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Determination of the domain of validity of the diffuse field assumption in plate vibration using the Variational Theory of Complex Rays.

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Résumé — L'objectif de ce papier est de déterminer le domaine de validité de l'hypothèse de champ vibratoire diffus dans une plaque. L'idée est de résoudre un problème de vibration sur un ensemble de plaques en utilisant la TVRC. La diffusivité du champ de vibration sur l'ensemble des plaque peut être mesuré soit en utilisant la fonction de corrélation du champ de déplacement, ou en utilisant une description ondulatoire du champ diffus. Le domaine de validité de champ diffus est directement relié au domaine de validité de la SEA.

Mots clés — Champ diffus, vibrations, hautes fréquences, Monte Carlo, TVRC, SEA

1 Introduction

The problem of calculating the response of a complex engineering structure to dynamic loading may generally be approached using the Finite Element Method [1]. This technique allows a great flexibility in the definition of the system geometry but unfortunately leads to very large numerical models as the frequency increases. Therefore its applications are in practice limited to the low-frequency range. In the mid frequency regime there are alternatives, such as the Trefftz method [2]. These methods differ from the FEM by employing shape functions that are exact solutions of the governing differential equations. They lead to a model size and computational effort that are considerably reduced compared to element-based methods. One of these methods is the Variational Theory of Complex Rays (VTCR) [3] where the vibrational field is sought as an integral distribution of plane waves (Herglotz function) and the boundary conditions are satisfied using a weak formulation. The unknown of the problem is the amplitude distribution of the waves.

It is, in principle, possible to extend deterministic methods to higher frequencies, at the expense of increasing demands in term of the size of the model and the consequent computational time. However, there is a fundamental physical reason why such extension is ultimately doomed to failure : it is associated with an unavoidable uncertainty about the precise dynamic properties of the structure. As the frequency increases, the solution becomes more and more sensitive to small changes in the structural details and therefore a deterministic solution becomes more and more unreliable.

The alternative is to treat the response prediction from the outset as a probabilistic problem. The high frequency response of a population of similar systems of which individual members differ in unpredictable detail may be characterized by their ensemble-average behavior. For example Statistical Energy Analysis [4] seeks the mean energy and its variance in the different part of the system. The theoretical justification of the SEA equations is normally based on either a diffuse wave field approach or a modal approach to the system dynamics. Therefore, the domain of validity of the SEA is directly linked to the domain of validity of the diffuse field assumption. In [5], some work has been done to determine the domain of validity using the modal approach. The goal of this paper is to determine the domain of validity of the diffuse field assumption in the particular case of plate vibrations.

2 The reference problem

The reference problem being considered is the steady-state vibration of a plates at a fixed circular frequency ω . This plate is assumed to have five edges and to be thin, homogeneous, isotropic and governed by the elastic Kirchhoff-Love theory. Classically, all quantities are defined as complex quantities, *i.e.* an amplitude $Q(\mathbf{x})$ is associated with $Q(\mathbf{x})e^{i\omega t}$. Let S denotes the mean surfaces of the plate and ∂S its boundaries. The thickness, stiffness tensor (including the damping η) and density are designated respectively by h , \mathbf{D} and ρ . The plate is excited by a point force F located at \mathbf{x}_F . The boundary conditions are free edges (prescribed zero bending moment and shear force) with three clamped points $C_{i,e}$ ($i \in \{1, 2, 3\}$) on each edge e (prescribed zero deflection and slope).

The reference problem is : find $(w, \mathbf{M}) \in \mathcal{W} \times \mathcal{M}$ (set of the finite-energy fields) such that :

$$\left\{ \begin{array}{ll} \text{div}(\mathbf{div} \mathbf{M}) + \rho_e h \omega^2 w_e = F \delta(\mathbf{x}_F - \mathbf{x}) & \text{in } S \\ \mathbf{M} = \mathbf{D} \mathbf{X}(w) & \text{in } S \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{ll} w = 0 & \text{on } C_{i,e}, e \in \{1..5\}, i \in \{1, 2, 3\} \\ w_{,n_e} = 0 & \text{on } C_{i,e}, e \in \{1..5\}, i \in \{1, 2, 3\} \\ \mathbf{n}_e \mathbf{M}_e \mathbf{n}_e = 0 & \text{along } \partial S \\ K_e = \mathbf{n}_e \text{div}(\mathbf{M}_e) + (\mathbf{n}_e \mathbf{M}_e \mathbf{n}_e)_{,t} = 0 & \text{along } \partial S \\ [[\mathbf{t}_e \mathbf{M}_e \mathbf{n}_e]] = 0 & \text{at the corner of } \partial S_e, e \in \{1, 2\} \end{array} \right. \quad (2)$$

where \mathbf{n}_e and \mathbf{t}_e are respectively the outward normal and the tangent to S on the edge e . $[[\square]]$ denotes the value of the jump of quantity \square and \mathbf{X} is the curvature operator. If \mathbf{D} is positive definite and the damping factors η of the plate is positive, the solution of the reference problem is unique.

3 The Variational Theory of Complex Rays applied to plate vibration

3.1 The variational formulation associated with the VTCR

The VTCR uses a global formulation of Problem (1)-(2) in terms of both displacements and forces along with the boundary conditions and transmission conditions. First, one must define \mathcal{S}^{ad} , the functional space that satisfies the governing equations (1) :

$$(w, \mathbf{M}) \in \mathcal{S}^{ad} \Leftrightarrow \left\{ \begin{array}{ll} (w, \mathbf{M}) \in \mathcal{W} \times \mathcal{M} \\ \text{div}(\mathbf{div} \mathbf{M}) + \rho h \omega^2 w = F \delta(\mathbf{x}_F - \mathbf{x}) & \text{in } S \\ \mathbf{M} = \mathbf{D} \mathbf{X}(w) & \text{in } S \end{array} \right. \quad (3)$$

Similarly, let \mathcal{S}_0^{ad} be the functional space which satisfies the homogeneous governing equations (the same equations as (3), but without F). Then, the VTCR formulation of Problem (1)-(2) is : find $(w, \mathbf{M}) \in \mathcal{S}^{ad}$ such that :

$$\int_{\partial S} [(\mathbf{n}_e \mathbf{M}_e \mathbf{n}_e) \delta w_{,n_e}^* - K \delta w^*] dl - \sum_{\text{corners}} [[\mathbf{t}_e \mathbf{M}_e \mathbf{n}_e]] \delta w^* + \sum_{C_{i,e}} [\mathbf{n}_e \delta \mathbf{M}_e \mathbf{n}_e w_{,n_e}^* - \delta K w^*] = 0 \quad (4)$$

$\forall (\delta w, \delta \mathbf{M}) \in \mathcal{S}_0^{ad}$

where \square^* denotes the complex conjugate of \square .

Equation (4) can be shortened to : find $s = (w, \mathbf{M}) \in \mathcal{S}^{ad}$ such that :

$$a(s, \delta s) = l(\delta s) \quad \forall \delta s \in \mathcal{S}_0^{ad} \quad (5)$$

In order to find an approximate solution of (1)-(2), the basic idea is to seek a solution of (5) in a space $\mathcal{S}^{h,ad} \subset \mathcal{S}^{ad}$ (and, consequently, $\mathcal{S}_0^{h,ad} \subset \mathcal{S}_0^{ad}$) of finite dimension : find $s^h = (w^h, \mathbf{M}^h) \in \mathcal{S}^{h,ad}$ such that :

$$a(s^h, \delta s^h) = l(\delta s^h) \quad \forall \delta s^h \in \mathcal{S}_0^{h,ad} \quad (6)$$

3.2 Construction of admissible fields

In order to define the $\mathcal{S}^{h,ad}$ term in (6), we must find the exact solutions of the governing equations (1). The explicit expression of (1) is :

$$D \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] - \rho h \omega^2 w = F \delta(\mathbf{x}_F - \mathbf{x}) \quad (7)$$

where $D = \frac{E(1+i\eta)}{1-\nu^2} \frac{h^3}{12}$, where E is the Young modulus and η the damping coefficient.

The displacement w is sought in the form $w = w^0 + w^p$, where w^p is a known particular solution of (7) while $w^0 \in \mathcal{S}_{ad,0}^e$ is taken as a sum of *Herglotz wave functions*, i.e. an integral superposition of propagative or evanescent plane waves of the form :

$$w^0(\mathbf{x}) := \int_C A^p(\theta) e^{i\mathbf{k}^p(\theta) \cdot (\mathbf{x} - \mathbf{x}^p)} d\theta + \sum_e \left[\int_C A^e(\theta) e^{i\mathbf{k}^e(\theta) \cdot (\mathbf{x} - \mathbf{x}^e)} d\theta + \sum_i \int_C A^{e,i}(\theta) e^{i\mathbf{k}^e(\theta) \cdot (\mathbf{x} - C_{e,i})} d\theta \right]. \quad (8)$$

Any Herglotz wave function is a solution of homogeneous equation (7). Keeping the terminology from previous works on the VCTR, functions A^p , A^e , and $A^{e,i}$ are called the *amplitude portraits* of w .

In (8), C is the unit circle, \mathbf{x}^p is a reference point located in the centre of the plate, \mathbf{x}^e is a reference point along edge e , $A^p(\theta)$ is the amplitude of the propagative plane wave in direction θ which contributes to w , $A^e(\theta)$ and $A^{e,i}(\theta)$ are the amplitude of the evanescent plane wave in direction θ which contributes to w . The wave vectors $\mathbf{k}^p(\theta)$ and $\mathbf{k}^e(\theta)$ are defined by :

$$\mathbf{k}^p(\theta) = k \cos(\theta) \mathbf{e}_x + k \sin(\theta) \mathbf{e}_y \quad (9)$$

$$\mathbf{k}^e(\theta) = -ik \sqrt{1 + \cos^2(\theta)} \mathbf{n}_e + k \cos(\theta) \mathbf{t}_e \quad (10)$$

where $k = \left(\frac{\rho h \omega^2}{D} \right)^{1/4}$

In order to discretize and get the VCTR formulation (6), each amplitude portrait A^\square (where \square is either p , e or e, i) is discretized as a sum of N^\square Dirac distributions supported at angular locations θ_n . Then, w is the superposition of a finite number of plane waves :

$$w(\mathbf{x}) := w^p + \sum_{n=1}^{N_e^p} a_n^p e^{i\mathbf{k}^p(\theta_n^p) \cdot (\mathbf{x} - \mathbf{x}^p)} + \sum_e \left[\sum_{n=1}^{N_e^e} a_n^e e^{i\mathbf{k}^e(\theta_n^e) \cdot (\mathbf{x} - \mathbf{x}^e)} + \sum_i \sum_{n=1}^{N_{e,i}^e} a_n^{e,i} e^{i\mathbf{k}^e(\theta_n^{e,i}) \cdot (\mathbf{x} - C_{e,i})} \right] \quad (11)$$

where a_n^\square are the unknowns of the problem and $\theta_n^\square = n \frac{2\pi}{N_e^\square}$.

3.3 The discretized forme of the VTCR

With the displacement discretized according to (11), the VTCR formulation (6) leads to a linear system of equations in the complex domain :

$$\mathbf{A} \mathbf{a} = \mathbf{b} \quad (12)$$

Matrix \mathbf{A} corresponds to the discretization of the bilinear form of weak formulation (6), \mathbf{a} is the vector of the unknown quantities which approximate the distribution of the wave amplitudes and \mathbf{b} corresponds to the discretization of the linear form of (6). \mathbf{A} does not have any symmetry, but its symmetric part is related to the dissipation of the problem, which guarantees that it is invertible. The VTCR approximate solution of Problem (1)-(2) is given by Equation (11).

4 The concept of diffuse field over an ensemble of plates

The concept of a diffuse field is used widely in vibro-acoustics to describe the high-frequency response of an ensemble of structural components or acoustic volumes. The wave description of the SEA is based on the hypothesis of diffuse field over the ensemble of structure. Therefore, being able to measure the diffusivity of the displacement field across an ensemble of plate is the first step to define the domain of validity of the SEA. In this section, we define the considered ensemble of plates, and then suggest two measures of the diffusivity based on different approaches.

4.1 Ensemble of plates

In this paper, we study the response of an ensemble of plates. The ensemble is created by randomising either the geometry of the plate and the position of the clamped point $C_{i,e}$. The Figure 1 represents the reference geometry and 6 members of the ensemble of plates.

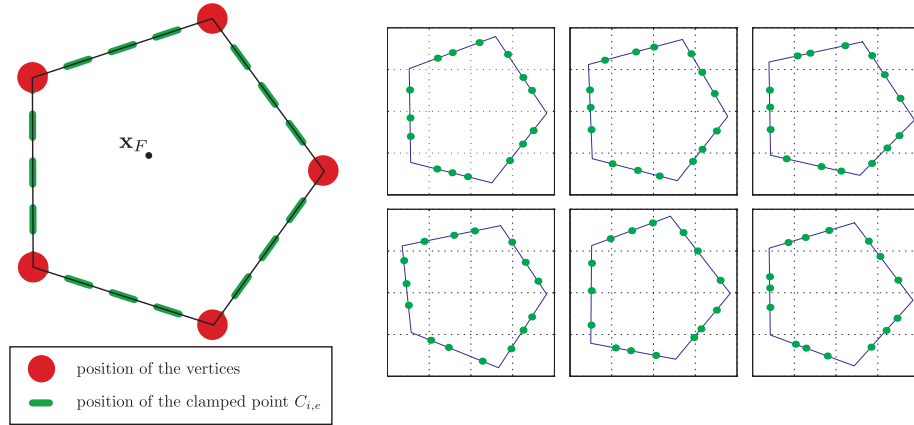


Fig. 1 – Left : geometry of reference for the ensemble of plates. Right : six members of the ensemble of plates.

The reference geometry is a regular pentagon with a circumradius of 1m. The position of the vertices is randomised by imposing a uniform distribution in a 0.05 m radius disk centered on the vertex of the reference pentagon. Then the geometry is scaled such as all the member of the ensemble have the same area.

The boundary conditions are randomised by changing the position of the clamped point. On each edge, the clamped point follow an uniform distribution over a 0.1 length. The three zones are uniformly distributed over the edge.

In this paper the ensemble of plate is composed of 8192 members. We consider 128 different geometries, and 64 different boundary conditions configurations. Each member is solved using the VTCR for a given frequency and level of damping. Using the VTCR allow us to have accrs to the displacement of the plate (like any other deterministic method) but also to a wave description of the vibrational field.

4.2 Displacement approach

The concept of a diffuse field can be expressed in terms of the correlation function $\chi^d(r)$ (defined in equation (13)) calculated on a set of lines over the ensemble of plates.

$$\chi^d(r) = \frac{[w(0) \cdot w(r)^*]}{\sqrt{[w(0) \cdot w(0)^*][w(r) \cdot w(r)^*]}} \quad (13)$$

where $[\]$ represents the mean over the set of lines over the ensemble of plates.

It is well known that the correlation function of a diffuse field $\chi^{d,ex}(r)$ (defined in equation (14)) along a line is related to the Green's function of the system [6].

$$\chi^{d,ex}(r) = J_0(kr) \quad (14)$$

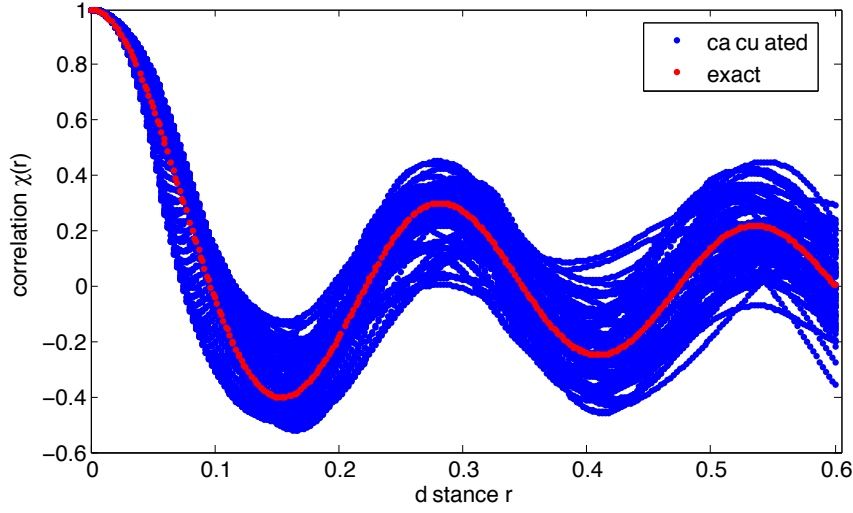


Fig. 2 – Measured correlation function over the set of line over the ensemble of plates at 400Hz, with a damping sets to 1.5% .

Therefore, by comparing the correlation function $\chi^d(r)$ (defined in equation (13)) and the close form $\chi^{d,ex}(r)$ (defined in equation (14)), it is possible to measure the diffusivity of the field across the ensemble of plates. Let ϵ^d be such a measure :

$$\epsilon^d = \int_r \left[|\chi^d(r) - \chi^{d,ex}(r)| \right] dr \quad (15)$$

4.3 Wave approach

A wave viewpoint of the diffuse field is adopted in this sub-section, and in this context a number of textbook definitions of a diffuse wavefield in an acoustic volume are as follows : “plane waves are incident from all directions with equal probability and random phase” , page 157 in [7] ; “the average energy density is the same throughout the volume of the enclosure, and all directions of propagation are equally probable” page 313 in [8]. A mathematical, rather than descriptive, statement of this concept has been given by Morse and Ingard page 576 in [9] and Pierce page 255 in [10] ; in the former case the response is represented as an integral over a continuum of plane wave headings, while in the latter case a sum over a large number of discrete wave headings is employed. Thus it is clear that plane wave motion is central to the concept of a diffuse field, and, moreover, the response of the system is considered to be a superposition of plane waves that propagate with equal intensity in all directions — a concept that is equally applicable to structural components.

To consider the field diffuse, the distribution of plane wave :

- should be uncorrelated,
- should have an uniform distribution of the phase in all the direction,
- should have the same distribution of magnitude in all the direction.

The next three subsection present how the verification of each these points are done.

4.3.1 Uncorrelation of the waves

The autocorrelation matrix χ^w of the amplitudes of the wave is defined as :

$$\chi^w(\theta_i, \theta_j) = \frac{[A^p(\theta_i) \cdot A^{p*}(\theta_j)]}{\sqrt{[A^p(\theta_i) \cdot A^{p*}(\theta_i)][A^p(\theta_j) \cdot A^{p*}(\theta_j)]}} \quad (16)$$

A diffuse field will produce a autocorrelation matrix equal to the identity matrix.

The measure of the correlation of the wave consists into fitting an exponential model of correlation

to χ^w . An exponential correlation matrix is defined by :

$$\chi^{w,e}(\theta_i, \theta_j) = \exp\left(-\frac{(|\theta_i - \theta_j| \bmod 2\pi)}{\Delta\theta_{cor}}\right) \quad (17)$$

where $\Delta\theta_{cor}$ represents the correlation angle. One could remark, that when the correlation angle tends to zero, the matrix $\chi^{w,e}$ tends to the identity.

The measure of the diffusivity is defined by :

$$\epsilon^{w,c} = \iint ||\chi^{w,e}(\theta_i, \theta_j) - \chi^w(\theta_i, \theta_j)|| d\theta_i d\theta_j \quad (18)$$

The correlation angle is chosen that it minimised the quantity $\epsilon^{w,c}$.

The figure 3 represents the autocorrelation matrices $\chi^{w,e}(\theta_i, \theta_j)$ and $\chi^w(\theta_i, \theta_j)$ calculated across the ensemble of plate at 400Hz with the damping set to 1.5

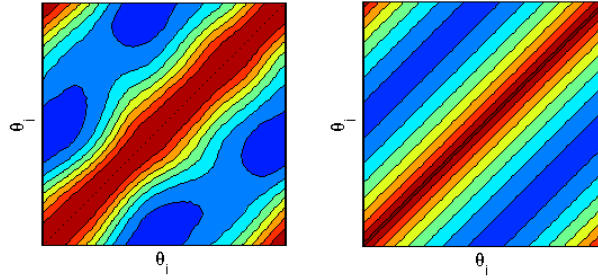


Fig. 3 – autocorrelation matrices $\chi^w(\theta_i, \theta_j)$ and $\chi^{w,e}(\theta_i, \theta_j)$ (respectively left and right) calculated across the ensemble of plate at 400Hz with the damping set to 1.5%.

4.3.2 Equipartition of the amplitude phase

For each propagative wave in the direction θ , the distribution of phase across the ensemble of plates should be uniform.

Let $d^\Phi(\theta)$ be the calculated histogram of the phase across the ensemble of plate of the wave propagating in the direction θ . The measure of the equidistribution if done by calculating :

$$\epsilon^{w,p} = \int |d^\Phi(\theta) - \frac{1}{2\pi}| d\theta \quad (19)$$

The figure 4 represents the calculated histogram of the phase in all the direction. It also represent the evolution of the mean square error of the distribution with the direction of propagation.

4.3.3 Equidistribution of the amplitude magnitude

The distribution of the magnitude of the propagative wave has to be identical for all the direction. It means that both of the mean and variance of the magnitude distribution in a given direction has to be constant over all the direction.

Therefore we define the measure :

$$\epsilon^{w,a} = \int \frac{1}{2} |[A(\theta)] - [[A]]| + \frac{1}{2} |(|A(\theta)|) - ((|A|))| d\theta \quad (20)$$

where $[A(\theta)]$ and $[[A]]$ represent respectively the mean value of the magnitude in the direction θ , and the mean value across all the direction of $[A(\theta)]$. Similar notation are used for the variance.

The first line of the figure 5 represents the histogram of the magnitude over all the direction, and the evolution of the mean and variance of the distribution with the direction of propagation.

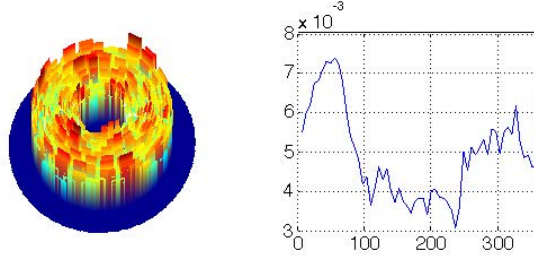


Fig. 4 – calculated histogram $d^h(\theta)$ and evolution of the mean square error of the distribution calculated across the ensemble of plate at 400Hz with the damping set to 1.5%.

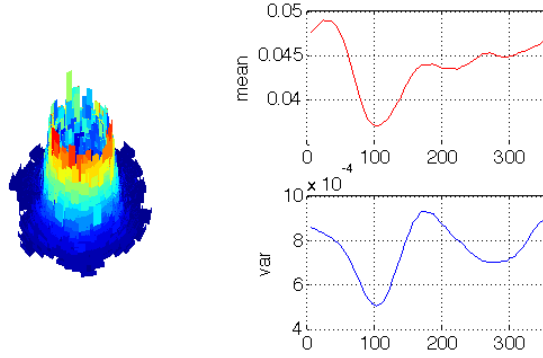


Fig. 5 – histogram of the magnitude over all the direction, and the evolution of the mean (red) and variance (blue) of the distribution with the direction of propagation calculated across the ensemble of plate at 400Hz with the damping set to 1.5%.

4.3.4 Global measure of the diffusivity

The measure of the diffusivity is done by combining the previous measures :

$$\epsilon^w = \sqrt[3]{\epsilon^{w,c} \times \epsilon^{w,p} \times \epsilon^{w,a}} \quad (21)$$

5 Results

In [5], Le Bot and Cotonni suggested that the domain of validity of the SEA could be represented in the plan frequency-damping. The domain of validity of the SEA is directly linked to the domain of validity of the diffuse field assumption. The idea of this section is to measure the diffusivity of the field across an ensemble of plates on the frequency range 100-1000Hz, and a damping range of 10^{-3} - $10^{-0.5}$.

The figure 4 and 5 represent the diffusivity of the field across the ensemble of plate measured respectively using the displacement approach end the wave approach. The color blue represents a better diffusivity of the field across the ensemble of plates.

As one can see, the two approaches give similar results. At low frequency, the field across the ensemble of plate is not diffuse, this could be expalin by the fact that the wave length if much bigger that the dimension of the plate and its variations, and therefore the response is not sensitive to the change of geometry. At higher frequency, one could remarks that only low value of damping produce a diffuse field across the ensemble of plates. When the damping is to high, the direct field is damped before it could reach the edges and produce a reverberant field, the response is then close to an infinite plate, and it not affected by the change of geometry and boundary conditions.

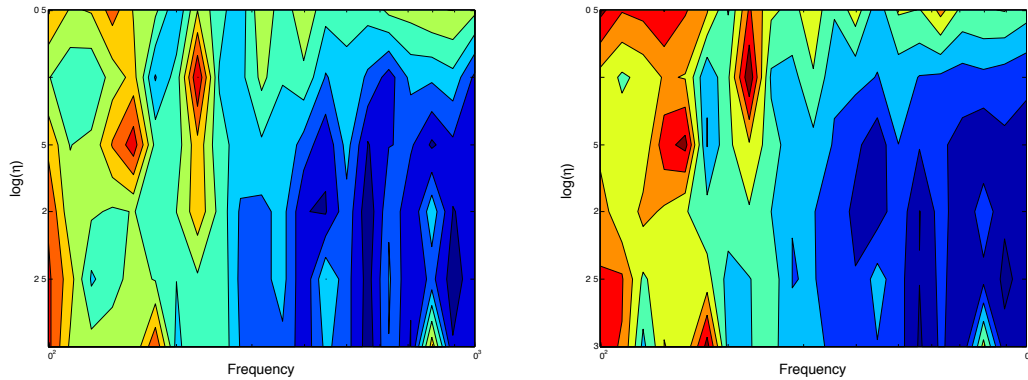


Fig. 6 – diffusivity across an ensemble of plates using a displacement and wave approach (respectively left and right).

6 Conclusion

In this paper the VTCR has been combined with a Monte Carlo simulation, to simulate the response of an ensemble of plates. The response could be obtained either in term of displacement or in term of waves. Then the result has been used to measured the diffusivity of the field across the ensemble of plates. It is important to highlight the fact that, the field is diffuse across the ensemble, whereas one realisation does not have a diffuse field. Two measures of the diffusivity has been proposed, one using a displacement approach the second one using a wave approach. It has been shown that the two approach gives similar results. The results could be used to determine the domain of validity of the SEA. It might be interesting to investigate the relation between the correlation angle of the waves and the geometry and investigate an wider frequency band. Further works will include the study of assembly of plates.

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