M. (2010). Families of prudent self-avoiding walks. *Journal of combinatorial theory Series A* **117** 313 - 344.

Brak, R., Guttman, A. J. and Whittington, S. G. (1992). A collapse transition in a directed walk model. *J. Phys. A: Math. Gen.* **25** 2437–2446.

Brak, R., Owczarek, A. L., Prellberg, T. and Guttmann, A. J. (1993). Finite-length scaling of collapsing directed walks. *Phys. Rev. E* **48** 2386–2396.

CARMONA, P., NGUYEN, G. B. and PÉTRÉLIS, N. (2016). Interacting partially directed self avoiding walk. From phase transition to the geometry of the collapsed phase. *Ann. Probab.* **44** 3234–3290.
CARMONA, P. and PÉTRÉLIS, N. (2016). Interacting partially directed self avoiding walk : scaling limits. **21** no. 49, 1–52 (electronic). Electronic Journal of Probability.
CARMONA, P. and PÉTRÉLIS, N. (2017a). A shape theorem for the scaling limit of the IPDSAW at criticality.
CARMONA, P. and PÉTRÉLIS, N. (2017b). Limit theorems for random walk excursion conditioned to have a typical area. https://arxiv.org/abs/1709.06448.
DENISOV, D., KOLB, M. and WACHTEL, V. (2015). Local asymptotics for the area of random walk excursions. *J. Lond. Math. Soc. (2)* **91** 495–513. MR3355112
DOBRUSHIN, R. and HRYNIV, O. (1996). Fluctuations of shapes of large areas under paths of random walks. *Probab. Theory Related Fields* **105** 423–458. MR1402652 (97h:60029)
DUPLANTIER, B. and SALEUR, H. (1987). Exact tricritical exponents for polymers at the FTHETA point in two dimensions. *Phys. Rev. Lett.* **59** 539–542.
DURRETT, R. (2010). *Probability: theory and examples*, fourth ed. *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge. MR2722836 (2011e:60001)
FLORY, P. J. (1953). *Principles of Polymer chemistry*. Cornell University Press, Ithaca (NY).
FOSTER, D. P. (1990). Exact evaluation of the collapse phase boundary for two-dimensional directed polymers. *Journal of Physics A: Mathematical and General* **23** L1135.
FOSTER, D. P. and YEOMANS, J. (1991). Competition between self-attraction and adsorption in directed self-avoiding polymers. *Physica A: Statistical Mechanics and its Applications* **177** 443 - 452.
GHERARDI, M. (2013). Theta-point polymers in the plane and Schramm-Loewner evolution. *Phys. Rev. E* **88** 032128.
GIACOMIN, G. (2007). *Random polymer models*. Imperial College Press, London. MR2380992 (2009c:82025)
HAMMOND, A. and HELMUTH, T. Self-avoiding walk with self-attraction. Preprint.
JANSON, S. (2007). Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probab. Surv.* **4** 80–145. MR2318402 (2008d:60106)
KAC, M. (1946). On the average of a certain Wiener functional and a related limit theorem in calculus of probability. *Trans. Amer. Math. Soc.* **59** 401–414. MR0016570 (8,37b)
NGUYEN, G. B. and PÉTRÉLIS, N. (2013). A variational formula for the free energy of the partially directed polymer collapse. *J. Stat. Phys.* **151** 1099–1120. MR3063498
ORR, W. J. C. (1947). Statistical treatment of polymer solutions at infinite dilution. *Trans. Faraday Soc.* **43** 12-27.
PÉTRÉLIS, N., SUN, R. and TORRI, N. (2017). Scaling limit of the uniform prudent walk. *Electron. J. Probab.* **22** 19 pp.
PÉTRÉLIS, N. and TORRI, N. (2016+). Collapse transition of the interacting prudent walk. *Ann. Instit. H. Poincaré D* to appear.
SALEUR, H. (1986). Collapse of two-dimensional linear polymers. *Journal of Statistical Physics* **45** 419–438.
TURBAN, L. and DEBIERRE, J. M. (1987). Self-directed walk: a Monte Carlo study in two dimensions. *J. Phys. A* **20** 679 – 686.
UELTSCHI, D. (2002). A self-avoiding walk with attractive interactions. *Probability Theory and Related Fields* **124** 189–203.
VAN DER HOFSTAD, R. and KLENKE, A. (2001). Self-attractive random polymers. *Ann. Appl. Probab.* **11** 1079–1115. MR1878291
VERNIER, E., JACOBSEN, J. L. and SALEUR, H. (2015). A new look at the collapse of two-dimensional polymers. *Journal of Statistical Mechanics: Theory and Experiment* **2015** P09001.
ZWANZIG, R. and LAURITZEN, J. I. (1968). Exact calculation of the partition function for a model of two dimensional polymer crystallization by chain folding. *J. Chem. Phys.* **48** 3351.