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## Modeling and analyzing the long-time behavior of random chemostat models

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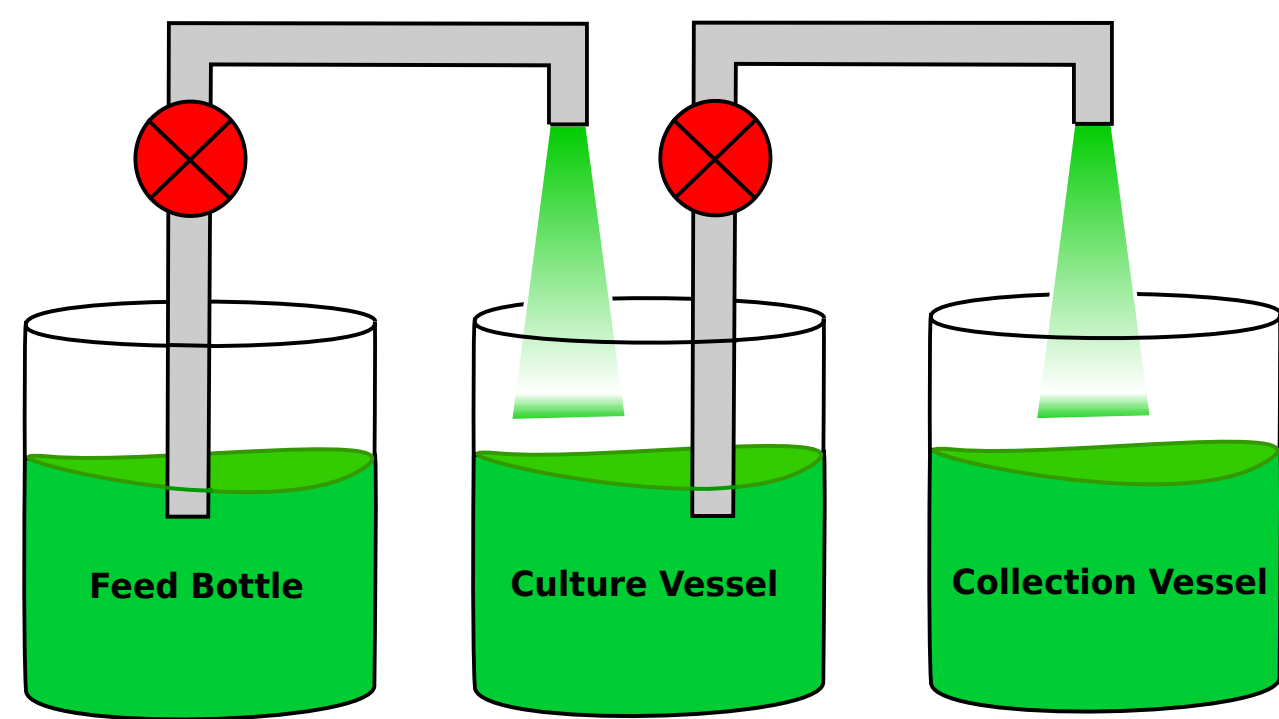
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## INTRODUCTION

Chemostat refers to a laboratory device used for growing microorganisms in a cultured environment and has been regarded as an idealization of nature to study microbial ecosystems at steady state, which is a really important and interesting problem since they can be used to study genetically altered microorganisms, waste water treatment and play an important role in ecology.



The simplest chemostat device consists of three interconnected tanks called *feed bottle*, *culture vessel* and *collection vessel*. The nutrient is pumped from the first tank to the culture vessel, where the interactions between the species and the nutrient take place, and there is also another flow being pumped from the culture vessel to the third tank such that the volume in the culture vessel remains constant. The mathematical model is given by the following system of ordinary differential equations

$$\begin{aligned} \frac{ds}{dt} &= (s_{in} - s)D - \frac{msx}{a+s}, & s(t), x(t): \text{concentration of nutrient and micrs.} \\ & & D: \text{dilution rate} \\ \frac{dx}{dt} &= -Dx + \frac{msx}{a+s}. & s_{in}: \text{input concentration of nutrient} \\ & & a: \text{half-saturation constant} \\ & & m: \text{max. consumption rate of microorganisms.} \end{aligned}$$

In this work, we are interested in modeling and analyzing disturbances on the input flow in the chemostat model. This kind of phenomena could explain, for instance, the presence of particles of dirt inside the pumps or temporary clogs at the input or output of the chemostat. There are many different ways to introduce randomness and/or stochasticity in some deterministic model. The most common way is to use the standard Wiener process,  $\omega$ , and replace  $D$  by  $D + \alpha\omega(t)$ , where  $\alpha > 0$  represents the intensity of the noise, as made in [1]. However, several drawbacks can be found when using this way to model disturbances on the input flow in the chemostat model. As a consequence, we are interested in considering another approach which consists on replacing  $D$  by  $D + \alpha z_{\beta, \nu}^*(\theta_t \omega)$ , where  $z_{\beta, \nu}^*(\theta_t \omega)$  represents a suitable Ornstein-Uhlenbeck process. We refer the interested readers to [1, 2] for more detailed explanations concerning this work.

## RANDOM DYNAMICAL SYSTEMS

**Definition 1** Let  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  be a separable Banach space. An random dynamical system (RDS) on  $\mathcal{X}$  consists of two ingredients: (a) a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ , where  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space and a family of mappings  $\theta_t: \Omega \rightarrow \Omega$  verifying

- (1)  $\theta_0 = \text{Id}_{\Omega}$  and  $\theta_s \circ \theta_t = \theta_{s+t}$  for all  $s, t \in \mathbb{R}$ ,
- (2) the mapping  $(t, \omega) \mapsto \theta_t \omega$  is measurable,
- (3) the probability measure  $\mathbb{P}$  is preserved by  $\theta_t$ , i.e.,  $\theta_t \mathbb{P} = \mathbb{P}$ .

and (b) a measurable mapping  $\varphi: [0, +\infty) \times \Omega \times \mathcal{X} \rightarrow \mathcal{X}$  such that for each  $\omega \in \Omega$ ,

- (i) the mapping  $\varphi(t, \omega): \mathcal{X} \rightarrow \mathcal{X}$ ,  $x \mapsto \varphi(t, \omega)x$  is continuous for every  $t \geq 0$ ,
- (ii)  $\varphi(0, \omega)$  is the identity operator on  $\mathcal{X}$ ,
- (iii) (cocycle property)  $\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \varphi(s, \omega)$  for all  $s, t \geq 0$ .

**Definition 2** A bounded random set  $K(\omega) \subset \mathcal{X}$  is said to be tempered respect to  $\{\theta_t\}_{t \in \mathbb{R}}$  if for a.e.  $\omega \in \Omega$ ,

$$\lim_{t \rightarrow +\infty} e^{-\eta t} \sup_{x \in K(\theta_{-t} \omega)} \|x\|_{\mathcal{X}} = 0, \quad \text{for all } \eta > 0.$$

We will denote the set of all tempered random sets of  $\mathcal{X}$  by  $\mathcal{E}(\mathcal{X})$ .

**Definition 3** A random set  $B(\omega) \subset \mathcal{X}$  is called a tempered random absorbing set if for any  $E \in \mathcal{E}(\mathcal{X})$  and a.e.  $\omega \in \Omega$ , there exists  $T_E(\omega) > 0$  such that

$$\varphi(t, \theta_{-t} \omega) E(\theta_{-t} \omega) \subset B(\omega), \quad \forall t \geq T_E(\omega).$$

**Definition 4** Let  $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$  be an RDS over  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$  with state space  $\mathcal{X}$  and let  $A(\omega) \subset \mathcal{X}$  be a random set. Then  $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega}$  is called a global random  $\mathcal{E}$ -attractor (or pullback  $\mathcal{E}$ -attractor) for  $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$  if

- (i)  $A(\omega)$  is a compact set of  $\mathcal{X}$  for any  $\omega \in \Omega$ ;
- (ii) For any  $\omega \in \Omega$  and all  $t \geq 0$ , it holds  $\varphi(t, \omega) A(\omega) = A(\theta_t \omega)$ ;
- (iii) For any  $E \in \mathcal{E}(\mathcal{X})$  and a.e.  $\omega \in \Omega$ ,  $\lim_{t \rightarrow \infty} \text{dist}_{\mathcal{X}}(\varphi(t, \theta_{-t} \omega) E(\theta_{-t} \omega), A(\omega)) = 0$ , where  $\text{dist}_{\mathcal{X}}(G, H) = \sup_{g \in G} \inf_{h \in H} \|g - h\|_{\mathcal{X}}$  is the Hausdorff semi-metric for  $G, H \subseteq \mathcal{X}$ .

**Proposition 1** Let  $B \in \mathcal{E}(\mathcal{X})$  be a closed absorbing set for the continuous RDS  $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ . Then  $\varphi$  has a unique pullback random attractor  $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega}$ .

For more detailed information about the theory of RDSs and pullback attractors, see [3].

## THE ORNSTEIN-UHLENBECK PROCESS

Let us consider the metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ , where

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}, \quad \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

$\mathcal{F}$  is the Borel  $\sigma$ -algebra on  $\Omega$  and  $\mathbb{P}$  is the corresponding Wiener measure on  $\mathcal{F}$ .

Now, we will introduce the Ornstein-Uhlenbeck (O-U) process as the random variable given by

$$z_{\beta, \nu}^*(\theta_t \omega) = -\beta \nu \int_{-\infty}^0 e^{\beta s} \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \quad \omega \in \Omega, \quad \beta, \nu > 0,$$

which solves the following Langevin equation

$$dz + \beta z dt = \nu d\omega(t), \quad t \in \mathbb{R}.$$

For every given  $b_2 > b_1 > 0$ , we define the probability space  $(\Omega_{\beta}, \mathcal{F}_{\beta}, \mathbb{P}_{\beta})$  as follows: the new set of events

$$\Omega_{\beta} = \{\omega \in \Omega : b_1 \leq D + \alpha z_{\beta, \nu}^*(\theta_t \omega) \leq b_2, \text{ for all } t \in \mathbb{R}\},$$

which is measurable, the new  $\sigma$ -algebra  $\mathcal{F}_{\beta} = \{A \cap \Omega_{\beta}, A \in \mathcal{F}\}$  and the probability measure defined as  $\mathbb{P}_{\beta}(F_{\beta}) = \mathbb{P}(F_{\beta})/\mathbb{P}(\Omega_{\beta})$ , for all  $F_{\beta} \in \mathcal{F}_{\beta}$ .

It is not difficult to check that  $(\Omega_{\beta}, \mathcal{F}_{\beta}, \mathbb{P}_{\beta}, \{\theta_t\}_{t \in \mathbb{R}})$  is also a metric dynamical system.

## RANDOM CHEMOSTAT MODEL

We are interested in investigating the following random chemostat model

$$\frac{ds}{dt} = (s_{in} - s)(D + \alpha z_{\beta, \nu}^*(\theta_t \omega)) - \frac{msx}{a+s}, \quad (1)$$

$$\frac{dx}{dt} = -(D + \alpha z_{\beta, \nu}^*(\theta_t \omega))x + \frac{msx}{a+s}. \quad (2)$$

**Theorem 1** For any  $\omega \in \Omega_{\beta}$  and any  $v_0 := (s_0, x_0) \in \mathcal{X} = \{(s, x) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ , system (1)-(2) possesses a unique global solution  $v(\cdot; 0, \omega, v_0) := (s(\cdot; 0, \omega, s_0), x(\cdot; 0, \omega, x_0)) \in \mathcal{C}^1([0, +\infty), \mathcal{X})$  with  $v(0; 0, \omega, v_0) = v_0$ , where  $s_0 := s(0; 0, \omega, v_0)$  and  $x_0 := x(0; 0, \omega, v_0)$ . Moreover, the solution mapping generates a random dynamical system  $\varphi_v: \mathbb{R}^+ \times \Omega_{\beta} \times \mathcal{X} \rightarrow \mathcal{X}$  defined by

$$\varphi_v(t, \omega)v_0 := v(t; 0, \omega, v_0), \quad \text{for all } t \in \mathbb{R}^+, v_0 \in \mathcal{X}, \omega \in \Omega_{\beta},$$

the value at time  $t$  of the solution of system (1)-(2) with initial state  $v_0$  at time zero.

**Theorem 2** There exists a tempered compact random absorbing set  $B_0(\omega) \in \mathcal{E}(\mathcal{X})$  for the RDS  $\{\varphi_v(t, \omega)\}_{t \geq 0, \omega \in \Omega_{\beta}}$  given by  $B_0(\omega) := \{(s, x) \in \mathcal{X} : s + x = s_{in}\}$ .

Therefore, thanks to Proposition 1, the RDS generated by system (1)-(2) possesses a unique random pullback attractor  $\mathcal{A} = \{A(\omega)\}_{\omega \in \Omega_{\beta}} \subset B_0(\omega)$ .

**Proposition 2** Assume that  $\bar{D} > \mu(s_{in})$  is fulfilled. Then, the random pullback attractor of the chemostat model (1)-(2) is reduced to a singleton component given by  $\mathcal{A} = \{(s_{in}, 0)\}$ .

**Theorem 3** Assume that

$$\bar{s} < s_{in}$$

holds true, where  $\bar{s} = \mu^{-1}(b_2)$  and  $\mu(s) = \frac{ms}{a+s}$ . Then, there exists a tempered compact random absorbing set, which is strictly contained in the positive cone  $\mathcal{X}$ , for the RDS  $\{\varphi_v(t, \omega)\}_{t \geq 0, \omega \in \Omega_{\beta}}$ .

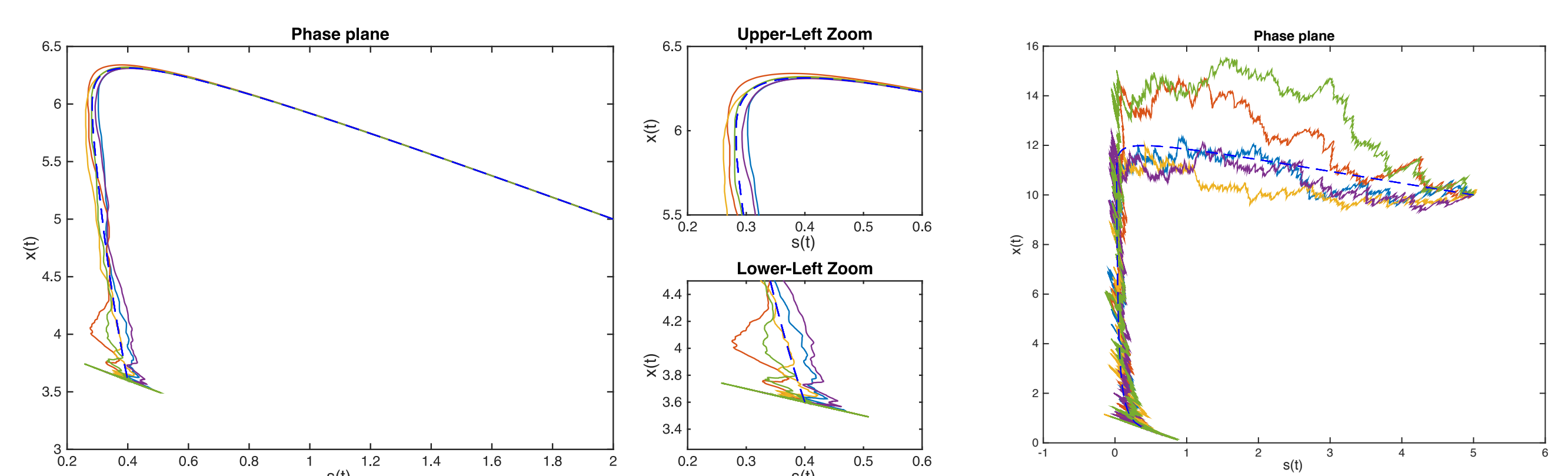


Figure 1. Persistence of the microbial biomass in the random and stochastic models

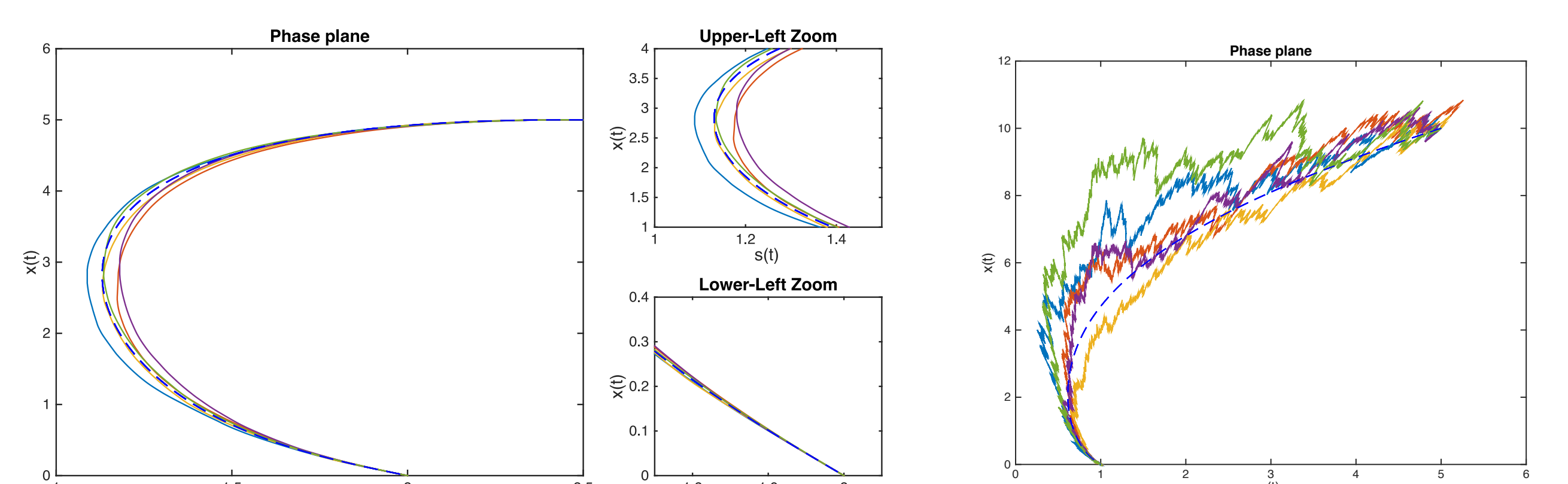


Figure 2. Extinction of the microbial biomass in the random and stochastic models

We refer the readers to [2] for the proof of the results in this work.

## REFERENCES

- [1] T. Caraballo, M. J. Garrido-Atienza, J. López-de-la-Cruz, and A. Rapaport. Corrigendum to "Some aspects concerning the dynamics of stochastic chemostats". *arXiv:1710.00774 [math.DS]*, 2017.
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