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GORENSTEIN FANO GENERIC TORUS ORBIT CLOSURES IN G/P

PIERRE-LOUIS MONTAGARD AND ALVARO RITTATORE

ABSTRACT. Given a reductive group G and a parabolic subgroup $P \subset G$, with maximal torus T , we consider (following Dabrowski's work) the closure X of a generic T -orbit in G/P , and determine in combinatorial terms when the toric variety X is \mathbb{Q} -Gorenstein Fano, extending in this way the classification of smooth Fano generic closures given by Voskresenskiĭ and Klyachko. As an application, we apply the well known correspondence between Gorenstein Fano toric varieties and reflexive polytopes in order to exhibit which reflexive polytopes correspond to generic closures — this list includes the reflexive root polytopes.

1. INTRODUCTION

Toric varieties — that is, normal varieties X over an algebraically closed field \mathbb{k} , on which an algebraic torus T acts effectively and with an open orbit — have been thoroughly studied since the beginning of the 1970's. Since the geometric properties of a toric variety can be described in combinatorial terms (by means of its associated fan), this family of algebraic varieties provides a nice framework in which to study either their geometric properties or the combinatorial properties of their associated fans.

In the early beginning of the theory of toric varieties, D. Mumford considered the toric variety associated to the fan obtained by considering the weight lattice and the set of all closed Weyl chambers of a root system R (see [1]). Afterwards, the geometry of this variety was intensively studied by several authors (see [2, 3, 4, 5]). In [6], V.E. Voskresenskiĭ and A.A. Klyachko considered a larger family of fans constructed by fixing a set I of simple roots of R and “gluing together” selected adjacent Weyl chambers that correspond to a choice of a proper subset $L \subsetneq I$ (see Definition 4). The invariance properties of these fans (w.r.t. the action of the Weyl group of R) allow the authors to characterize the pairs (R, L) such that the associated toric variety $X_{R,L}$ is smooth Fano. A remarkable result of R. Dabrowski proves that the toric varieties $X_{R,L}$ can be constructed as the closure of an orbit of a maximal torus on a flag variety — a “generic torus orbit closure”, see [7] and Theorem 3.

In this paper we generalize Voskresenskiĭ and Klyachko results: we describe all pairs (R, L) such that the associated toric variety $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano and which varieties among them are Gorenstein Fano (see Definition 2). The smoothness condition (as in [6]) imposes restrictions of the combinatorics, in such a way that (in the case of irreducible root systems) smooth Fano varieties are obtained only for root systems of type A_n (two infinite series), C_n (one additional infinite series) and G_2 (one exceptional case). By relaxing the smoothness constraint, we obtain \mathbb{Q} -Gorenstein Fano generic closures for all types of irreducible root systems except

E_7 and E_8 (see Section 4). More precisely, we exhibit twelve infinite series plus five exceptional cases of \mathbb{Q} -Gorenstein Fano varieties.

From a combinatorial perspective, the exhibition of toric Gorenstein Fano varieties is interesting because each of these varieties is naturally associated with a couple of dual reflexive polytopes (see Definition 1) — this duality allowed Batyrev to give a rigorous construction of mirror symmetry in the toric context, see [8]. Applying in our setting this well known correspondence, we produce the list of dual reflexive polytopes associated to each Gorenstein Fano toric variety $X_{R,L}$. As a minor by-product, we describe the list of root polytopes which are reflexive.

We briefly describe now the content of this paper. In Section 2 we establish the basic notations and we present some known results on Gorenstein Fano varieties and closures of generic orbits. In Section 3 we study some combinatorial properties of the cone $\sigma_{R,L}$ and we characterize the \mathbb{Q} -Gorenstein Fano generic closures in terms of the combinatorics of $\sigma_{R,L}$ (see Theorem 10). In Section 4 we present our main result (Theorem 12), namely a classification of all \mathbb{Q} -Gorenstein Fano, Gorenstein Fano and smooth Fano generic closures in terms of their defining set of roots $L \subset I$ (see Definition 2). The proof of this classification result relies heavily in the criterion established in Section 3. In Section 6 we exhibit the reflexive polytopes associated to the Gorenstein Fano generic closures; as an application of our combinatorial description, we classify the reflexive root polytopes.

In [9], the interested reader can find a SAGE (see [10]) package that allows to perform explicit calculations for the cone $\sigma_{R,L}$ — we use GAP3 (version maintained by Jean Michel, [11]) in order to use the package Chevie (see [12] and [13]).

2. PRELIMINARIES

2.1. \mathbb{Q} -Gorenstein Fano toric varieties.

All along this work, by a toric variety we mean a normal toric variety over an algebraically closed field \mathbb{k} ; our general reference for toric varieties is [14].

If T is an algebraic torus, we denote by Λ the characters group of T and by Λ^\vee the \mathbb{Z} -dual of Λ . We denote by $\Lambda_{\mathbb{Q}}$ (resp. $\Lambda_{\mathbb{Q}}^\vee$) the \mathbb{Q} -vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ (resp. $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda^\vee$), and if $(u, v) \in \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}}^\vee$ then $\langle u, v \rangle = v(u) \in \mathbb{Q}$ is the natural pairing of u and v .

If X is a subset of a finite dimensional \mathbb{Q} -vector space V , we denote by $\text{Conv}(X)$ the convex hull of X , by $\langle X \rangle$ the vector space generated by X , and by $\langle X \rangle_{\text{aff}}$ the affine space generated by X ; $\dim_{\text{aff}}(X)$ denotes the affine dimension of $\langle X \rangle_{\text{aff}}$. We denote by \mathbb{Q}^+X the positive cone generated by X (with the origin as vertex); the “dual cone” of X is defined as

$$X^\vee := \{\varphi \in V^\vee : \forall x \in X, \langle x, \varphi \rangle \geq 0\}.$$

If Σ is a fan in $\Lambda_{\mathbb{Q}}^\vee$ (see [14, Definition 3.1.2]), then $\Sigma(r)$ is the set of r -dimensional cones in Σ . For each $\rho \in \Sigma(1)$, u_ρ is the *primitive element* of the monoid $\rho \cap \Lambda^\vee$. The set of primitive elements of $\sigma \in \Sigma$ is denoted by

$$\text{Prim}(\sigma) = \{u_\rho : \rho \in \Sigma(1) \text{ and } \rho \subset \sigma\}.$$

We set $\text{Prim}(\Sigma) = \bigcup_{\sigma \in \Sigma} \text{Prim}(\sigma)$.

A fan Σ has associated a toric variety that we denote as X_Σ .

Recall that if $\sigma \subset \Lambda_{\mathbb{Q}}^\vee$ is a polyhedral strictly convex cone, then the *relative interior* of σ , denoted by $\overset{\circ}{\sigma}$, is the complement in σ of the union of the facets of σ .

Definition 1. If Λ is a lattice, a *lattice polytope* is the convex hull in $\Lambda_{\mathbb{Q}}$ of a finite subset $X \subset \Lambda$.

Assume now that $\mathcal{P} \subset \Lambda_{\mathbb{Q}}$ is a lattice polytope of maximal rank containing the origin in its strict interior. If \mathcal{Q} is a proper facet of \mathcal{P} , the *interior normal* of \mathcal{Q} , denoted by $\varphi_{\mathcal{Q}}$, is the unique element of $(\Lambda_{\mathbb{Q}})^{\vee}$ such that $\varphi_{\mathcal{Q}}(\mathcal{Q}) = -1$ and $\varphi_{\mathcal{Q}}(\mathcal{P} \setminus \mathcal{Q}) > -1$; the *exterior normal* is defined as $-\varphi_{\mathcal{Q}}$. The convex hull of the set of the interior normals of \mathcal{P} is called the *dual polytope* of \mathcal{P} and is denoted by $\mathcal{P}^{\circ} = \{u \in \Lambda_{\mathbb{Q}}^{\vee} : \langle v, u \rangle \geq -1 \ \forall v \in \mathcal{P}\}$.

It is clear that \mathcal{P}° is a polytope and that $(\mathcal{P}^{\circ})^{\circ} = \mathcal{P}$; we say that \mathcal{P} is a *reflexive polytope* if \mathcal{P}° is a lattice polytope.

Definition 2. Let X be a normal variety and denote by $-K_X$ the anti-canonical divisor. We say that X is \mathbb{Q} -Gorenstein Fano if $-K_X$ is an ample \mathbb{Q} -Cartier divisor; if moreover $-K_X$ is an ample Cartier divisor, we say that X is Gorenstein Fano. If X is a smooth Gorenstein Fano variety, we say that X is smooth Fano.

If X is \mathbb{Q} -Gorenstein Fano, the *Gorenstein index* of X , denoted by j_X , is the smallest positive integer j such that jK_X is Cartier — thus, a Gorenstein Fano variety is a \mathbb{Q} -Gorenstein Fano variety of Gorenstein index 1.

A fan $\Sigma \subset \Lambda_{\mathbb{Q}}^{\vee}$ is \mathbb{Q} -Gorenstein Fano, Gorenstein Fano or smooth Fano if the associated toric variety X_{Σ} has the corresponding property.

The following equivalences are well known (see for example [14, Theorem 4.2.8 and Lemma 6.1.13]):

Proposition 1. *Let Σ be a complete fan in $\Lambda_{\mathbb{Q}}^{\vee}$. The following assertions are equivalent:*

- (1) X_{Σ} is a \mathbb{Q} -Gorenstein Fano toric variety;
- (2) $\{\text{Conv}(\text{Prim}(\sigma)) : \sigma \in \Sigma(s), s = 1, \dots, n\}$ is the set of proper faces of the lattice polytope $\text{Conv}(\text{Prim}(\Sigma))$;
- (3) for every cone $\sigma \in \Sigma(n)$, the polytope $\text{Conv}(\text{Prim}(\sigma))$ is $(n-1)$ -dimensional; let $\varphi_{\sigma} \in \Lambda_{\mathbb{Q}}$ be such that $\langle \varphi_{\sigma}, v \rangle = -1$ for $v \in \text{Prim}(\sigma)$. Then $\langle \varphi_{\sigma}, w \rangle > -1$ for every $w \in \text{Prim}(\Sigma) \setminus \text{Prim}(\sigma)$.

Moreover, if X_{Σ} is a \mathbb{Q} -Gorenstein Fano toric variety, then

$$j_X = \min\{j \in \mathbb{N}^* : \forall u \in \Lambda^{\vee}, \forall \sigma \in \Sigma(n), \langle j\varphi_{\sigma}, u \rangle \in \mathbb{Z}\}.$$

In particular, if X_{Σ} is Gorenstein Fano then $\text{Conv}(\text{Prim}(\Sigma))$ is a reflexive lattice polytope. \square

2.2. Fans defined by root systems and generic orbits.

In this section we establish our notations on fans defined by root systems, and formulate Dabrowski's results accordingly, associating to a generic orbit the corresponding combinatorial data. When dealing with root systems, we follow Bourbaki's notations (see [15, 16]).

Notation 1. In what follows, R designs a root system of rank n and Λ_R its associated *root lattice*. We denote by R^+ a chosen set of positive roots; $S = \{\alpha_i : i \in I\}$ is the set of the *simple roots* associated to R^+ . We denote by Λ_P the lattice of weights and by $\{\omega_i : i \in I\}$ the set of *fundamental weights* associated to S .

If $\alpha \in R$, we denote by $s_{\alpha} : (\Lambda_P)_{\mathbb{Q}} \rightarrow (\Lambda_P)_{\mathbb{Q}}$ the associated reflection, and by W the Weyl group generated by the reflections associated to R . Recall that W acts on $(\Lambda_P)_{\mathbb{Q}}$ with the Weyl chamber $\mathcal{D} = \mathbb{Q}^+ \{\omega_i : i \in I\}$ as fundamental domain.

The root system dual to R is denoted by R^\vee . Recall that the *simple co-roots* $\{\alpha_i^\vee : i \in I\} \subset (\Lambda_P)_{\mathbb{Q}}^\vee$ and the *fundamental co-weights* $\{\omega_i^\vee : i \in I\} \subset (\Lambda_R)_{\mathbb{Q}}^\vee$ are such that $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$, $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$ for all $(i, j) \in I^2$. Also, the reflections $s_{\alpha_i^\vee}$ induce an action of W on $(\Lambda_P)_{\mathbb{Q}}^\vee$, with the dominant Weyl Chamber of R^\vee (denoted by \mathcal{D}^\vee) as fundamental domain.

If $L \subset I$, we will abuse notations and identify $S_L = \{\alpha_i : i \in L\}$ with L . We denote by W_L the subgroup of W generated by the corresponding reflections $\{s_{\alpha_i^\vee} : i \in L\}$.

We denote by $R = \prod_{k=1}^r R_k$ the decomposition of the root system R in irreducible root systems; the set of simple roots of R_k is denoted by $S(k) \subset S$ and we denote by $I(k) \subset I$ the corresponding subset of indexes.

We choose W -invariant scalar products in $(\Lambda_P)_{\mathbb{Q}}$ and $(\Lambda_P)_{\mathbb{Q}}^\vee$; these scalar products will be denoted by (\cdot, \cdot) in both cases.

Definition 3. With the previous notations, if $\lambda = \sum_{i=1}^n a_i \omega_i \in \Lambda_P$ is a weight (resp. $\lambda^\vee = \sum_{i=1}^n a_i \omega_i^\vee \in \Lambda_R^\vee$ is a co-weight), we define the *support* of λ (resp. λ^\vee) as the set $I_\lambda = \{\alpha_i : a_i \neq 0\} \subset S$.

From now on, G is a semi-simple group over \mathbb{k} and $T \subset G$ a maximal torus, such that R is the root system associated to the couple (G, T) ; we denote by $B \subset G$ the Borel subgroup associated to R^+ . To each subset $L \subset I$, we associate the parabolic subgroup P_L containing the opposite Borel subgroup B^- and such that the Weyl group of P_L is equal to W_L .

Recall that if $\lambda \in \Lambda_P$ is a dominant weight with support I_λ contained in $L^c = I \setminus L$, then λ can be extended to P_L . We denote by $V(\lambda)$ the *Weyl G -module* associated to λ .

Definition 4. Let $L \subsetneq I$ be a proper subset of roots. Following [6], we define the *cone associated to L* as

$$\sigma_{R,L} = \bigcup_{w \in W_L} w \mathcal{D}^\vee \subset (\Lambda_R)_{\mathbb{Q}}^\vee.$$

If $\sigma_{R,L}$ is strictly convex, we consider the complete fan having as maximal cones the translates $w \cdot \sigma_{R,L}$, where $w \in W$; we denote this fan by $\Sigma_{R,L} \subset (\Lambda_R)_{\mathbb{Q}}^\vee$.

We define $-\Sigma_{R,L} := \{-\sigma : \sigma \in \Sigma_{R,L}\}$; the corresponding toric variety is denoted by $X_{R,L} := X_{-\Sigma_{R,L}}$.

Remark 1. (1) The geometric meaning of the use of the co-weight lattice and the fan $-\Sigma_{R,L}$ in the definition of $X_{R,L}$ (e.g. instead of the weight lattice and $\Sigma_{R,L}$), will become evident in the next section (see Theorem 3 and Remark 2).

(2) Let $R = \prod_{k=1}^r R_k$ be a decomposition of the root system R in irreducible root systems. Then $\mathcal{D}^\vee = \prod_{k=1}^r \mathcal{D}_k^\vee$, where $\mathcal{D}_k^\vee \subset (\Lambda_k)_{\mathbb{Q}}^\vee$ is the dominant Weyl chamber of R_k^\vee — here Λ_k denotes the lattice generated by R_k . In particular, $\sigma_{R,L} = \prod_{k=1}^r \sigma_{R_k, L \cap I_k}$, $\Sigma_{R,L} = \prod_{k=1}^r \Sigma_{R_k, L \cap I_k}$, and $X_{R,L} \cong \prod_{k=1}^r X_{R_k, L \cap I_k}$.

Dabrowski proved in [7] that the toric varieties $X_{R,L}$ can be realized as closures of “generic” T -orbits in G/P_L . We briefly recall his construction, filling some minor gaps in the proofs presented in *op.cit.* for the sake of completeness.

Definition 5 (see [7, §1]). Let $L \subset I$ be a subset of roots, $\Pi_\lambda = \{\mu \in \Lambda_P : V(\lambda)_\mu \neq 0\}$ the set of T -weights of $V(\lambda)$, and \mathcal{A}_λ be the list of the T -weights counted with

multiplicity. A set of Plücker coordinates $\{f_\mu : \mu \in \mathcal{A}_\lambda\}$ is a choice of a basis of T -semi-invariants functions $f_\mu \in V(-\lambda)_\mu$.

If $x = uP \in G/P$, we consider $\Pi_\lambda(x) := \{\mu \in \Pi_\lambda : f_\mu(x) \neq 0 \text{ for some } f_\mu\}$. We say that the T -orbit $T \cdot x$ is *generic* in the sense of Dabrowski if $W \cdot \lambda \subset \Pi_\lambda(x)$ and the set $\lambda - w\Pi_\lambda(x)$ generates S^L as a sub-monoid — notice that if all the Plücker coordinates of x are non zero, then $T \cdot x$ is generic.

Proposition 2. *Let $(R^L)^+$ be the set of positive roots which are not sum of simple roots in L , and S^L be the sub-monoid generated by $(R^L)^+$. Then $\sigma_{R,L}$ is the dual cone of the convex cone generated by S^L . In particular, $\sigma_{R,L}$ is a strictly convex cone if and only if $L \cap I(k) \neq I(k)$ for all $k = 1, 2, \dots, r$.*

Proof. Since $\mathbb{Q}^+(S^L) = \mathbb{Q}^+((R^L)^+)$, it follows that

$$(\mathbb{Q}^+(S^L))^\vee = (\mathbb{Q}^+((R^L)^+))^\vee = \bigcap_{\beta \in (R^L)^+} \{\chi^\vee \in (\Lambda_R^\vee)_\mathbb{Q} : \langle \beta, \chi^\vee \rangle \geq 0\}.$$

It is clear that $(R^L)^+$ is W_L -stable; hence, $\mathbb{Q}^+((R^L)^+)$ is also W_L -stable and, as it contains the dominant chamber \mathcal{D}^\vee , we have the inclusion $\sigma_{R,L} \subset \mathbb{Q}^+((R^L)^+)$. In order to prove the equality, it suffices to prove that $s_{\alpha_i} \cdot \sigma_{R,L}$ is not contained in $\mathbb{Q}^+((R^L)^+)$ for all $i \notin L$. But if $\langle \alpha_i, \chi^\vee \rangle > 0$, then $\langle \alpha_i, s_{\alpha_i}(\chi^\vee) \rangle = -\langle \alpha_i, \chi^\vee \rangle < 0$.

In order to prove the converse, in view of Remark 1, we can assume that R is irreducible. In this case, it is clear that $\sigma_{R,I} = (\Lambda_R)_\mathbb{Q}^\vee$. If $L \neq I$, then it is well-known that $(R^L)^+$ generates a space of maximal dimension and the result follows. \square

We are in condition now to state Dabrowski's main result concerning the generic T -orbits in G/P_L and their closures.

Theorem 3 ([7, Theorem 3.2]). *If $L \subset I$ is such that $L \cap I(k) \neq I(k)$ for all k (see Proposition 2) and $T \cdot x \subset G/P_L$ is a generic orbit, then $\overline{T \cdot x}$ is a toric variety isomorphic to the toric variety $X_{R,L}$.* \square

Remark 2. (1) Since the torus T does not act effectively on G/P , in general the generic orbit associate to L has strictly lower dimension than T — that is $T \not\cong T \cdot x$.

However, it is easy to show that in the hypothesis of Proposition 2 we have that $T_x = Z(G)$. Since Λ_R is the lattice of characters of $T/Z(G)$, the description of $\overline{T \cdot x}$ as a $T/Z(G)$ -toric variety is given by a complete fan in the space $(\Lambda_R^\vee)_\mathbb{Q}$.

(2) When the root system is simply laced, one has some leeway for the combinatorial description of a closure of a generic orbit as a toric variety, since $\Lambda_R^\vee = \Lambda_P$ under the identification of R and R^\vee . This discreteness appears in the literature, sometimes by omission, e.g. in [7] the lattice is not explicitly mentioned; however, the reader should be aware that if the root system is not simply laced, then Λ_P and Λ_R^\vee are distinct lattices. The distinction between Λ_P and Λ_R^\vee must be taken into account in order to give a correct combinatorial description of the fan associated to adherence of a generic orbit, see for example [17].

3. A CRITERIA FOR $X_{R,L}$ TO BE \mathbb{Q} -GORENSTEIN FANO

In this section we characterize when the closure of a generic orbit is a Gorenstein Fano toric variety in terms of the combinatorial properties of the core of the associated cone $\sigma_{R,L}$. From now on we assume that $L \subset I$ is such that $\sigma_{R,L}$ is a strictly convex cone.

3.1. Various combinatorial properties of $\sigma_{R,L}$.

Definition 6. If $\lambda \in (\Lambda_R)_{\mathbb{Q}}$, the *Weyl polytope associated to λ* is defined as $\mathcal{WP}(\lambda) = \text{Conv}(W \cdot \lambda) \subset (\Lambda_P)_{\mathbb{Q}}$. The set of facets of $\mathcal{WP}(\lambda)$ containing λ is denoted by $C_{\lambda}(n-1)$.

In [7], Dabrowski showed that $\Sigma_{R,L}$ can be obtained as the fan dual of the Weyl polytope of a dominant weight with support $L^c = I \setminus L$:

Proposition 4. *Let λ be a dominant weight with support L^c , and assume that $\sigma_{R,L}$ is strictly convex. Then*

$$\Lambda_R \cap (\lambda - \mathcal{WP}(\lambda)) = S^L \cap (\lambda - \mathcal{WP}(\lambda)).$$

Moreover, the fan $\Sigma_{R,L}$ is dual to the polytope $\mathcal{WP}(\lambda)$. \square

The cones $\sigma_{R,L}$ being stable by the action of W_L , we can use this action in order to describe their combinatorics and geometry as follows.

Proposition 5. *Let $L \subset I$ and consider $\mathcal{F} \in \sigma_{R,L}(r)$. Then there exist unique pairs $(\gamma_i, w_i) \in \mathcal{D}^{\vee}(r) \times W_L$, $i = 1, \dots, s$, such that $\mathcal{F} = \bigcup_{i=1}^s w_i \cdot \gamma_i$. Moreover,*

- (1) for all $i, j = 1, \dots, s$, $i \neq j$, $w_i \cdot \gamma_i \cap w_j \cdot \gamma_j$ is a common proper face of $w_i \cdot \gamma_i$ and $w_j \cdot \gamma_j$;
- (2) if $w_i = w_j$ then $\gamma_i = \gamma_j$.

In particular, $\text{Prim}(\sigma_{R,L}) = W_L \cdot (\text{Prim}(\sigma_{R,L}) \cap \{\omega_1^{\vee}, \dots, \omega_n^{\vee}\})$.

Proof. By construction, $\sigma_{R,L}$ is stable under the (linear) action of W_L ; hence $\sigma_{R,L}(r)$ is stable under the action of W_L . Since \mathcal{D}^{\vee} is a fundamental domain of the action of W it follows that there exist $w_1, \dots, w_s \in W_L$ and $\gamma_1, \dots, \gamma_s \in \mathcal{D}^{\vee}(r)$ such that $\mathcal{F} = \bigcup_{i=1}^s w_i \cdot \gamma_i$. In particular, it follows that \mathcal{D}^{\vee} is also a fundamental domain for action of W_L on $\sigma_{R,L}$.

Let $w \in W_L$, $\gamma \in \mathcal{D}^{\vee}$ be such that $w \cdot \gamma \subset \mathcal{F}$. Since the affine dimension of $w \cdot \gamma$ and $w_i \cdot \gamma_i$, $i = 1, \dots, s$, is r , it follows that there exists $i \in \{1, \dots, s\}$ such that $\dim\langle w_i \cdot \gamma_i \cap w \cdot \gamma \rangle_{\text{aff}} = r$. Since \mathcal{D}^{\vee} is a fundamental domain for the action of W_L on $\sigma_{R,L}$, it follows that $w_i \cdot \gamma_i = w \cdot \gamma$ and therefore $w_i = w$ and $\gamma_i = \gamma$.

It is now easy to prove assertion (1). In order to prove (2), observe that if $\gamma_i \neq \gamma_j$, then $\dim\langle \gamma_i \cup \gamma_j \rangle_{\text{aff}} > r$ and it follows that $w \cdot (\gamma_i \cup \gamma_j)$ is not included in $\sigma_{R,L}(r)$ for any $w \in W_L$.

Finally, recall that $\text{Prim}(\sigma_{R,L})$ consists of the primitive elements of the rays in $\sigma_{R,L}(1)$ and that the lattice of co-roots is stable under the action of W . \square

Notation 2. We denote $J_L = \text{Prim}(\sigma_{R,L}) \cap \{\omega_1^{\vee}, \dots, \omega_n^{\vee}\}$ — by Proposition 5 above, J_L is a fundamental domain for the action of W_L on $\text{Prim}(\sigma_{R,L})$. In order to simplify the notations, we will often also denote the set of indexes $\{i : \omega_i^{\vee} \in J_L\}$ by J_L .

Once we have described $\text{Prim}(\sigma_{R,L})$ as a set of W_L -orbits, we are in condition to give a simple description of its affine support space.

Proposition 6. *If $L \subset I$ then*

$$\begin{aligned} \langle \text{Prim}(\sigma_{R,L}) \rangle_{\text{aff}} &= \omega_k^{\vee} + \langle (\bigcup_{j \in J_L} W_L \cdot (\omega_j^{\vee}) - \omega_j^{\vee}) \cup \{\omega_i^{\vee} - \omega_j^{\vee} : i, j \in J_L\} \rangle \\ &= \omega_k^{\vee} + \langle \{\alpha_i^{\vee} : i \in L\} \cup \{\omega_i^{\vee} - \omega_k^{\vee} : i \in J_L\} \rangle \end{aligned}$$

for any $\omega_k^{\vee} \in J_L$.

Proof. By Proposition 5, $\text{Prim}(\sigma_{R,L}) = W_L \cdot J_L$. If $\omega_i^\vee, \omega_j^\vee \in J_L$ and $w, w' \in W_L$, then $w \cdot (-\omega_i^\vee) - w' \cdot (-\omega_j^\vee) = w \cdot (-\omega_i^\vee) - \omega_i + \omega_i^\vee - \omega_j^\vee + \omega_j^\vee - w' \cdot (-\omega_j^\vee)$, and the first equality follows.

For the second equality, let $w = s_\ell \cdots s_1 \in W_L$, with $s_j \in \{s_{\alpha_i^\vee} : i \in L\}$. Then $w \cdot (-\omega_i^\vee) - \omega_i^\vee \in \langle \alpha_i^\vee : i \in L \rangle_{\mathbb{Q}}$ for all $\omega_i^\vee \in J_L$ and the inclusion \subset follows.

In order to prove the remaining inclusion, let $i \in L$. If $s_{\alpha_i^\vee} \cdot \nu = \nu$ for all $\nu \in \text{Prim}(\sigma_{R,L})$, then $s_{\alpha_i^\vee}$ acts trivially, since $\sigma_{R,L}$ is of maximal dimension; this is a contradiction. It follows that there exists $\nu \in \text{Prim}(\sigma_{R,L})$ such that $s_{\alpha_i^\vee} \cdot \nu \neq \nu$, and therefore $\alpha_i^\vee \in \langle \text{Prim}(\sigma_{R,L}) \rangle_{\text{aff}} - \omega_k^\vee$. \square

Since by Proposition 6 the set J_L determines $\Sigma_{R,L}$ and therefore $X_{R,L}$, we proceed to calculate J_L , by translating results by Khare to our context (see [18, Definition 3.1 and Theorem C]).

Definition 7. Let \mathcal{D} be the Dynkin diagram associated to the root system R , and consider a subset $L \subset I$ of simple roots. We say that a fundamental co-weight ω_i^\vee is *essential relatively to L* if each irreducible component of the graph $\mathcal{D} \setminus \{\omega_i^\vee\}$ contains a root in L .

Theorem 7. *If $L \subset I$ then J_L is the set of fundamental co-weights that are essential relatively to L .*

Proof. Under the duality between $\Sigma_{R,L}$ and $\mathcal{WP}(\lambda)$ (see Definition 6 and Proposition 4) the elements of $\text{Prim}(\sigma_{R,L})$ are the exterior normals of $C_\lambda(n-1)$.

On the other hand, by [18, Theorem C] an exterior normal of a facet in $C_\lambda(n-1)$ is in \mathcal{D} if and only if the corresponding co-weight is essential (see [18, Definition 3.1]). Since J_L corresponds to the facets in $C_\lambda(n-1)$ such that their exterior normal belongs to \mathcal{D} , the result follows. \square

We finish this section by presenting the notion of core of the cone $\sigma_{R,L}$. This construction, that exhibits a relationship between the faces of \mathcal{D}^\vee and the relative interior of $\sigma_{R,L}$, will be useful in our characterization of the Gorenstein Fano closures of generic orbits.

Definition 8. If $L \subset I$, we define the *core* of $\sigma_{R,L}$, denoted by $\mathcal{C}(\sigma_{R,L})$, as the face of \mathcal{D}^\vee generated by the set $\{\omega_i^\vee : i \in L^c\}$.

The core of a cone $\sigma_{R,L}$ can easily be characterized by its invariance properties:

Lemma 8. *If $L \subset I$, then $\mathcal{C}(\sigma_{R,L}) = (\mathcal{D}^\vee)^{W_L} = \bigcap_{w \in W_L} w\mathcal{D}^\vee = (\sigma_{R,L})^{W_L}$.* \square

Proposition 9. *If $L \subset I$ is such that $\sigma_{R,L}$ is strictly convex, then $\mathring{\mathcal{C}}(\sigma_{R,L}) = (\mathring{\sigma}_{R,L})^{W_L}$.*

Proof. By Proposition 2, $u \in \mathring{\sigma}_{R,L}$ if and only if $\langle \beta, u \rangle > 0$ for all $\beta \in (R^L)^+$. If $v \in \mathring{\mathcal{C}}(\sigma_{R,L})$ then $v = \sum_{i \in L^c} a_i \omega_i^\vee$, with $a_i > 0$ for all $i \in L^c$. Thus, $\langle \beta, v \rangle > 0$ for all $\beta \in (R^L)^+$ and therefore $\mathring{\mathcal{C}}(\sigma_{R,L}) \subset \mathring{\sigma}_{R,L}$. We deduce from Lemma 8 that $\mathring{\mathcal{C}}(\sigma_{R,L}) \subset (\mathring{\sigma}_{R,L})^{W_L}$.

Assume now that $v \in (\mathring{\sigma}_{R,L})^{W_L}$. Then v belongs to $(\mathcal{D}^\vee)^{W_L} \cap \mathring{\sigma}_{R,L}$ but, by Proposition 5, v does not belong to any of the facets of the cone $(\mathcal{D}^\vee)^{W_L}$. We apply again Lemma 8 and the result follows. \square

3.2. A criteria for $X_{R,L}$ to be Gorenstein Fano.

In this section we characterize when the closure of a generic orbit is a Gorenstein Fano toric variety in terms of the combinatorial properties of the core of the associated cone $\sigma_{R,L}$.

Definition 9. We define \mathcal{F}_L as the convex hull of the set $\text{Prim}(\sigma_{R,L})$ and $\mathcal{P}_{R,L}$ as the convex hull of $\text{Prim}(\Sigma_{R,L})$.

Clearly, $\dim_{\text{aff}} \mathcal{P}_{R,L} = n$ and $\dim_{\text{aff}}(\mathcal{F}_L)$ is n or $n - 1$ — in this last case \mathcal{F}_L is a facet of $\mathcal{P}_{R,L}$. When \mathcal{F}_L is a facet, we identify the exterior normal of \mathcal{F}_L (see Definition 1) with the unique element $n_L \in (\Lambda_R^\vee)_{\mathbb{Q}}$ such that $-\varphi_L(v) = (n_L, v)$. If \mathcal{F}_L is n -dimensional, we set $n_L = 0$.

Remark 3. (1) Notice that n_L is W_L -invariant, since $\text{Prim}(\sigma_{R,L}) = W_L \cdot J_L$.

(2) In particular, if $L^c = \{\omega_i^\vee\}$, then $n_L = a\omega_i^\vee$ — this easy remark will simplify some calculations.

(3) Using the W -invariance of n_L , we can characterize the affine dimension of $\text{Conv}(\text{Prim}(\sigma_{R,L}))$ as follows:

- (i) If there exists a W_L -invariant element $n_L \in (\Lambda_R^\vee)_{\mathbb{Q}}$ such that $(n_L, \nu) = 1$ for all $\nu \in J_L$, then $\text{Conv}(\text{Prim}(\sigma_{R,L}))$ is $(n - 1)$ -dimensional with exterior normal n_L .
- (ii) If such an element does not exist, then $\text{Conv}(\text{Prim}(\sigma_{R,L}))$ is n -dimensional.

Theorem 10. *Let $X_{R,L}$ be the closure of a generic orbit. Then $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano if and only if $n_L \in \mathring{\mathcal{C}}(\sigma_{R,L})$. Moreover, if $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano then*

$$j_{X_{R,L}} = \min\{j \in \mathbb{N}^* : \forall v \in \Lambda_R^\vee, (jn_L, v) \in \mathbb{Z}\}.$$

Proof. Since $X_{R,L} = X_{-\Sigma_{R,L}} \cong X_{\Sigma_{R,L}}$ (see Definition 4 and remark 1), it suffices to prove the assertion for $X_{\Sigma_{R,L}}$.

Assume that $n_L \in \mathring{\mathcal{C}}(\sigma_{R,L})$; then $\dim_{\text{aff}}(\mathcal{F}_L) = (n - 1)$ and, by Proposition 9, $n_L \in (\mathring{\sigma}_{R,L})^{W_L}$. If $\sigma \in \Sigma_{R,L}(n)$ then $\sigma = w \cdot \sigma_{R,L}$ for some $w \in W$. Therefore, $\text{Prim}(\sigma) = w \cdot \text{Prim}(\sigma_{R,L}) = wW_L \cdot J_L$ and it follows that

$$w \cdot \mathring{\mathcal{C}}(\sigma_{R,L}) = w \cdot (\mathring{\sigma}_{R,L})^{W_L} \subset w \cdot \mathring{\sigma}_{R,L} = \mathring{\sigma}.$$

In particular, $\text{Conv}(\text{Prim}(\sigma))$ is $(n - 1)$ -dimensional, with exterior normal $w \cdot n_L \in \mathring{\sigma}$.

By Proposition 1, it remains to prove that $(w \cdot n_L, v) < 1$ for all $v \in \text{Prim}(\Sigma_{R,L}) \setminus \text{Prim}(w \cdot \sigma_{R,L})$. By the W -invariance of $\Sigma_{R,L}(n)$ and $\text{Prim}(\Sigma)$ it suffices to prove that $(n_L, v) < 1$ for all $v \in \text{Prim}(w \cdot \sigma_{R,L}) \setminus \text{Prim}(\sigma_{R,L})$, where $w \in W$ is such that $(w \cdot \sigma_{R,L}) \cap \sigma_{R,L}$ is a common $(n - 1)$ -dimensional face.

Moreover, by the W_L -invariance of $\sigma_{R,L}$, we can assume that $w = s_{\alpha_i^\vee}$, where $i \in L$ is such that $(\alpha_i^\vee)^\perp$ is the support hyperplane of a $(n - 1)$ -dimensional face of $\sigma_{R,L}$. In this case $v = s_{\alpha_i^\vee} \cdot \nu$ for some $\nu \in \text{Prim}(\sigma_{R,L}) \setminus (\alpha_i^\vee)^\perp$ and we have that

$$(n_L, s_{\alpha_i^\vee} \cdot \nu) = (n_L, \nu) - \left(n_L, 2 \frac{(\alpha_i^\vee, \nu)}{(\alpha_i^\vee, \alpha_i^\vee)} \alpha_i^\vee \right) = 1 - \left(n_L, 2 \frac{(\alpha_i^\vee, \nu)}{(\alpha_i^\vee, \alpha_i^\vee)} \alpha_i^\vee \right).$$

Since $\nu \in \sigma_{R,L} \setminus (\alpha_i^\vee)^\perp$ and $n_L \in \mathring{\sigma}_{R,L}$, then $(\alpha_i^\vee, \nu) > 0$ and $(n_L, \alpha_i^\vee) > 0$, and the assertion follows.

Conversely, assume now $n_L \notin \mathring{\mathcal{C}}(\sigma_{R,L})$; since n_L is W_L -invariant, it follows that $n_L = \sum_{i \in L^c} a_i \omega_i^\vee$. If $n_L = 0$ then $\text{Conv}(\text{Prim}(\sigma))$ is n -dimensional and it follows from Proposition 1 that $X_{R,L}$ is not \mathbb{Q} -Gorenstein Fano. If $n_L \neq 0$, there exists

$i_0 \in L^c$ such that $a_{i_0} \leq 0$. Since $i_0 \in L^c$, it follows that $s_{\alpha_{i_0}^\vee} \notin W_L$ and therefore $s_{\alpha_{i_0}^\vee} \cdot \sigma_{R,L} \in \Sigma_{R,L}(n) \setminus \sigma_{R,L}$. Let $\nu \in \text{Prim}(\sigma_{R,L}) \setminus (\alpha_{i_0}^\vee)^\perp$. Then $(n_L, \nu) = 1$ and $(\nu, \alpha_{i_0}^\vee) > 0$. It follows that

$$\left(n_L, s_{\alpha_{i_0}^\vee} \cdot \nu \right) = \left(s_{\alpha_{i_0}^\vee} \cdot n_L, \nu \right) = (n_L, \nu) - \left(2 \frac{(\alpha_{i_0}^\vee, n_L)}{(\alpha_{i_0}^\vee, \alpha_{i_0}^\vee)} \alpha_{i_0}^\vee, \nu \right) = 1 - \left(2a_{i_0} \frac{(\alpha_{i_0}^\vee, \omega_{i_0}^\vee)}{(\alpha_{i_0}^\vee, \alpha_{i_0}^\vee)} \alpha_{i_0}^\vee, \nu \right) \geq 1.$$

Since $s_{\alpha_{i_0}^\vee} \cdot \nu \in \text{Prim}(s_{\alpha_{i_0}^\vee} \cdot \sigma_{R,L}) \setminus \text{Prim}(\sigma_{R,L})$, it follows from Proposition 1 that $X_{\Sigma_{R,L}}$ and hence $X_{R,L}$ are not \mathbb{Q} -Gorenstein Fano.

The last assertion follows straightforward from Proposition 1 — indeed, recall that $W \cdot n_L$ is set of exterior normals of the facets of $\text{Conv}(\text{Prim}(\Sigma))$. \square

The following (very) easy remark will be used several times in our classification of the Gorenstein Fano closures of generic orbits.

Lemma 11. *Let $b_{i,j,k} = (\omega_i^\vee, \omega_j^\vee - \omega_k^\vee)$, $i, j, k \in I$. Assume that $L \subset I$ is such that there exist $j, k \in J_L$ with $b_{i,j,k} \geq 0$ for all $i \in L^c$, and $\sum_{i \in L^c} b_{i,j,k}^2 \neq 0$. Then $n_L \notin \mathcal{C}(\sigma_{R,L})$.*

Proof. If $n_L \in \mathcal{C}(\sigma_{R,L})$, then $n_L = \sum_{i \in L^c} a_i \omega_i^\vee$, with $a_i > 0$. Since $\omega_j^\vee - \omega_k^\vee \in \langle \text{Prim}(\sigma_{R,L}) \rangle_{\text{aff}}$ (see Proposition 6), it follows that $0 < \sum_{i \in L^c} a_i b_{i,j,k} = (n_L, \omega_j^\vee - \omega_k^\vee) = 0$, and we obtain a contradiction. \square

4. \mathbb{Q} -GORENSTEIN FANO GENERIC CLOSURES

Theorem 12. *Let G be a simple affine algebraic group of root type R and $L \subsetneq I$ a proper subset of the set of simple roots I ; let T be a maximal torus and $T \subset P$ be the parabolic subgroup associated to L . Table 1 on page 10 gives a complete list of all the closures $X_{R,L} \subset G/P$ of a generic T -orbit that are \mathbb{Q} -Gorenstein Fano, Gorenstein Fano and smooth Fano varieties — e.g. if the table indicates that $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano, then $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano but not Gorenstein Fano.*

In the third column, we draw the Dynkin diagram of R^\vee ; the subscripts indicate the number of the corresponding simple roots in I , the elements of L are the roots drawn in black, the superscript J_L over a root α_i indicates that $\omega_i^\vee \in J_L$, the set of essential co-weights. In the fourth column we indicate the corresponding geometry — the Gorenstein index of the \mathbb{Q} -Gorenstein Fano variety $X_{R,L}$ is denoted by j . Finally, in the fifth column we exhibit the exterior normal $-\varphi_L \in (\Lambda_R)_\mathbb{Q}$ — recall that $\langle -\varphi_L, v \rangle = (n_L, v)$ for all $v \in (\Lambda_R^\vee)_\mathbb{Q}$.

Remark 4. (1) The description given in Table 1 is modulo automorphisms of the root system; for example, the varieties $X_{A_n, I \setminus \{1\}}$ and $X_{A_n, I \setminus \{n\}}$ are isomorphic (and both Fano) and the table only exhibits $X_{A_n, I \setminus \{1\}}$. In the same spirit, the conditions given on the rank are established in order to avoid repetition.

(2) In the proof of Theorem 12 we exhibit n_L when $X_{R,L}$ is \mathbb{Q} -Gorenstein Fano; in order to compute φ_L we use the matrix $(\langle \omega_i, \omega_j^\vee \rangle)_{i,j \in I}$, which is the inverse of the Cartan Matrix (see [19] for explicit calculations).

TABLE 1

type	rank	Dynkin(R^\vee), L and J_L	Geometry	$-\varphi_L \in (\Lambda_R)\mathbb{Q}$
A_n	$n \geq 1$		Smooth, Fano	$(n+1)\omega_1$
	$n \geq 2$		Gorenstein Fano	$\omega_1 + \omega_n$
	$n \geq 3$, odd		Gorenstein Fano	$2\omega_{\frac{n+1}{2}}$
	$n \geq 4$, even		Smooth, Fano	$(n+1)(\omega_{\frac{n}{2}} + \omega_{\frac{n}{2}+1})$
B_n	$n \geq 2$		Gorenstein Fano	ω_1
			Smooth, Fano	$2\omega_n$
C_n	$n \geq 3$		Gorenstein Fano	ω_1
			Gorenstein Fano	ω_2
			n even: Gor. Fano n odd: \mathbb{Q} -G.F., $j=2$	ω_n
D_n	$n \geq 4$		Gorenstein Fano	$2\omega_1$
			Gorenstein Fano	ω_2
E_6	6		Gorenstein Fano	ω_2
F_4	4		\mathbb{Q} -Gor. Fano, $j=2$	$\frac{\omega_1}{2}$
			Gorenstein Fano	ω_4
G_2	2		Smooth, Fano	ω_2
			\mathbb{Q} -Gor. Fano, $j=3$	$\frac{\omega_1}{3}$

5. PROOF OF THEOREM 12

5.1. Explicit calculations for ranks 1 and 2.

The classification for ranks 1 and 2 is done by direct examination.

Explicit calculations for G of type A_1 .

Clearly $\sigma_{A_1, \{1\}} = \mathbb{Q}^+$ and therefore $X_{A_1, \emptyset} = \mathrm{SL}_2(\mathbb{k})/B \cong \mathbb{P}^1$ is a smooth Fano variety.

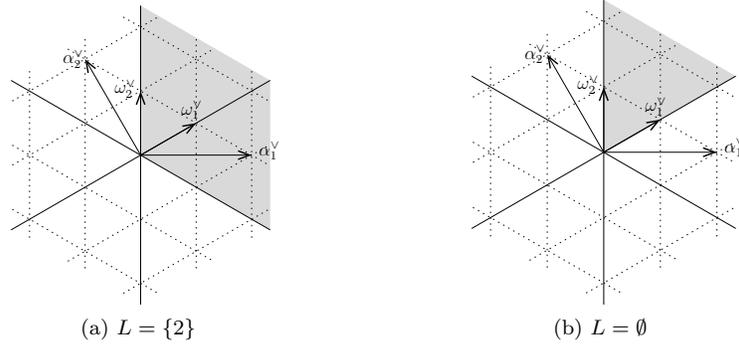
Explicit calculations for G of type A_2 .

FIGURE 1. G is of type A_2 ; the cone $\sigma_{R,L}$ is drawn in gray.

As follows from figures 1a and 1b, $\Sigma_{A_2, L}$ is a smooth Fano complete fan for all $L \subsetneq I = \{1, 2\}$ — recall that $R \cong R^\vee$. The varieties $X_{A_2, \{1\}}$ and $X_{A_2, \{2\}}$ are isomorphic to \mathbb{P}^2 with the canonical action of $T = (\mathbb{k}^*)^2 \subset \mathbb{P}^2$, and $X_{A_2, \emptyset}$ is isomorphic to the blowing up of three generic points in \mathbb{P}^2 .

Explicit calculations for G of type B_2 and C_2 .

Since $C_2 = (B_2)^\vee$, it suffices to describe the case $R = B_2$, which has Dynkin diagram $\bullet \rightleftarrows \bullet$.

If $L = \emptyset$, then $n_\emptyset = \omega_1^\vee$ and therefore $X_{B_2, \emptyset}$ is not \mathbb{Q} -Gorenstein Fano (see Figure 2a). It follows that $X_{C_2, \emptyset}$ is not \mathbb{Q} -Gorenstein Fano — in this case $n_\emptyset = \omega_2^\vee$.

If $L = \{1\}$ then $n_{\{1\}} = \omega_2^\vee$ and $X_{B_2, \{1\}}$ is a smooth Fano variety — $X_{B_2, \{1\}}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, see figure 2a. It follows that $X_{C_2, \{2\}} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

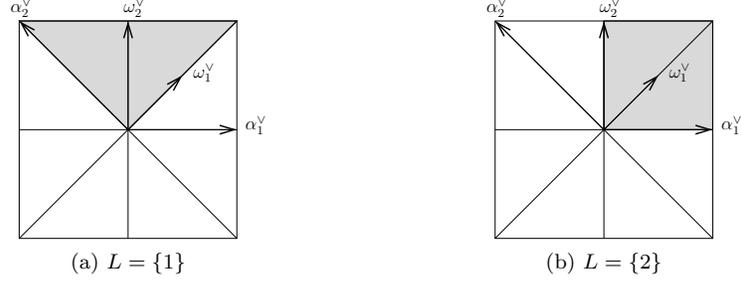
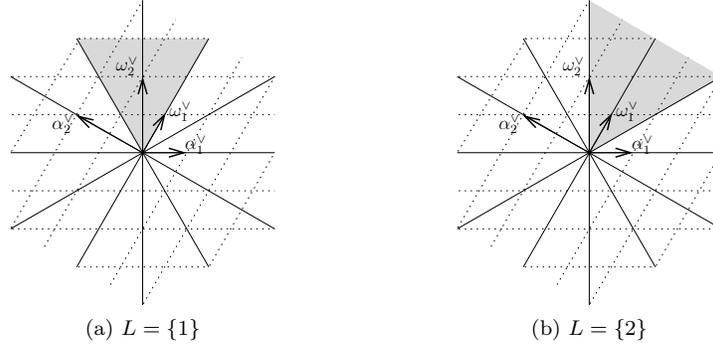
If $L = \{2\}$, then $n_{B_2, \{2\}} = \omega_1^\vee$ and $X_{B_2, \{2\}}$ and $X_{C_2, \{1\}}$ are Gorenstein Fano varieties (see figure 2b) — notice that these varieties are not smooth.

Explicit calculations for G of type G_2 .

Recall that G_2 has Dynkin diagram $\bullet \rightleftarrows \bullet$, with $G_2 \cong G_2^\vee$.

Since $\mathrm{Conv}(\mathrm{Prim}(\sigma_{G_2, \emptyset}))$ is not a proper face of $\mathrm{Conv}(\mathrm{Prim}(\Sigma_{G_2, \emptyset}))$ (see figure 3a), it follows that $X_{G_2, \emptyset}$ is not \mathbb{Q} -Gorenstein Fano.

An inspection of figure 3a shows that $X_{G_2, \{1\}}$ is a smooth Fano variety — in fact, $X_{G_2, \{1\}} \cong X_{A_2, \emptyset}$, the blowing up of three points in \mathbb{P}^2 . Also by inspection of figure 3b, we have that $X_{G_2, \{2\}}$ is \mathbb{Q} -Gorenstein Fano of index 3.

FIGURE 2. G is of type B_2 . The cone $\sigma_{R,L}$ is drawn in gray.FIGURE 3. G is of type G_2

5.2. Proof of theorem 12 for rank $n \geq 3$.

We follow a similar strategy for all cases: given R , (a multiple of) the transposed inverse of symmetrized Cartan Matrix associated to R^\vee determines a W -invariant scalar product on R^\vee (see for example [19, p. 295]). By Theorem 10, we need to determine when n_L belongs to $\mathcal{C}(\sigma_{R,L})$ (see Definition 9). Hence, for each L , we use Proposition 5 in order to compute J_L and then we deduce n_L .

Notation 3. Notice that we reason on L^c rather than on L . If $L \subset I = \{1, 2, \dots, n\}$, we denote by m (resp. M) the minimum (resp. the maximum) of the set L^c .

5.3. Explicit calculations for G of type A_n , $n \geq 3$.

We choose the W -invariant scalar product given by the matrix:

$$(1) \quad \begin{pmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \dots & 2 \cdot 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-i+1 & 2(n-i+1) & \dots & i(n-i+1) & i(n-i) & i \cdot 2 & i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 \cdot 2 & \dots & \dots & (n-1) \cdot 2 & n-1 \\ 1 & 2 & \dots & \dots & (n-1) & n \end{pmatrix}$$

5.3.1. Cases $L^c = \{1\}$ and $L^c = \{n\}$.

If $L^c = \{1\}$ then $J_L = \{\omega_n^\vee\}$, and it follows from Remark 3 that $n_L = \omega_1^\vee$. Hence, $X_{A_n, L}$ is Gorenstein Fano (see Theorem 10). Moreover, a direct computation (see

also [6, Remark 4]) shows that

$$\text{Prim}(\sigma_{R,L}) = W_L \cdot (\omega_n^\vee) = \{\omega_n^\vee, \omega_{n-1}^\vee - \omega_n^\vee, \omega_{n-2}^\vee - \omega_{n-1}^\vee, \dots, \omega_1^\vee - \omega_2^\vee\},$$

and therefore $X_{A_n,L}$ is a smooth Fano variety.

By symmetry of the Dynkin diagram, we deduce that $X_{A_n, I \setminus \{n\}}$ is also a smooth Fano variety.

5.3.2. *Case $L^c = \{i\}$, $i \neq 1, n$.*

In this case $J_L = \{\omega_1^\vee, \omega_n^\vee\}$, and $n_L = a\omega_i^\vee$ (see Remark 3). If $n \neq 2i - 1$ then $(\omega_1^\vee, \omega_i^\vee) = (\omega_n^\vee, \omega_i^\vee)$, and therefore $n_L = 0$. It follows that the polytope \mathcal{F}_L (see Definition 9) is n -dimensional and hence $X_{A_n,L}$ is not \mathbb{Q} -Gorenstein Fano (see Proposition 1).

If $n = 2i - 1$, then $n_L = \frac{\omega_i^\vee}{i}$ and $(n_L, \omega_j^\vee) \in \mathbb{Z}$ for all $j \in I$ (see Matrix (1)). It follows that $X_{A_n, I \setminus \{i\}}$ is Gorenstein Fano. Finally, since

$$\begin{aligned} \text{Prim}(\sigma_{R,L}) &= W_L \cdot \omega_1^\vee \cup W_L \cdot \omega_n^\vee = \\ &= \{\omega_1^\vee, \omega_2^\vee - \omega_1^\vee, \dots, \omega_i^\vee - \omega_{i-1}^\vee\} \cup \{\omega_n^\vee, \omega_{n-1}^\vee - \omega_n^\vee, \dots, \omega_i^\vee - \omega_{i+1}^\vee\} \end{aligned}$$

is not a simplicial set, it follows that $X_{A_n,L}$ is not smooth.

5.3.3. *Case $M = m + 1$.*

In this case $J_L = \{1, n\}$. If $X_{A_n,L}$ is \mathbb{Q} -Gorenstein Fano, it follows from Theorem 10 that $n_L = a\omega_m^\vee + b\omega_{m+1}^\vee$, with $a, b \in \mathbb{Q}_{>0}$. Hence,

$$0 = (n_L, \omega_1^\vee - \omega_n^\vee) = (a+b)(n-2m) + (a-b).$$

We deduce that $n = 2m$ and $n_L = (\omega_m^\vee + \omega_{m+1}^\vee)$. Moreover,

$$\begin{aligned} \text{Prim}(\sigma_{R,L}) &= W_L \cdot \omega_1^\vee \cup W_L \cdot \omega_n^\vee = \\ &= \{\omega_1^\vee, \omega_2^\vee - \omega_1^\vee, \dots, \omega_m^\vee - \omega_{m-1}^\vee\} \cup \{\omega_n^\vee, \omega_{n-1}^\vee - \omega_n^\vee, \dots, \omega_{m+1}^\vee - \omega_{m+2}^\vee\}, \end{aligned}$$

and therefore $\text{Prim}(\sigma_{R,L})$ is a basis of the co-weight lattice. It follows that the variety $X_{A_{2m}, \{m, m+1\}^c}$ is smooth Fano.

5.3.4. *Case $m = 1$ and $M = n$.*

In this case, $J_L = I$. If $X_{A_n,L}$ is \mathbb{Q} -Gorenstein Fano, then $n_L = \sum_{i \in L^c} a_i \omega_i^\vee$, $a_i > 0$, is defined up to a scalar by the equations $(n_L, \omega_j^\vee - \omega_{j'}^\vee) = 0$. Since $(\omega_1^\vee + \omega_n^\vee, \omega_j^\vee - \omega_{j'}^\vee) = 0$ we deduce that n_L is proportional to $\omega_1^\vee + \omega_n^\vee$, and therefore:

(i) If $L^c \neq \{1, n\}$, then $X_{A_n,L}$ is not \mathbb{Q} -Gorenstein Fano.

(ii) If $L^c = \{1, n\}$, then $n_L = \frac{\omega_1^\vee + \omega_n^\vee}{n+1}$ and therefore $X_{A_n, \{2, \dots, n-1\}}$ is Gorenstein Fano. Since $\text{Prim}(\sigma_{A_n, \{2, \dots, n-1\}}) = W_L \cdot \{\omega_j^\vee \mid j \in I\}$ is clearly not simplicial, we deduce that $X_{A_n, \{1, n\}^c}$ is not smooth Fano.

5.3.5. *Remaining cases.*

It remains to study the cases (i) $\#L^c \geq 2$, with $m \neq M - 1$ and $M < n$, and (ii) $\#L^c \geq 2$, with $m \neq M - 1$ and $1 < m$.

By symmetry, it suffices to study the case (i); in this case $J_L = \{1, m+1, \dots, M-1, n\}$. Assume that $n_L = \sum_{i \in L^c} a_i \omega_i^\vee$, $a_i > 0$; since n and $M-1$ belong to J_L , we deduce that

$$\sum_{i \in L^c} a_i (\omega_i^\vee, \omega_{M-1}^\vee - \omega_n^\vee) = 0.$$

The coefficients of the i -th row of Matrix (1) being a strict unimodal sequence with peak at the i -place, we deduce that $(\omega_i^\vee, \omega_{M-1}^\vee - \omega_n^\vee) > 0$ for all i such that $i \leq M-1$. Since $M < n$, we have that $(\omega_M^\vee, \omega_{M-1}^\vee - \omega_n^\vee) > 0$. Hence, since $L \subset \{m, \dots, M\}$, we obtain a contradiction (using Lemma 11); it follows that $X_{A_n, L}$ is not \mathbb{Q} -Gorenstein Fano.

5.4. Explicit calculations for G of type B_n , $n \geq 3$.

Recall that B_n has Dynkin diagram $\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$ and $(B_n)^\vee = C_n$. We choose the W -invariant scalar product on $(\Lambda_{B_n}^\vee)_{\mathbb{Q}} = (\Lambda_{C_n})_{\mathbb{Q}}$ given by the matrix

$$(2) \quad \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & i & \dots & i & i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & i & \dots & n-1 & n \end{pmatrix}$$

5.4.1. Case $\#L^c = 1$.

(i) If $L^c = \{i\}$ with $i \neq 1, n$, then $J_L = \{1, n\}$ and $n_L = a\omega_i^\vee$. Since $(\omega_i^\vee, \omega_1^\vee - \omega_n^\vee) = 1 - i$, it follows that $n_L = 0$ and $X_{B_n, \{i\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

(ii) If $L^c = \{1\}$, then $J_L = \{n\}$. Since $(\omega_1^\vee, \omega_n^\vee) = 1$, it follows that $n_L = \omega_1^\vee$ and therefore $X_{B_n, \{2, \dots, n\}}$ is Gorenstein Fano. Since $\#W_L \cdot \omega_n^\vee > n$, then $\sigma_{B_n, \{2, \dots, n\}}$ is not simplicial and we deduce that $X_{B_n, \{2, \dots, n\}}$ is not smooth.

(iii) If $L^c = \{n\}$, then $J_L = \{1\}$ and it follows that $n_L = \omega_n^\vee$. Hence, $X_{B_n, \{1, \dots, n-1\}}$ is Gorenstein Fano. Moreover, it is easy to see that $W_L \cdot J_L = W_L \cdot \omega_1^\vee = \{\omega_1^\vee, \omega_2^\vee - \omega_1^\vee, \dots, \omega_n^\vee - \omega_{n-1}^\vee\}$ (see for example [6, Remark (4)]) and therefore the variety $X_{B_n, \{1, \dots, n-1\}}$ is smooth Fano.

5.4.2. Case $\#L^c > 1$.

It is clear that $\{1, n\} \subset J_L$. Since $(\omega_i^\vee, \omega_n^\vee - \omega_1^\vee) = i - 1$ for all $i \in I$, we deduce from Lemma 11 that $X_{B_n, L}$ is not \mathbb{Q} -Gorenstein Fano.

5.5. Explicit calculations for G of type C_n .

Recall that C_n has Dynkin diagram $\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$. We choose the W -invariant scalar product in $(\Lambda_{C_n}^\vee)_{\mathbb{Q}} = (\Lambda_{B_n})_{\mathbb{Q}}$ given by the matrix

$$(3) \quad \begin{pmatrix} 2 & 2 & 2 & \dots & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & \dots & 4 & \dots & 4 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \dots & 2i & \dots & 2i & i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & i & \dots & n-1 & n/2 \end{pmatrix}$$

5.5.1. Case $\#L^c = 1$.

(i) If $L^c = \{1\}$ then $J_L = \{n\}$, and we deduce from Matrix (3) that $n_L = \omega_1^\vee$. Hence, $X_{C_n, \{2, \dots, n\}}$ is Gorenstein Fano. Since $\#W_L \cdot \omega_n^\vee > n$, it follows that $X_{C_n, \{2, \dots, n\}}$ is not smooth.

(ii) If $L^c = \{n\}$, then $J_L = \{1\}$ and $n_L = \omega_n^\vee$. Since $(\omega_n^\vee, \omega_i^\vee) = i$ if $i \neq n$ and $(\omega_n^\vee, \omega_n^\vee) = n/2$, we deduce that $X_{C_n, \{1, \dots, n-1\}}$ is Gorenstein Fano if n is even, and \mathbb{Q} -Gorenstein Fano of index 2 if n is odd.

It is easy to see that $W_L \cdot J_L = W_L \cdot \omega_1^\vee = \{\omega_1^\vee, \omega_2^\vee - \omega_1^\vee, \dots, 2\omega_n^\vee - \omega_{n-1}^\vee\}$ (see [6, Remark 5]); it follows that the fan $\Sigma_{C_n, \{1, \dots, n-1\}}$ is simplicial but $X_{C_n, \{1, \dots, n-1\}}$ is not smooth.

(iii) If $L^c = \{2\}$ then $J_L = \{1, n\}$. Since $(\omega_1^\vee, \omega_2^\vee) = (\omega_n^\vee, \omega_2^\vee) = 2$, it follows that $n_L = \frac{1}{2}\omega_2^\vee$. By inspection on Matrix (3) we deduce that $X_{C_n, \{2\}^c}$ is Gorenstein Fano. Since $W_L \cdot J_L > n$, it follows that $X_{C_n, \{2\}^c}$ is not smooth.

(iv) If $L^c = \{i\}$ with $i \neq 1, 2, n$, then $J_L = \{1, n\}$. Since $(\omega_i^\vee, \omega_1^\vee - \omega_n^\vee) = 2 - i$, it follows that $n_L = 0$ and the variety $X_{C_n, \{i\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

5.5.2. Case $M = m + 1$.

In this case $J_L = \{1, n\}$. Assume that $X_{C_n \{i-1, i\}^c}$, $2 \leq i \leq n$ is \mathbb{Q} -Gorenstein Fano.

(i) If $L^c = \{n-1, n\}$, then $n_L = a\omega_{n-1}^\vee + b\omega_n^\vee$, with $a, b > 0$. Since $n \geq 3$, it follows that $(n_L, \omega_1^\vee - \omega_n^\vee) = 2a - a(n-1) + b - \frac{n}{2}b = a(-n+3) - \frac{n-2}{2}b < 0$ and we obtain a contradiction.

(ii) If $L^c = \{i-1, i\}$, $i \neq n$, then $n_L = a\omega_{i-1}^\vee + b\omega_i^\vee$, with $a, b > 0$. Since $(n_L, \omega_1^\vee - \omega_n^\vee) = 2a - a(i-1) + 2b - bi = a(3-i) + b(2-i) \neq 0$ for all $2 \leq i < n$, we obtain a contradiction.

5.5.3. Case $m < M - 1$.

In this case $J_L = \{1, m+1, \dots, M-1, n\}$ and, since $m+1 < n$, it follows that $(\omega_i^\vee, \omega_{m+1}^\vee - \omega_1^\vee) \geq 0$ for all $i \in I$. We deduce from Lemma 11 that $X_{C_n, L}$ is not \mathbb{Q} -Gorenstein Fano.

5.6. Explicit calculations for G of type D_n , $n \geq 4$.

Recall that $D_n \cong (D_n)^\vee$, with Dynkin diagram . We choose

the W -invariant scalar product given by the matrix

$$(4) \quad \begin{pmatrix} 4 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 4 & 8 & 8 & \cdots & 8 & 4 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \cdots & 4i & 2i & 2i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-2) & n & n-2 \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n \end{pmatrix}$$

5.6.1. Case $\#L^c = 1$.

(i) If $L^c = \{i\}$, with $i \neq 1, 2, n-1, n$, then $J_L = \{1, n-1, n\}$ and $n_L = a\omega_i^\vee$. Since $(\omega_i^\vee, \omega_1^\vee - \omega_n^\vee) \neq 0$, we deduce that $n_L = 0$ and therefore the variety $X_{D_n, \{i\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

(ii) If $L^c = \{1\}$, then $J_L = \{n-1, n\}$. Clearly $n_L = \omega_1^\vee/2$, and $(n_L, \omega_i^\vee) \in \mathbb{Z}$ for all $i \in I$; therefore, $X_{D_n, \{1\}^c}$ is Gorenstein Fano. Since $\#W_L \cdot J_L > n$, it follows that $X_{D_n, \{1\}^c}$ is not smooth Fano.

(iii) If $L^c = \{n\}$, then $J_L = \{1, n-1\}$. If $n = 4$, we deduce by symmetry that $X_{D_4, \{1\}^c} \cong X_{D_4, \{4\}^c}$ and therefore $X_{D_4, \{4\}^c}$ is Gorenstein Fano.

If $n > 4$, then $(\omega_n^\vee, \omega_1^\vee - \omega_{n-1}^\vee) = 4 - n > 0$. It follows that $n_L = 0$ and therefore $X_{D_n, \{n\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

(iv) If $L^c = \{n-1\}$ we deduce by symmetry that the variety $X_{D_n, \{n-1\}^c}$ is Gorenstein Fano, and that $X_{D_n, \{n-1\}^c}$ is not \mathbb{Q} -Gorenstein Fano if $n > 4$.

(v) If $L^c = \{2\}$, then $J_L = \{1, n-1, n\}$. By inspection of Matrix (4), we deduce that $n_L = (\omega_2^\vee)/4$ and it follows that $X_{D_n, \{2\}^c}$ is Gorenstein Fano. Again, since $\#W_L \cdot J_L > n$, it follows that $X_{D_n, \{1\}^c}$ is not smooth.

5.6.2. Case $L^c = \{1, 2\}$.

Assume that $X_{D_n, \{1, 2\}^c}$ is \mathbb{Q} -Gorenstein Fano. Then, $n_L = a\omega_1^\vee + b\omega_2^\vee$, with $a, b > 0$. Since $J_L = \{1, n-1, n\}$ and that $(n_L, \omega_n^\vee - \omega_1^\vee) = -2a < 0$, we obtain a contradiction.

5.6.3. Case $\#L^c > 1$, $m > 1$.

It is easy to see that in this case $\{1, n-1, n\} \subset J_L$. Since $(\omega_i^\vee, \omega_{n-1}^\vee - \omega_1^\vee) \geq 0$ for $i \geq 2$, it follows from Lemma 11 that $X_{D_n, \{L\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

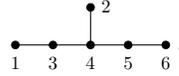
5.6.4. Remaining cases.

Since $\#L^c > 1$, $m = 1$ and $L^c \neq \{1, 2\}$, it follows that $2 < M$; hence, $\{1, 2\} \subset J_L = \{1, \dots, M-1, n-1, n\}$. Since $(\omega_i^\vee, \omega_2^\vee - \omega_1^\vee) \geq 0$, for all $i \in I$, it follows from Lemma 11 that $X_{D_n, \{L\}^c}$ is not \mathbb{Q} -Gorenstein Fano.

5.7. Explicit calculations for G of types E_6, E_7, E_8 and F_4 .

These cases can be calculated using Lemma 11 and Theorem 10, by direct inspection of the matrices associated to the W -invariant scalar product. We will treat in detail only the cases E_6 and F_4 .

5.7.1. G is of type E_6 .

The associated Dynkin diagram of E_6 is . We choose the W -invariant scalar product given by the matrix

$$\begin{pmatrix} 4 & 3 & 5 & 6 & 4 & 2 \\ 3 & 6 & 6 & 9 & 6 & 3 \\ 5 & 6 & 10 & 12 & 8 & 4 \\ 6 & 9 & 12 & 18 & 12 & 6 \\ 4 & 6 & 8 & 12 & 10 & 5 \\ 2 & 3 & 4 & 6 & 5 & 4 \end{pmatrix}$$

(i) If $L^c = \{2\}$, then $J_L = \{1, 6\}$, $n_L = \frac{1}{3}\omega_2^\vee$, and the variety $X_{E_6, \{2\}^c}$ is Gorenstein Fano and not smooth — because $\Sigma_{E_6, \{2\}^c}$ is not simplicial.

(ii) If $L^c = \{i\}$, $i \neq 2$, then $\#J_L \geq 2$, with $2 \in J_L$, and $n_L = a\omega_i^\vee$. If $j \in J_L \setminus \{2\}$, then $(\omega_i^\vee, \omega_j^\vee) \neq (\omega_i^\vee, \omega_2^\vee)$; therefore $n_L = 0$ and $X_{E_6, L}$ is not \mathbb{Q} -Gorenstein Fano.

(iii) If $L^c > 1$, then $\{1, 2, 6\} \subset J_L$.

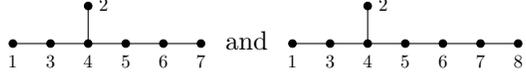
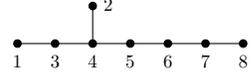
(iii-a) $1 \notin L^c$: since $(\omega_i^\vee, \omega_2^\vee - \omega_1^\vee) > 0$ for all $i \in L^c$, it follows from Lemma 11 that $X_{E_6, L}$ is not \mathbb{Q} -Gorenstein Fano.

(iii-b) $\{1, 6\} \subset L^c$: since $\{1, 2, 3, 5, 6\} \subset J_L$, it follows from Lemma 11 (applied to $j = 2$ and $k = 3$) that $X_{E_6, L}$ is not \mathbb{Q} -Gorenstein Fano.

(iii-c) $1 \in L^c$ and $6 \notin L^c$: it follows from Lemma 11 (applied to $j = 2$ and $k = 6$) that $X_{E_6, L}$ is not \mathbb{Q} -Gorenstein Fano.

5.7.2. G is of type E_7, E_8 .

The root systems E_7 and E_8 have associated associated Dynkin diagrams

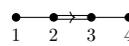
 and  respectively. Notice that in these cases (contrary to the case E_6) we have that $(\omega_2, \omega_1) \neq (\omega_2, \omega_n)$, where $n = 7$ or 8 .

We first deal with the case E_7 ; we set $n = 7$.

- (i) If $L^c = \{2\}$, then $\{1, n\} = J_L$ and $n_L = 0$; therefore $E_{n, \{2\}^c}$ is not \mathbb{Q} -Gorenstein Fano.
- (ii) If $L^c = \{i\}$, with $i \neq 2$ then $\#J_L > 1$ and $2 \in J_L$ and it follows as in the case of type E_6 (iii-a) that $X_{E_n, \{i\}^c}$ is not \mathbb{Q} -Gorenstein Fano.
- (iii) If $\#L^c > 1$, then either $\{1, 2\} \subset J_L$ or $\{2, n\} \subset J_L$. In both cases we deduce from Lemma 11 that that $E_{n, L}$ is not \mathbb{Q} -Gorenstein Fano.

In order to deal with the case E_8 , just substitute $n = 8$ in the previous discussion.

5.7.3. G is of type F_4 .

The Dynkin diagram of $F_4 \cong (F_4)^\vee$ is . We choose the W -invariant

scalar product in $(\Lambda_{F_4}^\vee)_{\mathbb{Q}}$ given by the matrix $\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 4 \end{pmatrix}$.

- (i) If $L^c = \{i\}$, with $i \neq 1, 4$, then $J_L = \{1, 4\}$ and it follows that $n_L = 0$.
- (ii) If $L^c = \{1\}$, then $J_L = \{4\}$, so $n_L = (\omega_1^\vee)/2$. It follows that $X_{F_4, \{2,3,4\}}$ is \mathbb{Q} -Gorenstein Fano of Gorenstein index 2.
- (iii) If $L^c = \{4\}$, then $J_L = \{1\}$, and $n_L = \frac{1}{2}\omega_4^\vee$. It follows that $X_{F_4, \{1,2,3\}}$ is Gorenstein Fano but not smooth, since $\#W_L \cdot \omega_1^\vee > 4$.
- (iv) If $\#L^c > 1$, then $\{1, 4\} \subset J_L$ and we deduce from Lemma 11 that $X_{F_4, L}$ is not \mathbb{Q} -Gorenstein Fano.

6. SOME COUPLE OF DUAL REFLEXIVE POLYTOPES

Once we have classified all Gorenstein Fano toric varieties $X_{R,L}$, we can apply [14, Theorem 8.3.4] (see Proposition 1) in order to produce a list of couples of reflexive polytopes $(\mathcal{P}_{R,L}, \mathcal{P}_{R,L}^\circ)$. Moreover, the W -invariance of $\mathcal{P}_{R,L}^\circ$ allows us to describe these polytopes as Weyl polytopes (see Definition 6).

Proposition 13. *Let R be a root system and $X_{R,L}$ a Gorenstein Fano generic closure. Then $\mathcal{P}_{R,L} \subset (\Lambda_R^\vee)_{\mathbb{Q}}$ is a dual reflexive polytope, with $\mathcal{P}_{R,L}^\circ = \mathcal{WP}(-\varphi_L) \subset (\Lambda_R)_{\mathbb{Q}}$, where $\varphi_L \in \Lambda_R$ is indicated in the last column of the corresponding row of Table 1.*

Proof. It remains to prove the last assertion. Since $\Sigma_{R,L}$ is stable by the W -action, it follows that $\mathcal{P}_{R,L}$ and therefore $\mathcal{P}_{R,L}^\circ$ are W -stable. Moreover, since W acts transitively on $\Sigma_{R,L}(n)$, W acts transitively on the maximal proper faces of $\mathcal{P}_{R,L}$. In particular, the set of proper faces of $\mathcal{P}_{R,L}$ is $\{w \cdot \text{Conv}(\text{Prim}(\sigma_{R,L})) : w \in W\}$. Recall that $\langle \varphi_L, u \rangle = (n_L, u)$ for all $u \in \Lambda_R^\vee$; since $\text{Conv}(\text{Prim}(\sigma_{R,L}))$ has n_L as exterior normal, it follows that

$$\mathcal{P}_{R,L}^\circ = \text{Conv}(W \cdot (-\varphi_L)) = \mathcal{WP}(-\varphi_L) \subset (\Lambda_P)_{\mathbb{Q}}.$$

□

Remark 5. The couple (L, J_L) determines completely the polytope $\mathcal{P}_{R,L}$. Indeed, $\mathcal{P}_{R,L} = \text{Conv}\langle W \cdot \{\omega_i^\vee : i \in J_L\} \rangle$. Notice that if J_L contains a single fundamental co-weight ω^\vee , then $\mathcal{P}_{R,L}$ is simply the Weyl polytope $\mathcal{WP}(\omega^\vee)$.

We finish this section with an application to the study of root polytopes — recall that if R is an (irreducible) root system, then the *root polytope* associated to R is the convex hull $\text{Conv}(R) \subset (\Lambda_R)_\mathbb{Q}$, see for example [20, 21, 22], where these polytopes are intensively studied.

Proposition 14. *Let R be an irreducible root system. Then the associated root polytope $\text{Conv}(R)$ (considered in the root lattice) is reflexive if and only if R is of type A_n, C_n, D_n, E_6 or G_2 .*

Proof. Let γ be the longest root of R ; then since γ is a dominant weight and $-\gamma$ is also a root, it follows that $\text{Conv}(R) = \mathcal{WP}(\gamma) = \mathcal{WP}(-\gamma)$.

Taking into account the description of γ as a dominant weight (see for example [15]), we deduce by inspection of Table 1 that there exists L such that $X_{R,L}$ is Gorenstein Fano, with $\varphi_L = \gamma$, if and only if R is of type A_n, C_n, D_n, E_6 or G_2 . If this is the case, then $\text{Conv}(R) = \mathcal{WP}(-\gamma)$ is a reflexive polytope.

Reciprocally, if $\text{Conv}(R)$ is a reflexive polytope, then the toric variety X associated to the normal fan of $\text{Conv}(R)$ is a Gorenstein Fano variety. Since $\Sigma_X = \Sigma_{R,L}$, where $\gamma = \sum_{i \in L} a_i \omega_i$, $a_i > 0$, the result follows. □

Example 1. We conclude with three explicit examples of reflexive polytopes (associated to generic closures) and their duals.

Type A_2 . The polytopes $\mathcal{P}_{A_2, \{1\}} \cong \mathcal{P}_{A_2, \{2\}}$ and $\mathcal{P}_{A_2, \emptyset}$ are reflexive. In the figure 4, the polytopes $\mathcal{P}_{A_2, L}$, with vertices in the weight lattice, are the interior polytopes whereas their duals are the exterior ones (with vertex in the root lattice).

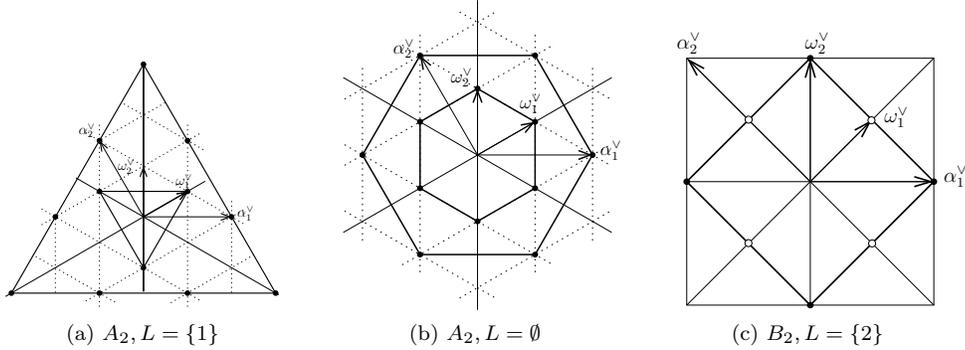


FIGURE 4. Three dual reflexive polygons associated to root systems of rank 2.

Type B_2 . The polytopes $\mathcal{P}_{B_2, \{2\}}$ and $\mathcal{P}_{B_2, \{1\}}$ are reflexive. In the figure 4, $\mathcal{P}_{B_2, \{2\}}$ is the exterior polytope, with vertex in the weight lattice of C_2 — it contains 9 lattice points; $\mathcal{P}_{B_2, \{2\}}^\vee$ is the same polytope but in the root lattice of B_2 , so it contains 5 lattice points.

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Email address: pierre-louis.montagard@umontpellier.fr

IMAG, UNIVERSITÉ DE MONTPELLIER, PLACE EUGÈNE BATAILLON, MONTPELLIER, 34090,
FRANCE

Email address: `alvaro.rittatore@cmat.edu.uy`

CENTRO DE MATEMÁTICA, UNIVERSIDAD DE MONTEVIDEO, IGUÁ 4225 ESQ. MATAOJO., MON-
TEVIDEO, URUGUAY