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ABSTRACT. In this paper we initiate the investigation of free boundary minimization problems ruled by general singular operators with A_2 weights. We show existence and boundedness of minimizers. The key novelty is a sharp $C^{1+\gamma}$ regularity result for solutions at their singular free boundary points. We also show a corresponding non-degeneracy estimate.

CONTENTS

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1. INTRODUCTION

We study local minimizers of singular, discontinuous functionals of the form

$$\mathcal{J}(u,\Omega)=\int_\Omega \left(\omega(x)|\nabla u|^2+\chi_{\{u>0\}}\right)dx \longrightarrow \min, \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 2$, and $\omega(x)$ is a measurable, singular A_2 weight in the sense that $0 \leq \omega \leq +\infty$, both ω and ω^{-1} are locally integrable, and

$$\left(\int_{B_r(z_0)} \omega\right)\left(\int_{B_r(z_0)} \omega^{-1}\right) \leq Cr^{2d},$$

for all balls $B_r(z_0) \subset \Omega$. Thus, ω may become zero or infinity along a lower-dimensional subset of Ω, hereafter denoted by:

$$\Lambda_0(\omega) := \omega^{-1}(0), \quad \Lambda_\infty(\omega) := \omega^{-1}(+\infty).$$

If will also be convenient to denote $\Lambda(\omega) := \Lambda_0(\omega) \cup \Lambda_\infty(\omega)$. The class of A_2 weight functions was introduced by Muckenhoupt [13] and is of central importance in modern harmonic analysis and its applications. A canonical example of an A_2 function is $|x|^\alpha$ with $-d < \alpha < d$ — having an isolated singularity at the origin. Recent results, see for instance [3], show A_2 weights play an important role in the theory of non-local diffusive problems. In particular, when $\omega(x) = |x_1|^\beta$, $-1 < \beta < 1$, and the axis $\{x_1 = 0\}$ is a subset of $\partial\Omega$, then (1.1) falls into the case studied in [2], where fractional cavitation problems are considered.

The mathematical analysis of free boundary cavitation problems goes back to the pioneering work Alt and Caffarelli [1], corresponding to the case $\omega \equiv 1$ in (1.1). In the present work, we start the investigation of free boundary problems of the type (1.1), for possibly singular weights, that is $\Lambda_\infty(\omega) \neq \emptyset$. Similarly, we can treat the degenerate case, i.e. when $\Lambda_0(\omega) \neq \emptyset$. However, for didactical purposes, in this paper we shall restrict the analysis to singular weights.

We show existence of a local minimizer u, and analyze analytic and weak geometric properties of the free boundary $\partial\{u > 0\} \cap \Omega$. The latter task is, in principle, a delicate issue. For instance, one notices, for singular weights, the existence of two distinct types of free boundary points:

(1) $\partial\{u > 0\} \cap \Lambda_\infty = \emptyset$;
(2) $\partial\{u > 0\} \cap \Lambda_\infty \neq \emptyset$.

Case (1) refers to a non-homogeneous version of the Alt-Caffarelli problem, [1], as, away from the singular set, the weight is uniformly elliptic, see for instance [4, 5, 9, 15]. Case (2) is rather more delicate, and in particular, a central result we prove in this current work classifies the geometric behavior of a local minimizer near a free boundary point $z_0 \in \partial\{u > 0\} \cap \Lambda_\infty$, in terms of its singularity rate near z_0 — a pure analytic information of the problem. Indeed, we show local minimizers are *precisely* $C^{1+\gamma}$ smooth along their corresponding free boundaries, where γ is half of the geometric blow-up rate of ω as it approaches the singular set Λ_∞; see condition (H3) for precise definitions.

The paper is organized as follows. In Section 2 we formally present the minimization problem we shall study. A brief description of the initial mathematical tools required in the investigation of A_2-singular free boundary problems is also delivered in that section. In Section 3 we discuss existence and L^∞ bounds for minimizers, whereas in Section 4 we establish various compactness properties for family of minimizers. In Section 5 we prove the key novel result of the paper, namely that solutions to A_2 cavitation problems are $C^{1+\gamma}$ regular at their singular free boundary points, where γ is a sharp prescribe value. In particular, if z_0 is a free boundary point and $\omega(z_0) < +\infty$, that is it is non-singular, then $\gamma(z_0) = 0$ and we recover the classical Alt-Caffarelli Lipschitz regularity estimate for cavitation problems. Finally in Section 6 we obtain a quantitative non-degeneracy estimate for solutions near its singular free boundary points.

2. Mathematical set-up

In this section we give a precise description of the minimization problem considered in this article and gather some of the main known results about elliptic equations involving A_2 weights, required in our study.

Given an open set $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{M}(\Omega)$ the set of all real-valued measurable functions defined on Ω. A nonnegative locally integrable function $\omega\colon \Omega \to \mathbb{R}$ is said to be an A_2 weight if ω^{-1} is also locally integrable and

$$\sup_{B\subset\Omega} \Big(\frac{1}{|B|}\int_B \omega\Big)\Big(\frac{1}{|B|}\int_B \omega^{-1}\Big) \leq C_1, \tag{2.1}$$

holds for a constant $C_1 > 0$ and any ball $B \subset \Omega$. Two weights are said to belong to the same A_2 class if condition (2.1) is verified for both functions with the same constant C_1.

Fixed an A_2 weight ω and $1 \leq p < \infty$, we define

$$L^p(\Omega,\omega) = \left\{ f \in \mathcal{M}(\Omega) \big|\ \|f\|_{L^p(\Omega,\omega)} := \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty \right\}.$$

Accordingly, we define the weighted Sobolev space as

$$W^{1,p}(\Omega,\omega) := \left\{ u \in L^p(\Omega,\omega) \big|\ D_i u \in L^p(\Omega,\omega),\ \text{for } i = 1, 2, \cdots d \right\},$$

and, by convention, we write $W^{1,2}(\Omega,\omega)$ as $H^1(\Omega,\omega)$.

An A_2 weight ω gives raise to the degenerate/singular elliptic operator

$$L_\omega(\cdot) = \operatorname{div}(\omega\nabla\cdot),$$

which acts on $H^1(\Omega,\omega)$. In a series of three papers, [6, 7, 8], Fabes, Jerison, Kenig, and Serapioni developed a systematic theory for this class of operators: existence of weak solutions, Sobolev embeddings, Poincaré inequality, Harnack inequality, local solvability in Hölder spaces, and estimates on the

Green's function. Below we state four results from [6], supporting, directly or indirectly, the framework developed in this present article.

Theorem 1 (Solvability in Sobolev spaces). *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, $h = (h_1, ..., h_n)$ satisfy $|h|/\omega \in L^2(\Omega, \omega)$, and $g \in H^1(\Omega, \omega)$. Then, there exists a unique solution $u \in H^1(\Omega, \omega)$ of $L_\omega u = -\operatorname{div} h$ in Ω with $u - g \in H^1_0(\Omega, \omega)$.*

Theorem 2 (Local Hölder regularity). *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and u a solution of $L_\omega u = -\operatorname{div} h$ in Ω, where $|h|/\omega \in L^2(\Omega, \omega)$. Then, u is Hölder continuous in Ω with a Hölder exponent depending only on d and the A_2 class of ω.*

Theorem 3 (Harnack inequality). *Let u be a positive solution of $L_\omega u = 0$ in $B_{4R}(x_0) \subset \mathbb{R}^d$. Then, $\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u$ for some constant C depending only on d and the A_2 class of ω — and in particular, independent of R.*

Theorem 4 (Poincaré inequality). *There is as positive constant C such that for all Lipschitz continuous function u defined on B_R, the following holds*

$$\Big(\frac{1}{\omega(B_R)} \int_{B_R} |u - A_R|^2 \omega\Big) \leq CR\Big(\frac{1}{\omega(B_R)} \int_{B_R} |\nabla u|^2 \omega\Big)$$

where $\omega(B_R) = \int_{B_R} \omega$ and A_R is either $\frac{1}{\omega(B_R)} \int_{B_R} u(x)\omega(x)\, dx$ or $⨍_{B_R} u(x)\, dx$.

Let us now turn to the mathematical description of the problem to be studied in the present article. Given a nonnegative boundary datum $f \in H^1(\Omega, \omega) \cap L^\infty(\Omega)$, we consider the minimization problem

$$\mathcal{J}(\omega, u, \Omega) = \int_\Omega \big(\omega(x)|\nabla u|^2 + \chi_{\{u>0\}}\big)\, dx \longrightarrow \min, \tag{2.2}$$

among functions $u \in H^1_f(\Omega, \omega) := f + H^1_0(\Omega, \omega)$, where χ_O stands for the characteristic function of the set O and ω is an A_2 weight.

We are mostly interested in local geometric properties of local minima near a singular free boundary point. Henceforth, as to properly carry out the analysis, it is convenient to localize the problem into the unit ball B_1 and assume the origin is a free boundary point, i.e., $0 \in \partial\{u > 0\}$. In addition we shall assume throughout the paper the following structural condition on the weight ω:

(H1) The weight ω belongs to A_2, and $0 < \tau_0 \leq \omega \leq +\infty$ a.e. in Ω.

It turns out that (H1) prescribes minimal condition under which one can develop an existence and regularity theory for corresponding singular cavitation problem.

We comment that we have chosen to develop the analysis of problems involving singular weights, rather than degenerate ones. In turn, we consider strictly positive weights that may blow-up along its singular set $\Lambda_\infty(\omega)$. We

could similarly treat bounded, degenerate weights of order $\gtrsim r^\sigma$, for some $\sigma < 2$.

A number of physical free boundary problems fall into the above mathematical set-up. Typical examples of weight functions we have in mind are

$$\omega(x) = |x'|^\alpha,$$

where $x = (x', x_{m+1}, \cdots, x_d)$, for $0 \leq m < d$ and $-m < \alpha \leq 0$. More generally, if $\mathcal{N}$ is an m-dimensional manifold properly embedded in $\mathbb{R}^d$, $0 \leq m < d$, we are interested in weights of the form

$$\omega(x) = \operatorname{dist}(x, \mathcal{N})^\alpha,$$

for some $-m < \alpha \leq 0$. This class of weight functions gives raise to the analysis of free boundary problems ruled by diffusion operators with an m-dimensional singular set. Anisotropic weights of the form

$$\omega(x) := \prod_{i=1}^{d} |x_i|^{\alpha_i},$$

also fall under the hypothesis considered in this work. In this case, condition (H1) is verified as long as $-1 < \alpha_i \leq 0$.

We are further interested in weights with possible distinct behaviors along sets of different dimensions, say

$$\omega(x) = |(x_1 \cdot x_2 \cdots x_m)|^{\alpha_1} \cdot |(x_{m+1} \cdots x_d)|^{\alpha_2},$$

where $-m < \alpha_1 \leq 0$, $-d + m < \alpha_2 \leq 0$. In this model, 0 is a singular free boundary point of degree $|\alpha_1 + \alpha_2|$, as we shall term later. If we label the cones $\mathcal{C}_1 := \{\Pi_{i=1}^m x_i = 0\}$ and $\mathcal{C}_2 := \left\{\Pi_{i=m+1}^d x_i = 0\right\}$, then any free boundary point in $\mathcal{C}_1 \setminus \mathcal{C}_2$ is singular of degree $|\alpha_1|$ and similarly, a point in $\partial\{u > 0\} \cap (\mathcal{C}_2 \setminus \mathcal{C}_1)$ is singular of degree $|\alpha_2|$. Any other free boundary point $z \in \partial\{u > 0\} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$ is a free boundary point of degree 0.

Notice that no continuity assumption has been required on ω. In particular, we are interested in the analysis of free boundary problems in possibly random media. Given a measurable function $0 < c_0 \leq \theta < c_0^{-1}$ defined on the unit sphere $\mathbb{S}^{d-1}$, $0 \leq m < d-2$ and $-m < \alpha \leq 0$, we can always define an α-homogeneous, A_2 weight function ω in $B_1 \setminus \{0\}$ as

$$\omega(x) := |x'|^\alpha \cdot \theta\left(\frac{x}{|x|}\right),$$

where, as above, $x = (x', x_{m+1}, \cdots, x_d)$. This is another important example of weights we have in mind as to motivate this work.

We conclude this section by setting a nomenclature convention: hereafter any constant that depends only upon dimension, d, and ω will be called universal.

3. Existence and local boundedness

In this preliminary section, we discuss about existence and local boundedness of minimizers. We also comment on the Euler-Lagrange equation satisfied by a local minimum. The proofs follow somewhat classical arguments, thus we simply sketch them here for the sake of completeness.

Fixed a nonnegative boundary datum $f \in H^1(\Omega,\omega)$, one can consider a minimizing sequence $v_j \in H^1_f(\Omega,\omega)$. Clearly,

$$C > \mathcal{J}(v_j) \geq \int_\Omega \omega(x)|\nabla v_j|^2 dx.$$

Thus, up to a subsequence, $v_j \to u$ weakly in $H^1(\Omega,\omega)$, for some function $u \in H^1_f(\Omega,\omega)$. By compactness embedding, $v_j \to u$ in $L^2(\Omega,\omega)$ and $v_j \to u$ a.e. in Ω. Passing to the limit as $j \to \infty$ (see [1, Section 1.3] to handle the term $\chi_{\{v_j>0\}}$), one concludes

$$\mathcal{J}(u) \leq \liminf_{j\to\infty} \mathcal{J}(v_j) = \min \mathcal{J}.$$

This shows the existence of a minimizer. To verify that u is nonnegative, one simply compares u with u^+ in the minimization problem. Also, if the boundary datum f is assumed to be in $H^1(\Omega,\omega) \cap L^\infty(\Omega)$, then for each $|t| \ll 1$, the function $u + t\,(\|f\|_\infty - u)^-$ competes with u in the minimization problem. Standard computations yield $\{u > \|f\|_\infty\}$ has measure zero, that is, $\{0 \leq u \leq \|f\|_\infty\}$ has total measure in Ω.

Now, if φ is a nonnegative test function in $C^\infty_0(\Omega)$, then $u+\varphi$ competes with u in the minimization problem. As $\{u+\varphi>0\} \supset \{u>0\}$, there holds,

$$-2\int_\Omega \omega(x)\nabla u\cdot\nabla\varphi dx = \mathcal{J}(u+\varphi) - \mathcal{J}(u) - \int_\Omega \left(\chi_{\{u+\varphi>0\}} - \chi_{\{u>0\}}\right) dx \geq 0.$$

This shows $\operatorname{div}(\omega(x)\nabla u)$ defines a nonnegative measure ν. If $B_\delta(x_0) \subset \{u>0\}$, then given a test function $\varphi \in C^\infty_0(B_\delta(x_0))$, for $0<|t|\ll 1$, $u+t\varphi$ is also positive in $B_\delta(x_0)$, thus we conclude $\nu \equiv 0$ in the interior of $\{u>0\}$.

Let us gather the information delivered above and state as a theorem for future reference.

Theorem 5. *Let ω be any A_2 weight and $f \in H^1(\Omega,\omega)$ nonnegative. Then there exists a minimizer $u \in H^1(\Omega,\omega)$ to problem (2.2) such that $u = f$ on $\partial\Omega$, in the trace sense. Furthermore, u is nonnegative, $\|u\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$, and there exists a non-negative Radon measure ν, supported on its free boundary $\partial\{u>0\}$, such that*

$$\operatorname{div}(\omega(x)\nabla u) = \nu$$

is verified in the distributional sense.

4. Compactness

In this section we discuss several compactness properties pertaining to the problem. In particular, we will show that bounded local minima are (universally) locally Hölder continuous, thus any family of bounded local minima is pre-compact in the uniform convergence topology.

We start with a lemma reminiscent of Caccioppoli inequality.

Lemma 6. *Let u be a bounded local minimizer of (2.2) in B_1, $y \in B_{1/2}$ and $r < 1/12$ be fixed. Denote by $m := \|u\|_{L^\infty(B_1)}$ and let h be the unique solution of*

$$\begin{cases} div(\omega(x)\nabla h) &= 0 \quad \text{in } B_{2r}(y) \\ h &= u \quad \text{on } \partial B_{2r}(y). \end{cases}$$

Then there exist universal constants $0 < \mu < 1$ and $C > 0$, such that

$$\int_{B_r(y)} \omega(x)|\nabla h|^2 dx \leq Cm^2 \cdot r^{d-2+2\mu}. \tag{4.1}$$

Proof. Following standard Caccioppoli type methods, one easily reaches

$$\int_{B_r(y)} \omega(x)|\nabla h|^2 \leq \frac{C}{r^2} \int_{B_{5r/4}(y)} \omega(x)\Big(u - \fint_{B_{5r/4}(y)} u\Big)^2.$$

By (H1), along with Reverse Hölder inequality for A_∞ weights, see for instance [14], we know $\omega \in L^q$ for some $q > 1$. Applying Hölder inequality

$$\begin{aligned} &\int_{B_{5r/4}(y)} \omega(x)\Big(u - \fint_{B_{5r/4}(y)} u\Big)^2 \\ &\quad \leq \Big(\int_{B_{5r/4}(y)} \omega^q(x)\Big)^{1/q} \Big(\int_{B_{5r/4}(y)} (u - \fint_{B_{5r/4}(y)} u)^{2q/q-1}\Big)^{(q-1)/q} \\ &\quad \leq C\Big(\int_{B_{5r/4}(y)} (u - \fint_{B_{5r/4}(y)} u)^{2q/q-1}\Big)^{(q-1)/q} \end{aligned}$$

Invoking Campanato theory in the last previous line, and denoting by μ the Hölder exponent of h (recall that h is Hölder continuous by Theorem 2), one has

$$\int_{B_{5r/4}(y)} \omega(x)\Big(u - \fint_{B_{5r/4}(y)} u\Big)^2 \leq Cm^2 r^{d+2\mu},$$

hence the result. □

Theorem 7 (Local $C^{0,\tau}$)**.** *Let u be local minimizer of (2.2) in B_1. There exist universal constants $0 < \tau \ll 1$ and $C > 1$ such that $u \in C^{0,\tau}(B_{1/2})$ and*

$$\|u\|_{C^{0,\tau}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Proof. Let u be a local minimizer of (2.2) in B_1, $y \in B_{1/2}$ and $r < 1/12$ be fixed. Let h be as in Lemma 6. Since h competes with u in the minimization problem, we can estimate

$$\int_{B_{2r}(y)} \omega(x) \left(|\nabla u|^2 - |\nabla h|^2 \right) dx \leq c_d r^d. \tag{4.2}$$

From the PDE satisfied by h, we have

$$\int_{B_{2r}(y)} \omega(x) \nabla h \cdot \nabla (h-u) dx = 0, \tag{4.3}$$

hence,

$$\int_{B_{2r}(y)} \omega(x) \left| \nabla u - \nabla h \right|^2 dx = \int_{B_{2r}(y)} \omega(x) \left(|\nabla u|^2 - |\nabla h|^2 \right) dx. \tag{4.4}$$

Combining (4.1), (4.2), (4.3), and (4.4), together with the triangle inequality, yields

$$\begin{array}{rcl} \|\nabla u\|_{L^2(B_r(y),\omega)} & \leq & \|\nabla (u-h)\|_{L^2(B_{2r}(y),\omega)} + \|\nabla h\|_{L^2(B_r(y),\omega)} \\ & \leq & c_d \cdot r^{\frac{d}{2}} + \sqrt{C} m \cdot r^{\frac{d}{2}-1+\mu}. \end{array} \tag{4.5}$$

By Hölder inequality and (H4), we can further estimate,

$$\begin{array}{rcl} \displaystyle\int_{B_r(y)} |\nabla u| dx & \leq & \displaystyle\left(\int_{B_r(y)} \omega^{-1}(x) dx \right)^{\frac{1}{2}} \cdot \left(\int_{B_r(y)} \omega(x) |\nabla u|^2 dx \right)^{\frac{1}{2}} \\ & \leq & \tau_0^{-1/2} r^{\frac{d}{2}} \cdot \left(c_d \cdot r^{\frac{d}{2}} + \sqrt{C} m \cdot r^{\frac{d}{2}-1+\mu} \right) \\ & \leq & C_4 r^{d-1+\mu}, \end{array} \tag{4.6}$$

for a constant $C_4 > 0$ depending only on universal parameters. Local Hölder continuity of u follows now by Morrey's Theorem, see [12]. □

Homogenization. We now turn our attention to limiting free boundary problems arising from homogenization. That is, hereafter we assume

(H2) Let 0 be a free boundary point. There exists $-d < \alpha \leq 0$ such that, as $\lambda \to 0+$,
$$\omega_\lambda(x) := \lambda^{|\alpha|} \omega(\lambda x),$$
converges locally in L^1 to a weight $\omega_0(x) \in A_2$.

The weight ω_0 is the homogenization limit, as it verifies $\omega_0(tx) = t^\alpha \omega_0(x)$, for all $t > 0$. Of course, should the original weight ω be homogeneous, then (H2) is immediately verified.

The examples discussed at the end of Section 2 are, essentially, α–homogeneous. In addition to classical homogenization procedures, condition (H2) further contemplates perturbations of those by terms $g(x) = \mathrm{o}(|x|^{-|\alpha|})$ as $|x| \to 0$, and products by bounded, positive functions $\theta(x) \in \mathrm{VMO}$. So typical, we have in mind weights of the form

$$\omega(x) = \theta(x) \omega_0(x) + g(x),$$

where ω_0 is α-homogeneous (as in the examples from Section 2), $\theta \in \mathrm{VMO}$ verifying $0 < \lambda_0 < \theta(x) < \lambda_0^{-1}$, and $|x|^{|\alpha|} g(x) \to 0$ as $|x| \to 0$.

Before we continue, let us make a comment on the scaling of the free boundary problem (2.2), which further substantiates (H2).

Remark 8. Let u be a local minimizer of

$$\mathcal{J}(\omega, u, \Omega) := \int_\Omega \omega(x)|\nabla u(x)|^2 + \chi_{\{u>0\}} dx.$$

Given $0 < \lambda < 1$, define

$$\beta = 1 - \frac{\alpha}{2}, \qquad u_\lambda(x) = \lambda^{-\beta} u(\lambda x), \quad \text{and} \quad \omega_\lambda(x) = \lambda^{|\alpha|}\omega(\lambda x),$$

then change of variables yields

$$\begin{aligned} \mathcal{J}(\omega, u, \Omega) &= \int_{\Omega/\lambda} \left[\omega(\lambda y)|\nabla u(\lambda y)|^2 + \chi_{\{u(\lambda y)>0\}}\right] \lambda^d dy \\ &= \int_{\Omega/\lambda} \left[\lambda^\alpha \omega_\lambda(y)\lambda^{2(-1+\beta)}|\nabla u_\lambda(y)|^2 + \chi_{\{u_\lambda(y)>0\}}\right] \lambda^d dy \\ &= \lambda^d \mathcal{J}(\omega_\lambda, u_\lambda, \Omega/\lambda). \end{aligned}$$

That is, u_λ is a local minimizer of functional $\mathcal{J}_\lambda$, ruled by an approximation of the homogenizing medium.

We start off with a weak compactness result.

Proposition 9 (Weak compactness in $W^{1,d+}$)**.** *Let λ_k be any sequence converging to zero and u_k local minima of $\mathcal{J}_k = \mathcal{J}(\omega_k, v, \Omega)$, with $\omega_k(x) := \lambda_k^{|\alpha|}\omega(\lambda_k x)$. There exists $\mu > 0$ such that $\{\nabla u_k\}_{k\geq 1}$ is locally weakly precompact in $L^{d+\mu}(\Omega)$.*

Proof. We revisit the proof of Theorem 7, for $\omega = \omega_{\lambda_k}$, $u = u_k$ as to reach the corresponding of (4.6), namely

$$\int_{B_r(y)} |\nabla u_k| dx \leq C r^{d-1+\tau}, \tag{4.7}$$

for positive constants C and $\tau > 0$ independent of k. Fixed $\Omega' \Subset \Omega$, by standard applications of Hölder inequality, we can bound $\{|\nabla u_k|\}_{k\in\mathbb{N}}$ in $L^{\frac{d}{1-\tau}}(\Omega')$, uniformly in k. We choose $\mu > 0$ such that $d(1-\tau)^{-1} = d + \mu$, and the result follows since $L^{d+\mu}$ is a reflexive space. □

In the sequel we indeed show that local minima of the functional $\mathcal{J}_\lambda$ converge to a minimizer of the singular homogenized problem ruled by ω_0.

Theorem 10. *Assume 0 is a free boundary point and that ω satisfies H1-H2. Let λ_k be any sequence converging to zero and u_k local minima of $\mathcal{J}_k = \mathcal{J}(\omega_k, v, \Omega)$, with $\omega_k(x) := \lambda_k^{|\alpha|}\omega(\lambda_k x)$. Then, up to a subsequence, u_k converges locally uniformly to a local minimum of $\mathcal{J}(\omega_0, v, \Omega)$.*

Proof. From Theorem 7 along with Proposition 9, up to a subsequence, $u_k \to u$ locally uniformly in Ω and locally weakly in $W^{1,d+\mu}$. By minimization, $\omega_k^{1/2}|\nabla u_k|$ is bounded in L^2. Also, from (H2), $\omega_k^{1/2} \to \omega_0^{1/2}$ strongly in L^2, and thus

$$\omega_k^{1/2}\nabla u_k \rightharpoonup \omega_0^{1/2}\nabla u,$$

locally in the weak topology of L^2. Lower weak semicontinuity of norm along with Fatou's Lemma imply, for any fixed subdomain $\Omega' \Subset \Omega$, there holds

$$\mathcal{J}(\omega_0, u_0, \Omega') \leq \liminf \mathcal{J}(\omega_k, u_k, \Omega'). \tag{4.8}$$

Now, fix a ball $B := B_r(x_0) \Subset \Omega$ and let φ be a smooth function in B satisfying $\varphi = u_0$ on ∂B. For $\varepsilon > 0$ small, consider

$$\psi(y) := \frac{|y - x_0| - r}{\varepsilon r}$$

and define $\varphi_k^\varepsilon \colon \Omega \to \mathbb{R}$ to be the linear interpolation between u_k and φ within $B_{(1+\varepsilon)r}(x_0)$, that is:

$$\varphi_k^\varepsilon(y) = \begin{cases} \varphi(y), & \text{if } |y - x_0| \leq r \\ u_k(y), & \text{if } |y - x_0| \geq (1+\varepsilon)r \\ (1 - \psi(y))u_0(y) + \psi(y)u_k(y), & \text{if } r < |y - x_0| < (1+\varepsilon)r. \end{cases} \tag{4.9}$$

Since u_k is a local minimizer of $\mathcal{J}_k$ over $\widetilde{B} := B_{(1+\varepsilon)r}(x_0)$, we have

$$\mathcal{J}(\omega_k, u_k, \widetilde{B}) \leq \mathcal{J}(\omega_k, \varphi_k^\varepsilon, \widetilde{B}),$$

that is

$$\mathcal{J}(\omega_k, u_k, \widetilde{B}) \leq \mathcal{J}(\omega_k, \varphi, B) + \mathcal{J}(\omega_k, \varphi_k^\varepsilon, \widetilde{B} \setminus B). \tag{4.10}$$

On the transition region, $r < |y - x_0| < (1+\varepsilon)r$, taking into account (4.9), we estimate for a.e. y,

$$|\nabla \varphi_k^\varepsilon(y)| \leq |\nabla u_0(y)| + |\nabla u_k(y)| + \frac{C}{\varepsilon}|u_k(y) - u_0(y)|. \tag{4.11}$$

Consequently, from (4.9), (4.10) and (4.11), taking into account that $|\widetilde{B} \setminus B| \sim \varepsilon$ and that $\omega_k(y)|\nabla u_k|^2$ is bounded in $L^{1+\delta}(\widetilde{B})$, for some $\delta > 0$, we obtain, for some $0 < \beta \ll 1$,

$$\mathcal{J}(\omega_k, u_k, \widetilde{B}) \leq \mathcal{J}(\omega_k, \varphi, B) + C\varepsilon^\beta + \varepsilon^{-2+\beta}\mathrm{o}(1), \tag{4.12}$$

as $k \to \infty$. Finally, letting $k \to \infty$, taking into account (4.8), we obtain

$$\mathcal{J}(\omega_0, u_0, B) \leq \mathcal{J}(\omega_0, \varphi, B) + C\varepsilon^\beta.$$

Since φ and $\varepsilon > 0$ were taken arbitrary, we conclude the proof of the Theorem. □

5. $C^{1+\gamma}$ REGULARITY AT THE FREE BOUNDARY

In the previous section we showed minimizers are locally Hölder continuous, which, in particular, yields a rough oscillation control of u near the free boundary. The heart of the matter, though, is to describe the precise geometric behavior of a local minimizer at free boundary points. Roughly speaking, solutions to singular free boundary problems should adjust their vanishing rate based upon the singularity of the medium. More precisely, in this section we shall work under the following assumption:

(H3) For some $-d < \alpha \leq 0$, there exists $L > 0$ such that

$$\fint_{B_r(0)} \omega(x)dx \leq Lr^{\alpha}.$$

Recall, for notation convenience, we have localized 0 as a free boundary point, thus, condition (H3) prescribes, in a way, the maximum blow-up rate of ω at a free boundary. In this case, we will say 0 is a singular free boundary point of degree $|\alpha|$. In particular, regular free boundary points from classical study of cavitation problems, that is such that $\omega(0)$ is finite, represent (singular) free boundary points of degree zero.

While there is no hope to obtain an estimate superior than Hölder continuity of local minima, at any other point, in this section we show that u behaves as a $C^{1+\gamma}$ function around a singular free boundary point. As usual, this implies higher order differentiability of u at free boundary points. In particular, if $1 + \gamma = N$ is an integer, then $u \in C^{N-1,1}$, in the sense it is $N - 1$ differentiable, and the Nth-Newtonian quotient remains bounded. When $1 + \gamma = N + \theta$, for $0 < \theta < 1$, then solutions are $C^{N,\theta}$ regular at free boundary points.

This is the contents of the following key result:

Theorem 11. *Assume* $(H1) - (H2) - (H3)$ *and let* u *be a local minimizer of* (2.2) *so that 0 is a free boundary point. Then*

$$\sup_{B_r} u \leq Cr^{1+\frac{|\alpha|}{2}},$$

for a universal constant $C > 0$*, independent of* u*.*

The proof of Theorem 11 will be based on a geometric flatness improvement technic. For that, we need a Lemma:

Lemma 12. *Let* $u \in H^1(B_1, \omega)$ *verify* $0 \leq u \leq 1$ *in* B_1*, with* $u(0) = 0$*. Given* $\delta > 0$*, there exists* $\varepsilon > 0$ *depending only on* δ *and universal constants, such that if* u *is a local minimizer of*

$$J_{\varepsilon}(u) := \int_{B_1} \omega_{\lambda}(x)|\nabla u(x)|^2 + \varepsilon\chi_{\{u>0\}}dx, \tag{5.1}$$

for any $0 < \lambda < 1$, then, in $B_{1/2}$, u is at most δ, that is,

$$\sup_{B_{1/2}} u \leq \delta. \tag{5.2}$$

Proof. Suppose, for the sake of contradiction, that the thesis of the Lemma does not hold. This means for some $\delta_0 > 0$, one can find a sequence of functions $u_j \in H^1(B_1, \omega_j)$ satisfying:

(1) $0 \leq u_j \leq 1$ in B_1;
(2) $u_j(0) = 0$;
(3) u_j is a local minimizer of $J_j(u) := \int_{B_1} \omega_{\lambda_j}(x)|\nabla u(x)|^2 + \frac{1}{j}\chi_{\{u>0\}}dx$;

however,

$$\sup_{B_{1/2}} u_j \geq \delta_0 \quad \forall j \in \mathbb{N}. \tag{5.3}$$

From compactness estimates proven in Section 4, up to a subsequence, u_j converges locally uniformly to a function a nonnegative function u_∞, with $u_\infty(0) = 0$. Also, from similar analysis as in Theorem 10, u_∞ is a local minimizer of

$$\mathcal{J}_0(v) := \int_{B_1} \omega_0(x)|\nabla v(x)|^2 dx.$$

Applying maximum principle (available for minimizers of the functional $\mathcal{J}_0$), we conclude $u_\infty \equiv 0$. That is, we have proven u_j converges locally uniformly to 0. Therefore, for $j \gg 1$, we reach a contradiction with (5.3). The proof of Lemma 12 is complete. □

Proof of Theorem 11. We will make few (universal) decisions. Initially we set

$$\delta_\star := 2^{\frac{\alpha}{2}-1}.$$

Lemma 7 assumes the existence of a positive (universal) constant $\varepsilon_0 > 0$ such that any normalized minimizer of J_{ε_0}, as defined in (5.1) verifies (5.2), for $\delta_\star$. Define

$$\varrho_0 := \sqrt[2-\alpha]{\varepsilon_0}, \quad \text{and} \quad \tilde{u}(x) := u(\varrho_0 x).$$

By remark 8, $\tilde{u}$ is a local minimizer of $\mathcal{J}_{\varepsilon_0}$, and hence, from Lemma 7, there holds:

$$\sup_{B_{1/2}} \tilde{u}(x) \leq 2^{\frac{\alpha}{2}-1}. \tag{5.4}$$

Next, by induction, we iterate the previous argument as to show

$$\sup_{B_{2^{-k}}} \tilde{u}(x) \leq 2^{k\left(\frac{\alpha}{2}-1\right)}. \tag{5.5}$$

Estimate (5.4) gives the first step of induction, $k = 1$. Now, suppose we have verified (5.5) for $k = 1, 2, \cdots p$. Define

$$\tilde{v}(x) := 2^{p\left(1-\frac{\alpha}{2}\right)} \cdot \tilde{u}\left(2^{-p}x\right).$$

It follows from induction hypothesis that $0 \leq v \leq 1$. Also, from scaling, we check that $\tilde{v}$ is too a minimizer of $\mathcal{J}_{\varepsilon_0}$. Applying Lemma 7 to $\tilde{v}$ we conclude

$$\sup_{B_{1/2}} \tilde{v} \leq 2^{\frac{\alpha}{2}-1},$$

which, in terms of $\tilde{u}$, gives the precisely the $(p+1)$ step of induction.

Now, given a (universally small) radius $r > 0$, choose $k \in \mathbb{N}$, such that

$$2^{-(k+1)} < \varrho_0^{-1} r \leq 2^{-k}.$$

We can then estimate

$$\begin{aligned}
\sup_{B_r} u &= \sup_{B_{\varrho_0^{-1} r}} \tilde{u} \\
&\leq \sup_{B_{2^{-k}}} \tilde{u} \\
&\leq 2^{k\left(\frac{\alpha}{2}-1\right)} \\
&\leq \left(\frac{2}{\varrho_0}\right)^{1-\frac{\alpha}{2}} \cdot r^{1+\frac{|\alpha|}{2}} \\
&= C r^{1+\frac{|\alpha|}{2}},
\end{aligned}$$

for $C > 1$ universal, as required. □

6. Nondegeneracy and weak geometry

In the previous section we show local minima are $C^{1+\gamma}$ smooth along the singular free boundary, for $\gamma = \frac{|\alpha|}{2}$. In this section we prove a competing inequality which assures that such a geometric decay is sharp. Naturally, such an estimate requires a corresponding lower bound for the degree of singularity of the free boundary, namely:

(H4) For some $-d < \alpha \leq 0$, there exists $\tau_\star > 0$ such that

$$\tau_\star r^\alpha \leq \fint_{B_r(0)} \omega(x) dx,$$

for all $0 < r \ll 1$.

Again, we recall 0 is the localized free boundary point we are analyzing, thus condition (H4) conveys the idea that the origin is a singular free boundary point of degree *at least* $|\alpha|$. Note that (H4) yields

$$\inf_{r>0} \fint_{B_1} \omega_r(x) dx \geq \tau_\star,$$

where, as before, $\omega_r(x) = r^{|\alpha|}\omega(rx)$.

Theorem 13. *Let u be a minimizer of f (2.2), 0 a free boundary point and assume (H4). Then*

$$\sup_{B_r} u(x) \geq 2 \cdot \sqrt{\tau_\star \frac{d^d}{(d+2)^{d+2}}} \cdot r^{1+\frac{|\alpha|}{2}}, \quad \forall 0 < r < 1,$$

where d is dimension.

Proof. The idea of the proof is to cut a family of concentric holes on the graph of u, compare the resulting functions with u in terms of the minimization problem $\mathcal{J}$, and finally optimize the cutting-hole parameter; here are the details. Let $0 < r < 1$ be a fixed radius and define $v_r \colon B_1 \to \mathbb{R}$ as

$$v_r(y) := r^{\frac{\alpha}{2}-1} u(ry).$$

The goal is to show that

$$\mathcal{S}_r := \sup_{B_1} v_r$$

is uniformly bounded from below, independently of r. From Remark 8, v_r is a local minimizer of $\mathcal{J}_r$ over B_1. Next, let us choose $0 < \sigma < 1$, $0 < \varepsilon \ll 1$ and craft a smooth, radially symmetric function $\varphi \colon B_1 \to \mathbb{R}$ satisfying:

$$0 \le \varphi \le 1, \quad \varphi \equiv 0 \text{ in } B_\sigma, \quad \varphi = 1 \text{ on } \partial B_1, \quad |\nabla \varphi| \le (1+\varepsilon)(1-\sigma)^{-1}. \tag{6.1}$$

In the sequel, let us consider the test function $\xi \colon B_1 \to \mathbb{R}$ given by

$$\xi(x) := \min\left\{v(x), (1+\varepsilon)\mathcal{S}_r \cdot \varphi(x)\right\}.$$

By construction, ξ competes with v in the minimization problem $\mathcal{J}_r$, and thus

$$\int_{B_1} \left(\omega_r(x)|\nabla \xi|^2 + \chi_{\{\xi>0\}}\right) dx \ge \int_{B_1} \left(\omega_r(x)|\nabla v|^2 + \chi_{\{v>0\}}\right) dx. \tag{6.2}$$

We can rewrite (6.2) as an inequality of the form $\mathcal{A} \ge \mathcal{B}$, for

$$\begin{aligned} \mathcal{A} &:= \int_{B_1} \omega_r(x) \left(|\nabla \xi|^2 - |\nabla v|^2\right) dx, \\ \mathcal{B} &:= \int_{B_1} \left(\chi_{\{v>0\}} - \chi_{\{\xi>0\}}\right) dx. \end{aligned} \tag{6.3}$$

As to estimate $\mathcal{B}$ from below we note that $\{v > 0\} \supset \{\xi > 0\}$, thus

$$\mathcal{B} = \int_{B_1} \chi_{\{\xi=0\}} dx \ge \mathcal{L}^n(B_\sigma). \tag{6.4}$$

Next we estimate $\mathcal{A}$ from above. For that, let us define

$$\Pi := \left\{x \in B_1 \;\middle|\; (1+\varepsilon)\mathcal{S}_r \cdot \varphi(x) < v(x)\right\}$$

and compute

$$\begin{aligned}\int_{B_1} \omega_r(y)\left(|\nabla\xi|^2-|\nabla v|^2\right)dx &= \int_{\Pi} \omega_r(x)\left(|\nabla\xi|^2-|\nabla v|^2\right)dx\\ &\le (1+\varepsilon)^2\mathcal{S}_r^2\int_{B_1}\omega_r(x)\,|\nabla\varphi|^2dx\\ &\le (1+\varepsilon)^4(1-\sigma)^{-2}\cdot\left(\int_{B_1}\omega_r(x)dx\right)\cdot\mathcal{S}_r^2.\end{aligned} \tag{6.5}$$

From the relation $\mathcal{A}\ge\mathcal{B}$, we obtain

$$\mathcal{S}_r^2 \ge \frac{\tau_\star}{(1+\varepsilon)^4}\sigma^d(1-\sigma)^2.$$

Letting $\varepsilon\to 0$ and selecting $\sigma=\dfrac{d}{d+2}$ yields the optimal lower bound. The proof of Theorem 13 is complete. □

A important consequence of Theorem 11 and Theorem 13 combined is that, around free boundary points of same homogeneity α, a local minimum detaches from its coincidence set, $\{u=0\}$, precisely as $\text{dist}^{1+\frac{|\alpha|}{2}}$.

Corollary 14. *Let u be a minimizer of (2.2) and assume all free boundary points in $B_{1/2}$ satisfy conditions (H1)—(H4). Then there exists a universal constant $C>1$ such that*

$$C^{-1}\text{dist}(x,\partial\{u>0\})^{1+\frac{|\alpha|}{2}} \le u(x) \le C\cdot\text{dist}(x,\partial\{u>0\})^{1+\frac{|\alpha|}{2}}$$

for any $x\in B_{1/4}\cap\{u>0\}$.

By standard arguments we then conclude that near free boundary points of same homogeneity, the set of positivity $\{u>0\}$ has uniform positive density:

Corollary 15. *Let u be a minimizer of (2.2) and assume all free boundary points in $B_{1/2}$ satisfy (H1)—(H4) Then, for any $0<r<\frac{1}{4}$,*

$$\frac{\mathcal{L}^n\left(B_r\cap\{u>0\}\right)}{\mathcal{L}^n\left(B_r\right)}\ge\mu,$$

where $0<\mu<1$ is a universal constant, independent of r.

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Sorbonne Universités, UPMC Univ Paris 06, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Univ Paris Diderot, Sorbonne Paris Cité, F-75005, Paris, France
E-mail address: jimmy.lamboley@imj-prg.fr

Johns Hopkins University, Krieger Hall, N. Charles St., Baltimore, MD 21218
E-mail address: sire@math.jhu.edu

University of Central Florida, 4393 Andromeda Loop N, Orlando, FL 32816
E-mail address: eduardo.teixeira@ucf.edu