Solutions of the Navier- Stokes equation in a bounded domain

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Résumé
On présente les conditions pour avoir la solution générale de l'équation de Navier-Stokes lorsque le domaine est borné et que les forces extérieures ne dépendent pas du temps. Le problème se décompose alors en deux équations : Une équation de Poisson pour la pression et une équation différentielle ordinaire pour la vitesse dépendant fortement de la distribution connue à l'instant initial. En général, quand des approximations polynomiales sont justifiées, la solution de l'équation différentielle ordinaire est donnée par des séries entières de fonctions de Weierstrass-Mandelbrot. On peut aisément étendre la méthode à des équations différentielles beaucoup plus générales.

Abstract.
In this paper, we exhibit the conditions to have the general solution of the Navier-Stokes equation, in particular when the domain is bounded and the external forces don't depend on time. Locally, the problem can be decomposed in two equations: A Poisson's equation for the pressure and an ordinary differential equation on the velocity depending strongly on the known initial distribution at the origin. In general, when polynomial approximations are justified, the solution is given by power series of Weierstrass-Mandelbrot functions. We extend the method to more general equations.

Keyword
Navier-Stokes equation, ω-time, almost periodic functions, bounded polynomial iterations, Weierstrass-Mandelbrot functions.

Introduction

Since the important J. Leray's paper on the Navier-Stokes equation (NSE) in 1934, many problems remain about the existence and the construction of a solution, at such a point that this existence constitutes one of the millennium's problems posed by Fefferman. But all the methods used to solve the question are

too much dependant of the linearization of the equation. An oversee of the question can be found in the Lemarie-Rieusset's book.

This paper presents a completely different point of view. It gives some new reflexions about the conditions and the understanding of the existence of the Navier-Stokes equation (**NSE**). But, here, the difficulties don't come from non-linearities but from linearities, especially for non autonomous equations.

The main chapter entitled the equation of Navier-Stokes presents our approach. An appendix of three chapters follows to prove the solution in power series of Weierstrass-Mandelbrot functions.

First, we improve and correct our first results about Navier-Stokes equation given in Cirier, especially oriented on probabilistic results. Here, we search deterministic solutions. We decompose this equation in two problems when the external forces don't depend on the time:
- The first problem is the existence of the initial pressure defined by a Poisson's equation when the velocity is known; this question seems well-known under Dirichlet's conditions.
- The second is the resolution of an ordinary differential equation defining the position of the velocity in function of the time. This equation is completely defined by initial conditions. The technics of iteration give some conditions to obtain an interesting solution: that is the subject of the appendix.

The following three chapters of the appendix give hints to understand the solution in terms of Weierstrass - Mandelbrot's series. They are a recall of some properties of the iterations applied to the differential iterations. The full demonstrations can be found in Cirier where probabilistic approaches are studied.
I - Composition of functions,
II - Deterministic method: linearization of iterations,
III - Application to differential equations.

The equation of Navier Stokes

Fefferman [6], writer of the problem of the equation of Navier Stokes for the Clay's foundation, seems very embarrassed to formulate it: in fact, he proposes four problems A, B, C, D. Carefully, he seems accept periodical solutions to represent the physically reasonable movement. We are not sure it is true.
On other hand, he exposes very clearly many ambiguities of the question. But, his paper is singularly imprisoned by many physical considerations, with many parameters and derivatives in relation with the non linearity of the equations.

Our approach would give a best new understanding. Naturally, many things

remain to improve. In conclusion, we formulate a solution of a more synthetic generalization of the problem. Now, place to the classical Navier's problem.

1 – Presentation of the equation

The Navier's equations concern the unknown velocity vector $\boldsymbol{u} \in \mathbb{R}^n$ and the pressure $p \in \mathbb{R}^+$ of an incompressible fluid in movement. The variables are the position $\boldsymbol{x}$ and the time t: $\mathbf{x}=(\boldsymbol{x}, t)$; $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.
Initially, we have as many equations as unknown functions with:

$$\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{t}} = \nu \Delta u_i - \sum_{j=1}^{j=n} u_j \frac{\partial u_i}{\partial \boldsymbol{x}_j} - \frac{\partial p}{\partial \boldsymbol{x}_j} + f_i(\boldsymbol{x}, t), \quad i = 1,..n$$

with the null divergence:

$$\text{div } \boldsymbol{u} = \sum_{i=1}^{i=n} \frac{\partial u_i}{\partial \boldsymbol{x}_i} = 0$$

and the known function: $\boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{u}_0(\boldsymbol{x})$ *at time:* $t = 0$.

Main hypothesis

We can have many problems with the linearity induced by the non autonomous externally forces $\boldsymbol{f}(\boldsymbol{x}, t)$. *So,* ***we suppose*** $\boldsymbol{f}(\boldsymbol{x}, t)$ ***free of*** t*:*

$$\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}).$$

Vectorial notation

- We note with vectors: $N_i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = \nu \Delta u_i - \sum_{j=1}^{j=n} u_j \frac{\partial u_i}{\partial \boldsymbol{x}_j} - \frac{\partial p}{\partial \boldsymbol{x}_j} + f_i(\boldsymbol{x})$

$$\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = [N_1, N_2, .. N_n]$$

Then: $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = \nu \Delta \boldsymbol{u} - \sum_{j=1}^{j=n} u_j \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}_j} - \boldsymbol{p} + \boldsymbol{f}(\boldsymbol{x})$

with $\boldsymbol{p} = \partial p / \partial \boldsymbol{x}$

So, the **NSE** is: $\partial \boldsymbol{u} / \partial t = \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$

Let $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = (\partial \boldsymbol{u} / \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$

And: $\partial \boldsymbol{u} \otimes \partial \boldsymbol{u} = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \frac{\partial u_j}{\partial \boldsymbol{x}_i} \frac{\partial u_i}{\partial \boldsymbol{x}_j}$

2 - Principle of demonstration

The demonstration is based on the following arguments:
At time $t = 0$, the function $\boldsymbol{u}$ of $\boldsymbol{x}$ is known: $\boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{u}_0(\boldsymbol{x})$. We deduce easily the pressure p_o. Then, substituting $\boldsymbol{u}, \boldsymbol{p}$ (and all the derivatives of $\boldsymbol{u}$) by $\boldsymbol{u}_0, \boldsymbol{p}_0$ as functions of $\boldsymbol{x}$ in the NSE $\partial \boldsymbol{u} / \partial t = \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$, we obtain an ordinary differential equation of $\boldsymbol{x}(t)$ in function of t in a neighbourhood $\mathcal{V}^+(O)$ of 0. By this transformation, the unknown function is now $\boldsymbol{x}(t)$ and the solution is $\boldsymbol{u}_0(\boldsymbol{x}(t))$.
But, we have to check at each time t the conditions of existence:
Now, if we suppose at an arbitrary fixed time $t_o \geqq 0$ that the velocity $\boldsymbol{u}$ is known in function of the position $\boldsymbol{x}$: $\boldsymbol{u}(\boldsymbol{x}, t_o) = \boldsymbol{u}_{to}(\boldsymbol{x})$. We translate easily the demonstration from 0 to t_o and find the solution in a neighbourhood $\mathcal{V}^+(t_o)$ of t_o under new conditions of existence at t_o. And so on.

First, we recall:

Lemma 1

If $\boldsymbol{u}(\boldsymbol{x})$ *is known, the condition* $div\ \boldsymbol{u} = 0$ *implies that the pressure p follows the Poisson's equation:*

$$\Delta p = \mathrm{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u} \otimes \partial\boldsymbol{u} \qquad \forall \boldsymbol{x} \in \mathbb{R}^n\ ,\ t_o \in \mathbb{R}^+$$

And this solution is unique under Dirichlet's conditions.

■ The divergence operator commutes with all derivations. If we apply this operator to the first *n* equations defining $\partial\boldsymbol{u}/\partial t$, we obtain:

$$\mathrm{div}\Delta\boldsymbol{u} = \Delta\mathrm{div}\boldsymbol{u} = 0\ ;\ \mathrm{div}\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial}{\partial t}\mathrm{div}\boldsymbol{u} = 0$$

With: $\mathrm{div}\sum_{j=1}^{j=n} u_j \frac{\partial}{\partial x_j}\ \boldsymbol{u} = \partial\boldsymbol{u} \otimes \partial\boldsymbol{u}$

And: $\mathrm{div}\boldsymbol{p} = \mathrm{div}\nabla p = \Delta p$

So: $0 = \partial\boldsymbol{u} \otimes \partial\boldsymbol{u} + \Delta p - \mathrm{div}\boldsymbol{f}(\boldsymbol{x})$

The pressure is only function of $\boldsymbol{x}$ and of $\boldsymbol{u}(\boldsymbol{x})$. If $\boldsymbol{u}(\boldsymbol{x})$ is known, the unicity of p depends on the unicity of the solution of the Poisson's equation:

$$\Delta p = \mathrm{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u} \otimes \partial\boldsymbol{u} = h(\boldsymbol{x})$$

But, the solution of the Poisson's equation is given modulo a function satisfying the Laplace's equation. Limit conditions such as Dirichlet's conditions are necessary to prove the uniqueness of this solution. This condition doesn't appear clearly in the text of Fefferman, but many authors, as Tao, know it. ■

The utility of the hypothesis **H** t_o will appear in the following.

2 - Hypothesis H t_o **at a fixed time** $t_o \geqq 0$**:**

- $\boldsymbol{u}(\boldsymbol{x}, t_o) = \boldsymbol{u}_{t_o}(\boldsymbol{x})$ *is a known function of* $\boldsymbol{x} \in \mathbb{R}^n$ *, at the fixed time* t_o.
- *We recall explicit and implicit initial conditions at* t_o*:*

  $\boldsymbol{u}_{t_o}(\boldsymbol{x})$ *has div* $\boldsymbol{u}_{t_o} = 0$ *and* $\Delta p_{t_o} = h_{t_o}(\boldsymbol{x})$ *has a unique solution.*
- *The jacobian matrix* $\partial\boldsymbol{u}_{t_o}/\partial\boldsymbol{x}$ *is invertible. If* $\partial\Omega_{t_o} = \{\boldsymbol{x}|\ \ |\partial\boldsymbol{u}_{t_o}/\partial\boldsymbol{x}| = 0\}$ *, that means:* $\boldsymbol{x} \notin \partial\Omega_{t_o}$.
- *The domain* Ω_{t_o} *of* $\boldsymbol{x}$ *is (Dirichlet) bounded under the previous condition for* t_o.

We note $\mathbf{H}t^*$ if $\mathbf{H}t_o$ is true at each time between 0 and t.

Example at time $t_o = 0$

At time $t = 0$, the known initial velocity $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ satisfies the initial conditions: $\mathrm{div}\ \boldsymbol{u}_0 = 0$ and the pressure must satisfy the condition: $\Delta p_0 = h_0(\boldsymbol{x})$ with a unique solution. A sufficient condition for the unicity is that the bounded domain Ω of $\boldsymbol{x}$ satisfies the Dirichlet's conditions

The matrix $\partial\mathbf{u}_{\mathrm{to}}/\partial\mathbf{x}$ is invertible means that the jacobian: $|\partial\boldsymbol{u}_o/\partial\boldsymbol{x}| \neq 0$. Let $\partial\Omega_o$ the set of $\boldsymbol{x}$:

$$\partial\Omega_o = \{\boldsymbol{x}|\ \ |\partial\boldsymbol{u}_o/\partial\boldsymbol{x}| = 0\}$$

We have interest to take Ω_o satisfying the Dirichlet's conditions strictly inside the frontier $\partial\Omega_o$. Here, the Poisson's equation will have generally a unique solution. (Every terrestrial position is bounded).

Lemma 2

Let $\boldsymbol{x}_t = \boldsymbol{x}(\boldsymbol{x}_{to}, t)$ the position of the velocity $\boldsymbol{u}_t = \boldsymbol{u}(\boldsymbol{x}_{to}, t)$ at time $t \geq t_o$ for the initial position $\boldsymbol{x}_{to}$. At time t_o, we suppose that the velocity $\boldsymbol{u}(\boldsymbol{x}_{to}, t_o) = \boldsymbol{u}_{to}(\boldsymbol{x}_{to})$ is a known function $\boldsymbol{u}_{to}$ of the position $\boldsymbol{x}_{to} = \boldsymbol{x}(\boldsymbol{x}_{to}, t_o)$.

Under **H**t_o*, for all $\boldsymbol{x}_t \notin \partial\Omega_{to}$, the solution $\boldsymbol{x}_t$ of the NSE for $\forall t \geq t_o$, is:*

$$\boldsymbol{u}_t = \boldsymbol{u}_{to}(\boldsymbol{x}_t)$$

where $\boldsymbol{u}_{to}$ is known and $\boldsymbol{x}_t$ is the solution of the ordinary differential equation with the known function $\boldsymbol{F}_{to}$ of $\boldsymbol{x}_t$:

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_{to}(\boldsymbol{x}_t) = (\partial\boldsymbol{u}_{to}(\boldsymbol{x}_t)/\,\partial\boldsymbol{x}_t\,)^{-1}\boldsymbol{N}_{to}(\boldsymbol{x}_t);$$

with initial condition: $\boldsymbol{x}_{to} = \boldsymbol{x}(\boldsymbol{x}_{to}, t_o)$

Then: $\Delta p_t = \mathrm{div} f(\boldsymbol{x}_t) - \partial\boldsymbol{u}_t \otimes \partial\boldsymbol{u}_t = h_t(\boldsymbol{x}_t)$

■ At time t_o, the pressure p_{to} must satisfy the Poisson's equation for $\boldsymbol{f}(\boldsymbol{x})$ and the known function $\boldsymbol{u}_{to}(\boldsymbol{x})$ at $\boldsymbol{x}_{to}$:

$$\Delta p_{to} = \mathrm{div}\boldsymbol{f}(\boldsymbol{x}_{to}) - \partial\boldsymbol{u}_{to} \otimes \partial\boldsymbol{u}_{to} = h_{to}(\boldsymbol{x}_{to})$$

If $\boldsymbol{x}$ has a good smooth frontier (as under Dirichlet's conditions), the solution is unique. And $\boldsymbol{N}_{\mathrm{to}}(\boldsymbol{x})$ is a known function of $\boldsymbol{x}$ **.**

Now, we study $\boldsymbol{x}_t = \boldsymbol{x}_{to}(\boldsymbol{x}_{to}, t)$ solution of the NSE for $\forall t \geq t_o$ and a initial position $\boldsymbol{x}_{to}$. As $\boldsymbol{u}_{to}(\boldsymbol{x}_t)$ is known but $\boldsymbol{x}_t = \boldsymbol{x}_{to}(\boldsymbol{x}_{to}, t)$ is unknown, the NSE becomes in a neighbourhood $\mathcal{V}^+(t_o)$ of t_o:

$$\partial\boldsymbol{u}_{to}(x_t)/\,\partial t = \boldsymbol{N}_{\mathrm{to}}(\boldsymbol{x}_t)$$

$$\partial\boldsymbol{u}_{to}(x_t)/\,\partial t = \partial\boldsymbol{u}_{to}(\boldsymbol{x}_t)/\,\partial\boldsymbol{x}_t\,.\,\partial\boldsymbol{x}_t/\,\partial t$$

Then:

$$\partial\boldsymbol{x}_t/\,\partial t = (\partial\boldsymbol{u}_{to}/\,\partial\boldsymbol{x}_t\,)^{-1}\boldsymbol{N}_{to}(\boldsymbol{x}_t).$$

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_{to}(\boldsymbol{x}_t)$$

Where $\boldsymbol{F}_{to}(\boldsymbol{x}_t)$ is a known function of only $\boldsymbol{x}_t$. The NSE is reduced to an ordinary differential equation (ONSE) near t_o with the initial position $\boldsymbol{x}_{to}$.

Let $\boldsymbol{x}_t = \boldsymbol{x}_{to}(\boldsymbol{x}_{to}, t)$ be the solution for the initial position $\boldsymbol{x}_{to} = \boldsymbol{x}_{to}(t_o)$ at time $t = t_o$. Then, the velocity at time $t \geqq t_o$ for the known initial position $\boldsymbol{x}_{to}$ is:

$$\boldsymbol{u}_{to}(\boldsymbol{x}_{to}, t) = \boldsymbol{u}_{to}\big(\boldsymbol{x}(\boldsymbol{x}_{to}, t)\big)$$

and the NSE at time t remains yet valid for the new velocity $\boldsymbol{u}_{to}(\boldsymbol{x}_t)$ in a neighbourhood $\mathcal{V}^+(t_o)$ of t_o . ■

Remark

We have obtained the result that, at each time t_o, the solution of the NSE is solution of the ONSE defining $\boldsymbol{x}(t)$. But, as this $\boldsymbol{x}(t)$is changing with t, we may have the case where $\boldsymbol{x}(t_o) \in \partial\Omega_{to}$ for some time t_o. So, we define the time ω:

Definition
Let ω-time the first time where $\boldsymbol{x}(\omega) \in \partial\Omega_{to}$.
It means that $\partial\Omega_{to} = \{\boldsymbol{x}_{to} |\ \ |\partial \boldsymbol{u}_{to}/\partial \boldsymbol{x}_{to}| = 0\}$.
We will clarify this condition in the next proposition.

3 - Proposition
Let $\boldsymbol{u}_o(\boldsymbol{x})$ *be the known function of the position* $\boldsymbol{x}$ *at time* $t = 0$.
For all $0 \leqq t < \omega$ *, under* $\boldsymbol{H}t^*$**,** *the solution of the NSE for* $\forall t \geq 0$*, is:*

$$\boldsymbol{u}(\boldsymbol{x}_t, t) = \boldsymbol{u}_o(\boldsymbol{x}(t))$$

$\boldsymbol{x}_t = \boldsymbol{x}(\boldsymbol{x}_o, t)$ *is solution of the ONSE with* $\boldsymbol{F}_o$ *known for an initial position* $\boldsymbol{x}_o$*:*

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_o(\boldsymbol{x}_t);$$

- *Then:* $\Delta p_t = \ \mathrm{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_t \otimes \partial \boldsymbol{u}_t = h_t(\boldsymbol{x}_t)$

Where: $\boldsymbol{F}_o(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o) = (\partial \boldsymbol{u}_o/\ \partial \boldsymbol{x}\ )^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o)$

With p_o*:* $\Delta p_o = \ \mathrm{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_o \otimes \partial \boldsymbol{u}_o = h_o(\boldsymbol{x})$

- *The ω -time is defined by the two conditions: :* $\boldsymbol{x}_t \notin \partial\Omega_o$ *and* $|\ \partial \boldsymbol{x}_t/\ \partial \boldsymbol{x}| \neq \boldsymbol{0}$
- *The solution of the ONSE is reduced to the study of the zero of* $\boldsymbol{F}_0(\boldsymbol{x})$ *(*$\boldsymbol{F}_o(\boldsymbol{\alpha}) = \boldsymbol{0}$*)and the eigen values at these points. More, if we suppose that* $\boldsymbol{F}_0(\boldsymbol{x})$ *can be uniformly approximated by polynomials, at least locally near fixed points (*$\boldsymbol{F}_o(\boldsymbol{\alpha}) = \boldsymbol{0}$*) , especially if they are hyperbolic, the solution is a power series of Weierstrass - Mandelbrot functions. The equality of Perceval is satisfied.*

■

- We take $t_o = 0$ and write under $\mathbf{H}t^*$ for all $0 \leqq t < \omega$,

$$\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}_o(\boldsymbol{x}(t))$$

with $\boldsymbol{x}(t)$ solution of: $\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_o(\boldsymbol{x}_t)$

Where: $\boldsymbol{F}_o(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o) = (\partial \boldsymbol{u}_o/\ \partial \boldsymbol{x}\ )^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o)$

And p_o*:* $\Delta p_o = \ \mathrm{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_o \otimes \partial \boldsymbol{u}_o = h_o(\boldsymbol{x})$

If we note $\boldsymbol{x}_o = \boldsymbol{x}(0)$ the initial position at time $t_o = 0$, the position $\boldsymbol{x}(t)$ at time t will be $\boldsymbol{x}(t) = \boldsymbol{x}(\boldsymbol{x}_o, t)$.
- We check the coherence of the solution $\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}_o(\boldsymbol{x}(\boldsymbol{x}_o, t))$, at every time:
$0 \leqq t < \omega$, to have a solution of the NSE:
Let the $(\mathrm{ONSE})_o$ be the ONSE with the known velocity $\boldsymbol{u}_o(\boldsymbol{x})$at time 0:

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = (\partial \boldsymbol{u}_o/\ \partial \boldsymbol{x}\ )^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o)$$

Now, we want to study the fluid not at time 0 but at time $t_o > 0$ with the position of the velocity $\boldsymbol{x}_{to} = \boldsymbol{x}_{to}(\boldsymbol{x})$ defined by the solution of the $(\mathrm{ONSE})_o$ at time t_o. The new $(\mathrm{ONSE})_{to}$ with the new position $\boldsymbol{x}_{to}$ and the new known velocity $\boldsymbol{u}_o(\boldsymbol{x}_{to})$ is: $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = (\partial \boldsymbol{u}_o/\ \partial \boldsymbol{x}\ \partial \boldsymbol{x}_{to}/\ \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\ \boldsymbol{x}_{to}, \boldsymbol{u}(\boldsymbol{x}_{to}))$
The solution is invariant under a translation of time only if $\partial \boldsymbol{u}_o/\ \partial \boldsymbol{x}\ \partial \boldsymbol{x}_{to}/\ \partial \boldsymbol{x}$ is

invertible. So, the conditions are: $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}|\,|\partial \boldsymbol{x}_{to}/\partial \boldsymbol{x}| \neq \boldsymbol{0}$. We have a new condition $|\partial \boldsymbol{x}_{to}/\partial \boldsymbol{x}| \neq \boldsymbol{0}$ with the previous $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| \neq \boldsymbol{0}$.

- The analysis of the ordinary differential equation will be studied in the appendix in terms of iteration: Under the hypothesis of the hyperbolicity of the fixed points, the solution is developed in power series of functions of Weierstrass - Mandelbrot. See appendix. ■

Remarks
Under our main assumptions, we find many situations where the NSE has no unique solution.

If $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$, the corresponding surface is a pole for the NSE and we have turbulences all along this surface and $\Delta p = \mathrm{div} f(\boldsymbol{x})$. So, before to do anything, we have to note the points such as $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$ and study the behaviour of the process far from these points as possible. But, we can have many other turbulences when $|\partial \boldsymbol{x}_t/\partial \boldsymbol{x}| = \boldsymbol{0}$.

Anyway, we have seen that the solutions are local near each fixed points and we can meet many other perturbations when the process cross from a basin of a fixed point to an another as in the Lorenz iteration.

4 - Conclusion
The paradigm of the problem has completely moved: the equations of the movement depend entirely on the given initial situation $\boldsymbol{u}_o(\boldsymbol{x})$! But the good solution is local near by the initial $\boldsymbol{u_o}(\boldsymbol{x})$.

New generalised equation $\boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$,
If our demonstrations are correct, we see easily that we can take more complicated derivable function $\boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$instead of $\nu\Delta\boldsymbol{u} - \sum_{j=1}^{j=n} u_j \frac{\partial u}{\partial x_j} - \frac{\partial p}{\partial x}$. The method will be the same: parameter ν does not have a great importance mathematically in this context. In $\boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$, we can put any functions (smooth functions, derivatives, even unique solution of an equation as the pressure p) of $\boldsymbol{u}(\boldsymbol{x})$.

Definition
Let $\boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$ be a smooth function of $\boldsymbol{u}(\boldsymbol{x})$, and of (any smooth functions, derivatives, even unique solution of a functional or integro-differential equation of) $\boldsymbol{u}$, where $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$.
Let: $\partial \boldsymbol{u}(x)/\partial t = \boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$

Theorem
At time $t = 0$, $\boldsymbol{u}_o(\boldsymbol{x}) = \boldsymbol{u}_o$ the solution is a smooth known function. Then, the

solution of:

$$\partial \boldsymbol{u}(x)/\,\partial t = \boldsymbol{N}(\boldsymbol{u},\boldsymbol{x})$$

Is, for all $0 \leqq t < \omega$: $\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}_o(\boldsymbol{x}(\boldsymbol{t}))$,
where $\boldsymbol{x}(\boldsymbol{t})$ *is the solution of the ordinary differential equation with* $\boldsymbol{F}_o(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}_o) = (\partial \boldsymbol{u}_o/\,\partial \boldsymbol{x}\,)^{-1}\boldsymbol{N}(\boldsymbol{u}_o,\boldsymbol{x})$*function of* $\boldsymbol{x}$:

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{F}_o(\boldsymbol{x});$$

■ As $\boldsymbol{u_o}(\boldsymbol{x})$ is function of $\boldsymbol{x}$, every smooth functions, derivatives and unique solutions of a functional or integro-differential equation of $\boldsymbol{u}$ can be explained as functions of $\boldsymbol{x}$ and we can write $\boldsymbol{N}(\boldsymbol{u}_o,\boldsymbol{x}) = \boldsymbol{N}_o(\boldsymbol{x})$.
The only important things to check are:
- the jacobian $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| \neq 0$,
- the unicity of the solution of the functionals in $\boldsymbol{N}(\boldsymbol{x})$,
- the veracity of $\mathbf{H}t^*$ at each time *t*, and especially the ω point,
- the border of the domain of $\boldsymbol{x}$.

And we have to solve the ordinary differential equation: $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{u}_o(\boldsymbol{x}))$ ■

But, if we are interested by an improbable stationary behaviour, i.e. $\partial \boldsymbol{x}_t/\,\partial t = O$, we have to search a solution of $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}) = 0$, with our very complicate hypothesis under condition $|\partial \boldsymbol{u}/\partial \boldsymbol{x}| = 0$.

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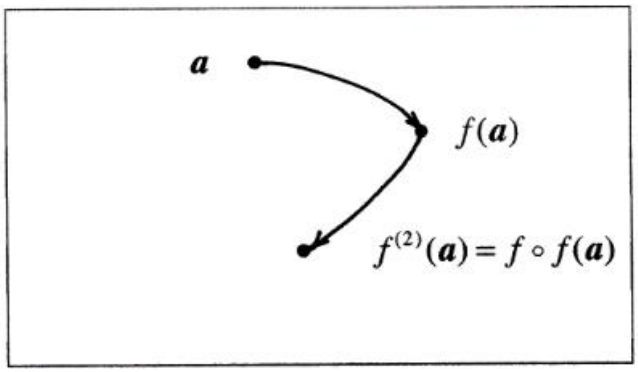
Appendix

I - Composition of functions

Let f a polynomial application of $\mathbb{R}^d$ in itself. This space can be occasionally $\mathbb{C}^d$. Geometrically, we represent it as a point transformation:



We start from a point $\boldsymbol{a}_0 \in C$. We construct : $\boldsymbol{a}_1 = f(\boldsymbol{a}_0)$ and we do it again indefinitely : $\boldsymbol{a}_p = f(a_{p-1})$. So, we have: $\boldsymbol{a}_p = f^{(p)}(\boldsymbol{a}_0)$

1 - Definition of a bounded iteration

We call iteration the data of a function f when we iterate it indefinitely with composition's law of functions noted $\circ$ and a starting point $\boldsymbol{a}_0$. the p- iterated $f^{(p)}$ of f is such as:

$$f^{(p)} = f \circ f^{(p-1)}.$$

We call *orbit of* $\boldsymbol{a}_0$ the set of the iterates of a point $\boldsymbol{a}_0$: $O(\boldsymbol{a}_0) = (\boldsymbol{a}_0, \boldsymbol{a}_1 .. \boldsymbol{a}_p)$.

In all this paper, f is always supposed defined on a bounded domain C with a bounded image in C : $f(C) \subset C \subset \mathbb{C}^d$. As f is polynomial, the extrema of f has just to belong in C. For instance, the logistic function : $ax(1-x)$ defined on $[0,1]$ has an image $[0, a/4]$ which remains bounded for $a \leq 4$. But, generally, it is often very difficult to prove this property. Here, we admit that it is always true, because proofs are more simple. On other hand, this hypothesis remains the implicit or explicit basis of many cases studied by users and physicists: The Lorenz's differential equations represent mathematically terrestrial atmosphere as a gas contained between a hot sphere (hearth) and a cold sphere (stratosphere). So:

2 – Hypothesis H1

Iteration f is polynomial with degree r and applies a bounded set C of $\mathbb{R}^d$ in itself.

In mathematics, a dynamic system is generally indexed on an additive semi - group rather than on integers. Here, differential equations will be seen as iterations rather than indexed on a continuous additive semi - group.

3 - Conjugation of iterations

As the study of the composition's law $f^{(p)}$ is our main goal, we have to search commutative invariants for this law. So, from a geometrical point of view, qualitative properties of an iteration do not vary when we operate a continuous invertible change of coordinates of $\mathbb{R}^d$ in $\mathbb{R}^d$: $\boldsymbol{a} = \gamma(\boldsymbol{t})$. Generally, this change is a diffeomorphism. This technique will be often used here.

Let $\boldsymbol{a}_1$ be the image of $\boldsymbol{a}$ by f:

$$\boldsymbol{a}_1 = f(\boldsymbol{a})$$

Changing of basis with: $\boldsymbol{a} = \gamma(t)$ and $\boldsymbol{a}_1 = \gamma(\boldsymbol{t}_1)$, we get:

$$\boldsymbol{a}_1 = \gamma(\boldsymbol{t}_1) = f(a) = f \circ \psi(\boldsymbol{t})$$

Then: $$\boldsymbol{t}_1 = \psi^{-1} \circ f \circ \psi(\boldsymbol{t}).$$

4 - Invariant sets

Definitions

A set A is invariant under f if : f(A) ⊂ A.

Let the chain: $f^{(p)}(C) \subset \ldots \subset f(C) \subset C$ and the adherence $C_\infty = f^{(p)}(C)$. We have the invariant set $C_\infty = f(C_\infty)$. An attractor is a compact sub-set $C_0 \subset C_-$ such as the orbit of any point of a given neighbourhood of a point converges to C_0.

So, the bounded domain C is invariant. Orbit $O(\boldsymbol{a}_0)$ is invariant under f.

Generally, an iteration has some remarkable invariants.

Fixed points and cycles

Definition

An invariant point $\mathbf{0}$ is called fixed point if $\mathbf{0} = f(\mathbf{0})$. Cycles are composed of invariant points α under $f^{(p)}$ such as $\alpha = f^{(p)}(\alpha)$ but $\alpha \neq f(\alpha)$.

Remarks

- Even if there are many fixed points or cycles, we take the origin at one point denoted $\mathbf{0}$ for all local study.

- All that we say about a fixed point $\mathbf{0}$ will be true for all the points of the collection of cycles or other fixed points. There is sometimes interactions between attraction's domains of these different points.
- If invariant points can be dense, we may have continuous differentiable set; but we don't study these sets here.

Eigen values at the fixed point
At the fixed point : $\mathbf{0}$, if f is linear, we have :$f(\boldsymbol{a}) = f'(\mathbf{0})\boldsymbol{a}$ where $f'(\mathbf{0})$ is a square matrix. Then, it is easy to calculate $f^{(p)}$ and forecast the behaviour when $p \to \infty$. But $f'(0)$must be diagonalisable. We have to research the eigen values λ of $f'(\mathbf{0})$, and study the characteristic polynomial:

$$|f'(\mathbf{0}) - \lambda\, I| = 0.$$

In order to diagonalize $f'(\mathbf{0})$, we suppose the eigen values distinct. We note that the eigen value $|\lambda| = 1$ may induce functional relationship between fixed points in a neighbourhood of $\mathbf{0}$ and a particular behaviour of the iteration. The importance of the role of the eigen values at the fixed point is illustrated by some most important theorems of convergence, often called theorems of the fixed point, especially for contracting iterations characterized by eigen values $|\lambda| < 1$. We have same results for dilating iterations.
Definition
In the intermediate case where we have $|\lambda| \neq (0,1)$, with eigen values larger and smaller than the unit, iteration and the fixed point are said hyperbolic.

justification
Let the linear application Λ of $\mathbb{C}^d$ ($d \geq 2$) in $\mathbb{C}^d$: $\boldsymbol{a} \to \Lambda\boldsymbol{a}$. Λ is the diagonal matrix of the eigen values λ, $|\lambda| \neq (0,1)$. the iteration is written: $\boldsymbol{a}_p = \Lambda^p \boldsymbol{a}_0$. Each axis of coordinates is globally invariant. If $\lambda_1 > 1 > \lambda_2 > 0$, we have a kind of invariant hyperbola. So, if the eigen values of the linear part of $f'(\mathbf{0})$ have this configuration, the fixed point is said hyperbolic. We don't speak here about complex eigen values, except if the argument is like $2k\pi/p$. then, $f^{(p)}$ has all the real positive eigen values.

To study this *intermediate case*, we have to examine the invariant functions. The case where some eigen values equal 1 is not studied here.

Invariant surfaces and curves
Definition
An invariant surface or variety is defined by the data of a vector G with k functions analytical, distinct of the identity, G_ℓ : $\mathbb{R}^d \to \mathbb{R}$ such as

$$G_\ell \circ f(\boldsymbol{a}) = G_\ell(\boldsymbol{a})\ ,\ \ell = 1,\ldots,k < d\ .$$

If $\boldsymbol{a}_0$ belongs to G , all the points of the orbit of $\boldsymbol{a}_0$ belong to the invariant surface : $G(\boldsymbol{a}) = G(\boldsymbol{a}_0)$.

Linearization of f in $\mathbb{R}^d$

The behaviour of a point $\boldsymbol{a}$ in the neighbourhood of a fixed point $\mathbf{0}$, is more easy when we can linearize its evolution. Two functional equations are useful:

- Construction of d ***functions of linearization*** $\varphi_\ell(\boldsymbol{a})$, $\mathbb{R}^d \to \mathbb{R}$:

$$\varphi_\ell \circ f(\boldsymbol{a}) = \lambda_\ell \, \varphi_\ell(\boldsymbol{a}) \, , \ \ell = 1,...,d$$

- Construction of d ***invariant functions*** $\boldsymbol{a}_\ell(\boldsymbol{t})$ coordinates of $\boldsymbol{a}(\boldsymbol{t})$: $\mathbb{R}^d \to \mathbb{R}$,

$$a_\ell(\lambda \boldsymbol{t}) = f_\ell(\boldsymbol{a}(\boldsymbol{t})) \, , \ \ell = 1,...,d,$$

sometimes called *separatrices*. They are coordinates of $\boldsymbol{a}(\boldsymbol{t})$:

$$\boldsymbol{a}(\lambda \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})).$$

$\boldsymbol{a}(\boldsymbol{t})$ is an invariant function under the action of f because every point transformed by f remains defined by the same parameterization. These invariant curves contain the origin. Some scholars call them linearizing functions.

In the both cases, applications φ and $\boldsymbol{a}$ are constructed locally with series.

We call study with more details $\boldsymbol{a}(\boldsymbol{t})$) in the paragraph on the invariant functions.

Invariant measures

Generally, a measure f-invariant is defined as the solution of the Perron - Frobenius operator:

Definition

Let an arbitrary measure of probability P and f a P - measurable application. For all borelian set B, the transformation of P by the Perron - Frobenius operator P_f is: $P \circ f^{-1}(B) = P_f(B)$.

The measure P it f-invariant if, for all borelian set B, we have:

$$P \circ f^{-1}(B) = P(B)$$

Such a functional equation is not easy to solve. We present solutions in our book.

II - Deterministic method: linearization of iterations

Following the works of Poincaré in 1879 on the linearization of differential equations and iterations, many results have been obtained on this question. Among the last one, the main theorem of Grobman - Hartman (1959) says: in a Banach's space B, if Λ is a linear hyperbolic application of B in B, f a continuous invertible application and if $f - \Lambda$ is a lipchitsian contraction, then f is linearizable in a neighbourhood of 0, fixed point of f.

In our case, Λ is a application linear hyperbolic, means that the eigen values at 0 are not in resonance : $\lambda_\ell \neq \lambda^n$, $\ell = 1,...,d$, and that we have eigen values with

modulus larger and smaller than 1. « f is linearizable in neighbourhood of 0 » means that we have:

$$\boldsymbol{a}(\Lambda\, \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})).$$

When we iterate indefinitely f, the eigen values with modulus smaller than 1 will tend to 0. So, it remains only the q eigen values with modulus larger than 1: $|\lambda_\ell| > 1$, denoted λ and we keep the notation $\boldsymbol{t}$ for the corresponding vector of dimension q : (λ_ℓ, t_ℓ), $\ell = 1,...,q \pounds d$,

The other coordinates corresponding to the eigen values of modulus smaller than 1 tend to 0 when the number of iterations tend to the infinite.

To continue, we use more restrictive hypothesis on the eigen values with modulus larger than 1 : $|\lambda_\ell| > 1,\ \ell = 1,...,q \pounds d$.

1 - Hypothesis H

Les eigen values are not in resonance: $\lambda_\ell \neq \lambda^n, \ell = 1,...,q \pounds d$. *All the q eigen values with modulus larger than 1:* $\lambda_\ell > 1,\ \ell = 1,...,q \pounds d$ *are positive transcendental. So:* $0 \neq \Sigma\, n_k \lambda_\ell{}^k$, $\forall n_k \in \mathbb{N}; \forall k \in \mathbb{Z};\ \ell = 1,...,q \pounds d$

The equation of linearization becomes **E**:

$$\boldsymbol{a}(\lambda\, \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})) \text{ where } \boldsymbol{t} \in \text{»}^{\,q} \text{ and } \lambda \in \text{»}^{+q}$$

Under these hypothesis, the theorem of Bohr comes without difficulties. The hypothesis $\lambda \in \text{»}^{+q}$ is not necessary because: if $\lambda < 0$, by iterating twice times f, we obtain positive eigen values. Positive eigen values $\lambda_\ell > 1$ are necessary to use the Weierstrass- Mandelbrot's function.

If $\boldsymbol{a}(\boldsymbol{t})$ is bounded, the image of $\boldsymbol{a}(\boldsymbol{t})$is relatively compact, so the solution of equation **E** is an almost periodic function of Bohr, then : $\boldsymbol{a}(\boldsymbol{t}) \in AP$.

2 - Theorem of Bohr (recall of the Bohr's approximation)

Let a trigonometric function:

$$P_n(\boldsymbol{t}) = \sum_{k=1}^{k=n} \boldsymbol{c}_m\ \mathrm{e}^{i\boldsymbol{\mu}_k \boldsymbol{t}}\ :$$

Where $\boldsymbol{\mu}_k$, $\boldsymbol{t} \in \text{»}^{\,q}$, $\boldsymbol{c}_k \in \text{»}^{\,d}$, $\boldsymbol{\mu}_k \boldsymbol{t}$ is the scalar product.

If $\boldsymbol{a}(\boldsymbol{t}) \in AP$, for $\forall \varepsilon > 0$, it exists a sequence of $P_n(\boldsymbol{t})$ such as:

$\boldsymbol{c}(\boldsymbol{\mu}) = \boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}}(\boldsymbol{a})$ where $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}}$ is the transformation of Fourier- Bohr

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}}(\boldsymbol{a}) = \lim_{T\to\infty} \frac{1}{(2T)^q} \int \ldots \int_{-T}^{+T} \boldsymbol{a}(\boldsymbol{t})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{\mu}\boldsymbol{t}}\, d\boldsymbol{t}$$

Let $\boldsymbol{c}(\boldsymbol{\mu})$ *the vector coefficient of Fourier-Bohr for the almost period* $\boldsymbol{\mu}$.

Let $\boldsymbol{\Lambda}(\boldsymbol{a}) = \{\boldsymbol{\mu}: \boldsymbol{c}(\boldsymbol{\mu}) \neq \boldsymbol{0}\}$*the set of the almost period (p.p.)* $\boldsymbol{\Lambda}$ *is countable.* $\boldsymbol{a}(\boldsymbol{t})$ *is uniformly continuous with a representation with series of Bohr Fourier:*

$$\boldsymbol{a}(\boldsymbol{t}) \sim \sum_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}} \boldsymbol{c}(\boldsymbol{\mu})\ \mathrm{e}^{i\boldsymbol{\mu}\boldsymbol{t}} \quad .$$

This representation is unique. The convergence in quadratic mean holds. The equality of Perceval is satisfied.

Remark
The theorem of Bohr generalizes the Fourier's series. The continuity is uniform.

3 - Solution of E in the space of the almost periodic functions

Under H3, the solution of the equation **E** is:

$$\boldsymbol{a}(\lambda\,\boldsymbol{t})=f\big(\boldsymbol{a}(\boldsymbol{t})\big) \text{ where } \boldsymbol{t}\in\,\text{»}^{\,q} \text{ and } \lambda>1\in\,\text{»}^{+q}$$

As its image is relatively compact in »^{d}, its solution belongs to the set of the almost periodic *(A.P.)* functions.
But, equation **E** implies conditions on the almost periods (*a.p.*) $\boldsymbol{\mu}$ and on the corresponding coefficients $\boldsymbol{c}(\boldsymbol{\mu})$.
Now, we search the solution of the equation **E** in the set *(A.P.)*.

Definition
Let the polynomial finite formal sums defined for all the q positive eigen values with modulus larger than 1 $\lambda_\ell>1,\ \ell=1,\ldots,q\pounds d$

$$\varphi_m(\lambda)=\Sigma_{jk}\,m_j\lambda^k\mid m_j\in\mathbb{N};m_j\le m;k,j\in\mathbb{Z}$$

Here, only the integer m is important, and not the degrees of λ *which are arbitrary in* $\mathbb{Z}$. *We call m the* ***index****.*
■ If $\Phi(\lambda)$ is the set $\Phi(\lambda)=\boldsymbol{[}\varphi_m(\lambda)|m\in\mathbb{N}\,\boldsymbol{]}$. We have the following algebra for all the elements:

If $\varphi_m(\lambda),\varphi^{1}m(\lambda),\varphi^{2}m(\lambda)\in\Phi(\lambda)$, then:
$\lambda\varphi_m(\lambda)\in\Phi(\lambda)$, $\lambda^{-1}\varphi_m(\lambda)\in\Phi(\lambda)$, $n\varphi^{1}_m(\lambda)+n\varphi^{2}_m(\lambda)\in\Phi(\lambda)$, $\forall n,n\in\mathbb{N}$.

As f is invertible, if μ is an a.p. , then $\mu\lambda^k$ and $\mu\lambda^{-k}$ are still an a.p. Then, the μ are multiplies by all the coefficients of the polynomial. So, all the a.p. belong to $\Phi(\lambda)$. ■

4 - Theorem (linearization)
Under H, the p.p. μ *solutions of* **E** *are* $\mu=\varphi_m(\lambda)$ *and the solutions* $\boldsymbol{a}(\boldsymbol{t})$ *can be written ;*

$$\boldsymbol{a}(\boldsymbol{t})=\Sigma_m\Sigma_\varphi\,\Re e\boldsymbol{[}\boldsymbol{c}(\varphi_{\,m}(\lambda))(1-\exp(\varphi_m(\lambda)\boldsymbol{t}))\boldsymbol{]}$$

If the index $m=1$, *then :* $\varphi_{\,m}(\lambda)=\lambda^k|k\in\mathbb{Z}$
and $\boldsymbol{a}_1(\boldsymbol{t})=\boldsymbol{c}\boldsymbol{w}(\boldsymbol{t})$
Where:
- $\boldsymbol{c}$ *is the matrix of the eigen vectors of :* $f'(\boldsymbol{0})\boldsymbol{c}=\boldsymbol{c}\boldsymbol{\lambda}$ *at the fixed point* $\boldsymbol{0}$, *corresponding only to the q eigen values with modulus larger than 1 and of zero elsewhere.*
- $\boldsymbol{w}(\boldsymbol{t})$ *is the diagonal matrix compound of a diagonal bloc* $\mathbb{R}^{q}\times\mathbb{R}^{q}$ *of the q eigen*

values λ and of zero elsewhere. The diagonal bloc $q \times q$ is composed of Weierstrass-Mandelbrot's functions(WMF):

$$w_\ell(t_\ell) = \Re e\left[\Sigma_k r_\ell{}^k (1 - \exp i\left(\lambda_\ell{}^k t_\ell\right)\right]$$

with: $r_\ell{}^{-1} = \lambda_\ell \qquad \ell = 1,...,q \pounds d$

■ hints of proof

** the exponents of* $\mathbf{E}$ *and the almost periods for each coordinate of* $\boldsymbol{t}$ *are:*

$$\lambda\varphi_m(\lambda) = \Sigma_{m'} n_{m'}\varphi_{m'}(\lambda),$$

** for the coefficients* of *Fourier-Bohr*

The coefficients of Fourier-Bohr $\boldsymbol{a}(\boldsymbol{t})$ are obtained by Fourier's transformation for each *p.p.* $\varphi_m(\lambda)$. Applying this transformation $\boldsymbol{\mathcal{F}}\varphi_m(\lambda)$ to equation **E**:

$$\mathbf{c}(\varphi_m/\lambda) = \boldsymbol{\mathcal{F}}\varphi_m(\boldsymbol{a})$$

We order the calculus with the increasing index m beginning with $m = 1$

**First, we consider* $m = 1$.

Now, the *a.p.* for a coordinate of $\boldsymbol{t}$ is $\varphi_1(\lambda) = \lambda^k$. The condition on the *a.p.* becomes:

$$\lambda\lambda^k = \Sigma n_{k'}\lambda^{k'}.$$

If the eigen values are transcendental, all the coefficients $n_{k'}$ satisfy: $n_{k'} = 0$ except $n_{k+1} = 1$. So, the solution unique is : $\mu_k = \lambda^k, k \in \mathbb{Z}$. The following is easy. ■

Remarks:

The hypothesis of the transcendence of λ is perhaps too strong because, λ cannot be algebraic, which is very restrictive. We think that there is less (For example, the eigen values are algebraic in the Henon's case).

All the other cases are more difficult. Indeed, if we don't use the hypothesis of transcendence and we want to obtain $\mu_k = \lambda^k | k \in \mathbb{Z}$, we have to prove that all the resultants of the characteristic equation of the eigen values and $\lambda\lambda^k = \Sigma n_{k'}\lambda^{k'}$ are not null. In the other cases, the relations between the coefficients are not linear.

Construction of the solution $a(\boldsymbol{w})$

Note $\boldsymbol{w}$ the vector of coordinates:

$$w_\ell = w_\ell(t_\ell) \text{ for } \ell = 1,...,q \pounds d$$

and $\qquad w_\ell = 0 \qquad \text{for } \ell = q+1,...,d.$

Proposition

Under H, We can write: $\boldsymbol{a}(\boldsymbol{t}) \sim \sum_m \boldsymbol{a}_m \boldsymbol{w}^m$ *and the* $\boldsymbol{a}_m$ *are determines by recurrence with* $\boldsymbol{a}_1, \boldsymbol{a}_2 \ldots \boldsymbol{a}_{m-1}$. *The power series converges as an exponential.*

5 - Self similarity (*S.S.*) and function of WM

Definition

Let $\lambda, t \in \mathbb{R}^{+q}$ *with* $\lambda > 1$ *and* $0 < r \leq 1$, *the real function* $\boldsymbol{w}(\boldsymbol{t})$ *is* self similar *if:*

$$\boldsymbol{w}(\lambda \boldsymbol{t}) = r^{-1}\boldsymbol{w}(\boldsymbol{t}) \ .$$

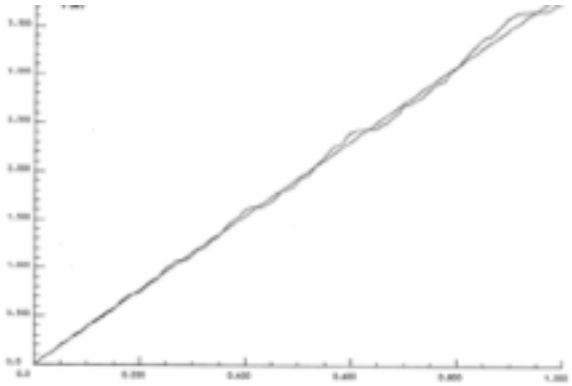
Example: Let the pavement $I_k = \prod_\ell [\lambda_\ell{}^{k_\ell}, \lambda_\ell{}^{k_\ell + 1}[, k_\ell \in \mathbb{Z}$ and $\mathbb{R}^q = \bigcup_k I_k$

Weierstrass-Mandelbrot's function (WM) in $\mathbb{R}^+$

The WMF: $w_\lambda(t) = \Re e\left[\sum_{k \in \mathbb{Z}} r^k(1 - \exp i(\lambda^k t))\right]$

is defined $r=\lambda^{D-2} \leq 1$ and converges (See Berry p 463). The $\lambda^k \mid \lambda \geq 1$ are the almost periods *a.p.* of $w_\lambda(t)$ and $\lambda, t \in \mathbb{R}^+$.

Berry presents an interesting simulation of the function WM when $D = 1$ with: $\lambda = 1{,}5$. The result is not a straight line.



The WMF has the proprieties of self-similarity : $w_\lambda(\lambda t) = r^{-1} w_\lambda(t)$

and also a kind of linearity, for $m, n \in \mathbb{N}$: $w_\lambda(t) = (m/n) w_{\lambda^{m/n}}(t)$

Some unsettled problems

- When f is partially linear or when the eigen values are integers or fractions, we can encounter some difficulties
- When there are many fixed points, we can compute the invariant functions for each fixed point, but it seems that the iteration can swing from a domain of attraction to another.
- When there is resonance, we can apply the same method with theorem of Poincaré Dulac.

III - Application to differential equations

1- Introduction: equations with partial derivatives

Many problems can be studied with a partial differential equation such as:

$$\partial \boldsymbol{a} / \partial \boldsymbol{x} = F(\boldsymbol{a}).$$

i.e. : $\partial \boldsymbol{a}_\ell / \partial \boldsymbol{x}_\ell = F_\ell(\boldsymbol{a})$, $\ell = 1, \ldots, k < d$,

where $F(\boldsymbol{a})$ is a vector of polynomials of $\boldsymbol{a} \in \mathbb{C}$ bounded of $\mathbb{R}^d$ in C of $\mathbb{R}^d$ with as many equations than unknown functions $\boldsymbol{a}(\boldsymbol{x})$. The number of equations equals the dimension of $\boldsymbol{a}$.

The dimension of the vector $\boldsymbol{x} \in \mathbb{R}^d$ is equal à the dimension du vector $\boldsymbol{a} \in \mathbb{R}^d$.
The equations of Navier Stokes satisfy this situation with some conditions.
Here, the problems of frontiers are excluded, although, by hypothesis, the iteration is always supposed to belong in a bounded domain by its own inertia.
The problem is to find a function $\boldsymbol{a}(\boldsymbol{x})$ obeying this equation with an initial condition $\boldsymbol{a}(\boldsymbol{x}_0) = \boldsymbol{a}_0$.
But, the number d of coordinates of the variable $\boldsymbol{x}$ can be choose arbitrary smaller according the studied problem. By example, If $\boldsymbol{x} \in \mathbb{R}$, we have a differential ordinary equation. In other words, all the coordinates of $\boldsymbol{x}$ are equal.

We study theses differential equations with the results that we have obtained about the iterations: we have to translate the equation en words of iteration. But, there is two ways to approach the question. The first is deterministic and centred on the almost periodic functions, the second is probabilistic with the operator of Perron Frobenius. Here, we only study the first way. Probabilistic results can be found in Cirier.

2 - Definition of a differential iteration

We call iteration differential the data of a starting point $\boldsymbol{a}_0$ and *of the application* $f(\boldsymbol{a}, \delta)$ of $\mathbb{R}^d$ in a bounded set of $\mathbb{R}^d$ *defined by :*

$$f(\boldsymbol{a}, \delta) = \boldsymbol{a} + \delta \boldsymbol{F}(\boldsymbol{a})$$

with : $$\delta \in \boldsymbol{\Delta} = \{\mathbf{0} < \delta < \delta_0\} \cap \mathbb{R}^d$$

δ is *generally related to* $\boldsymbol{x}$ *by the relation* $\delta = \boldsymbol{x}/n;\ n \in \mathbb{N}$. *We note* $\boldsymbol{xF} = \{x_\ell F_\ell\};\ \ell = 1,...,d$, *so:*

$$f(\boldsymbol{a}, \boldsymbol{x}/n) = \boldsymbol{a} + \boldsymbol{x}\, \boldsymbol{F}(\boldsymbol{a})/n$$

We suppose now, for all $\delta \in \boldsymbol{\Delta}$ *, that the polynomial function* $f(\boldsymbol{a}, \delta)$ *applies a bounded set C of* $\mathbb{R}^d$ *in itself.* $F(\boldsymbol{a})$ is *of degree r.*

Generally, even if it means to modify $\boldsymbol{x}$ in δ, the integer n is common for all the coordinates of $\boldsymbol{x}$ and is lied to the number of iterations we want to do and the solution of the equation is the limit of:

$$\boldsymbol{a}_n(x) = f^{(n)}(\boldsymbol{a}_0, \boldsymbol{x}/n)$$

when $n \to \infty$.
This property will be used by for instance to define the limit cycles.
For all small $\delta \in \boldsymbol{\Delta}$ $f(\boldsymbol{a}, \delta)$ is invertible in C; so, $f(\boldsymbol{a}, \delta)$ *is a*n analytic diffeomorphism. We can construct for points fixed and cycles, a function $\boldsymbol{a}_n(x)$ and an invariant measure for fixed $\delta < \delta_0$.

But, even if we can apply all the previous results to $f(\boldsymbol{a},\delta)$ for all fixed δ, the invariant distribution will depend on $\delta \in \boldsymbol{\Delta}$. (If the domain C remains invariant).

The essential is question to check the convergence or the invariance of the solutions. More, we should like to determine the conditions of convergence to an asymptotic solution and check if this solution exists. We begin to study the fixed points and cycles.

3- Fixed points and cycles

We define the fixed points and the cycles as being the fixed points and the cycles of the differential iteration for $\forall\delta \in \boldsymbol{\Delta}$ fixed. Then δ vary and $\delta \to 0$.

Definition

For δ fixed, a fixed point α is defined by $f(\alpha,\delta) = \alpha$ *and a cycle* $K_{\delta,p}$ *of order p is defined by* $f(\alpha,\delta) = \alpha$

Proposition

- Pour$\forall\delta$ *, the zero* α *of* $\boldsymbol{F}$ *,* $\boldsymbol{F}(\alpha) = 0$ *are the fixed points of* $f(\alpha,\delta)$*. If* ρ *is a eigen value of* $\boldsymbol{F}'(\alpha)$ *, then* **:** λ_ℓ *is eigen value of* $f'(\alpha,\delta)$ *with* **:**

$\lambda_\ell = 1 + \rho_\ell \delta_\ell = 1 + \rho_\ell x_\ell/n,\ .\ \ell=1,...,d$

If $\boldsymbol{F}'(\alpha)$ *is diagonalisable, we write in the base of the eigen vectors* $\boldsymbol{F}'(\alpha) = \Lambda$ *, and the linear part of* $f(\boldsymbol{a},\delta)$ *is.* $(1 + \rho\,\boldsymbol{x}/n)\boldsymbol{a}$

- For δ fixed, a cycle $K_{n,\delta}$ *of order n defined by p points* $\alpha_1, \alpha_2, \ldots \alpha_n$ *will depend on δ and verify :*

$$\boldsymbol{\alpha}_{k+1} = f(\boldsymbol{\alpha}_k,\delta), \qquad k=1,2..,n.$$

and $\sum_{k=1}^{k=n} \partial^m \boldsymbol{F}(\boldsymbol{\alpha}_k)/\partial\boldsymbol{a}^m = 0,\ s=1,..,n\ ;m=1,..,r$

- If $n \to \infty$, $\delta \to 0$, *it can exist a limit cycle K. The points of K must check asymptotically the cyclical orbits of uniform density satisfying:*

$$\int_0^1 \partial^m\ \boldsymbol{F}(\boldsymbol{a}(\boldsymbol{x}(t)))/\partial\boldsymbol{a}^m dt = 0;\ m=1,..,r$$

et satisfying to: $\partial\boldsymbol{a}/\,\partial\boldsymbol{x} = F(\boldsymbol{a})$ a cycle off order *n for* δ

where the index of iteration $t = \lim_{n\to\infty} s/n,\ s=1,..,n$ *determine a uni-dimensional cyclical path.*

■ Let $K_{n\cdot\delta}$ a cycle of order *n for* δ fixed. For $\forall\boldsymbol{a} \in K_{n\cdot\delta}$:

$$f^{(n)}(\boldsymbol{a},\boldsymbol{x}/n) = \boldsymbol{a}$$

For *k=1,..,n* , $f^{(k+1)}(\boldsymbol{a},\frac{\boldsymbol{x}}{n}) = f^{(k)}(\boldsymbol{a},\frac{\boldsymbol{x}}{n}) + \delta\,\boldsymbol{F}\circ f^{(k)}(\boldsymbol{a},\frac{\boldsymbol{x}}{n})$

By summation of these equalities, then by derivation with respect to δ, we obtain the condition, if a cycle limit K exists. ■

4 - Deterministic approach by invariant functions

We situate the approach in the scoop of the previous paragraph, near by a hyperbolic fixed point, where we have an invariant function $\boldsymbol{a}(\boldsymbol{t},\boldsymbol{x}/n) : \mathbb{R}^d \to \mathbb{R}^d$

satisfying the relation $\boldsymbol{a}(\lambda \boldsymbol{t}, \boldsymbol{x}/n) = f(\boldsymbol{a}(\boldsymbol{t}, \boldsymbol{x}/n))$ for all fixed $\delta = \frac{\boldsymbol{x}}{n}; n \in \mathbb{N}$. For a cycle, we cannot use the same way because the points are not well isolated.
We have seen that, for study invariant functions , we have only to consider the q eigen values $\lambda_\ell = 1 + \rho_\ell \delta_\ell = 1 + \rho_\ell x_\ell / n$ larger than the unit, so :

$$\rho_\ell x_\ell / n > 0 .$$

Convention

Here, we prefer use the letter $\boldsymbol{u}$ instead of the letter $\boldsymbol{t}$ for not confuse the parameter of fluctuation $\boldsymbol{t}$ with some variables $\boldsymbol{x}$ which can be misinterpreted with the time. Only the coordinates of $\boldsymbol{u}$ corresponding to $\frac{\rho_\ell x_\ell}{n} > 0$are interesting, the others are null. $\boldsymbol{u}$ is *call* ***parameter of fluctuation***. The equation E is now:

$$\boldsymbol{a}((1 + \rho \boldsymbol{x}/n)\, \boldsymbol{u}, \boldsymbol{x}/n) = f(\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{x}/n))$$

Hypothesis H becomes H':

Hypotheses H'

For all δ fixed, the polynomial function $f(\boldsymbol{a}, \boldsymbol{x}/n)$ applies a bounded set $\mathbb{C}$ of $\mathbb{R}^d$ in itself. The q eigen values larger than 1:

$$\lambda_\ell = 1 + \rho_\ell \delta_\ell > 1$$

are real and transcendental, all the others are not in resonance.

Apparently, in many cases this hypothesis is not necessary.
First, we recall that, for δ fixed, then we have an analytic of frame of the solution by absolutely convergent series. More, the solution belongs to the set of the functions almost periodic. We have now to clarify what are the periodic functions.

Proposition

Under H', the equation $\boldsymbol{E}_\delta$*:*

$$\boldsymbol{a}((1 + \rho \boldsymbol{x}/n)\, \boldsymbol{u}, \boldsymbol{x}/n) = f(\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{x}/n))$$

of the differential iteration has an asymptotic almost periodic solution. The a.p. belong to $\Phi(\lambda)$, composed with sums of $m(1 + \rho\delta)^k | m \in \mathbb{N}; k \in \mathbb{Z}$.
We write:

$$\boldsymbol{a}(\boldsymbol{u}) = \Sigma_m \boldsymbol{a}_m(\boldsymbol{u}) | m \in \mathbb{N}$$

with :

$$\boldsymbol{a}_m(\boldsymbol{u}) = \Sigma_{\varphi} \boldsymbol{c}\,(\varphi_m(\lambda))(1 - \exp(\varphi_m(\lambda)\boldsymbol{t}))$$

For $m = 1$, the a.p. are : λ^k ; the coefficients $\boldsymbol{c}(\lambda^k)$ satisfy:

$$\boldsymbol{c}_\ell\left(\lambda_\ell{}^k\right) = (1 + \rho_\ell \delta_\ell)^{-k} \boldsymbol{c}_\ell$$

where $\boldsymbol{c}_\ell$ is eigen vector of ***F'*** *for the eigen value $\rho_\ell \delta_\ell > 0$.*
When $n \to \infty$: $\boldsymbol{a}_{1\ell}(\boldsymbol{u}) = \boldsymbol{c}_\ell \boldsymbol{w}_\ell(\boldsymbol{u})$
$\boldsymbol{w}_\ell(\boldsymbol{u})$ is the function of WM with:

$$\lambda_\ell = \exp(x_\ell \rho_\ell) \text{ and : } r_\ell = \exp - (x_\ell \rho_\ell)$$

■ For all δ fixed, we have just to take the previous demonstrations at the neighbourhood of a fixed point or of a cycle and change only the notation. The limit for $n \to \infty$ is not difficult. ■

Remarks
- What happen when we start from any position? Supposing the iteration without resonance or smalls divisors, we do the same computation, but allowing that the eigen values are negatives.
- The solution is local. It is not always so simple when we have many points fixed, as in the case of Lorenz. We have observed with probabilistic methods, the moving from a basin of domination to another.

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