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Application of the Bending-Gradient theory to a beam lattice

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Résumé — The Bending-Gradient theory for thick plates is the extension to heterogeneous plates of Reissner-Mindlin theory originally designed for homogeneous plates. The Bending-Gradient theory was extended to in-plane periodic structures made of connected beams (space frames) which can be considered macroscopically as a plate in [1]. Its application to a square beam lattice reveals that classical Reissner-Mindlin theory cannot properly model such microstructures. Comparisons with exact solutions show that only the Bending-Gradient theory captures second order effects in both deflection and local stress fields.

1 Introduction

The classical theory of plates, known also as Kirchhoff-Love plate theory is based on the assumption that the normal to the mid-plane of the plate remains normal after transformation. It enables to have a first order estimate of the macroscopic deflection as well as local stress fields. In most applications the first order deflection is accurate enough. However, this theory does not capture the local effect of shear forces on the microstructure because shear forces are one higher-order derivative of the bending moment in equilibrium equations ($Q_\alpha = M_{\alpha\beta,\beta}$).

Because shear forces are part of the macroscopic equilibrium of the plate, their effect is also of great interest for engineers when designing structures. However, modeling properly the action of shear forces is still a controversial issue. Reissner [2] suggested a model for homogeneous plates based on a parabolic distribution of transverse shear stress through the thickness (Reissner-Mindlin theory). This model performs well for homogeneous plates and gives more natural boundary conditions than those of Kirchhoff-Love theory. Thus, it is appreciated by engineers and broadly used in applied mechanics. However, the direct extension of this model to more complex microstructures raised many difficulties. Many suggestions were made for laminated plates [3, 4] as well as in-plane periodic plates [5, 6], leading to more complex models.

Revisiting the approach from Reissner [2] directly with laminated plates, Lebéé and Sab [7, 8] showed that the transverse shear static variables which come out when the plate is heterogeneous are not shear forces Q_α but the full gradient of the bending moment $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$. Using conventional variational tools, they derived a new plate theory – called Bending-Gradient theory – which is actually turned into Reissner-Mindlin theory when the plate is homogeneous. This new plate theory is seen by the authors as an extension of Reissner’s theory to heterogeneous plates which preserves most of its simplicity. Originally designed for laminated plates, it was also extended to in-plane periodic plates using averaging considerations such as Hill-Mandel principle and successfully applied to sandwich panels [9, 10].

In order to give a more comprehensive illustration of the features of this new theory, its homogenization scheme was extended to space frames in [1]. Space frames are large roofings made of many identical unit-cells. Numerous illustration are given in Buckminster Fuller’s achievements. In this work, a space frame is a unit-cell made of connected beams periodically reproduced in a plane and which “from far” can be considered as a plate. Many devices fall into this category : space trusses [11], tensegrity, nexorade [12], gridshells [13], lattices, expanded metal, gratings, *etc.*

In Section 2 we recall briefly the main features of the Bending-Gradient theory and some key point related to space frame homogenization. Then, in Section 3 we consider a square beam lattice (Figure 1) and apply both Bending-Gradient and Cecchi and Sab [6] Reissner-Mindlin homogenization scheme. This very simple pattern will enable the derivation of closed-form solutions of the auxiliary problems

which are easy to interpret.

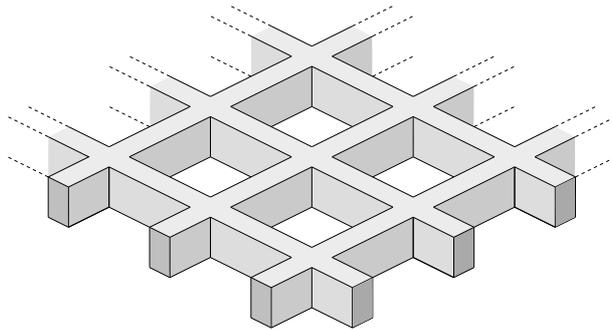


Fig. 1 – A square beam lattice

In order to check this prediction, a comparison with exact solutions of the cylindrical bending of the lattice is performed in Section 4. It reveals that only the Bending-Gradient theory is able to capture second order effects both in terms of deflection and local stress fields.

2 Homogenization of a periodic space frame as a thick plate

Full details about the Bending-Gradient plate theory are provided in [7, 8, 9]. In this section we recall shortly its main features and gives the main steps of the homogenization scheme dedicated to space frames.

We consider a linear elastic plate which mid-plane is the 2D domain $\omega \subset \mathbb{R}^2$. Cartesian coordinates (x_1, x_2, x_3) in the reference frame (e_1, e_2, e_3) are used to describe macroscopic fields. At this stage, the microstructure of the plate is not specified.

The membrane stress $N_{\alpha\beta}$, the bending moment $M_{\alpha\beta}$, and shear forces Q_α ($\alpha, \beta, \gamma \dots = 1, 2$) are the usual generalized stresses for plates. Moreover, the main feature of the Bending-Gradient theory is the introduction of an additional static unknown : the gradient of the bending moment $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$. The 2D third-order¹ tensor $\underline{\underline{R}}$ complies with the following symmetry : $R_{\alpha\beta\gamma} = R_{\beta\alpha\gamma}$. It is possible to derive shear forces \underline{Q} from $\underline{\underline{R}}$ with : $Q_\alpha = R_{\alpha\beta\beta}$.

The full bending gradient $\underline{\underline{R}}$ has six components (taking into account symmetries of indices) whereas \underline{Q} has two components. Thus, using the full bending gradient as static unknown introduces four additional static unknowns. More precisely : R_{111} and R_{222} are respectively the cylindrical bending part of shear forces Q_1 and Q_2 , R_{121} and R_{122} are respectively the torsion part of these shear forces and R_{112} and R_{221} are linked to strictly self-equilibrated stresses. Equilibrium equations and stress boundary conditions are very similar to those of Reissner-Mindlin theory where $Q_\alpha = M_{\alpha\beta,\beta}$ is replaced by $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$.

The main difference between Reissner-Mindlin and Bending-Gradient plate theories is that the latter enables the distinction between each component of the gradient of the bending moment whereas they are mixed into shear forces with Reissner-Mindlin theory. Consequently, in addition to the usual Kirchhoff-Love stiffness tensors $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{D}}$, a generalized shear compliance tensor is introduced : $\underline{\underline{f}}$. This six order tensor generalizes the usual Reissner-Mindlin constitutive law.

When the plate is laminated, closed-form expression for $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{D}}$ and $\underline{\underline{f}}$ where derived in [7]. When the plate is periodic these tensors must be derived using an homogenization procedure. This procedure relies on the fundamental assumption that loadings as well as macroscopic variables ($\underline{\underline{M}}, \underline{\underline{M}}, \underline{\underline{R}}$) varies slowly compared to the size of the microstructure (the unit-cell). This means that at the scale of the unit-cell the macroscopic fields can be considered as uniform. Hence, it is possible to define auxiliary problems which enable the derivation of the actual microscopic fields as a linear superposition of macroscopic fields. Once this is done, it is straightforward to take the average of the elastic energy stored inside the unit-cell and equate it to the macroscopic elastic energy density (namely the macroscopic constitutive

1. Vectors and higher-order tensors are boldfaced and different underlinings are used for each order : vectors are straight underlined, \underline{u} . Second order tensors are underlined with a tilde : $\underline{\tilde{M}}$. Third order tensors are underlined with a parenthesis : $\underline{\underline{R}}$. Fourth order tensors are doubly underlined with a tilde : $\underline{\underline{\tilde{D}}}$. Sixth order tensors are doubly underlined with a parenthesis : $\underline{\underline{\underline{f}}}$.

equation). This analogy between micro and macroscopic energy is usually called Hill-Mandel principle. When the plate microstructure is constituted of a 3D continuum, the auxiliary problems were introduced in [9]. In the present case, we consider periodic plates which unit-cell is constituted of connected beams. This necessitates the adaptation of the homogenization scheme for 3D continuum which is detailed in [1]. Basically, in addition to the translation displacement field \mathbf{u} we must also take into account the rotation field $\boldsymbol{\theta}$ of the beams.

The homogenization procedure is done as follows. First the unit-cell is defined and the Kirchhoff-Love auxiliary problem is solved. This enables the derivation of the beam stress inside the unit-cell as function of the macroscopic bending moment $\bar{\mathbf{M}}$ and membrane stress $\bar{\mathbf{N}}$. Then a second auxiliary problem is solved for deriving the beam stress related to the bending-gradient $\bar{\mathbf{R}}$. Since the localization of stress is known it is possible to compute the plates constitutive tensors $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$ and $\bar{\mathbf{f}}$ equating the microscopic energy stored in the unit-cell with the plate energy density.

3 Homogenization of a square lattice as a Bending-Gradient plate

In order to illustrate the homogenization scheme, we consider a very simple structure : a square beam lattice (Figure 1). In this section, we detail the homogenized behavior of the lattice as a Bending-Gradient plate and also as a Reissner-Mindlin plate.

3.1 The unit-cell

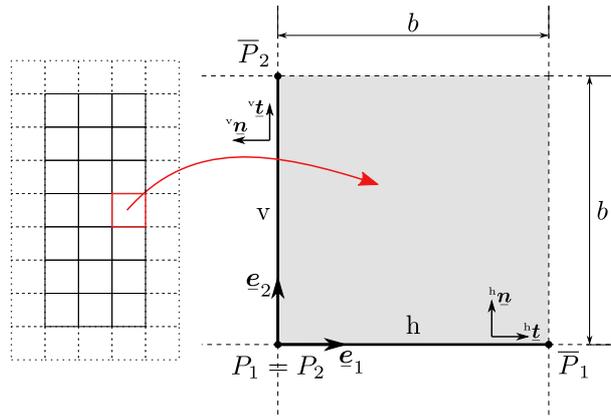


Fig. 2 – The beam lattice unit-cell

Figure 2 shows the unit-cell of the lattice idealization. It is constituted of two beams (“h” : horizontal, “v” : vertical) which lie in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane. Hence their local reference frames write as $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for beam “h” and $(\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_3)$ for beam “v”. Both beams are identical with length b . We assume the beam section has two orthogonal axis of symmetry. One axis is \mathbf{e}_3 .

3.2 Kirchhoff-Love auxiliary problem

We give directly the generalized stress field localization as function of $\bar{\mathbf{N}}$ and $\bar{\mathbf{M}}$ in the local reference frame of each beam :

$${}^h\mathbf{r}^{KL} = \begin{pmatrix} bN_{11} \\ bN_{12} \\ 0 \end{pmatrix}_h \quad \text{and} \quad {}^h\mathbf{m}^{KL} = \begin{pmatrix} -bM_{12} \\ bM_{11} \\ bN_{12}(s_1 - \frac{b}{2}) \end{pmatrix}_h \quad (1a)$$

$${}^v\mathbf{r}^{KL} = \begin{pmatrix} bN_{22} \\ -bN_{12} \\ 0 \end{pmatrix}_v \quad \text{and} \quad {}^v\mathbf{m}^{KL} = \begin{pmatrix} bM_{12} \\ bM_{22} \\ -bN_{12}(s_1 - \frac{b}{2}) \end{pmatrix}_v \quad (1b)$$

Where \mathbf{r} is the beam resultant (traction and shear forces) and \mathbf{m} is the bending moment (torsion and bending moments). This result is rather intuitive : membrane traction in Direction 1 generates only uniform traction of beam "h", as well as N_{22} generates traction in beam "v". In plane shear N_{12} puts both beams in non-uniform in-plane bending. Similarly, pure bending in Direction 1 generates pure out-of-plane bending of beam "h" as well as M_{22} generates pure out-of-plane bending in beam "v". Finally, plate torsion M_{12} generates pure torsion in both beams.

The corresponding Kirchhoff-Love stiffness tensors are :

$$\left[\underset{\approx}{\mathbf{A}} \right] = \begin{pmatrix} \frac{ES}{b} & 0 & 0 \\ 0 & \frac{ES}{b} & 0 \\ 0 & 0 & \left(\frac{b}{G_{S,2}} + \frac{b^3}{12EI_3} \right)^{-1} \end{pmatrix}, \quad \left[\underset{\approx}{\mathbf{B}} \right] = 0 \quad \text{and} \quad \left[\underset{\approx}{\mathbf{D}} \right] = \begin{pmatrix} \frac{Eb}{b} & 0 & 0 \\ 0 & \frac{Eb}{b} & 0 \\ 0 & 0 & \frac{GJ}{b} \end{pmatrix} \quad (2)$$

where we introduced Voigt notation $\left[\underset{\approx}{\mathbf{A}} \right]$ for fourth order tensors $\underset{\approx}{\mathbf{A}}$ as :

$$\left[\underset{\approx}{\mathbf{A}} \right] = \begin{pmatrix} A_{1111} & A_{2211} & \sqrt{2}A_{1211} \\ A_{2211} & A_{2222} & \sqrt{2}A_{1222} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & 2A_{1212} \end{pmatrix} \quad (3)$$

There is no coupling between $\underset{\approx}{\mathbf{N}}$ and $\underset{\approx}{\mathbf{M}}$ ($\underset{\approx}{\mathbf{B}} = 0$) and there is no Poisson's effect ($A_{1122} = D_{1122} = 0$).

3.3 Bending-Gradient and Reissner-Mindlin auxiliary problems

The generalized stress localization related to $\underset{\approx}{\mathbf{R}}$ in the local reference frame of each beam is :

$${}^h\mathbf{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b(R_{111} + R_{122}) \end{pmatrix}_h \quad \text{and} \quad {}^h\mathbf{m}^{(R)} = \begin{pmatrix} bR_{121} \left(s_1 - \frac{b}{2} \right) \\ bR_{122} \left(s_1 - \frac{b}{2} \right) \\ 0 \end{pmatrix}_h \quad (4a)$$

$${}^v\mathbf{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b(R_{121} + R_{222}) \end{pmatrix}_v \quad \text{and} \quad {}^v\mathbf{m}^{(R)} = \begin{pmatrix} -bR_{122} \left(s_1 - \frac{b}{2} \right) \\ bR_{121} \left(s_1 - \frac{b}{2} \right) \\ 0 \end{pmatrix}_v \quad (4b)$$

Let us recall that $Q_1 = R_{111} + R_{122}$ and $Q_2 = R_{121} + R_{222}$. Thus the resultants ${}^h\mathbf{r}^{(R)}$ and ${}^v\mathbf{r}^{(R)}$ are directly function of the shear forces. However, the torsion part of shear forces (R_{122} and R_{121}) generates additional torsion and bending in ${}^h\mathbf{m}^{(R)}$ and ${}^v\mathbf{m}^{(R)}$. Thus, shear forces related to cylindrical bending and torsion do not generate the same local stress field.

In order to compare in details the Bending-Gradient theory with Reissner-Mindlin theory, we also considered a Reissner-Mindlin homogenization scheme. We chose Cecchi and Sab [6] method for deriving a Reissner-Mindlin constitutive equation. Following the work from Whitney [3], Cecchi and Sab [6] assume the plate is under cylindrical bending for deriving local stress fields and a Reissner-Mindlin constitutive equation. The related Reissner-Mindlin field localization writes as follows :

$${}^h\mathbf{r}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_1 \end{pmatrix}_h \quad \text{and} \quad {}^h\mathbf{m}^{(Q)} = \mathbf{0} \quad (5a)$$

$${}^v\mathbf{r}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_2 \end{pmatrix}_v \quad \text{and} \quad {}^v\mathbf{m}^{(Q)} = \mathbf{0} \quad (5b)$$

Thus, there is no more correction of \mathbf{m} .

4 Cylindrical bending of a square beam lattice

In order to give a clear demonstration of the additional warping effects included in the Bending-Gradient theory we compare Kirchhoff-Love, Reissner-Mindlin and Bending-Gradient models to the exact solution in cylindrical bending configuration where the lattice direction is rotated by 45° with respect to the bending direction.

Figure 3 shows the lattice configuration. It is bent in Direction x_1 along $2n$ diamond shaped unit-cells and is infinitely periodic in direction x_2 . The beams are transversely loaded with the sine shaped load $\mathbf{k} = -k_0 \sin \kappa x_1 \mathbf{e}_3$. The span is : $L = 2\sqrt{2}nb$ (Figure 4).

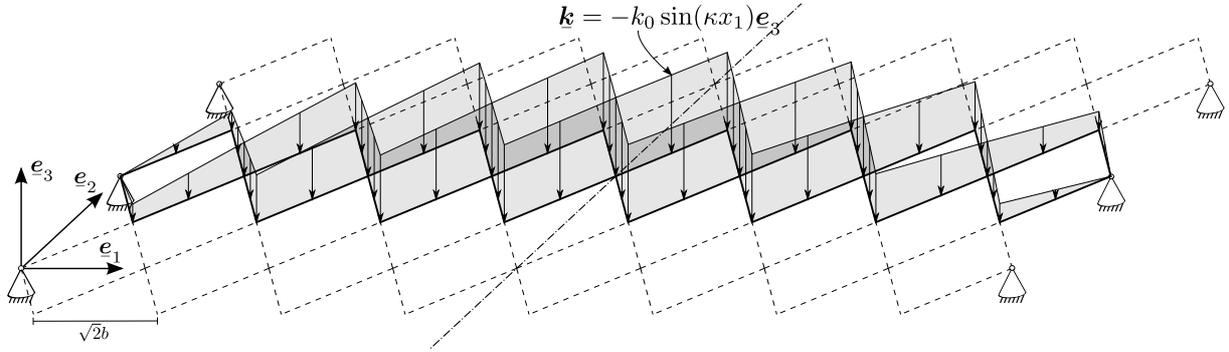


Fig. 3 – Beam lattice configuration at 45° in cylindrical bending configuration

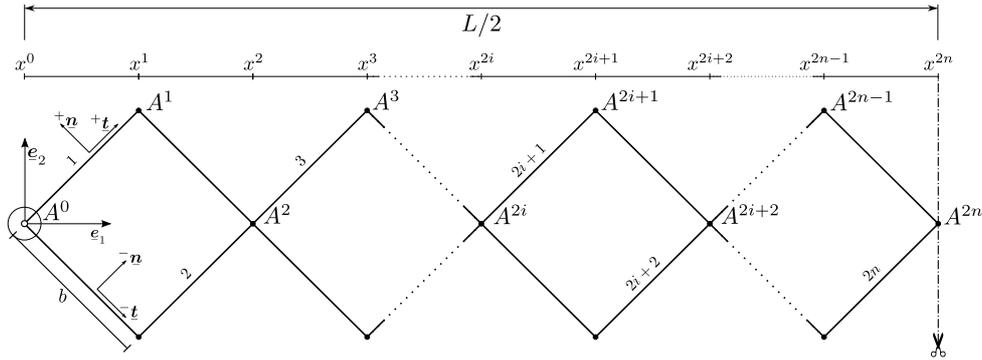


Fig. 4 – Top view of the beam lattice at 45° and detailed parameters (only half a span is shown)

There are two sets of beams. The first set “+” is numbered $1..2n$ and is rotated by $+\pi/4$ in $(\mathbf{e}_1, \mathbf{e}_2)$ plane and the second set “-” is rotated by $-\pi/4$. It is symmetric to set + with respect to any $(A^j, \mathbf{e}_1, \mathbf{e}_3)$ plane. Hence the focus is on set + in the following.

Beams are perfectly connected at nodes A_j located at $x_1 = x^j = jb/\sqrt{2}$. Simple support condition occurs at node A^0 and A^{4n} : both beams are perfectly connected together, however the node is free to rotate with respect to the support.

The derivation of the exact solution of this problem, though cumbersome, does not raise any difficulty. The homogenized plate surrogate for the configuration in Figure 3 is a simply supported plate under cylindrical bending where the distributed load is $\mathbf{p} = \frac{2k_0}{b} \sin \kappa x_1 \mathbf{e}_3$ in order to have the same loading per unit surface.

In order to compare the models, the bending moment m_2 and the torsion m_1 along beams “+” are plotted respectively in Figures 5 and 6 for the exact solution and homogenization approximations. Kirchhoff-Love and Reissner-Mindlin solutions are identical and plotted as the same continuous curve which follows the exact solution on average. By contrast, the Bending-Gradient approximation follows closely the exact solution. Especially, it captures the change of sign of the bending moment m_2 close to the support. In case the lattice is made of concrete, this kind detail is useful for an efficient design. This is not included in the other approximations. The errors $\frac{m_1 - m_1^{\text{ex}}}{m_1^*}$ and $\frac{m_2 - m_2^{\text{ex}}}{m_2^*}$ are also plotted in Figures 7 and 8. It shows that the maximum error is 1% for the Bending-Gradient and 9% for the other approaches.

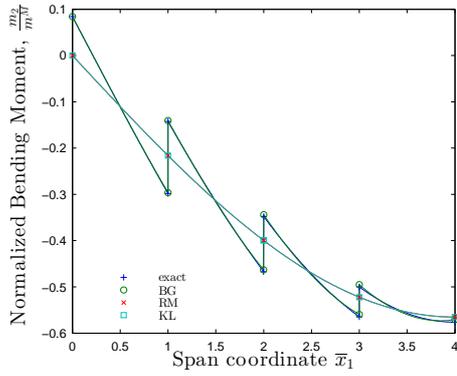


Fig. 5 – Bending moment m_2 along Beams “+” for a 4 unit-cell lattice (only half the span is plotted)

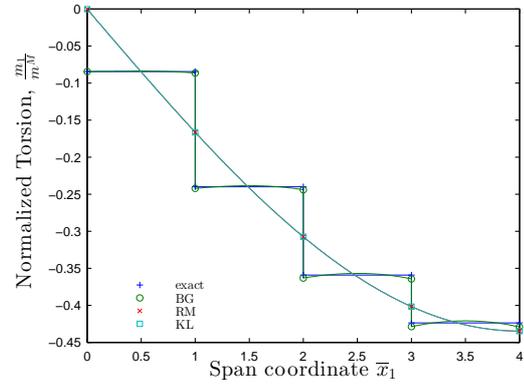


Fig. 6 – Torsion m_1 along Beams “+” for a 4 unit-cell lattice (only half the span is plotted)

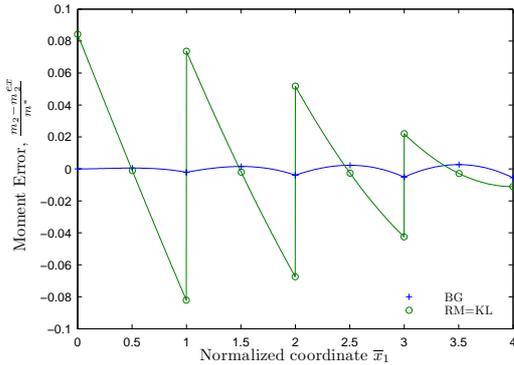


Fig. 7 – Bending moment error along Beams “+” for a 4 unit-cell lattice (only half the span is plotted)

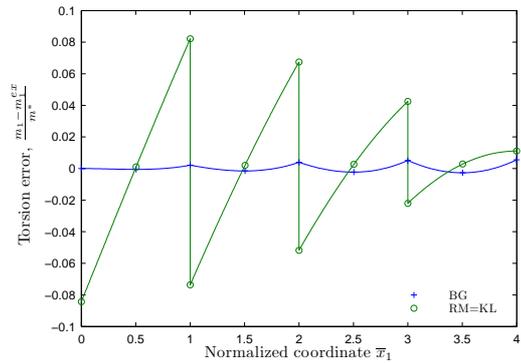


Fig. 8 – Torsion error along Beams “+” for a 4 unit-cell lattice (only half the span is plotted)

5 Discussion

In the previous section, three homogenization approximations of the cylindrical bending of a beam lattice were derived. The first one, leading to a Kirchhoff-Love plate model is also known to be the first order of the asymptotic expansion [14]. This approach enables to have a first order estimate of the macroscopic deflection as well as local stress fields. Except with sandwich panels and laminated plates, in most application the first order deflection is accurate enough. However, this approach does not capture the local effect of shear forces on the microstructure and the common practice for deriving the actual effect of transverse shear appears clearly as not reliable in more general configurations.

The Reissner-Mindlin homogenization approach used in this paper clearly misses some part of their effect. This is because of the assumption of cylindrical bending which is made in most approach for deriving a Reissner-Mindlin-like constitutive equation. This assumption enforces $Q_1 = M_{11,1}$ and precludes any possible effect of the other derivatives of the bending moment on the local stress on the microstructure.

In the present case, the Bending-Gradient approximation captures all second order effects and gives a very satisfying description of both the deflection and the local stress fields.

Our approach still presents some limitations. First, in the present case, we chose supports conditions for the lattice so that edge effects remain limited. It is well known in homogenization that these effects might be not negligible in some configurations. This point is far out of the scope of this paper and raises the question – common to all models – regarding the mechanical meaning of plate boundary conditions. Second, the local load distribution on the microstructure was arbitrarily chosen. At first order, this choice does not affect the fields localization (Kirchhoff-Love case). However, different load distributions could generate different localization of stress fields when looking for a higher order approximation (see [15])

for instance).

6 Conclusion

In this paper, the Bending-Gradient homogenization scheme was extended to space frames. Its application to a square beam lattice led to closed form solutions of the auxiliary problems. It also revealed that this microstructure cannot be properly modeled with Reissner-Mindlin theory. This observation was confirmed when comparing homogenization approaches with the exact solution of the cylindrical bending of the lattice. Only the Bending-Gradient theory is able to capture second order effects both in macroscopic deflection and in local fields.

Many other periodic structures might not be correctly modeled with Reissner-Mindlin theory such as concrete waffle slabs, orthotropic decks or composite floor systems and requires more accurate approaches such as the Bending-Gradient model.

Finally, provided the limitations pointed out in Section 5 are kept in mind, the Bending-Gradient plate theory is considered by the authors as a sound extension of the Reissner-Mindlin theory to the case of heterogeneous plates. More precisely, it is an attempt to concentrate most of the results given by the second order of the asymptotic expansion [5] into a model which keeps most aspects of Reissner-Mindlin model, well understood by engineers.

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