CONTROLLABILITY AND OBSERVABILITY FOR NON-AUTONOMOUS EVOLUTION EQUATIONS: THE AVERAGED HAUTUS TEST

El-Maati Ouhabaz, Bernhard Haak, Duc-Trung Hoang

CONTROLLABILITY AND OBSERVABILITY FOR NON-AUTONOMOUS EVOLUTION EQUATIONS: THE AVERAGED HAUTUS TEST

BERNHARD HAAK, DUC-TRUNG HOANG, AND EL MAATI OUHABAZ

Abstract. We consider the observability problem for non-autonomous evolution systems (i.e., the operators governing the system depend on time). We introduce an averaged Hautus condition and prove that for skew-adjoint operators it characterizes exact observability. Next, we extend this to more general class of operators under a growth condition on the associated evolution family. We give an application to the Schrödinger equation with time dependent potential and the damped wave equation with a time dependent damping coefficient.

1. Introduction

Observability is an important concept in system and control theory. It treats the question to which extent an observation, i.e., partial knowledge of the solution of an evolution equation, determines its initial or final state. The theory has been studied for several decades for systems of the form:

$$\left\{ \begin{array}{lcl} x'(t) + Ax(t) & = & 0 \quad t \in [0,T] \\ x(0) & = & x_0 \\ y(t) & = & Cx(t) \end{array} \right. \tag{1.1}$$

in which the two operators A and C are independent of time t and satisfy appropriate conditions such as $-A$, with domain $\mathscr{D}(A)$, generates a strongly continuous semigroup on a Hilbert space H and C is bounded from $\mathscr{D}(A)$ into another Hilbert space Y.

Observability consists of unique determination or recovery of the initial (or final) time state under the knowledge of the observed solution $y(\cdot)$. Recall that in the case of matrices A and C (finite dimensional setting), all observation concepts coincide and can be characterized in various manners. The Kalman rank condition is certainly the most known version; it states that C is observable if and only if the matrix

$$[C \,|\, CA \,|\, CA^2 \,|\, \dots \,|\, CA^{n-1}]$$

has full rank. An equivalent statement is the Hautus lemma: it characterizes observability by the condition

$$\forall \lambda \in \mathbb{C}: \qquad \operatorname{rank}[\lambda I - A, C] = n$$

that clearly is equivalent to the condition

$$\|Cx\|^2 + \|(\lambda I - A)x\|^2 \geq \kappa \|x\|^2. \tag{1.2}$$

In an infinite-dimensional setting with operators A, C, instead of matrices, rank conditions are not appropriate. However, the Hautus test in the form (1.2) can be generalized, and has actually been proposed in [24] as a criterion for observability. Russell and Weiss conjectured in [24] that this inequality characterizes exact observability. They proved in [24] that the conjecture is valid for bounded and invertible operators A. Later, Jacob and Zwart [12] showed equivalence for diagonal semigroup generator on a Riesz basis if the output space Y is finite dimensional. The general

Institut de Mathématiques de Bordeaux, UMR CNRS 5251, Université de Bordeaux, 351 cours de la Liberation, 33405 Talence, France

E-mail addresses: bernhard.haak@math.u-bordeaux.fr , duc-trung.hoang@math.u-bordeaux.fr , elmaati.ouhabaz@math.u-bordeaux.fr.

Date: August 26, 2019.

The first named author is partially supported by the Collaborative Research DFG and ANR project INFIDHEM ANR-16-CE92-0028 .

conjecture was later proved to be wrong, see [13]. Note however, that if C is admissible and A has a bounded H^∞-calculus on a suitable sector (which is, in turn a consequence of admissibility and exact observation, see Proposition 5.1 in [11]), then it does not seem to be known whether the Hautus condition implies observability. There exist other formulations of the Hautus condition (or spectral condition) and there are several cases where it implies exact observability. This holds for example if A generates a unitary group. We refer to [18, 27] for early results with bounded observations, and [5, 19] for successive extensions. These have subsequently been generalized (see [14]) to groups with certain growth bounds. See also [26] for more information and references on this subject.

In this paper we consider first order non-autonomous evolution equations of the following form:

$$\text{(A,C)} \qquad \begin{cases} x'(t)+A(t)x(t) &= 0 \quad t\in[0,\tau] \\ x(0) &= x_0 \\ y(t) &= C(t)x(t). \end{cases}$$

The difference with (1.1) is that we allow operators A and C to depend on time t. To be precise, let $\tau>0$ and assume that for $t\in[0,\tau]$, the operator $A(t)$ generates a strongly continuous contraction semigroup $(e^{-sA(t)})_{s\geq 0}$ on the Hilbert space H. We suppose further that there exists a densely and continuously embedded subspace $\mathscr{D}\hookrightarrow H$ such that that for all $t\in[0,T]$, $\mathscr{D}(A(t))=\mathscr{D}$ and that $t\mapsto A(t)v$ is continuously differentiable in H for every $v\in\mathscr{D}$. These assumptions are sufficient to guarantee that the Cauchy problem $x'(t)=A(t)x(t)$, $x(0)=x_0$ admits a solution, see e.g. [21, Sections 5.3 and 5.4]. For each t, $C(t):\mathscr{D}\to Y$ is a bounded operator. Then, for initial data $x_0\in\mathscr{D}$, the solution x to (A,C) satisfies $x(t)\in\mathscr{D}$ for each $t\geq 0$ and hence $y(t)$ is well defined. We define observability concepts (and controllability concepts for the adjoint system) as in the autonomous case (1.1).

In the case of time-dependent matrices, a famous result of Silverman and Meadows [25] characterizes exact observability and controllability. Their arguments have been adapted to certain infinite dimensional settings, see for example [1, 2, 3]. Our main objective is different. We seek to prove observability from a certain Hautus type condition. In order to do this, we introduce the following averaged Hautus conditions:

$$\|x\|^2\leq m^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|C(s)e^{\lambda s}x\big\|^2\,\mathrm{d}s\Big)+M^2\Big(\tfrac{1}{\tau}\int_0^\tau e^{\mathrm{Re}\lambda.s}\big\|(\lambda+A(s))x\big\|\,\mathrm{d}s\Big)^2$$

for all $\lambda\in\mathbb{C}$ and all $x\in\mathscr{D}$, or

$$\|x\|^2\leq m^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|C(s)x\big\|^2\,\mathrm{d}s\Big)+M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi+A(s))x\big\|^2\,\mathrm{d}s\Big)$$

for all $\xi\in\mathbb{R}$ and $x\in\mathscr{D}$. These inequalities do coincide with the usual Hautus conditions if the operators A and C are independent of t. We prove that these averaged Hautus conditions imply exact observability when the operators $A(t)$ are skew-adjoint. This result is refined to the case of invertible evolution families (not necessarily unitary) under certain growth constraints. We apply these results to Schrödinger equations with time dependent potentials and to a damped wave-equation with time-dependent damping.

Finally, we mention the papers [8], [2]and the references therein on observability (or controllability) of parabolic equations (with time dependent coefficients). The approach in these papers is based on Carleman estimates and it differs from ours.

2. Preliminary results

Recall that we suppose $A(t):\mathscr{D}\to H$ to have a fixed domain, that $t\mapsto A(t)v$ is continuously differentiable in H for every $v\in\mathscr{D}$ and each semigroup $e^{-sA(t)}$ is a contraction on H. By [21, Sections 5.3 and 5.4] there exists a unique evolution family $(U(t,s))_{0\leq s\leq t\leq\tau}$ on H generated by $A(t)_{0\leq t\leq\tau}$. This evolution family satisfies the following properties.

(1) $\|U(t,s)\|\leq Me^{-\omega(t-s)}$ for some $\omega\in\mathbb{R}$
(2) For all $v\in\mathscr{D}$, $\frac{\partial^+}{\partial t}U(t,s)v|_{t=s}=-A(s)v$, $\quad \frac{\partial^+}{\partial t}U(t,s)v=-A(t)U(t,s)v$.
(3) For all $v\in\mathscr{D}$, $\frac{\partial}{\partial s}U(t,s)v=U(t,s)A(s)v$.

(4) $U(t,s)\mathscr{D} \subseteq \mathscr{D}$
(5) For all $v \in \mathscr{D}$, $(s,t) \mapsto U(t,s)v$ is continuous in $\mathscr{D}$ for $0 \le s \le t \le T$.

For every $v \in \mathscr{D}$, the evolution equation

(CP) $$\begin{cases} \frac{d}{dt}\eta(t) + A(t)\eta(t) & = & 0 \quad 0 \le s \le t \le \tau \\ \eta(s) & = & v \end{cases}$$

has a unique solution. This solution is given by $\eta(t) = U(t,s)v$. For $f \in L^1(0,\tau;H)$, the non homogeneous problem

(NHCP) $$\begin{cases} \frac{d}{dt}\eta(t) + A(t)\eta(t) & = & f(t) \quad 0 \le s \le t \le \tau \\ \eta(s) & = & v \in H. \end{cases}$$

has then a mild solution given by

$$\eta(t) = U(t,s)v + \int_s^t U(t,r)f(r)\,\mathrm{d}r, \tag{2.1}$$

see e.g. [21, p.146]. If, in addition to the standing assumptions, $f \in C^1([s,\tau],H)$ then (NHCP) has a unique classical solution which coincides with the mild solution, see for example [21, Theorem 5.2, p.146].

We associate with (A,C) the operator

$$(\Psi_{s,\tau}x)(t) = \begin{cases} C(t)U(t,s)x & t \in [s,\tau] \\ 0 & t > \tau \end{cases}$$

and define the following notions:

Definition 2.1 (Averaged admissible observations). *Let $(C(t))_{t\in[0,\tau]}$ be a family of bounded operators in $\mathscr{L}(\mathscr{D},Y)$, where Y is some Hilbert space. We say that $(C(t))_t$ are averaged admissible observations for $(A(t))_{t\in[0,\tau]}$ if there exists a constant $M_\tau > 0$ such that*

$$\int_s^\tau \left\|C(t)U(t,s)x\right\|^2 \mathrm{d}t \le M_\tau^2 \|x\|^2 \quad \forall x \in \mathscr{D}, s \in [0,\tau].$$

(one can also consider a weaker admissibility notion by requiring the above inequality for $s = 0$, only). For a single operator $C(t_0)$ such that

$$\int_0^\tau \left\|C(t_0)U(t,s)x\right\|^2 \mathrm{d}t \le M_\tau \|x\|^2 \quad \forall x \in \mathscr{D}$$

we say that $C(t_0)$ is admissible for $(A(t))_{t\in[0,T]}$.

For averaged admissible observations, $\Psi_{s,\tau}$ extends to a bounded operator from H to $L_2(s,\tau;Y)$ which we denote again by $\Psi_{s,\tau}$.

In this definition the norm inside the integral is taken in Y and the norm of x is taken in H. We always use the same notation $\|\cdot\|$ for both, the difference will be clear from the context.

Definition 2.2. *Suppose that $(C(t))_t$ is an averaged admissible observation for $(A(t))_t$. We say that the system (A,C) is*

a) exactly averaged observable in time τ *if the map $\Psi_{s,\tau}$ is bounded from below in the sense that there exists a constant $\kappa_\tau > 0$ such that for all $x \in \mathscr{D}$*

$$\int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t \ge \kappa_\tau \|x\|^2.$$

For a given $t_0 \in [0,\tau]$, the system $(A, C(t_0))$ is exactly observable at time τ if

$$\int_0^\tau \left\|C(t_0)U(t,0)x\right\|^2 \mathrm{d}t \ge \kappa_\tau \|x\|^2.$$

b) final-time averaged observable in time τ *if there exists a constant $\kappa_\tau > 0$ such that*

$$\int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t \ge \kappa_\tau \|U(\tau,0)x\|^2 \quad \forall x \in \mathscr{D}.$$

As above we define final observability for the simple operator $C(t_0)$ for some t_0 as

$$\int_0^\tau \left\|C(t_0)U(t,0)x\right\|^2 \,\mathrm{d}t \geq \kappa_\tau \|U(\tau,0)x\|^2.$$

c) approximately averaged-observable in time τ *if* $\ker \Psi_{s,\tau} = \{0\}$ *for all* $0 \leq s < \tau$*. Again we define approximate observability for a single operator* $C(t_0)$ *if* $(A, C(t_0))$ *is approximate observable in average as above.*

In order to justify the use of the term "averaged" in the previous notions of observability, we note that it might be possible that $(A, C(t_0))$ is not exactly (or final or approximately) observable for some $C(t_0)$ or even for all $t_0 \in J$ for some subset J of $[0,\tau]$ but (A, C) is exactly (or final or approximately) observable in average. In order to see this, we consider the autonomous case $A(t) = A$ and an observation operator C such that the autonomous system is exactly (or null or approximately) observable at time τ_0. Define

$$C(t) = \begin{cases} C, & t \in [0,\tau_0] \\ 0, & t \in (\tau_0,\tau]. \end{cases}$$

Then

$$\int_0^\tau \left\|C(t)e^{-tA}x\right\|^2 \mathrm{d}t \geq \int_0^{\tau_0} \left\|C(t)e^{-tA}x\right\|^2 \mathrm{d}t \geq \kappa_\tau \|x\|^2.$$

Hence the averaged observability property for $(A, C(t))$ at time τ holds but the system $(A, C(t_0))$ is not observable for $t_0 \in (\tau_0, \tau]$ at any time. The same observation is valid for null and approximate average observability.

Along with (A,C) we consider a controlled evolution equation. First, we recall the following: one can construct an extrapolation space H_{-1} and extrapolated operators $A_{-1}(t)$ such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{\quad A_{-1}(t) \quad} & H_{-1}(t) \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle i}\big\uparrow \\ \mathscr{D} & \xrightarrow{\quad A(t) \quad} & H \end{array}$$

One way to realize $H_{-1}(t)$ is to take the completion of H with respect to a resolvent norm $\|(\lambda - A(t))^{-1}x\|_H$ or via its identification with $\mathscr{D}(A(t)^*)'$. For all this we refer to [9, Chapter II.5].

In order to keep the abstract setting simple we will suppose for the rest of this section that $\mathscr{D}(A(t)^*) =: \mathscr{D}^*$ is independent of time as well and equivalent norms with constants independent of t. Note that if for all $t \in [0,\tau]$, $A(t) = A(0) + R_t$ with a bounded operator on H, then $A(t)^* = A(0)^* + R_t^*$ with domain $\mathscr{D}^* := \mathscr{D}(A(0)^*)$ independent of t. In the setting of the averaged Hautus test we consider later, we will make the assumption $A(t) = A(0) + R_t$ with a family of uniformly bounded operators R_t on H. In this case $H_{-1}(t) = H_{-1}$ and have equivalent norms with constants independent of t.

Let U be another Hilbert space and let $B(t) : U \to H_{-1}$ is bounded for each $t \in [0,\tau]$. We consider in H_{-1} the evolution equation

$$\text{(A,B)} \qquad \begin{cases} x'(t) + A(t)x(t) &= B(t)u(t) \qquad t \in [0,\tau] \\ x(s) &= 0. \end{cases}$$

Since the mild solution is of the form (2.1), we have the naturally associated operator

$$\Phi_{s,\tau} u = \int_s^\tau U(\tau,r)B(r)u(r)\,\mathrm{d}r \qquad (\tau \leq \tau) \tag{2.2}$$

to (A,B).

Definition 2.3 (Averaged admissible controls)**.** *Let $(B(t))_{t\in[0,\tau]}$ be a family of bounded operators in $\mathscr{L}(U;H_{-1})$. We say that $(B(t))_t$ are averaged admissible controls for $(A(t))_{t\in[0,\tau]}$ if there exists a constant $M_\tau > 0$ such that the solution x to (A,B) satisfies $x(t)\in H$ and for all $s\in[0,\tau)$*

$$\Big\|\int_s^\tau U(\tau,r)B(r)u(r)\,\mathrm{d}r\Big\|^2 \le M_\tau^2\big\|u\big\|^2_{L_2(s,\tau;U)}$$

for all $u\in\mathscr{D}(0,\tau;U)$ (one can also consider a weaker admissibility notions by requiring the above inequality for $s=0$, only).

Let us consider the retrograde final-value problem

$$\left\{\begin{array}{rcl} z'(t) - A(t)^* z(t) & = & 0 \\ z(\tau) & = & z_\tau. \end{array}\right. \tag{2.3}$$

Observe that for $x\in\mathscr{D}$ and $x^*\in\mathscr{D}^*$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle x, U(\tau,t)^*x^*\rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\langle U(\tau,t)x, x^*\rangle = -\langle U(\tau,t)A(t)x, x^*\rangle = \langle x, -A(t)^*U(\tau,t)^*x^*\rangle$$

so that $z(t) = U(\tau,t)^* z_\tau$ solves the retrograde equation (2.3) on $[s,\tau]$ for all $0\le s<\tau$.

Lemma 2.4. *The family $(B(t))_{t\in[0,\tau]}$ are admissible controls for $(A(t))_{t\in[0,\tau]}$ if and only if the family $(B(t)^*)_{t\in[0,\tau]}$ are admissible observations for the retrograde equation (2.3).*

Proof. The following calculation is standard.

$$\begin{aligned}\sup_{\|u\|_2\le 1}\Big\|\int_s^\tau U(\tau,r)B(r)u(r)\,\mathrm{d}r\Big\| &= \sup_{\|u\|_2\le 1}\sup_{\|x^*\|\le 1}\Big|\int_s^\tau \langle U(\tau,r)B(r)u(r), x^*\rangle\,\mathrm{d}r\Big| \\ &= \sup_{\|x^*\|\le 1}\sup_{\|u\|_2\le 1}\Big|\int_s^\tau \langle u(r), B(r)^*U(\tau,r)^*x^*\rangle\,\mathrm{d}r\Big| \\ &= \sup_{\|x^*\|\le 1}\Big(\int_s^\tau \big\|B(r)^*U(\tau,r)^*x^*\big\|^2\,\mathrm{d}r\Big)^{1/2}.\end{aligned}$$ □

Definition 2.5. *Let $(B(t))_t$ be averaged admissible controls for $(A(t))_{t\in[0,\tau]}$. We say that (A,B) is*

a) Exactly averaged controllable in time τ *if for any $s\in[0,\tau)$ and $x_s, x_\tau\in H$, there exist $u\in L^2(s,\tau;U)$ such that the mild solution x satisfies $x(s)=x_s$ and $x(\tau)=x_\tau$.*
This definition coincides with the usual one in the autonomous case, that is, given two states $x_s, x_\tau\in H$ we find a control u such that the solution takes the value x_s at the initial time $t=s$ and the value x_τ at time $t=\tau$.

b) approximately averaged controllable in time τ *if for any $0\le s<\tau$ and any $x_s, x_\tau\in H$ and $\varepsilon>0$, there exist $u\in L^2(0,\tau;U)$ such that $x(s)=x_s$ and $\|x(\tau)-x_\tau\|<\varepsilon$.*

c) averaged null controllable in time τ *if for every $0\le s<\tau$ and every $x_s\in H$, there exist $u\in L^2(s,\tau;U)$ such that the mild solution x satisfies $x(s)=x_s$ and $x(\tau)=0$.*

Since the mild solution is given by

$$x(t) = U(t,s)x_s + \int_s^t U(t,r)B(r)u(r)\,\mathrm{d}r$$

it is clear that in order to obtain exact averaged controllability it suffices to consider the case where $x(s)=0$.

Proposition 2.6. *Let $B(t)\in\mathscr{L}(U,H_{-1})$ be a family of averaged admissible controls for $(A(t))_{t\in[0,\tau]}$. Then*

a) *Exact averaged controllability for* (A,B) *in time τ is equivalent to exact averaged observability of the retrograde final-value problem (2.3) with the observation operators $C(t)=B(t)^*$.*

b) *Approximate averaged controllability for* (A,B) *in time τ is equivalent to approximate averaged observability of the retrograde final-value problem (2.3) with the observation operators $C(t)=B(t)^*$.*

c) *Averaged null controllability for* (A,B) *in time* τ *is equivalent to averaged observability of* $z(s)$, $0 \le s < \tau$ *where* z *is the solution of the retrograde final-value problem (2.3) with the observation operators* $C(t) = B(t)^*$.

Proof. First note that $(\Phi_{s,\tau}^* z_s)(t) = B(t)^* U^*(\tau,t) z_s$ for $t \in [s,\tau]$. For simplicity we extend this function by zero for other values of t. Exact averaged controllability for (A,B) at τ is equivalent to range$(\Phi_{s,\tau}) = H$ for all s. Since these operators are bounded, the latter property is equivalent to the fact that their adjoints $\Phi_{s,\tau}^*$ is bounded from below on $L^2(s,\tau;H)$, i.e., there exists $\kappa_{s,\tau}$ such that

$$\int_s^\tau \|B(t)^* U(\tau,t)^* z_s\|^2 \,\mathrm{d}t \ge \kappa_{s,\tau} \|z_s\|^2$$

for all $z_s \in \mathscr{D}^*$. Approximate averaged controllability is equivalent to range$(\Phi_{s,\tau})$ being dense for all $s \in [0,\tau)$, or, equivalently, the respective adjoints being injective. Finally, averaged null controllability in time τ is equivalent to range$(U(\tau,s)) \subset$ range$(\Phi_{s,\tau})$ for all $0 \le s < \tau$. Applying [26, Proposition 12.1.2], averaged null controllability is equivalent to

$$\|U(\tau,s)^* z_\tau\|^2 \le \delta^2 \|\Phi_{s,\tau}^* z_\tau\|^2 = \delta^2 \int_s^\tau \left\|B(t)^* U(\tau,t)^* z_\tau\right\|^2 \mathrm{d}t$$

for some constant $\delta > 0$. But $U(\tau,s)^* z_\tau = z(s)$ where $z(\cdot)$ is the solution of the retrograde equation (2.3). □

3. The averaged Hautus test: skew-adjoint operators

Throughout this section, the family of operators $A(t)_{0\le t\le\tau}$ is as before. Let $C(t)_{0\le t\le\tau}$ be a family of bounded operators from $\mathscr{D}$ to a Hilbert space Y. In the autonomous case $A(t) = A$ and $C(t) = C$ for all t, it is well known that for admissible C the exact observability of the system (A,C) implies the so-called Hautus test (or spectral condition)

$$\|x\|^2 \le m^2 \|Cx\|^2 + M^2 \|(i\xi + A)x\|^2 \tag{3.1}$$

for some positive constants m and M and all $\xi \in \mathbb{R}$ and $x \in \mathscr{D}(A)$. There is also another condition with $\lambda \in \mathbb{C}$ in place of $i\xi$, see below. In the general non-autonomous situation we introduce an integrated (or averaged) version of this test. We also study, as in the autonomous case, when the averaged Hautus test is necessary and/or sufficient for averaged observability. We start with the "necessary" part.

Proposition 3.1. *Suppose that* $(C(t))$ *is averaged admissible for* $(A(t))$. *If the system* (A,C) *is exactly averaged observable at time* $\tau > 0$ *then there exist positive constants* m *and* M *such that:*

$$\|x\|^2 \le m^2 \Big(\tfrac{1}{\tau}\int_0^\tau \left\|C(s)e^{\lambda s}x\right\|^2 \mathrm{d}s\Big) + M^2 \Big(\tfrac{1}{\tau}\int_0^\tau e^{Re\lambda.s} \|(\lambda + A(s))x\| \,\mathrm{d}s\Big)^2 \tag{AH.1}$$

for all $\lambda \in \mathbb{C}$ *and all* $x \in \mathscr{D}$,

$$\|x\|^2 \le m^2 \Big(\tfrac{1}{\tau}\int_0^\tau \left\|C(s)x\right\|^2 \mathrm{d}s\Big) + M^2 \Big(\tfrac{1}{\tau}\int_0^\tau \left\|(i\xi + A(s))x\right\|^2 \mathrm{d}s\Big) \tag{AH.2}$$

for all $\xi \in \mathbb{R}$ *and* $x \in \mathscr{D}$.

Remark 3.2. *If* $C(s) = C$ *for all* s *then* (AH.1) *can be written as:*

$$\|x\|^2 \le \tfrac{e^{2\tau Re(\lambda)} - 1}{2\tau Re(\lambda)} m^2 \|Cx\|^2 + M^2 \big(\tfrac{1}{\tau}\int_0^\tau e^{Re\lambda.s} \left\|(\lambda + A(s))x\right\| \mathrm{d}s\big)^2 \tag{AH.3}$$

If, in addition, $A(s)$=A *then both assertions coincide with the classical Hautus (or spectral) conditions. We call the conditions* (AH.1) *and* (AH.2) *averaged Hautus tests.*

Proof. The proof is similar to the autonomous case. We start from $\frac{d}{\mathrm{d}s}\big(e^{\lambda s}C(t)U(t,s)x\big) = \lambda e^{\lambda s}C(t)U(t,s)x + e^{\lambda s}C(t)U(t,s)A(s)x$ for $x \in \mathscr{D}$. Integrating on $[0,\tau]$ yields

$$e^{\lambda t}C(t)x - C(t)U(t,0)x = \int_0^t C(t)U(t,s)(A(s) + \lambda)x e^{\lambda s} \,\mathrm{d}s.$$

Hence,

$$\int_0^\tau \big\|C(t)U(t,0)x\big\|^2\,\mathrm{d}t \le 2\int_0^\tau \big\|C(t)xe^{\lambda t}\big\|^2\,\mathrm{d}t + 2\int_0^\tau \Big\|\int_0^t C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}\,\mathrm{d}s\Big\|^2\,\mathrm{d}t$$

Since (A,C) is exactly averaged observable on $[0,\tau]$, the left hand side is bounded below by $m_0\|x\|^2$ for some constant $m_0>0$. We estimate the second term on the right hand side

$$\begin{aligned} I &:= \left(\int_0^\tau \Big\|\int_0^t C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}\,\mathrm{d}s\Big\|^2\,\mathrm{d}t\right)^{1/2} \\ &= \sup\left\{\Big|\int_0^\tau\int_0^t \Big\langle C(t)U(t,s)(\lambda{+}A(s))xe^{\lambda s}, g(t)\Big\rangle_H \,\mathrm{d}s\,\mathrm{d}t\Big| \;:\; \|g\|_{L_2(0,\tau;H)}\le 1\right\} \\ &= \sup_{\|g\|_{L_2}\le 1}\left|\int_0^\tau \Big\langle (\lambda{+}A(s))x\,e^{\lambda s}, \int_s^\tau U(t,s)^*C(t)^*g(t)\,\mathrm{d}t\Big\rangle_H \,\mathrm{d}s\right| \\ &\le \sup_{\|g\|_{L_2}\le 1}\Big(\int_0^\tau \big\|(\lambda{+}A(s))x\,e^{\lambda s}\big\|_H \,\Big\|\int_s^\tau U(t,s)^*C(t)^*g(t)\,\mathrm{d}t\Big\|_H \,\mathrm{d}s\Big). \end{aligned}$$

By Lemma 2.4 and the admissibility assumption of $(C(t))$, there exists a constant $K_\tau>0$ such that

$$I \le K_\tau\int_0^\tau \big\|(\lambda{+}A(s))xe^{\lambda s}\big\|\,\mathrm{d}s = K_\tau\int_0^\tau \big\|(\lambda{+}A(s))x\big\|e^{\mathrm{Re}\lambda.s}\,\mathrm{d}s.$$

and (AH.1) follows. The second assertion is obtained from the first one by taking $\lambda=i\xi$ and using the Cauchy-Schwarz inequality. □

Now we study the converse. In the autonomous case i.e., $A(s)=A$ and $C(t)=C$, it is well known that condition (AH.2) implies the exact observability if the single operator A is skew-adjoint. We extend this result to our more general situation.

Theorem 3.3. *Suppose that $A(t)\in\mathscr{L}(\mathscr{D};H)$ be a family of skew-adjoint operators generating an evolution family $U(t,s)_{0\le s\le t\le\tau}$. Suppose that the* differences *of the operators $A(t)$ are bounded and satisfy the estimate*

$$\big\|A(t)-A(s)\big\|_{\mathscr{L}(H)} \le L \qquad \forall t,s\in[0,\tau]$$

for some constant $L<\frac{1}{\sqrt{2}M}$. Assume that $C(t)\in\mathscr{L}(\mathscr{D};Y)$ is a family of averaged admissible observation operators and that the second averaged Hautus condition (AH.2) *holds with positive constants m and M. Then, for all $\tau>\tau^*:=\frac{2\pi M}{\sqrt{1-2L^2M^2}}$ there exists $\kappa_\tau>0$ depending on M,L and τ such that, for all $x\in\mathscr{D}$ the exact averaged observability estimate*

$$\frac{1}{\tau}\int_0^\tau\int_0^\tau \big\|C(s)U(t,0)x\big\|^2\,\mathrm{d}t\,\mathrm{d}s \ge \frac{\kappa_\tau}{m^2}\|x\|^2 \tag{3.2}$$

holds. In particular, if $C(s)=C$ is constant, then the system (A,C) *is exactly averaged observable for $\tau>\tau^*$, i.e, for all $x\in\mathscr{D}$,*

$$\int_0^\tau \big\|CU(t,0)x\big\|^2\,\mathrm{d}t \;\ge\; \frac{\kappa_\tau}{m^2}\,\|x\|^2.$$

Proof. We proceed in a similar way as in the autonomous case. Let $\tau>0$, $\varphi\in H_0^1(0,\tau)$ and $x\in\mathscr{D}$. For $t,s\in[0,\tau]$, let $h(t):=\varphi(t)U(t,0)x$ and $f(t,s):=h'(t)+A(s)h(t)$. Note that h and $f(.,s)$ can be extended continuously by zero outside $(0,\tau)$ since $\varphi\in H_0^1(0,\tau)$. We write $\widehat{f}(\xi,s)$ for the partial Fourier transform of f with respect to the first variable, and observe that

$$\widehat{f}(\xi,s) = \int_{\mathbb{R}} e^{-it\xi}f(t,s)\,\mathrm{d}t = \int_{\mathbb{R}} e^{-it\xi}h'(t)\,\mathrm{d}t + \int_{\mathbb{R}} e^{-it\xi}A(s)h(t)\,\mathrm{d}t \;=\; i\xi\widehat{h}(\xi)+A(s)\widehat{h}(\xi)$$

where we use the fact that each operator $A(s)$ is closed in order to have $\widehat{A(s)h}(\xi)=A(s)\widehat{h}(\xi)$. We apply (AH.2) with $z_0=\widehat{h}(\xi)$ to obtain

$$\|\widehat{h}(\xi)\|^2 \le \frac{m^2}{\tau}\int_0^\tau \big\|C(s)\widehat{h}(\xi)\big\|^2\,\mathrm{d}s + \frac{M^2}{\tau}\int_0^\tau \big\|(i\xi+A(s))\widehat{h}(\xi)\big\|^2\,\mathrm{d}s$$

$$= \tfrac{m^2}{\tau} \int_0^\tau \big\|C(s)\widehat{h}(\xi)\big\|^2\,\mathrm{d}s + \tfrac{M^2}{\tau} \int_0^\tau \big\|\widehat{f}(\xi,s)\big\|^2\,\mathrm{d}s.$$

We integrate over all $\xi \in \mathbb{R}$ and use Plancherel's theorem together with the fact that $C(s)\widehat{h}(\xi) = \widehat{C(s)h(\xi)}$ to deduce

$$\int_0^\tau \big\|h(t)\big\|^2\,\mathrm{d}t \le \tfrac{m^2}{\tau}\int_0^\tau\int_0^\tau \big\|C(s)h(t)\big\|^2\,\mathrm{d}t\,\mathrm{d}s + \tfrac{M^2}{\tau}\int_0^\tau\int_0^\tau \big\|f(t,s)\big\|^2\,\mathrm{d}t\,\mathrm{d}s. \tag{3.3}$$

We estimate the last term on the right hand side as follows

$$\begin{aligned}
&\int_0^\tau\int_0^\tau \big\|f(t,s)\big\|^2\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^\tau\int_0^\tau \big\|h'(t) + A(s)h(t)\big\|^2\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_0^\tau\int_0^\tau \big\|\varphi'(t)U(t,0)x - \varphi(t)A(t)U(t,0)x + \varphi(t)A(s)U(t,0)x\big\|^2\,\mathrm{d}t\,\mathrm{d}s\\
&\le 2\tau\int_0^\tau \big\|U(t,0)x\big\|^2|\varphi'(t)|^2\,\mathrm{d}t + 2\int_0^\tau\int_0^\tau \big\|(A(t)-A(s))U(t,0)x\big\|^2|\varphi(t)|^2\,\mathrm{d}t\,\mathrm{d}s. \qquad (3.4)
\end{aligned}$$

By skew-adjointness,

$$\frac{d}{dt}\big\|U(t,s)x\big\|^2 = -2\mathrm{Re}\langle A(t)U(t,s)x, U(t,s)x\rangle = 0$$

for $x \in \mathscr{D}$ and so $U(t,s)$ is unitary for $0 \le s \le t \le \tau$. Therefore (3.3) can be rewritten as

$$\begin{aligned}
\|x\|^2\int_0^\tau \big|\varphi(t)\big|^2\,\mathrm{d}t \le \tfrac{m^2}{\tau}\int_0^\tau\int_0^\tau \big\|C(s)U(t,0)x\big\|^2\varphi(t)^2\,\mathrm{d}t\,\mathrm{d}s + 2M^2\|x\|^2\int_0^\tau \big|\varphi'(t)\big|^2\,\mathrm{d}t\\
+ 2L^2M^2\|x\|^2\int_0^\tau \big|\varphi(t)\big|^2\,\mathrm{d}t.
\end{aligned}$$

Hence

$$\kappa(\varphi)\,\|x\|^2 \le \tfrac{m^2}{\tau}\int_0^\tau\int_0^\tau \big\|C(s)U(t,0)x\big\|^2\big|\varphi(t)\big|^2\,\mathrm{d}t\,\mathrm{d}s$$

where

$$\kappa(\varphi) = \Big((1-2L^2M^2)\int_0^\tau \big|\varphi(t)\big|^2\,\mathrm{d}t - 2M^2\int_0^\tau \big|\varphi'(t)\big|^2\,\mathrm{d}t\Big).$$

We have to chose φ such that the constant $\kappa(\varphi)$ is positive. Taking the first eigenfunction of the Dirichlet Laplacian on $(0,\tau)$, i.e., $\varphi(t) := \sin\big(\frac{t\pi}{\tau}\big)$, we maximize $\kappa(\varphi)$ and obtain from $\|\varphi\|_\infty = 1$

$$\tfrac{\kappa\tau}{m^2}\,\|x\|^2 \le \int_0^\tau\int_0^\tau \big\|C(s)U(t,0)x\big\|^2\,\mathrm{d}t\,\mathrm{d}s$$

where $\kappa = \big((1-2L^2M^2)\frac{\tau}{2} - \frac{\pi^2M^2}{\tau}\big)$. To ensure $\kappa > 0$ we need $L^2 < \frac{1}{2M^2}$ and $\tau > \tau^*$. □

Remark 3.4. (1) *In* (3.4) *we have used for simplicity the inequality* $(a+b)^2 \le 2(a^2+b^2)$ *but we could instead use* $(a+b)^2 \le (1+r)a^2 + (1+r^{-1})b^2$ *for any* $r > 0$*. In this case, we obtain the theorem (with the same proof) with the conditions* $L < \frac{1}{M\sqrt{1+r}}$ *and* $\tau^* = \frac{\pi M\sqrt{1+r^{-1}}}{\sqrt{1-M^2(1+r)L^2}}$.

(2) *If* $A(t) = A$ *and hence* $L = 0$ *we obtain (from the previous remark) as minimal control time* $\tau^* = \pi M$*. This is the usual minimal time in the case of unitary groups.*

(3) *In the last assertion of theorem, if instead of* $C(s) = C$*, we assume that*

$$\|C(s) - C(t)\| \le L_0|t-s|^\alpha$$

for some positive constants α *and* L_0 *we obtain that for* L_0 *small enough, the system* (A,C) *is exactly averaged observable. Indeed, we have from* (3.2)

$$\kappa\|x\|^2 \le 2\int_0^\tau\int_0^\tau \big\|(C(t)-C(s))U(t,0)x\big\|^2\,\mathrm{d}s\,\mathrm{d}t + 2\int_0^\tau\int_0^\tau \big\|C(t)U(t,0)x\big\|^2\,\mathrm{d}s\,\mathrm{d}t$$

$$\leq 2L_0 \int_0^\tau \int_0^\tau |t-s|^{2\alpha}\,\mathrm{d}s\,\mathrm{d}t \|x\|^2 + 2\tau \int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t$$
$$= \frac{2L_0\tau^{2\alpha+2}}{(2\alpha+1)(\alpha+1)}\|x\|^2 + 2\tau \int_0^\tau \left\|C(t)U(t,0)x\right\|^2 \mathrm{d}t.$$

(4) *If we define*

$$\widetilde{C}x := \tfrac{1}{\tau}\int_0^\tau C(s)x\,\mathrm{d}s$$

then we can apply Proposition 3.1 and Theorem 3.3 to the time independent operator $\widetilde{C}$. *We obtain equivalence between*

$$\kappa_\tau \|x\|^2 \leq \int_0^\tau \left\| \int_0^\tau C(s)U(t,0)x\,\mathrm{d}s \right\|^2 \mathrm{d}t$$

and

$$\|x\|^2 \leq m^2 \left\| \Big(\tfrac{1}{\tau}\int_0^\tau C(s)\,\mathrm{d}s \Big) x \right\|^2 + M^2 \Big(\tfrac{1}{\tau}\int_0^\tau \left\|(i\xi - A(s))x\right\|^2 \mathrm{d}s \Big).$$

(5) *We have assumed in the theorem that* $A(t)$ *are skew-adjoint operators in order to have* $U(t,s)$ *is a unitary operator on* H. *The previous proof works under the assumption that*

$$K_0\|x\| \leq \|U(t,0)x\| \leq K_1\|x\|,\ x \in H$$

for some positive constants K_0 *and* K_1. *The statement of the theorem holds with different conditions* L *and* τ^* *(depending on* K_0 *and* K_1*).*

4. The averaged Hautus test: a more general class of operators

In this section we extend Theorem 3.3 to a more general class of operators. More precisely, we consider operators $A(t)$ for which the corresponding evolution family $U(t,s)$ is not necessarily an isometry but satisfies an estimate of the form

$$ke^{\alpha(t-s)}\|x\| \leq \|U(t,s)x\| \leq Ke^{\beta(t-s)}\|x\|,\ x \in H \tag{4.1}$$

for some constants k, K, α and β. This question was considered in the autonomous case $A(t) = A$ and $C(t) = C$ by Jacob and Zwart [14]. We shall follow similar ideas as in their paper. Note however, even in this autonomous case, the result is very much less precise than in the case of unitary groups. In particular, the minimal time for observability obtained in [14] is $\frac{1}{\beta-\alpha}$. This value becomes large as α and β are close and this is not consistent with the result on unitary groups.

The main tool is the following optimal Hardy inequality.

Theorem 4.1 (Gurka [10], Opic-Kufner [20])**.** *Let* $v, w \geq 0$ *be weight functions on* $[0,\tau]$. *Then the weighted Hardy inequality*

$$\left\|\varphi\right\|_{L_2(0,\tau;w(x)dx)} \leq C_H \left\|\varphi'\right\|_{L_2(0,\tau;v(x)dx)} \tag{4.2}$$

holds for all $\varphi \in H_0^1(0,\tau)$ *if and only if*

$$B := \sup\left\{ \left(\int_x^y w(t)\,\mathrm{d}t \right) \min\left(\int_0^x \tfrac{1}{v(t)}\,\mathrm{d}t, \int_y^\tau \tfrac{1}{v(t)}\,\mathrm{d}t \right) : \quad 0 < x, y < \tau \right\}$$

is finite. In this case, the optimal constant C_H *in* (4.2) *satisfies* $\frac{B}{\sqrt{2}} \leq C_H \leq 4B$.

We make a basic remark on evolution families $U(t,s)_{0\leq s\leq t}$. Given $U(t,s)$ which is exponentially bounded, i.e., $\|U(t,s)x\| \leq Ke^{\beta(t-s)}\|x\|$. If in addition each $U(t,s)$ is invertible then writing $V(t) := U(t,0)$ gives

$$V(t) = U(t,0) = U(t,s)U(s,0) = U(t,s)V(s) \quad \Longleftrightarrow \quad U(t,s) = V(t)V(s)^{-1}.$$

Then $I = V(t)V(t)^{-1}$ gives $\|x\| \leq Ke^{\beta t}\|V(t)^{-1}x\|$ and so $\|V(t)^{-1}x\| \geq \frac{1}{K}e^{-\beta t}\|x\|$ so that

$$ke^{\alpha(t-s)}\|x\| \leq \|U(t,s)x\| \leq Ke^{\beta(t-s)}\|x\|. \tag{4.3}$$

holds for $\alpha = -\beta$ and $k = \frac{1}{K}$. If A is 'shifted', i.e., replaced by $A+\omega$, this symmetry $\alpha = -\beta$ will break, and we will therefore use only (4.3) for *some* constants $k, K > 0$ and $\alpha \leq \beta$.

Theorem 4.2. *Let $A(t)_{0\leq t\leq\tau} \in \mathscr{L}(\mathscr{D}; H)$ be a family of operators generating an evolution family $U(t,s)$ and let $0 < k \leq K$ and $\alpha < \beta$ be such that* (4.3) *holds. We suppose that the differences $A(t) - A(s)$ are bounded operators with $\|A(t) - A(s)\| \leq L$ for some L such that $L < \dfrac{k}{\sqrt{2}KMe^{(\beta-\alpha)\tau}}$. Let $C \in \mathscr{L}(\mathscr{D}; Y)$. Then the averaged Hautus condition* (AH.3) *implies exact observability for all $\tau > \tau^{**}$, i.e.,*

$$\int_0^\tau \left\|CU(t,0)x\right\|^2 \mathrm{d}t \geq \tfrac{\kappa}{m^2} \left\|x\right\|^2 \quad \forall x \in H$$

for some $\tau^{**} > 0$ provided that there exist $0 \leq x \leq y \leq \tau^{**}$ such that

$$f(x,y) := \left(\tfrac{k^2}{4K^2M^2(\beta-\alpha)}(e^{-2(\beta-\alpha)x} - e^{-2(\beta-\alpha)y}) + L^2(x-y)\right)\min(x, \tau - y) > 2.$$

Proof. Observe that exact (averaged) observability is invariant under spectral shifts (replacing A by $A+\omega$), which in turn allows to assume $\beta = 0$ and $\alpha = -\omega$ for $\omega = \beta - \alpha > 0$. We follow the lines of the proof of Theorem 3.3 until (3.4). Using (4.3) instead of unitarity leads to consider a new function

$$\kappa(\varphi) := \int_0^\tau |\varphi(t)|^2 (k^2 e^{-2\omega t} - 2K^2M^2L^2)\, dt \;-\; 2K^2M^2 \int_0^\tau |\varphi'(t)|^2\, dt.$$

Then $\kappa(\varphi) > 0$ is equivalent to

$$\int_0^\tau |\varphi'(t)|^2\, dt < \int_0^\tau |\varphi(t)|^2 (\tfrac{k^2}{2K^2M^2} e^{-2\omega t} - L^2)\, dt. \tag{4.4}$$

This is an 'inverse Hardy inequality', when compared to (4.2). To establish such an estimate for at least one function $\varphi \in H_0^1(0,\tau)$, we consider on $[0,\tau]$ the weight function

$$w(t) = \tfrac{k^2}{2K^2M^2} e^{-2\omega t} - L^2.$$

Observe that w is positive if

$$0 \leq L < \frac{k}{\sqrt{2}KMe^{(\beta-\alpha)\tau}}. \tag{4.5}$$

In order to obtain (4.4) we use the optimality statement in Theorem 4.1 with $v(x) = 1$: if $\sqrt{2} < B < \infty$, the optimal constant guaranteeing (4.2) is larger than one. Hence, for any $C < 1$ there exists a $\varphi \in H_0^1([0,\tau])$ for which (4.2) fails. This function will then satisfy (4.4), and provides a strictly positive constant $\kappa(\varphi)$, yielding exact averaged observability with $\kappa := \kappa(\varphi)$, as in the proof of Theorem 3.3 (by rescaling we may suppose $\|\varphi\|_\infty = 1$). Clearly, $\sqrt{2} < B$ is equivalent to our condition on $f(x,y)$ to be larger than 2 for some $0 \leq x \leq y$. □

On the compact set $T = \{0 \leq x \leq y \leq \tau\} \subset \mathbb{R}^2$ we consider the function

$$\begin{aligned} f(x,y) &:= \left(\int_x^y w(t)\, dt\right) \min\left(\int_0^x \mathrm{d}t, \int_y^\tau dt\right) \\ &= \left(\tfrac{k^2}{4K^2M^2\omega}(e^{-2\omega x} - e^{-2\omega y}) + L^2(x-y)\right)\min(x, \tau - y). \end{aligned}$$

It is continuous and satisfies $f|_{\partial T} = 0$ so that the maximum is taken inside T. However, due to the many parameters and the mixture of power-type functions with exponentials it may be difficult to calculate explicitly the maximum of f in T. We therefore concentrate on a sufficient condition that ensures $f(x,y) > 2$ for some x and y. We consider for example the case where $f(\frac{1}{4}\tau, \frac{3}{4}\tau) > 2$, i.e.,

$$\tau e^{-\frac{\omega\tau}{2}} \left(\tfrac{1-e^{-\omega\tau}}{\omega\tau}\right) > \tfrac{1}{\tau}\left(\tfrac{32K^2M^2}{k^2}\right) + \tau\left(\tfrac{2L^2K^2M^2}{k^2}\right).$$

By numerical calculations*, we see that if $\omega\tau \leq \frac{1}{\sqrt{2}}$, then the left hand side is larger than $\frac{\tau}{2}$, so that for $\frac{2L^2K^2M^2}{k^2} < \frac{1}{4}$, $\tau^2 = \frac{128K^2M^2}{k^2}$ gives a concrete observation time. We obtain the following corollary.

Corollary 4.3. *Suppose that $L < \frac{k}{2\sqrt{2}KM}$ and $0 \leq \beta - \alpha \leq \frac{k}{16KM}$. Then we have exact observability at time $\tau > \tau^{**}$ where $\tau^{**} = \frac{8\sqrt{2}KM}{k}$. In particular, if k=K=1 and L, M are such that $8L^2M^2 < 1$ and $0 \leq \beta - \alpha \leq \frac{1}{16M}$, then we have exact observability at time $\tau > \tau^{**}$ where $\tau^{**} = 8\sqrt{2}M$.*

In the autonomous case $A(t) = A$ with A is a generator of a group we have $L = 0$, hence for $0 \leq \beta - \alpha \leq \frac{k}{16KM}$ we obtain exact observability at time $\tau > \tau^{**} = \frac{8\sqrt{2}KM}{k}$. This might be better than the observation time given in [14] which is $\frac{1}{\beta-\alpha}$.

5. Applications to the wave and Schrödinger equations with time dependent potentials

In this section we give applications of our results to observability of the Schrödinger and wave equations both with time dependent potentials. We also consider the damped wave equation with time dependent damped term. Before going into these examples we explain the general idea. It is based on a perturbation argument which shows that the Hautus test carries over from the time independent operator to time dependent ones. Once the Hautus test is satisfied by the perturbed operator we appeal to the results of the previous sections and obtain observability of the system.

Let A be the generator of unitary group on H. We assume that $C : \mathscr{D}(A) \to Y$ is an admissible operator and such that the system (A, C) is exactly observable at time τ_0. Therefore the Hautus test is satisfied by the operators A and C. Now let $R(t)_{0\leq t\leq \tau}$ be a family of uniformly bounded operators on H. By classical bounded perturbation argument (see, e.g., [9, Theorem 9.19]). the operators given by $A(t) = A+R(t)$, $t \in [0, \tau]$, generate an evolution family $U(t, s)$ on H. Note that for every $x \in H$

$$e^{-\beta(t-s)}\|x\| \leq \|U(t,s)x\| \leq e^{\beta(t-s)}\|x\| \tag{5.1}$$

with $\beta = \sup_{t\in[0,\tau]} \|R(t)\|$. Indeed, one has for every $x \in \mathscr{D}(A)$, $\mathrm{Re}\langle (A+R(t))x, x\rangle = \mathrm{Re}\langle R(t)x, x\rangle$ ans hence

$$-\beta\|x\|^2 \leq \mathrm{Re}\langle (A + R(t))x, x\rangle \leq \beta\|x\|^2.$$

We apply this with $U(t, s)x$ at the place of x and obtain

$$-\beta\|U(t,s)x\|^2 \leq \tfrac{1}{2}\tfrac{\partial}{\partial t}\big\|U(t,s)x\big\|^2 \leq \beta\|U(t,s)x\|^2.$$

We integrate and obtain (5.1). Note that if $\mathrm{Re}\langle R(t)x, x\rangle = 0$, then $U(t, s)$ is unitary.

Let now $x \in \mathscr{D}(A)$ and $\xi \in \mathbb{R}$. The Hautus test for (A, C) gives

$$\begin{aligned}\|x\|^2 &\leq m^2\|Cx\|^2 + M^2\|(i\xi + A)x\|^2 \\ &\leq m^2\|Cx\|^2 + 2M^2\|(i\xi + A + R(s))x\|^2 + 2M^2\|R(s)\|^2\|x\|^2.\end{aligned}$$

Integrating on $[0, \tau]$ with respect to s gives

$$\|x\|^2 \leq m^2\|Cx\|^2 + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi + A + R(s))x\big\|^2\,\mathrm{d}s\Big) + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|R(s)\big\|^2\,\mathrm{d}s\Big)\|x\|^2.$$

Suppose in addition that there exists $\tau_1 > 0$ and $\mu < 1$ such that for $\tau \geq \tau_1$

$$2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|R(s)\big\|^2\,\mathrm{d}s\Big) \leq \mu. \tag{5.2}$$

Then we obtain

$$(1-\mu)\|x\|^2 \leq m^2\|Cx\|^2 + 2M^2\Big(\tfrac{1}{\tau}\int_0^\tau \big\|(i\xi + A + R(s))x\big\|^2\,\mathrm{d}s\Big). \tag{5.3}$$

*The function $g(x) = e^{-x/2}(\frac{1-e^{-x}}{x})$ is larger than 1/2 for $x \leq 0.7143$ and $\frac{1}{\sqrt{2}} \leq 0.70711$.

Note that we could also replace $i\xi$ by $\lambda \in \mathbb{C}$ and obtain the Hautus test (AH.3). Next we assume that C is admissible for the unitary group e^{tA} generated by A. That is there exists a constant $K_\tau > 0$ such that

$$\int_0^\tau \left\| Ce^{tA}x \right\|^2 \mathrm{d}t \le K_\tau \|x\|^2, \; x \in \mathscr{D}(A). \tag{5.4}$$

We prove that C is admissible for $(A + R(t))$. In order to do so, we start from Duhamel's formula[†]

$$U(t,s)x - e^{(t-s)A}x = \int_s^t e^{(t-r)A}R(r)U(r,s)x \,\mathrm{d}r. \tag{5.5}$$

We use (5.4) so that

$$\begin{aligned} \int_0^\tau \left\| CU(t,s)x \right\|^2 \mathrm{d}t &\le 2\int_0^\tau \left\| Ce^{(t-s)A}x \right\|^2 \mathrm{d}t + 2\int_0^\tau \left\| \int_s^t Ce^{(t-r)A}R(r)U(r,s)x \,\mathrm{d}r \right\|^2 \mathrm{d}t \\ &\le 2K_\tau \|x\|^2 + 2\tau \int_s^\tau \int_r^\tau \left\| Ce^{(t-r)A}R(r)U(r,s)x \right\|^2 \mathrm{d}t \,\mathrm{d}r \\ &\le 2K_\tau \|x\|^2 + 2K_\tau \int_s^\tau \|R(r)U(r,s)x\|^2 \,\mathrm{d}r \le \; K'_\tau \|x\|^2, \end{aligned}$$

where we use the fact that the operators $R(r)$ are uniformly bounded and $U(t,s)$ is exponentially bounded.

We have admissibility of C and the averaged Hautus test (5.3). Now we conclude either by Theorem 3.3 or Corollary 4.3 that, as soon as $\|R(t) - R(s)\|$ are small enough, we have exact observability of the system $(A + R(.), C)$ at time $\tau > \tau^*$ for some $\tau^* > 0$. Note that (5.2) holds if $R(t) = 0$ for $t \ge t_0$ for some $t_0 > 0$.

The Schrödinger equation. Let Ω be a bounded domain of $\mathbb{R}^d$ with a C^2-boundary Γ. Let Γ_0 be an open subset of Γ and $Y = L^2(\Gamma_0)$. It is known that for appropriate condition on Γ_0, the Schrödinger equation

$$\begin{cases} z'(t,x) &=& i\Delta z(t,x) \quad (t,x) \in [0,\tau] \times \Omega \\ z(0,.) &=& z_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ z(t,x) &=& 0 \quad (t,x) \in [0,\tau] \times \Gamma \end{cases} \tag{5.6}$$

satisfies the observability inequality

$$\int_0^\tau \int_{\Gamma_0} |\tfrac{\partial z}{\partial \nu}(t,x)|^2 d\sigma \,\mathrm{d}t \ge \kappa_\tau \|z_0\|^2_{H_0^1(\Omega)} \tag{5.7}$$

for every $\tau > 0$, see for example [26, Chapter 7]. Let C be the normal derivative $\frac{\partial}{\partial \nu}$ on Γ_0, $Y = L^2(\Gamma_0, d\sigma)$ and Δ_D the Laplacian with Dirichlet boundary conditions. The previous inequality means that the system $(i\Delta_D, C)$ is exactly observable at time τ. Let now $R(t)f = iV(t)f$ where $V(t,.) \in W^{1,\infty}(\Omega)$ is a real-valued potential which depends on time. Then under appropriate conditions on V we obtain from the discussion above that the non-autonomous system $(i(\Delta_D + V(t)), C)$ is exactly observable at time $\tau > \tau^*$ for some $\tau^* > 0$. This means that (5.7) is satisfied for the solution of the Schrödinger equation with time dependent potential

$$\begin{cases} z'(t,x) &=& i\Delta z(t,x) + iV(t)z(t,x) \quad (t,x) \in [0,\tau] \times \Omega \\ z(0,.) &=& z_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ z(t,x) &=& 0 \quad (t,x) \in [0,\tau] \times \Gamma. \end{cases} \tag{5.8}$$

Note however that our method does not give observability at any time $\tau > 0$. If $V(t) = V$ is independent of t then observability for the Schrödinger equation perturbed by the potential V holds at any time $\tau > 0$, see [26, Chapter 7] and the references there.

[†] in order to prove this formula one takes the derivative of $f(r) := e^{(t-r)A}U(r,s)x$ for $s \le r \le t$ and then integrate from s to t.

The wave equation. Let again Ω be a bounded smooth domain of $\mathbb{R}^d$. We consider the wave equation

$$\left\{\begin{array}{lcl} z''(t,x) & = & \Delta z(t,x) \in [0,\tau]\times\Omega \\ z(0,.) & = & z_0 \in H_0^1(\Omega),\ z'(0,.) = z_1 \in L^2(\Omega) \\ z(t,x) & = & 0 \quad (t,x)\in[0,\tau]\times\Gamma. \end{array}\right. \tag{5.9}$$

Let Γ_0 be a part of the boundary Γ. Observability for the wave equation with the observation operator $C = \frac{\partial}{\partial\nu}_{|\Gamma_0}$ have been intensively studied. Under appropriate geometric conditions on Γ_0, there exists $\tau_0 > 0$ such that for $\tau > \tau_0$ there exists a positive constant κ_τ such that

$$\kappa_\tau\Big(\int_\Omega |z_1|^2 + \int_\Omega |\nabla z_0|^2\Big) \le \int_0^\tau\int_{\Gamma_0} |\tfrac{\partial z}{\partial\nu}|^2 d\sigma\,\mathrm{d}t. \tag{5.10}$$

We refer to [4, 17, 15] and the references therein. Let $A_0 = \begin{pmatrix} 0 & I \\ -\Delta_D & 0\end{pmatrix}$ on $H := H_0^1(\Omega)\times L^2(\Omega)$. It is a standard fact that A_0 generates a unitary group $U(t)_{t\in\mathbb{R}}$ on H. Set $\widetilde{C}(f,g) := (\frac{\partial f}{\partial\nu}_{|\Gamma_0}, 0)$. Then the energy estimate (5.10) is precisely the observability inequality

$$\kappa_\tau\|(z_0,z_1)\|_H^2 \le \int_0^\tau \big\|\widetilde{C}U(t)(z_0,z_1)\big\|_{L^2(\Gamma_0)}^2\,\mathrm{d}t. \tag{5.11}$$

Now we consider the damped wave equation without a potential

$$\left\{\begin{array}{lcl} z''(t,x) & = & \Delta z(t,x) + b(t,x)z'(t,x) + V(t,x)z(t,x) \in [0,\tau]\times\Omega \\ z(0,.) & = & z_0 \in H_0^1(\Omega),\ z'(0,.) = z_1 \in L^2(\Omega) \\ z(t,x) & = & 0 \quad (t,x)\in[0,\tau]\times\Gamma. \end{array}\right. \tag{5.12}$$

Going to the first order system on H, the wave equation (5.12) can be rewritten as $Z' = A(t)Z$ with $A(t) = \begin{pmatrix} 0 & I \\ \Delta + V(t) & b(t)\end{pmatrix} = A_0 + R(t)$ where $R(t) = \begin{pmatrix} 0 & 0 \\ V(t) & b(t)\end{pmatrix}$. As in the case of the Schrödinger equation we can apply the previous discussion to see that the Hautus test for A_0 implies our averaged Hautus test for $(A(t))_t$. In order to do so we need to verify (5.2). This property holds if

$$\frac{1}{\tau}\int_0^\tau \Big(\|V(t)\|_{W^{1,\infty}(\Omega)}^2 + \|b(t)\|_{L^\infty(\Omega)}^2\Big)\,\mathrm{d}t$$

is small enough. The norms $\|R(t) - R(s)\|$ are small if the quantities $\|V(t) - V(s)\|_{W^{1,\infty}(\Omega)} + \|b(t) - b(s)\|_{L^\infty(\Omega)}$ are small. In this case, we obtain exact averaged observability for (5.12). That is, we obtain the energy estimate (5.10) for τ large enough for solution z to (5.12). If V and b are independent of t then observability results are known (see [26]). If $b(t) = 0$ and V depends on t, then a more precise result can be found in [22] for a special class of Γ_0. The proof in [22] is different from ours and it is based on Carleman estimates.

References

[1] F. Alabau-Boussouira, J.-M. Coron, G. Olive *Internal controllability of first order quasi-linear hyperbolic systems with a reduced number of controls.* SIAM J. Control Optim. 55 (2017), no. 1, 300–323.

[2] F. Ammar Khodja, A. Benabdallah, C. Dupaix, M. González-Burgos *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems.* Differ. Equ. Appl. 1 (2009), no. 3, 427–457.

[3] I. Aksikas, J. Winkin, D. Dochain *Optimal LQ-feedback control for a class of first-order hyperbolic distributed parameter systems.* ESAIM Control Optim. Calc. Var. 14 (2008), no. 4, 897–908.

[4] C. Bardos, G. Lebeau and J. Rauch *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, *SIAM J. Control Optim.* 30 (1992), no. 5, 1024-1065.

[5] N. Burq and M. Zworski *Geometric control in the presence of a black box.* J. Amer. Math. Soc. 17 (2004), no. 2, 443–471.

[6] J.M. Coron, *Control and Nonlinearity, Mathematical Surveys and Monographs*, vol. 136, American Mathematical Society, Providence, RI, 2007.

[7] S. Dolecki and D. L. Russell, *A general theory of observation and control, SIAM J. Control Optimization*, 15 (1977), no. 2, pp. 185–220.

[8] O.Yu. Èmanuilov, *Controllability of parabolic equations* (Russian), Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879?900.

[9] K.J. Engel and R. Nagel *One-parameter semigroups for linear evolution equations*, *Graduate Texts in Mathematics*, 194. Springer-Verlag, New York, 2000. xxii+586 pp. ISBN: 0-387-98463-1.
[10] P. Gurka, Petr *Generalized Hardy's inequality.* Časopis Pěst. Mat. 109 (1984), no. 2, 194–203.
[11] B. Haak and E.M. Ouhabaz, *Exact observability, square functions and spectral theory*, *J. Funct. Anal.* 262 (2012), no. 6, 2903-2927.
[12] B. Jacob and H. Zwart, *Observability of diagonal systems with a finite-dimensional output operator*, *Control Lett.*, 43 (2001), pp. 101-109.
[13] B. Jacob and H. Zwart *Counterexamples concerning observation operators for C0-semigroups.* SIAM J. Control Optim. 43 (2004), no. 1, 137–153.
[14] B. Jacob and H. Zwart, *On the Hautus test for exponentially stable C_0-groups*, *SIAM J.Control Optim*, vol. 48, No.3, (2009) pp 1275-1288.
[15] V. Komornik, *Exact controllability and stabilization (the multiplier method)*, *Wiley, Masson*, Paris 1995.
[16] T. Li, *Controllability and Observability for Quasilinear Hyperbolic Systems*, *AIMS Series on Applied Mathematics, Vol. 3, American Institute of Mathematical Sciences (AIMS)*, Springfield, MO, 2010.
[17] J. L. Lions, *Controlabilité exacte, perturbations et stabilisation de systèmes distribués*, *Tome 1, Recherches en Mathématiques Appliquées*, vol. 8, Masson, Paris, 1988.
[18] K. Liu *Locally distributed control and damping for the conservative systems.* SIAM J. Control Optim. 35 (1997), no. 5, 1574–1590.
[19] L. Miller *Controllability cost of conservative systems: resolvent condition and transmutation.* J. Funct. Anal. 218 (2005), no. 2, 425–444.
[20] B. Opic and A. Kufner *Hardy-type inequalities*, Czechoslovak Academy of Sciences, Longman Sciences / Wiley (1990), ISBN 0-563-05135-3.
[21] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, *Applied mathematical sciences AMS*, vol 44, 1983.
[22] J.P. Puel, *Global Carleman inequalities for the wave equation and applications to controllability and inverse problems*, see `http://www1.univ-ag.fr/aoc/pub/gdt/co/Puel-Inverse-problems.pdf`.
[23] D.L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, *Stud. Appl. Math*, 52 (1973), 189–221.
[24] D. L. Russell and G. Weiss, *A general necessary condition for exact observability*, *SIAM J. Control Optim*, Vol 32 (1), pp. 1–23, (1994).
[25] L. M. Silverman and H. E. Meadows, *Controllability and observability in time-variable linear systems*, *SIAM J. Control Optim*, vol. 5, no.1 , 64–73, 1967.
[26] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Basel, 2009.
[27] Q. Zhou and M. Yamamoto *Hautus condition on the exact controllability of conservative systems.* Internat. J. Control 67 (1997), no. 3, 371–379.