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Abstract: In this work, we examine the benefit of having periodic dilution rate in the chemostat model in terms of averaged conversion rate. We compare the effect of bringing a same substrate quantity by a periodic rate with a constant rate. We show that for the classical chemostat model with a Contois growth function, the performance of the averaged conversion rate can be improved under certain conditions. Using Pontryagin's Principle, we characterize the extremals of the problem which minimizes the averaged substrate concentration among periodic trajectories of a given period.

Keywords: Chemostat Model, Optimal Periodic Control, Pontryagin Maximum Principle, Over-yielding.

1. INTRODUCTION

The foundations of the chemostat theory was originally given by Monod (1950) and Novick and Szilard (1950). Many works related to the chemostat model have been published in mathematical, biological and chemical engineering journals (see for instance Smith and Waltman (1995); Harmand et al. (2017); Ziv et al. (2013)). The concept of continuous culture is that micro-organisms are grown in a fixed volume that is continually diluted by addition of new nutrient with a simultaneous removal of micro-organisms and nutrients. Typical examples of the use of continuous culture techniques have been in the field of biotechnology (see for instance Grasman et al. (2005); Xu et al. (2013); Zhao and Yuan (2016); Wang et al. (2016); Zhao and Yuan (2017)). Moreover, it plays an important role in modeling natural ecosystems such as lakes, lagoons (Wright and Hobbie (1965)) and also in modeling the waste-water treatment processes (Gaudy and Gaudy (1966)) where the objective is to guarantee, in nominal operating conditions, the best water quality. It is defined in terms of small concentration of undesirable chemicals such as nitrate.

Many forms of the consumption term and the growth rate function in the chemostat model have been introduced. Monod (1950) assumed that no nutrient other than the substrate are limiting and that no toxic by-products of metabolism build up. He has proposed the following formulation

$$\mu(s) = \frac{\mu_{max} s}{k_s + s} \tag{1}$$

which describes the specific growth rate and the consumption of the substrate where s denotes the substrate concentration. There are two parameters: μ_{max}, the maximum specific growth rate, and k_s , the half-saturation constant. An other formulation was proposed by Andrews using Haldane function, in order to illustrate the inhibition of microorganisms by high substrate concentration (Andrews (1968)):

$$\mu(s) = \frac{\mu_{max} s}{k_s + s + s^2/k_I},$$

where k_I is the inhibition constant. Whereas, the Contois expression developed by Contois (1959) for Aerobacter aerogenes indicates that the growth function depends on the microorganism concentration denoted by x as well as the concentration of the limiting nutrient:

$$\mu(s, x) = \frac{\mu_{max} s}{k_c x + s}. \tag{2}$$

There are two ways to create a periodic environment in a chemostat with the operating parameters: making the input nutrient concentration vary periodically or the flow rate (see Smith and Waltman (1995)). Both configurations have been studied in literature, for example in Butler et al. (1985), Peng and H.I.Freedman (2000). The novelty of our work is to we consider a chemostat model with a Contois growth function. We show first that the averaged conversion rate can be improved by a periodic dilution rate, under the constraint of a given amount of nutrient to be brought during the period. Then, using tools of optimal control theory, we determine the extremals of the optimal control problem satisfying the input constraint. Finally, we study numerically the problem with the software `bocop` (Team Commands (2017)) and compare with the extremals determined analytically. For applications in waste-water treatments, this means that the average water quality can be improved by a periodic non constant flow rate.

2. CONDITIONS FOR HAVING AN OVER-YIELDING

In this paper, we consider the following classical chemostat model:

$$\begin{cases} \dot{s} = -\mu(s,x)x + (s_{in} - s)u(t), \\ \dot{x} = (\mu(s,x) - u(t))x. \end{cases} \tag{3}$$

In the above equations, x refers to the microorganism concentration, s to the substrate concentration, $s_{in} > 0$ is the input substrate concentration, $u(\cdot)$ is the dilution rate which is the control variable. Furthermore, $\mu(s,x)$ is the Contois growth rate expression given by (2). Notice that Contois kinetics is C^1, where the function $x \mapsto \mu(s,x)$ is strictly decreasing for all s and $\mu(0,x) = 0$ for all x and (see Wang and Li (2014)).

Given $T > 0$, we consider the admissible control set defined as:

$\mathcal{U} := \{u : \mathbb{R}_+ \to [u_{min}, u_{max}]$ s.t. $u(\cdot)$ meas., $T-$periodic$\}$,

where u_{min} and u_{max} denote respectively the minimal and the maximal dilution rates.

Our aim in this section is to determine whether the averaged conversion rate can be improved by a periodic dilution rate, comparing with a constant rate $\bar{u} \in (u_{min}, u_{max})$. Here, we suppose the following constraint:

$$\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t = \bar{u} \tag{4}$$

which means that we impose to have the substrate quantity brought by a periodic rate, during a period, equal to the quantity brought by the constant rate $\bar{u}$.

Remark 1. We look for T-periodic solutions of (3).

Recall the classical result of the chemostat (see Harmand et al. (2017)):

Lemma 2. All trajectories of system (3) with $u \in \mathcal{U}$ such that (4) is satisfied and $x(0) > 0$ converge asymptotically to the invariant set $s + x = s_{in}$.

Therefore, a periodic solution has to belong to the set $s + x = s_{in}$. We consider the dynamics in this set which amounts to reduce the dynamic (3) to a one-dimensional system given by

$$\dot{s} = (-\nu(s) + u(t))(s_{in} - s), \tag{5}$$

where the function ν is defined as

$$\nu(s) := \mu(s, s_{in} - s) = \frac{\mu_{max}\ s}{k_c(s_{in} - s) + s},$$

k_c and μ_{max} are positive.

We suppose that

$$\bar{u} < \nu(s_{in}) = \mu_{max}.$$

Then, the non-trivial equilibrium point $\bar{s}$, solution of $\nu(s) = \bar{u}$ exists and is given by

$$\bar{s} = \frac{\bar{u}k_c s_{in}}{\bar{u}(k_c - 1) + \mu_{max}}.$$

This steady state is positive as $\bar{u} \in (0, \mu_{max})$. Moreover, it is necessarily globally stable because ν is increasing.

The optimization problem consists in finding a control $u \in \mathcal{U}$ such that the mean value of the substrate

$$J_T(u) := \frac{1}{T}\int_0^T s_u(t)\,\mathrm{d}t, \tag{6}$$

is minimized, where

$$s_u \in \mathcal{S}_T := \{s(\cdot) : [0,T] \to [0, s_{in}],\ \text{solution of (5) with}\ s(0) = s(T)\ \text{with}\ u \in \mathcal{U}\ \text{satisfying (4)}\}.$$

Definition 3. We say that the system exhibits an over-yielding if the value of the performance index J_T is less than the value of the performance index at the steady state $\bar{s}$, i.e. there exists $u \in \mathcal{U}$ such that:

$$J_T(u) < \bar{s} = J_T(\bar{u}),$$

Proposition 4. If $k_c > 1$ then ν is strictly convex and an *over-yielding* exists.

Proof. Note first that $(0, s_{in})$ is invariant by (5). We consider a periodic function $s \in \mathcal{S}_T$ and define ξ as $\xi(t) := \log(s_{in} - s(t))$. If we differentiate ξ w.r.t t, we get

$$\xi'(t) = \frac{-s'(t)}{s_{in} - s(t)}.$$

The integration of ξ' from 0 to T using (5) gives

$$\int_0^T \xi'(t)\,\mathrm{d}t = \int_0^T (\nu(s(t)) - u(t))\,\mathrm{d}t = 0.$$

Therefore,

$$\int_0^T (\nu(s(t)) - \nu(\bar{s}))\,\mathrm{d}t = 0. \tag{7}$$

Let us prove first that Contois's function is convex. The first and the second derivatives of ν are:

$$\frac{d}{ds}\nu(s) = s_{in}\frac{\mu_{max}\ k_c}{(k_c(s_{in} - s) + s)^2} > 0,$$

$$\frac{d^2}{ds^2}\nu(s) = 2s_{in}\ \frac{\mu_{max}\ k_c\ (k_c - 1)}{(k_c(s_{in} - s) + s)^3}.$$

One can easily conclude that ν is strictly convex exactly when $k_c > 1$. Thus, when this condition is verified, we get, by applying Jensen's inequality

$$\nu\Big(\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t\Big) < \frac{1}{T}\int_0^T \nu(s(t))\,\mathrm{d}t = \nu(\bar{s}),$$

and as ν is strictly monotonic, one obtains

$$\frac{1}{T}\int_0^T s(t)\,\mathrm{d}t < \bar{s},$$

therefore,

$$J_T(u) < J_T(\bar{u}).$$

3. OPTIMAL SYNTHESIS AND NUMERICAL SIMULATIONS

In this section, we suppose that the condition

$$k_c > 1$$

holds true, so that an over yielding exists.

For convenience, we reformulate the constraint (4) and consider the following dynamics

$$\begin{cases} \dot{s} = (-\nu(s) + u)(s_{in} - s), \\ \dot{z} = u, \end{cases} \tag{8}$$

where $s \in \mathcal{S}_T$. As ν is monotonic, it is clear that equation (7) is satisfied only if $s(t) - \bar{s}$ changes sign on $[0,T]$. Then, necessarily there exists $t_0 \in [0,T]$ such that

$$s(t_0) = \bar{s}.$$

We then consider, without any loss of generality, the following boundary conditions

$$s(0), z(0) = (\bar{s}, 0), \quad (s(T), z(T)) = (\bar{s}, \bar{u}T). \tag{9}$$

Thus, the optimal control problem can be stated as follows

$$\inf_{u \in \mathcal{U}} J_T(u)\ \text{satisfying (8) – (9).} \tag{10}$$

We define the break-even concentration associated to the function ν as

$$\lambda(u) = \sup\{s < s_{in} \text{ s.t. } \nu(s) < u\}$$

One can straightforwardly check that the interval $I := (\lambda(u_{min}), \lambda(u_{max}))$ is invariant and that $\bar{s}$ belongs to this interval. Therefore we shall look in the sequel for periodic solutions $s(\cdot)$ that belong I.

Let $H : \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (10):

$$\begin{aligned} H &= H(s, z, \lambda_s, \lambda_z, \lambda_0, u) \\ &= \lambda_0 s - \nu(s)\lambda_s(s_{in} - s) + u(\lambda_s(s_{in} - s) + \lambda_z), \end{aligned}$$

where $\lambda := (\lambda_s, \lambda_z)^T$ denotes the adjoint vector. Let $u \in \mathcal{U}$ be an optimal control corresponding to a trajectory $(s(\cdot), z(\cdot))$. Then, there exists $\lambda_0 \le 0$ and an absolutely continuous map $\lambda : [0, T] \to \mathbb{R}^2$ satisfying the following adjoint equations for a.e. $t \in [0, T]$:

$$\begin{cases} \dot\lambda_s = -\lambda_0 + \lambda_s(\nu'(s)(s_{in} - s) + u(t) - \nu(s)), \\ \dot\lambda_z = 0, \end{cases} \tag{11}$$

where $(\lambda_0, \lambda(\cdot))$ is non identically null. The control u satisfies the maximization condition almost everywhere on $[0, T]$:

$$u(t) \in \underset{v\in[u_{min},u_{max}]}{\arg\max}\ H(s, z, \lambda_s, \lambda_z, \lambda_0, v). \tag{12}$$

The switching function ψ associated to the control is defined by:

$$\psi := \frac{\partial H}{\partial u} = \lambda_s(s_{in} - s) + \lambda_z.$$

From (12), we obtain the following control law:

$$\begin{cases} \psi(t) > 0 \Rightarrow u(t) = u_{max} \\ \psi(t) < 0 \Rightarrow u(t) = u_{min} \\ \psi(t) = 0 \Rightarrow u(t) \in [u_{min}, u_{max}] \end{cases}$$

As ν is increasing then one has $u_{min} < \nu(s) < u_{max}$ on I. Therefore, when $\psi > 0$ (resp. $\psi < 0$) on some time interval, s increases (resp. decreases).
A singular arc occurs if ψ vanishes on some time interval $[t_1, t_2]$ with $t_1 < t_2$ (see Bonnard and Chyba (2002)) and a switching time $t_s \in (0, T)$ is such that $s(\cdot)$ is not C^1 at t_s. Using (11), the derivative of ψ w.r.t t is

$$\dot\psi = (s_{in} - s)(-\lambda_0 + \lambda_s\nu'(s)(s_{in} - s)).$$

In this paper, we assume that an optimal trajectory is normal ($\lambda_0 \neq 0$). We take in the sequel $\lambda_0 = -1$.

Remark 5. It follows, by the periodicity of s and equation (7), that ψ changes sign and is zero at least twice. This implies the existence of two switching times at least.

Let us define s_{max}, s_{min}, t_{max}, t_{min} by

$$s_{max} := \max_{[0,T]} s(t);\ t_{max} := \min\{t \text{ s.t. } s(t) = s_{max}\},$$

$$s_{min} := \min_{[0,T]} s(t);\ t_{min} := \min\{t \text{ s.t. } s(t) = s_{min}\}.$$

As $s(\cdot)$ changes it monotony at t_{max} and t_{min}, they are necessarily two switching points.

Proposition 6. If $\tilde t \in [0, T]$ is a switching time different from t_{max} and t_{min} then

$$s(\tilde t) \in \{s_{max}, s_{min}\}$$

Proof. Note first that the times t_{max} and t_{min} necessarily satisfy

$$\psi(t_{max}) = \psi(t_{min}) = 0,$$

hence

$$\lambda_s(t_{max})(s_{in} - s_{max}) = \lambda_s(t_{min})(s_{in} - s_{min}) = -\lambda_z.$$

Since the Hamiltonian is conserved along any extremal trajectory, one has

$$H = -s_{max} + \lambda_z\nu(s_{max}) = -s_{min} + \lambda_z\nu(s_{min}),$$

which implies that λ_z is positive and

$$\frac{1}{\lambda_z} = \frac{\nu(s_{max}) - \nu(s_{min})}{s_{max} - s_{min}}.$$

Let us define the function g as

$$g(s) := \frac{\nu(s) - \nu(a)}{s - a},\ s \neq a,$$

where $a \in I$ is a parameter. One can easily conclude, from the Chordal Slope lemma, that g is strictly increasing whatever is a. Therefore, if $a = s_{min}$, the equation $g(s) = \frac{1}{\lambda_z}$ has a unique solution $s = s_{max}$ and $s = s_{min}$ if $a = s_{max}$.
Let us now suppose by contradiction that there exists a switching time $\tilde t \in [0, T]$ such that

$$\tilde t \notin \{t_{max}, t_{min}\},$$

and

$$s(\tilde t) = \tilde s \notin \{s_{max}, s_{min}\}.$$

Since $\psi(\tilde t) = 0$ then one has

$$\frac{1}{\lambda_z} = \frac{\nu(\tilde s) - \nu(s_{min})}{\tilde s - s_{min}} = \frac{\nu(\tilde s) - \nu(s_{max})}{\tilde s - s_{max}}.$$

Thus $\tilde s$ is solution of $g(s) = \frac{1}{\lambda_z}$ with $a = s_{min}$ and $a = s_{max}$ which is a contradiction. Therefore, we get $\tilde s \in \{s_{max}, s_{min}\}$.

Proposition 7. The optimal trajectory has no singular arc.

Proof. Assume by contradiction that there exists a time interval $[t_1, t_2]$, $t_1 < t_2$ where ψ is zero, then one has

$$\psi(t) = \dot\psi(t) = 0, \tag{13}$$

for any time $t \in I$. The equality (13) implies $\nu'(s(t)) = \frac{1}{\lambda_z}$ and $u(t) = \nu(s(t))$ over $[t_1, t_2]$. Thus, s is constant over the singular arc. Let $s^* \in I$ be such that for any time $t \in [t_1, t_2]$, one has $s(t) = s^*$. Proposition 6 implies that s^* takes two possible values s_{min} or s_{max}.
If $s^* = s_{max}$ then

$$\begin{aligned} H &= -s_{max} - \nu(s_{max})\lambda_s(t_{max})(s_{in} - s_{max}) \\ &= -s_{max} + \nu(s_{max})\lambda_z \\ &= -s_{min} + \nu(s_{min})\lambda_z, \end{aligned}$$

hence, we get

$$\frac{1}{\lambda_z} = \frac{\nu(s_{max}) - \nu(s_{min})}{s_{max} - s_{min}} = \nu'(s_{max}),$$

which is a contradiction with the strict convexity of ν. Therefore $s^* = s_{min}$. Similarly, we get

$$\frac{1}{\lambda_z} = \frac{\nu(s_{max}) - \nu(s_{min})}{s_{max} - s_{min}} = \nu'(s_{min}),$$

which is also a contradiction with the strict convexity of ν. One then concludes that $s^* \notin \{s_{min}, s_{max}\}$ and the system has no singular arc. This ends the proof.

At this stage, we have thus proved that if u is an optimal control then it is of bang-bang type *i.e.* it is a succession of arcs $u = u_{min}$ and $u = u_{max}$. Moreover, the number of switching times is necessarily even (otherwise a switch will have to occur at $s(T) = \bar{s}$ in contradiction with Proposition 6).

We have computed the optimal cost J_T associated to the optimal control containing 2, 4, 6 and 8 switchings, for a fixed period T. The numerical values are given in the following table:

$2n$	2	4	6	8
J_T	1.5817	1.7749	1.8148	1.8356

with $s_{in} = 3$, $k_c = 2.5$, $\mu_{max} = 1$, $\bar{s} = 1.8377$, $u_{min} = 0$ and $u_{max} = 2$.
We observe that the cost is minimal for $n = 1$.
Notice that, for a given period T, a solution with $2n$ switches is necessarily T/n periodic and thus its cost is equal to the cost of the two switches solution on the interval $[0, T/n]$. We have plotted the cost of the periodic solution containing exactly two switches and observe that is decreasing w.r.t the period T (see Figure 3).

We then conjecture the following optimality result:

Conjecture 8. The optimal periodic control of the problem (10), has exactly two switching times and is expressed as follows

$$u(t) = \begin{cases} u_{max} & \text{if } t \in [0, t_1) \cup (t_2, T] \\ u_{min} & \text{if } t \in (t_1, t_2) \end{cases} \tag{14}$$

where t_1 and t_2 are uniquely defined as solution of the system

$$\begin{aligned} (t_1 + T - t_2)u_{max} + (t_2 - t_1)u_{min} &= \bar{u}T \\ \int_0^T \nu(s(t))\,\mathrm{d}t &= \bar{u}T \end{aligned}$$

Solving the problem numerically using the software `bocop` (Team Commands (2017)) with different initial guess allows us to affirm that an optimal periodic control for the problem (10) contains only two switching times (see Figures 1 and 2). Note that `bocop` implements a local optimization method. It is done by a discretization in time applied to the state and control variables and the dynamic equation. Finally, as the optimal cost is decreasing w.r.t T then the improvement of the conversion rate with a periodic forcing, given by (14), can be significant and increases with the period.

4. CONCLUSION

In this work, we have considered the specific chemostat model with Contois kinetics and determined whether the averaged conversion rate can be improved by a periodic flow rate. We have analyzed the optimal control problem using Pontryagin Maximum Principle and we have shown that the best periodic control is bang-bang with $2n$ switching times, $n \geq 1$. Numerical simulations indicate that the optimal periodic strategy contains exactly two switching times during a period. We shall look for generalization of this result for more general growth functions.

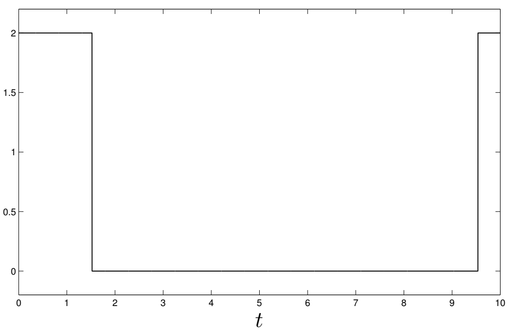

Fig. 1. Plot of the optimal control u of type bang-bang given by (14) with $T = 10$, $u_{min} = 0$ and $u_{max} = 2$.

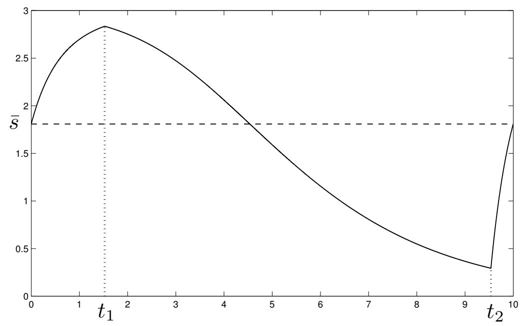

Fig. 2. Plot of the substrate concentration s corresponding to (14) with $T = 10$.

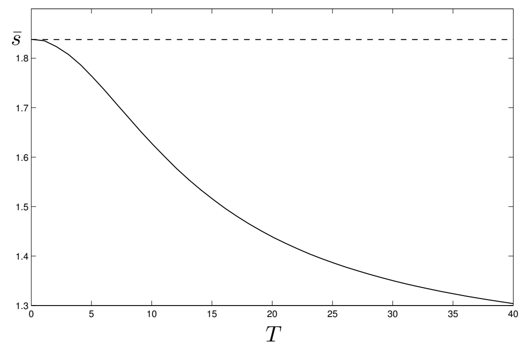

Fig. 3. Plot of the optimal J_T w.r.t. the period T.

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