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► **To cite this version:**

Eduardo Abi Jaber, Omar El Euch. Markovian structure of the Volterra Heston model. *Statistics and Probability Letters*, 2019. hal-01716696v2

HAL Id: hal-01716696

<https://hal.science/hal-01716696v2>

Submitted on 28 Feb 2018

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Markovian structure of the Volterra Heston model*

Eduardo Abi Jaber [†] Omar El Euch[‡]

February 24, 2018

Abstract

We characterize the Markovian and affine structure of the Volterra Heston model in terms of an infinite-dimensional adjusted forward process and specify its state space. More precisely, we show that it satisfies a stochastic partial differential equation and displays an exponentially-affine characteristic functional. As an application, we deduce an existence and uniqueness result for a Banach-space valued square-root process and provide its state space. This leads to another representation of the Volterra Heston model together with its Fourier-Laplace transform in terms of this possibly infinite system of affine diffusions.

Keywords: Affine Volterra processes, stochastic Volterra equations, Markovian representation, stochastic invariance, Riccati-Volterra equations, rough volatility.

MSC2010 Classification: 60H20, 45D05, 91G99.

1 Introduction

The Volterra Heston model is defined by the following dynamics

$$dS_t = S_t \sqrt{V_t} dB_t, \quad S_0 > 0, \quad (1.1)$$

$$V_t = g_0(t) + \int_0^t K(t-s) \left(-\lambda V_s ds + \nu \sqrt{V_s} dW_s \right), \quad (1.2)$$

with $K \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, $g_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\lambda, \nu \in \mathbb{R}_+$ and $B = \rho W + \sqrt{1-\rho^2} W^\perp$ such that (W, W^\perp) is a two-dimensional Brownian motion and $\rho \in [-1, 1]$. It has been introduced in [2] for the purpose of financial modeling following the literature on so-called rough volatility models [10]. Hence S_t typically represents a stock price at time t with instantaneous stochastic variance V_t .

This model nests as special cases the Heston model for $K \equiv 1$, and the rough Heston model of [8], obtained by setting $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha \in (\frac{1}{2}, 1)$ and

$$g_0(t) = V_0 + \int_0^t K(s) \lambda \theta ds, \quad t \geq 0, \quad \text{for some } V_0, \theta \geq 0, \quad (1.3)$$

so that the only model parameters are $V_0, \theta, \lambda, \rho, \nu, \alpha$. Recall that the rough Heston model does not only fit remarkably well historical and implied volatilities of the market, but also enjoys a semi-closed formula for the characteristic function of the log-price in terms of a solution of a deterministic Riccati-Volterra integral equation.

*We would like to thank Bruno Bouchard and Mathieu Rosenbaum for very fruitful discussions and comments.

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In [7], the authors highlight the crucial role of (1.3) in the design of hedging strategies for the rough Heston model. Here we consider more general input curves g_0 . Our motivation is twofold. In practice, the function g_0 is intimately linked to the forward variance curve $(\mathbb{E}[V_t])_{t \geq 0}$. More precisely, taking the expectation in (1.2) leads to the following relation

$$\mathbb{E}[V_t] + \lambda \int_0^t K(t-s)\mathbb{E}[V_s]ds = g_0(t), \quad t \geq 0.$$

Thus, allowing for more general input curves g_0 leads to more consistency with the market forward variance curve. From a mathematical perspective, this enables us to understand the general picture behind the Markovian and affine nature of the Volterra Heston model (1.1)-(1.2).

More precisely, adapting the methods of [2], we provide a set of admissible input curves \mathcal{G}_K defined in (2.5) such that (1.1)-(1.2) admits a unique \mathbb{R}_+^2 -valued weak solution for any $g_0 \in \mathcal{G}_K$. In particular, we show that the Fourier-Laplace transform of $(\log S, V)$ is exponentially affine in $(\log S_0, g_0)$. Then we prove that, conditional on \mathcal{F}_t , the shifted Volterra Heston model (S_{t+}, V_{t+}) still has the same dynamics as in (1.1)-(1.2) provided that g_0 is replaced by the following adjusted forward process

$$g_t(x) = \mathbb{E} \left[V_{t+x} + \lambda \int_0^x K(x-s)V_{t+s}ds \mid \mathcal{F}_t \right], \quad x \geq 0. \quad (1.4)$$

This leads to our main result which states that \mathcal{G}_K is stochastically invariant with respect to the family $(g_t)_{t \geq 0}$. In other words, if we start from an initial admissible input curve $g_0 \in \mathcal{G}_K$, then g_t belongs to \mathcal{G}_K , for all $t \geq 0$, see Theorem 3.1. This in turn enables us to characterize the Markovian structure of (S, V) in terms of the stock price and the adjusted forward process $(g_t)_{t \geq 0}$. Furthermore, $(g_t)_{t \geq 0}$ can be realized as the unique \mathcal{G}_K -valued mild solution of the following stochastic partial differential equation of Heath–Jarrow–Morton-type

$$dg_t(x) = \left(\frac{d}{dx}g_t(x) - \lambda K(x)g_t(0) \right) dt + K(x)\nu\sqrt{g_t(0)}dW_t, \quad g_0 \in \mathcal{G}_K,$$

and displays an affine characteristic functional.

As an application, we establish the existence and uniqueness of a Banach-space valued square-root process and provide its state space. This leads to another representation of $(V_t, g_t)_{t \geq 0}$. Moreover, the Fourier-Laplace transform of $(\log S, V)$ is shown to be an exponential affine functional of this process. These results are in the spirit of the Markovian representation of fractional Brownian motion, see [3, 13].

The paper is organized as follows. In Section 2, we prove weak existence and uniqueness for the Volterra Heston model and provide its Fourier-Laplace transform. Section 3 characterizes the Markovian structure in terms of the adjusted forward variance process. Section 4 establishes the existence and uniqueness of a Banach-space valued square-root process and provides the link with the Volterra framework. In Appendix A we derive general existence results for stochastic Volterra equations. Finally, for the convenience of the reader we recall in Appendix B the framework and notations regarding stochastic convolutions as in [2].

Notations : Elements of \mathbb{C}^m are viewed as column vectors, while elements of the dual space $(\mathbb{C}^m)^*$ are viewed as row vectors. For $h \geq 0$, Δ_h denotes the shift operator, i.e. $\Delta_h f(t) = f(t+h)$. If the function f on \mathbb{R}_+ is right-continuous and of locally bounded variation, the measure induced by its distribution derivative is denoted df , so that $f(t) = f(0) + \int_{[0,t]} df(s)$ for all $t \geq 0$. Finally, we use the notation $*$ for the convolution operation, we refer to Appendix B for more details.

2 Existence and uniqueness of the Volterra Heston model

We study in this section the existence and uniqueness of the Volterra Heston model given by (1.1)-(1.2) allowing for arbitrary curves g_0 as input. When g_0 is given by (1.3), [2, Theorem 7.1(i)] provides the existence of a \mathbb{R}_+^2 -valued weak solution to (1.1)-(1.2) under the following mild assumptions on K :

$$K \in L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{R}), \text{ and there is } \gamma \in (0, 2] \text{ such that } \int_0^h K(t)^2 dt = O(h^\gamma) \text{ and} \quad (H_0)$$

$$\int_0^T (K(t+h) - K(t))^2 dt = O(h^\gamma) \text{ for every } T < \infty,$$

$$K \text{ is nonnegative, not identically zero, non-increasing and continuous on} \quad (H_1)$$

$$(0, \infty), \text{ and its } \textit{resolvent of the first kind } L \text{ is nonnegative and non-increasing}^1$$

$$\text{ in the sense that } s \rightarrow L([s, s+t]) \text{ is non-increasing for all } t \geq 0.$$

We show in Theorem 2.1 below that weak existence in \mathbb{R}_+^2 continue to hold for (1.1)-(1.2) for a wider class of admissible input curves g_0 . Since S is determined by V , it suffices to study the Volterra square-root equation (1.2). Theorem A.1(ii) in the Appendix guarantees the existence of an unconstrained continuous weak solution V to the following modified equation

$$V_t = g_0(t) + \int_0^t K(t-s) \left(-\lambda V_s ds + \nu \sqrt{V_s^+} dW_s \right), \quad (2.1)$$

for any locally Hölder continuous function g_0 , where $x^+ : x \rightarrow \max(0, x)$. Clearly, one needs to impose additional assumptions on g_0 to ensure the nonnegativity of V and drop the positive part in (2.1) so that V solves (1.2). Hence, weak existence of a nonnegative solution to (1.2) boils down to finding a set \mathcal{G}_K of admissible input curves g_0 such that any solution V to (2.1) is nonnegative.

To get a taste of the admissible set \mathcal{G}_K , we start by assuming that g_0 and K are continuously differentiable on $[0, \infty)$. In that case, V is a semimartingale such that

$$dV_t = (g'_0(t) + (K' * dZ)_t - K(0)\lambda V_t) dt + K(0)\nu \sqrt{V_t^+} dW_t, \quad (2.2)$$

where $Z = \int_0^\cdot (-\lambda V_s ds + \nu \sqrt{V_s^+} dW_s)$. Relying on Lemma B.1 in the Appendix², we have

$$K' = (K' * L)(0)K + d(K' * L) * K,$$

so that $K' * dZ$ can be expressed as a functional of (V, g_0) as follows

$$K' * dZ = (K' * L)(0)(V - g_0) + d(K' * L) * (V - g_0). \quad (2.3)$$

Since $V_0 = g_0(0)$, it is straightforward that $g_0(0)$ should be nonnegative. Now, assume that V hits zero for the first time at $\tau \geq 0$. After plugging (2.3) in the drift of (2.2), a first-order Euler scheme leads to the formal approximation

$$V_{\tau+h} \approx (g'_0(\tau) - (K' * L)(0)g_0(\tau) - (d(K' * L) * g_0)(\tau) + (d(K' * L) * V)_\tau) h,$$

for small $h \geq 0$. Since $K' * L$ is non-decreasing and $V \geq 0$ on $[0, \tau]$, it follows that $(d(K' * L) * V)_\tau \geq 0$ yielding the nonnegativity of $V_{\tau+h}$ if we impose the following additional condition

$$g'_0 - (K' * L)(0)g_0 - d(K' * L) * g_0 \geq 0.$$

¹We refer to Appendix B for the definition of the *resolvent of the first kind* and some of its properties.

²Under (H_1) one can show that $K' * L$ is right-continuous, non-decreasing and of locally bounded variation (as in Remark B.3 in the Appendix), thus the associated measure $d(K' * L)$ is well defined.

In the general case, V is not necessarily a semimartingale, and a delicate analysis should be carried on the integral equation (2.1) instead of the infinitesimal version (2.2). This suggests that the infinitesimal derivative operator should be replaced by the semigroup operator of right shifts leading to the following condition on g_0

$$\Delta_h g_0 - (\Delta_h K * L)(0)g_0 - d(\Delta_h K * L) * g_0 \geq 0, \quad h \geq 0,^3 \quad (2.4)$$

and to the following definition of the set \mathcal{G}_K of admissible input curves

$$\mathcal{G}_K = \left\{ g_0 \in \mathcal{H}^{\gamma/2} \text{ satisfying (2.4) and } g_0(0) \geq 0 \right\}, \quad (2.5)$$

where $\mathcal{H}^\alpha = \{g_0 : \mathbb{R}_+ \rightarrow \mathbb{R}, \text{ locally Hölder continuous of any order strictly smaller than } \alpha\}$. Recall that γ is the exponent associated with K in (H_0) .

The following theorem establishes the existence of a \mathbb{R}_+^2 -valued weak continuous solution to (1.1)-(1.2) on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ for any admissible input curve $g_0 \in \mathcal{G}_K$. Since S is determined by V , the proof follows directly from Theorems A.1-A.2.

Theorem 2.1. *Assume that K satisfies (H_0) - (H_1) . Then, the stochastic Volterra equation (1.1)-(1.2) has a \mathbb{R}_+^2 -valued continuous weak solution (S, V) for any positive initial condition S_0 and any admissible input curve $g_0 \in \mathcal{G}_K$. Furthermore, the paths of V are locally Hölder continuous of any order strictly smaller than $\gamma/2$ and*

$$\sup_{t \leq T} \mathbb{E}[|V_t|^p] < \infty, \quad p > 0, \quad T > 0. \quad (2.6)$$

Example 2.2. *The following classes of functions belong to \mathcal{G}_K .*

- (i) $g \in \mathcal{H}^{\gamma/2}$ non-decreasing such that $g(0) \geq 0$. Since K is non-increasing and L is nonnegative, we have $0 \leq \Delta_h K * L \leq 1$ for all $h \geq 0$ (see the proof of [2, Theorem 3.5]) yielding, for all $t, h \geq 0$, that $\Delta_h g(t) - (\Delta_h K * L)(0)g(t) - (d(\Delta_h K * L) * g)(t)$ is equal to

$$\int_0^t (g(t) - g(t-s))(\Delta_h K * L)(ds) + g(t+h) - g(t) + g(t)(1 - (\Delta_h K * L)(t)) \geq 0$$

- (ii) $g = V_0 + K * \theta$, with $V_0 \geq 0$ and $\theta \in L_{loc}^2(\mathbb{R}_+, \mathbb{R})$ such that $\theta(s)ds + V_0 L(ds)$ is a nonnegative measure. First, $g \in \mathcal{H}^{\gamma/2}$ due to (H_0) and the Cauchy-Schwarz inequality

$$(g(t+h) - g(t))^2 \leq 2 \left(\int_0^t (K(s+h) - K(s))^2 ds + \int_0^h K(s)^2 ds \right) \int_0^{t+h} \theta(s)^2 ds.$$

Moreover, $g(0) = V_0 \geq 0$ and

$$\Delta_h g - (\Delta_h K * L)(0)g - d(\Delta_h K * L) * g \quad (2.7)$$

is equal to

$$V_0(1 - \Delta_h K * L) + \Delta_h(K * \theta) - (\Delta_h K * L)(0)K * \theta - d(\Delta_h K * L) * K * \theta.$$

(2.4) now follows from Lemma B.2 with $F = \Delta_h K$, after noticing that (2.7) becomes

$$\Delta_h(K * (V_0 L + \theta)) - \Delta_h K * (V_0 L + \theta) = \int_{\cdot}^{\cdot+h} K(\cdot + h - s)(V_0 L(ds) + \theta(s)ds) \geq 0.$$

³ Recall that under (H_1) one can show that $\Delta_h K * L$ is right-continuous and of locally bounded variation (see Remark B.3 in the Appendix), thus the associated measure $d(\Delta_h K * L)$ is well defined.

We now tackle the weak uniqueness of (1.1)-(1.2) by characterizing the Fourier-Laplace transform of the process $X = (\log S, V)$. Indeed, when g_0 is of the form (1.3), X is a two-dimensional affine Volterra process in the sense of [2, Definition 4.1]. For this particular g_0 , [2, Theorem 7.1(ii)] provides the exponential-affine transform formula

$$\mathbb{E}[\exp(uX_T + (f * X)_T)] = \exp\left(\psi_1(T) \log S_0 + u_2 g_0(T) + \int_0^T F(\psi_1, \psi_2)(s) g_0(T-s) ds\right) \quad (2.8)$$

for suitable $u \in (\mathbb{C}^2)^*$ and $f \in L^1([0, T], (\mathbb{C}^2)^*)$ with $T > 0$, where $\psi = (\psi_1, \psi_2)$ solves the following system of Riccati-Volterra equations

$$\psi_1 = u_1 + 1 * f_1, \quad (2.9)$$

$$\psi_2 = u_2 K + K * F(\psi_1, \psi_2), \quad (2.10)$$

with

$$F(\psi_1, \psi_2) = f_2 + \frac{1}{2} (\psi_1^2 - \psi_1) + (\rho\nu\psi_1 - \lambda)\psi_2 + \frac{\nu^2}{2} \psi_2^2. \quad (2.11)$$

A straightforward adaptation of [2, Theorems 4.3 and 7.1] shows that the affine transform (2.8) carries over for any admissible input curve $g_0 \in \mathcal{G}_K$ with the same Riccati equations (2.9)-(2.10).

Theorem 2.3. *Assume that K satisfies (H_0) and that the shifted kernels $\Delta_h K$ satisfy (H_1) for all $h \in [0, 1]$. Fix $g_0 \in \mathcal{G}_K$, $S_0 > 0$ and denote by (S, V) a \mathbb{R}_+^2 -valued continuous weak solution to (1.1)-(1.2). For any $u \in (\mathbb{C}^2)^*$ and $f \in L_{\text{loc}}^1(\mathbb{R}_+, (\mathbb{C}^2)^*)$ such that*

$$\text{Re } \psi_1 \in [0, 1], \text{ Re } u_2 \leq 0 \text{ and } \text{Re } f_2 \leq 0, \quad (2.12)$$

with ψ_1 given by (2.9), the Riccati-Volterra equation (2.10) admits a unique global solution $\psi_2 \in L_{\text{loc}}^2(\mathbb{R}_+, \mathbb{C}^*)$. Moreover, the exponential-affine transform (2.8) is satisfied. In particular, weak uniqueness holds for (1.1)-(1.2).

3 Markovian structure

Using the same methodology as in [7], we characterize the Markovian structure of the Volterra Heston model (1.1)-(1.2) in terms of the \mathbb{F} -adapted infinite-dimensional adjusted forward curve $(g_t)_{t \geq 0}$ given by (1.4) which is well defined thanks to (2.6). Furthermore, we prove that the set \mathcal{G}_K is stochastically invariant with respect to $(g_t)_{t \geq 0}$.

Theorem 3.1. *Under the assumptions of Theorem 2.1, fix $g_0 \in \mathcal{G}_K$. Denote by (S, V) the unique solution to (1.1)-(1.2) and by $(g_t)_{t \geq 0}$ the process defined by (1.4). Then, (S^{t_0}, V^{t_0}) satisfies*

$$\begin{aligned} dS_t^{t_0} &= S_t^{t_0} \sqrt{V_t^{t_0}} dB_t^{t_0}, \quad S_0^{t_0} = S_{t_0}, \\ V_t^{t_0} &= g_{t_0}(t) + \int_0^t K(t-s) \left(-\lambda V_s^{t_0} ds + \nu \sqrt{V_s^{t_0}} dW_s^{t_0} \right), \end{aligned}$$

where $(B^{t_0}, W^{t_0}) = (B_{t_0+} - B_{t_0}, W_{t_0+} - W_{t_0})$ are two Brownian motions independent of \mathcal{F}_{t_0} such that $d\langle B^{t_0}, W^{t_0} \rangle_t = \rho dt$. Moreover, \mathcal{G}_K is stochastically invariant with respect to $(g_t)_{t \geq 0}$, that is

$$g_t \in \mathcal{G}_K, \quad t \geq 0.$$

Proof. The part for V^{t_0} is immediate after observing that

$$g_{t_0}(t) = g_0(t_0 + t) - \int_0^{t_0} K(t + t_0 - s)\lambda V_s ds + \int_0^{t_0} K(t + t_0 - s)\nu\sqrt{V_s}dW_s, \quad (3.1)$$

for all $t_0, t, h \geq 0$. The part for S^{t_0} is straightforward. We move to proving the claimed invariance. Fix $t_0, t, h \geq 0$ and define $Z = \int_0^\cdot (-\lambda V_s ds + \nu\sqrt{V_s}dW_s)$. By Lemma B.2 and Remark B.3 in the Appendix,

$$\Delta_h K = (\Delta_h K * L)(0)K + d(\Delta_h K * L) * K, \quad (3.2)$$

so that

$$(\Delta_h K * dZ) = (\Delta_h K * L)(0)(V - g_0) + d(\Delta_h K * L) * (V - g_0).$$

Hence,

$$\begin{aligned} V_{t+h}^{t_0} &= g_0(t_0 + t + h) + (\Delta_h K * dZ)_{t_0+t} + \int_0^h K(h-s)dZ_{t_0+t+s} \\ &= g_0(t_0 + t + h) + (\Delta_h K * L)(0)(V_t^{t_0} - g_0(t_0 + t)) \\ &\quad + (d(\Delta_h K * L) * (V - g_0))_{t_0+t} + \int_0^h K(h-s)dZ_{t_0+t+s} \\ &= g_0(t_0 + t + h) - (\Delta_h K * L)(0)g_0(t_0 + t) - (d(\Delta_h K * L) * g_0)(t_0 + t) \\ &\quad + (\Delta_h K * L)(0)V_t^{t_0} + (d(\Delta_h K * L) * V)_{t_0+t} + \int_0^h K(h-s)dZ_{t_0+t+s} \\ &\geq (\Delta_h K * L)(0)V_t^{t_0} + (d(\Delta_h K * L) * V)_{t_0+t} - \int_0^h K(h-s)\lambda V_{t+s}^{t_0} ds \\ &\quad + \int_0^h K(h-s)\nu\sqrt{V_{t+s}^{t_0}}dW_{t+s}^{t_0}, \end{aligned}$$

since $g_0 \in \mathcal{G}_K$. We now prove (2.4). Set $G_h^{t_0} = \Delta_h g_{t_0} - (\Delta_h K * L)(0)g_{t_0} - d(\Delta_h K * L) * g_{t_0}$. The previous inequality combined with (1.4) yields

$$\begin{aligned} G_h^{t_0}(t) &= \mathbb{E} \left[V_{t+h}^{t_0} + (\lambda K * V^{t_0})_{t+h} - (\Delta_h K * L)(0)(V_t^{t_0} + (\lambda K * V^{t_0})_t) \mid \mathcal{F}_{t_0} \right] \\ &\quad - \mathbb{E} \left[\left(d(\Delta_h K * L) * (V^{t_0} + \lambda K * V^{t_0}) \right)_t \mid \mathcal{F}_{t_0} \right] \\ &\geq \mathbb{E} \left[(d(\Delta_h K * L) * V)_{t_0+t} - \left(d(\Delta_h K * L) * V^{t_0} \right)_t - \int_0^h K(h-s)\lambda V_{t+s}^{t_0} ds \mid \mathcal{F}_{t_0} \right] \\ &\quad + \mathbb{E} \left[(\lambda K * V^{t_0})_{t+h} - \left(((\Delta_h K * L)(0)K + d(\Delta_h K * L) * K) * \lambda V^{t_0} \right)_t \mid \mathcal{F}_{t_0} \right]. \end{aligned}$$

Relying on (3.2), we deduce

$$\begin{aligned} G_h^{t_0}(t) &\geq \mathbb{E} \left[\int_t^{t_0+t} (d(\Delta_h K * L))(ds)V_{t_0+t-s} - \int_0^h K(h-s)\lambda V_{t+s}^{t_0} ds \mid \mathcal{F}_{t_0} \right] \\ &\quad + \mathbb{E} \left[\int_t^{t+h} K(t+h-s)\lambda V_s^{t_0} ds \mid \mathcal{F}_{t_0} \right] \\ &= \mathbb{E} \left[\int_t^{t_0+t} (d(\Delta_h K * L))(ds)V_{t_0+t-s} \mid \mathcal{F}_{t_0} \right]. \end{aligned}$$

Hence (2.4) holds for g_{t_0} , since $V \geq 0$ and $d(\Delta_h K * L)$ is a nonnegative measure, see Remark B.3. Finally, by adapting the proof of [2, Lemma 2.4], we can show that for any $p > 1, \epsilon > 0$ and $T > 0$, there exists a positive constant C_1 such that

$$\mathbb{E} [|V_{t+h} - V_t|^p] \leq C_1 h^{p(\gamma/2 - \epsilon)}, \quad t, h \geq 0, t + h \leq T + t_0,$$

Relying on (H₀), (1.4) and Jensen inequality, there exists a positive constant C_2 such that

$$\mathbb{E} [|g_{t_0}(t+h) - g_{t_0}(t)|^p] \leq C_2 h^{p(\gamma/2 - \epsilon)}, \quad t, h \geq 0, t + h \leq T,$$

By Kolmogorov continuity criterion, $g_{t_0} \in \mathcal{H}^{\gamma/2}$ so that $g_{t_0} \in \mathcal{G}_K$ since $g_{t_0}(0) = V_{t_0} \geq 0$. \square

Theorem 3.1 highlights that V is Markovian in the state variable $(g_t)_{t \geq 0}$. Indeed, conditional on \mathcal{F}_t for some $t \geq 0$, the shifted Volterra Heston model (S^t, V^t) can be started afresh from (S_t, g_t) with the same dynamics as in (1.1)-(1.2). Notice that g_t is again an admissible input curve belonging to \mathcal{G}_K . Therefore, applying Theorems 2.1 and 2.3 with (S^t, V^t, g_t) yields that the conditional Fourier-Laplace transform of $X = (\log S, V)$ is exponentially affine in $(\log S_t, g_t)$:

$$\mathbb{E} \left[\exp(uX_T + (f * X)_T) \mid \mathcal{F}_t \right] = \exp(\psi_1(T-t) \log S_t + (u_2 g_t + F(\psi_1, \psi_2) * g_t)(T-t)), \quad (3.3)$$

for all $t \leq T$, where F is given by (2.11), under the standing assumptions of Theorem 2.3.

Moreover, it follows from (3.1) and the fact that $g(0) = V$ that the process $(g_t)_{t \geq 0}$ solves

$$g_t(x) = \Delta_t g_0(x) + \int_0^t \Delta_{t-s} (-\lambda K g_s(0))(x) ds + \int_0^t \Delta_{t-s} \left(K \nu \sqrt{g_s(0)} \right) (x) dW_s. \quad (3.4)$$

Recalling that $(\Delta_t)_{t \geq 0}$ is the semigroup of right shifts, (3.4) can be seen as a \mathcal{G}_K -valued mild solution of the following Heath–Jarrow–Morton-type *stochastic partial differential equation*

$$dg_t(x) = \left(\frac{d}{dx} g_t(x) - \lambda K(x) g_t(0) \right) dt + K(x) \nu \sqrt{g_t(0)} dW_t, \quad g_0 \in \mathcal{G}_K. \quad (3.5)$$

The following proposition provides the characteristic functional of $(g_t)_{t \geq 0}$ leading to the strong Markov property of $(g_t)_{t \geq 0}$. Define $\langle g, h \rangle = \int_{\mathbb{R}_+} g(x) h(x) dx$, for suitable functions f and g .

Theorem 3.2. *Under the assumptions of Theorem 2.3. Let $h \in C_c^\infty(\mathbb{R}_+)$ and $g_0 \in \mathcal{G}_K$. Then,*

$$\mathbb{E} [\exp(i \langle g_t, h \rangle)] = \exp(\langle H_t, g_0 \rangle), \quad t \geq 0, \quad (3.6)$$

where H solves

$$H_t(x) = ih(x-t) \mathbf{1}_{\{x > t\}} + \mathbf{1}_{\{x \leq t\}} \left(-\lambda \langle H_{t-x}, K \rangle + \frac{\nu^2}{2} \langle H_{t-x}, K \rangle^2 \right), \quad t, x \geq 0. \quad (3.7)$$

In particular, weak uniqueness holds for (3.4) and $(g_t)_{t \geq 0}$ is a strong Markov process on \mathcal{G}_K .

Proof. Consider $\tilde{S}_t = 1 + \int_0^t \tilde{S}_u \sqrt{V_u} dW_u$, for all $t \geq 0$. Then, (\tilde{S}, V) is a Volterra Heston model of the form (1.1)-(1.2) with $\rho = 1$ and $\tilde{S}_0 = 1$. Fix $t \geq 0$, $\langle g_t, h \rangle$ is well defined since $x \rightarrow g_t(x)$ is continuous. It follows from (3.1) together with stochastic Fubini theorem, see [15, Theorem 2.2], which is justified by (2.6), that

$$\begin{aligned} \langle g_t, h \rangle &= \langle g_0(t + \cdot), h \rangle + \left(\frac{\nu}{2} - \lambda \right) \int_0^t \langle K(t-s + \cdot), h \rangle V_s ds + \nu \int_0^t \langle K(t-s + \cdot), h \rangle d(\log \tilde{S})_s \\ &= \langle g_0, h(-t + \cdot) \rangle + \left(\frac{\nu}{2} - \lambda \right) \int_0^t \langle K, h(s-t + \cdot) \rangle V_s ds \\ &\quad + \nu \langle K, h \rangle \log \tilde{S}_t - \nu \int_0^t \langle K, h'(s-t + \cdot) \rangle \log \tilde{S}_s ds, \end{aligned}$$

where the last identity follows from an integration by parts. Hence, setting

$$\begin{aligned} u_2 &= 0, & u_1 &= i\nu\langle K, h \rangle, & f_1(t) &= -i\nu\langle K, h'(-t + \cdot) \rangle, \\ \psi_1(t) &= u_1 + (1 * f_1)(t) = i\nu\langle K(t + \cdot), h \rangle, \\ f_2(t) &= i\left(\frac{\nu}{2} - \lambda\right)\langle K(t + \cdot), h \rangle, & \psi_2 &= K * F(\psi_1, \psi_2), \end{aligned}$$

with F as in (2.11), the characteristic functional follows from Theorem 2.3

$$\mathbb{E}[\exp(i\langle g_t, h \rangle)] = e^{i\langle h(-t + \cdot), g_0 \rangle} \mathbb{E}\left[\exp\left(u_1 \log \tilde{S}_t + (f_1 * \log \tilde{S})_t + (f_2 * V)_t\right)\right] = \exp(\langle H_t, g_0 \rangle)$$

where

$$H_t(x) = h(x - t)\mathbb{1}_{\{x > t\}} + \mathbb{1}_{\{0 \leq x \leq t\}}F(\psi_1, \psi_2)(t - x), \quad x \geq 0,$$

and (2.11) reads

$$\begin{aligned} F(\psi_1, \psi_2)(t) &= -\lambda\langle K(t + \cdot), h \rangle + \frac{\nu^2}{2}\langle K(t + \cdot), h \rangle^2 \\ &\quad + (\nu^2\langle K(t + \cdot), h \rangle - \lambda)\psi_2(t) + \frac{\nu^2}{2}\psi_2(t)^2. \end{aligned} \quad (3.8)$$

Now observe that

$$\langle H_t, K \rangle = \langle h(-t + \cdot), K \rangle + \int_0^t F(\psi_1, \psi_2)(t - x)K(x)dx = \langle h, K(t + \cdot) \rangle + \psi_2(t).$$

Hence, after plugging $\psi_2(t) = \langle H_t, K \rangle - \langle h, K(t + \cdot) \rangle$ back in (3.8) we get that

$$F(\psi_1, \psi_2)(t) = -\lambda\langle H_t, K \rangle + \frac{\nu^2}{2}\langle H_t, K \rangle^2,$$

yielding (3.7). Weak uniqueness now follows by standard arguments. In fact, thanks to (2.6) and stochastic Fubini theorem, $(g_t)_{t \geq 0}$ solves (3.5) in the weak sense, that is

$$\langle g_t, h \rangle = \langle g_0, h \rangle + \int_0^t (\langle g_s, -h' \rangle - \lambda\langle K, h \rangle g_s(0)) ds + \int_0^t \nu\langle K, h \rangle \sqrt{g_s(0)} dW_s, \quad h \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Therefore, combined with Theorem 3.1, $(g_t)_{t \geq 0}$ solves a martingale problem on \mathcal{G}_K . In addition, (3.6) yields uniqueness of the one-dimensional distributions which is enough to get weak uniqueness for (3.4) and the strong Markov property by [9, Theorem 4.4.2]. \square

We notice that (3.6)-(3.7) agree with [11, Proposition 4.5] when $\lambda = 0$. Moreover, one can lift (3.7) to a non-linear partial differential equation in duality with (3.5). Indeed, define the measure-valued function $\bar{H} : t \rightarrow \bar{H}_t(dx) = H_t(x)\mathbb{1}_{\{x \geq 0\}}dx$. Then, it follows from (3.7) that

$$\bar{H}_t(dx) = ih(x - t)\mathbb{1}_{\{x > t\}}dx + \int_0^t \delta_0(dx - (t - s))(-\lambda\langle \bar{H}_s, K \rangle + \frac{\nu^2}{2}\langle \bar{H}_s, K \rangle^2)ds$$

which can be seen as the mild formulation of the following partial differential equation

$$d\bar{H}_t(dx) = \left(-\frac{d}{dx}\bar{H}_t(dx) + \delta_0(dx)(-\lambda\langle \bar{H}_t, K \rangle + \frac{\nu^2}{2}\langle \bar{H}_t, K \rangle^2)\right)dt, \quad \bar{H}_0(dx) = ih(x)\mathbb{1}_{\{x > t\}}dx. \quad (3.9)$$

We refer to [4, 5] for similar results in the discontinuous setting. The previous results highlight not only the correspondence between stochastic Volterra equations of the form (1.2) and stochastic partial differential equations (3.5) but also between their dual objects, that is the Riccati-Volterra equation (2.10) and the non-linear partial differential equation (3.9). One can establish a correspondence between (1.2) and other related stochastic partial differential equations which, unlike $(g_t)_{t \geq 0}$, do not necessarily have a financial interpretation but for which the dual object satisfies a nicer non-linear partial differential equation than (3.9), see [14].

4 Application: square-root process in Banach space

As an application of Theorems 2.1, 2.3, 3.1, we obtain conditions for weak existence and uniqueness of the following (possibly) infinite-dimensional system of stochastic differential equations

$$dU_t(x) = \left(-xU_t(x) - \lambda \int_0^\infty U_t(z)\mu(dz) \right) dt + \nu \sqrt{\int_0^\infty U_t(z)\mu(dz)} dW_t, \quad x \in \text{supp}(\mu), \quad (4.1)$$

for a fixed positive measure of locally bounded variation μ^4 . This is achieved by linking (4.1) to a stochastic Volterra equation of the form (1.2) with the following kernel

$$K(t) = \int_0^\infty e^{-xt}\mu(dx), \quad t > 0. \quad (4.2)$$

We will assume that μ is a positive measure of locally bounded variation such that

$$\int_0^\infty (1 \wedge (xh)^{-1/2})\mu(dx) \leq Ch^{(\gamma-1)/2}, \quad \int_0^\infty x^{-1/2}(1 \wedge (xh))\mu(dx) \leq Ch^{\gamma/2}; \quad h > 0, \quad (H_2)$$

for some $\gamma \in (0, 2]$ and positive constant C . The reader may check that in that case K satisfies (H_0) . Furthermore, [12, Theorem 5.5.4] guarantees the existence of the resolvent of the first kind L of K and that (H_1) is satisfied for the shifted kernels $\Delta_h K$ for any $h \in [0, 1]$. Hence, K satisfies assumptions of Theorems 2.1 and 2.3.

By a solution U to (4.1) we mean a family of continuous processes $(U(x))_{x \in \text{supp}(\mu)}$ such that $x \rightarrow U_t(x) \in L^1(\mu)$ for any $t \geq 0$, $(\int_0^\infty U_t(x)\mu(dx))_{t \geq 0}$ is a continuous process and (4.1) holds a.s. on some filtered probability space. If such solution exists, we set $V = \int_0^\infty U_t(x)\mu(dx)$ and $g_0 = \int_0^\infty U_0(x)e^{-x(\cdot)}dx$. Thanks to (H_2) , the stochastic Fubini theorem yields for each $t \geq 0$

$$V_t = g_0(t) + \int_0^t K(t-s)(-\lambda V_s ds + \nu \sqrt{V_s} dW_s). \quad (4.3)$$

The processes above being continuous, the equality holds in terms of processes. Thus, provided that g_0 belongs to \mathcal{G}_K , Theorem 2.3 leads to the weak uniqueness of (4.1) because for each $x \in \text{supp}(\mu)$,

$$U_t(x) = e^{-xt}U_0(x) + \int_0^t e^{-x(t-s)}(-\lambda V_s ds + \nu \sqrt{V_s} dW_s), \quad t \geq 0. \quad (4.4)$$

On the other hand, if we assume that $g_0 = \int_0^\infty U_0(x)e^{-x(\cdot)}\mu(dx) \in \mathcal{G}_K$ for some initial family of points $(U_0(x))_{x \in \text{supp}(\mu)} \in L^1(\mu)$, there exists a continuous solution V for (4.3) by Theorem 2.1. In that case, we define for each $x \in \text{supp}(\mu)$, the continuous process $U(x)$ as in (4.4). Thanks to (H_2) and (2.6), another application of the stochastic Fubini theorem combined with the fact that V satisfies (4.3) yields that, for each $t \geq 0$, $(U_t(x))_{x \in \text{supp}(\mu)} \in L^1(\mu)$ and

$$V_t = \int_0^\infty U_t(x)\mu(dx). \quad (4.5)$$

Moreover, by an integration by parts, we get for each $x \in \text{supp}(\mu)$,

$$U_t(x) = e^{-xt}U_0(x) + Z_t e^{-xt} + \int_0^t x e^{-x(t-s)}(Z_s - Z_t) ds,$$

⁴We use the notation $\text{supp}(\mu)$ to denote the support of a measure μ , that is the set of all points for which every open neighborhood has a positive measure. Here we assume that the support is in \mathbb{R}_+ .

with $Z = \int_0^\cdot (-\lambda V_s ds + \nu \sqrt{V_s} dB_s)$. We know that for fixed $T > 0$, $\eta \in (0, 1/2)$ and for almost any $\omega \in \Omega$ there exists a positive constant $C_T(\omega)$ such that $|Z_s - Z_t| \leq C_T(\omega)|t - s|^\eta$ for all $t, s \in [0, T]$. Hence for any $t \in [0, T]$ and $x \in \text{supp}(\mu)$

$$|U_t(x)| \leq |U_0(x)| + C_T(\omega)e^{-xt}t^\eta + C_T(\omega)x \int_0^t e^{-xs}s^\eta ds = |U_0(x)| + C_T(\omega)\eta \int_0^t e^{-xs}s^{\eta-1} ds.$$

Then,

$$\sup_{t \in [0, T]} |U_t(x)| \leq |U_0(x)| + C_T(\omega)\eta \int_0^T e^{-xs}s^\eta ds \in L^1(\mu).$$

Therefore by dominated convergence theorem, the process $(\int_0^\infty U_t(x)\mu(dx))_{t \geq 0}$ is continuous. In particular, (4.5) holds in terms of processes and it follows from (4.4) that U is a solution of (4.1).

This leads to the weak existence and uniqueness of (4.1) if the initial family of points $(U_0(x))_{x \in \text{supp}(\mu)}$ belongs to the following space \mathcal{D}_μ defined by

$$\mathcal{D}_\mu = \{(u_x)_{x \in \text{supp}(\mu)} \in L^1(\text{supp}(\mu)); \int_0^\infty u_x e^{-x(\cdot)} \mu(dx) \in \mathcal{G}_K\}, \quad (4.6)$$

with K given by (4.2). Notice that for fixed $t_0 \geq 0$ and for any $t \geq 0$ and $x \in \text{supp}(\mu)$,

$$U_{t+t_0}(x) = U_{t_0}(x)e^{-xt} + \int_0^t e^{-x(t-s)} \left(-\lambda \int_0^\infty U_{s+t_0}(z)\mu(dz) + \nu \sqrt{\int_0^\infty U_{s+t_0}(z)\mu(dz)} dW_{s+t_0} \right)$$

and then by stochastic Fubini theorem

$$\int_0^\infty U_{t+t_0}(y)\mu(dy) = g_{t_0}(t) + \int_0^t K(t-s) \left(-\lambda \int_0^\infty U_{s+t_0}(z)\mu(dz) + \nu \sqrt{\int_0^\infty U_{s+t_0}(z)\mu(dz)} dW_{s+t_0} \right),$$

with $g_{t_0}(t) = \int_0^\infty U_{t_0}(y)e^{-yt}\mu(dy)$. Thanks to Theorem 3.1, we deduce that $g_{t_0} \in \mathcal{G}_K$ and therefore $(U_{t_0}(x))_{x \in \text{supp}(\mu)}$ belongs to \mathcal{D}_μ . As a conclusion, the space \mathcal{D}_μ is stochastically invariant with respect to the family of processes $(U(x))_{x \in \text{supp}(\mu)}$.

Theorem 4.1. *Fix μ a positive measure of locally bounded variation satisfying (H₂). There exists a unique weak solution U of (4.1) for each initial family of points $(U_0(x))_{x \in \text{supp}(\mu)} \in \mathcal{D}_\mu$. Furthermore for any $t \geq 0$, $(U_t(x))_{x \in \text{supp}(\mu)} \in \mathcal{D}_\mu$.*

	$K(t)$	Parameter restrictions	$\mu(d\gamma)$
Fractional	$c \frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$\alpha \in (1/2, 1)$	$c \frac{x^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} dx$
Gamma	$ce^{-\lambda t} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$\lambda \geq 0, \alpha \in (1/2, 1)$	$c \frac{(x-\lambda)^{-\alpha} \mathbf{1}_{(\lambda, \infty)}(x)}{\Gamma(\alpha)\Gamma(1-\alpha)} dx$
Exponential sum	$\sum_{i=1}^n c_i e^{-\gamma_i t}$	$c_i, \gamma_i \geq 0$	$\sum_{i=1}^n c_i \delta_{\gamma_i}(dx)$

Table 1: Some measures μ satisfying (H₂) with their associated kernels K . Here $c \geq 0$.

Remark 4.2 (Representation of V in terms of U). *In a similar fashion one can establish the existence and uniqueness of the following time-inhomogeneous version of (4.1)*

$$dU_t(x) = (-xU_t(x) - \lambda(g_0(t) + \langle 1, U_t \rangle_\mu)) dt + \nu \sqrt{g_0(t) + \langle 1, U_t \rangle_\mu} dW_t, \quad x \in \text{supp}(\mu), \quad (4.7)$$

whenever

$$g_0 = \tilde{g}_0 + \int_0^\infty e^{-x(\cdot)} U_0(x) \mu(dx) \in \mathcal{G}_K,$$

with $\tilde{g}_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$. In this case,

$$\tilde{g}_0(t + \cdot) + \int_0^\infty e^{-x(\cdot)} U_t(x) \mu(dx) \in \mathcal{G}_K, \quad t \geq 0.$$

In particular, for $U_0 \equiv 0$, $g_0 = \tilde{g}_0 \in \mathcal{G}_K$ and K as in (4.2), the solution V to the stochastic Volterra equation (1.2) and the forward process $(g_t)_{t \geq 0}$ admit the following representations

$$V_t = g_0(t) + \langle 1, U_t \rangle_\mu, \quad g_{t_0}(t) = g_0(t_0 + t) + \langle e^{-t(\cdot)}, U_t \rangle_\mu, \quad t, t_0 \geq 0, \quad (4.8)$$

where we used the notation $\langle f, g \rangle_\mu = \int_0^t f(x)g(x)\mu(dx)$. These results are in the spirit of [3, 13].

When μ has finite support, (4.7) is a finite dimensional diffusion with an affine structure in the sense of [6]. This underlying structure carries over to the case of infinite support and is the reason behind the tractability of the Volterra Heston model.

Remark 4.3 (Affine structure of $(\log S, V)$ in terms of U). *Let the notations and assumptions of Remark 4.2 be in force. Relying on the existence and uniqueness of the Riccati-Volterra equation (2.10) one can establish the existence and uniqueness of a differentiable (in time) solution χ_2 to the following (possibly) infinite-dimensional system of Riccati ordinary differential equations*

$$\partial_t \chi_2(t, x) = -x\chi_2(t, x) + F(\psi_1(t), \langle \chi_2(t, \cdot), 1 \rangle_\mu), \quad \chi_2(0, x) = u_2, \quad x \in \text{supp} \mu, t \geq 0, \quad (4.9)$$

such that $\chi_2(t, \cdot) \in L^1(\mu)$, for all $t \geq 0$ and $t \rightarrow \langle \chi_2(t, \cdot), 1 \rangle_\mu \in L^2_{\text{loc}}(\mathbb{R}_+)$ with ψ_1 given by (2.9) and F by (2.11). Moreover, the unique global solution $\psi_2 \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^*)$ to the Riccati-Volterra equation (2.10) admits the following representation

$$\psi_2 = \int_0^\infty \chi_2(\cdot, x) \mu(dx),$$

where χ_2 is the unique solution to (4.9). In particular, combining the equality above with (3.3) and the representation of $(g_t)_{t \geq 0}$ in (4.8) leads to the exponentially-affine functional

$$\mathbb{E} \left[\exp(uX_T + (f * X)_T) \mid \mathcal{F}_t \right] = \exp(\phi(t, T) + \psi_1(T - t) \log S_t + \langle \chi_2(T - t, \cdot), U_t \rangle_\mu)$$

for all $t \leq T$ where $\phi(t, T) = (u_2 \Delta_t g_0 + F(\psi_1, \psi_2) * \Delta_t g_0)(T - t)$, (u, f) as in (2.12) and U solves (4.7).

The representations of this section lead to a generic approximation of the Volterra Heston model by finite-dimensional affine diffusions, see [1] for the rigorous treatment of these approximations.

A Existence results for stochastic Volterra equations

In this section, we consider the following d -dimensional stochastic Volterra equation

$$X_t = g(t) + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s, \quad (\text{A.1})$$

where $K \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^{d \times d})$, W is a m -dimensional Brownian motion, $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are continuous with linear growth. By adapting the proofs of [2, Appendix A] (there, g is constant), we obtain the following existence results. Notice how the domain \mathcal{G}_K defined in (2.5) enters in the construction of constrained solutions in Theorem A.2 below.

Theorem A.1. *Under (H_0) , assume that $g \in \mathcal{H}^{\gamma/2}$.*

- (i) *If b and σ are Lipschitz continuous, (A.1) admits a unique continuous strong solution X .*
- (ii) *If b and σ are continuous with linear growth and K admits a resolvent of the first kind L , then (A.1) admits a continuous weak solution X .*

In both cases, X is locally Hölder continuous of any order strictly smaller than $\gamma/2$ and

$$\sup_{t \leq T} \mathbb{E}[|X_t|^p] < \infty, \quad p > 0, \quad T > 0. \quad (\text{A.2})$$

Theorem A.2. *Assume that $d = m = 1$ and that the scalar kernel K satisfies (H_0) - (H_1) . Assume also that b and σ are continuous with linear growth such that $b(0) \geq 0$ and $\sigma(0) = 0$. Then (A.1) admits a nonnegative continuous weak solution for any $g \in \mathcal{G}_K$.*

Proof. Theorem A.1(ii) yields the existence of an unconstrained continuous weak solution X to the following modified equation $X_t = g(t) + \int_0^t K(t-s)b(X_s^+)ds + \int_0^t K(t-s)\sigma(X_s^+)dW_s$. As in the proof of [2, Theorem 3.5], it suffices to prove the nonnegativity of X under the stronger condition, that, for some fixed $n \in \mathbb{N}$,

$$x \leq n^{-1} \text{ implies } b(x) \geq 0 \text{ and } \sigma(x) = 0. \quad (\text{A.3})$$

Set $Z = \int(b(X)dt + \sigma(X)dW)$ and $\tau = \inf\{t \geq 0 : X_t < 0\}$. Since $g(0) \geq 0$, $\tau \geq 0$. On $\{\tau < \infty\}$,

$$X_{\tau+h} = g(\tau+h) + (K * dZ)_{\tau+h} = g(\tau+h) + (\Delta_h K * dZ)_\tau + \int_0^h K(h-s)dZ_{\tau+s}, \quad h \geq 0. \quad (\text{A.4})$$

Using Lemma B.2 and Remark B.3 below, together with the fact that $X \geq 0$ on $[0, \tau]$,

$$\begin{aligned} g(\tau+h) + (\Delta_h K * dZ)_\tau &= g(\tau+h) + (\Delta_h K * L)(0)(X-g)(\tau) \\ &\quad + (d(\Delta_h K * L) * X)_\tau - (d(\Delta_h K * L) * g)(\tau) \\ &\geq g(\tau+h) - (d(\Delta_h K * L) * g)(\tau) - (\Delta_h K * L)(0)g(\tau), \end{aligned}$$

which is nonnegative. In view of (A.4) it follows that

$$X_{\tau+h} \geq \int_0^h K(h-s)(b(X_{\tau+s})ds + \sigma(X_{\tau+s})dW_{\tau+s}) \quad (\text{A.5})$$

on $\{\tau < \infty\}$ for all $h \geq 0$. Now, on $\{\tau < \infty\}$, $X_\tau = 0$ and $X_{\tau+h} < 0$ for arbitrarily small h . On the other hand, by continuity there is some $\varepsilon > 0$ such that $X_{\tau+h} \leq n^{-1}$ for all $h \in [0, \varepsilon]$. Thus (A.3) and (A.5) yield $X_{\tau+h} \geq 0$ for all $h \in [0, \varepsilon]$. This shows that $\tau = \infty$, ending the proof. \square

B Reminder on stochastic convolutions and resolvents

For a measurable function K on \mathbb{R}_+ and a measure L on \mathbb{R}_+ of locally bounded variation, the convolutions $K * L$ and $L * K$ are defined by

$$(K * L)(t) = \int_{[0,t]} K(t-s)L(ds), \quad (L * K)(t) = \int_{[0,t]} L(ds)K(t-s)$$

whenever these expressions are well-defined. If F is a function on \mathbb{R}_+ , we write $K * F = K * (Fdt)$. We can show that $L * F$ is almost everywhere well-defined and belongs to $L_{loc}^p(\mathbb{R}_+)$, whenever $F \in L_{loc}^p(\mathbb{R}_+)$. Moreover, $(F * G) * L = F * (G * L)$ *a.e.*, whenever $F, G \in L_{loc}^1(\mathbb{R}_+)$, see [12, Theorem 3.6.1 and Corollary 3.6.2] for further details.

For any continuous semimartingale $M = \int_0^\cdot b_s ds + \int_0^\cdot a_s dB_s$ the convolution $(K * dM)_t = \int_0^t K(t-s)dM_s$ is well-defined as an Itô integral for every $t \geq 0$ such that $\int_0^t |K(t-s)||b_s|ds + \int_0^t |K(t-s)|^2|a_s|^2ds < \infty$. By stochastic Fubini Theorem, see [2, Lemma 2.1], we have $(L * (K * dM)) = ((L * K) * dM)$, *a.s.* whenever $K \in L_{loc}^2(\mathbb{R}_+, \mathbb{R})$ and a, b are locally bounded *a.s.*

We define the *resolvent of the first kind* of a $d \times d$ -matrix valued kernel K , as the $\mathbb{R}^{d \times d}$ -valued measure L on \mathbb{R}_+ of locally bounded variation such that $K * L = L * K \equiv \text{id}$, where id stands for the identity matrix, see [12, Definition 5.5.1]. The *resolvent of the first kind* does not always exist. The following results are shown in [2, Lemma 2.6].

Lemma B.1. *Let $K \in L_{loc}^2(\mathbb{R}_+)$ and $Z = \int_0^\cdot b_s ds + \int_0^\cdot \sigma_s dW_s$ a continuous semimartingale with b and σ locally bounded. Assume that X and $K * dZ$ are continuous processes and that K admits a resolvent of the first kind L . Then $X = K * dZ$ if and only if $L * X = Z$.*

Lemma B.2. *Assume that $K \in L_{loc}^1(\mathbb{R}_+)$ admits a resolvent of the first kind L . For any $F \in L_{loc}^1(\mathbb{R}_+)$ such that $F * L$ is right-continuous and of locally bounded variation one has*

$$F = (F * L)(0)K + d(F * L) * K.$$

Remark B.3. *The previous lemma will be used with $F = \Delta_h K$, for a fixed $h \geq 0$. If K is continuous on $(0, \infty)$, then $\Delta_h K * L$ is right-continuous. Moreover, if K is nonnegative and L is non-increasing in the sense that $s \rightarrow L([s, s+t])$ is non-increasing for all $t \geq 0$, then $\Delta_h K * L$ is non-decreasing since $\Delta_h K * L = 1 - \int_{(0,h]} K(h-s)L(\cdot + ds)$, $t \geq 0$.*

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