Vortex solutions for the compressible Navier-Stokes equations with general viscosity coefficients in 1D: regularizing effects or not on the density

Boris Haspot

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Abstract

We consider Navier-Stokes equations for compressible viscous fluids in the one-dimensional case with general viscosity coefficients. We prove the existence of global weak solution when the initial momentum $\rho_0 u_0$ belongs to the set of the finite measure $\mathcal{M}(\mathbb{R})$ and when the initial density ρ_0 is in the set of bounded variation functions $BV(\mathbb{R})$. In particular it includes at the same time the case of initial momentum which are Dirac masses and initial density which admit shocks. We can observe in particular that this type of initial data have infinite energy. Furthermore we show that if the coupling between the density and the velocity is sufficiently strong then the initial density which admits initially shocks is instantaneously regularizing and becomes continuous. This coupling is expressed via the regularity of the so called effective velocity $v = u + \frac{\mu(\rho)}{\rho^2}\partial_x \rho$ (see [20, 18, 3]) with $\mu(\rho)$ the viscosity coefficient. Inversely if the coupling between the initial density and the initial velocity is to weak (it means $\rho_0 v_0 \in \mathcal{M}(\mathbb{R})$) then we prove the existence of weak energy in finite time but the density remains a prirori discontinuous on the time interval of existence.

1 Introduction

In this paper we wish to investigate the existence of global weak solutions of the following Navier-Stokes equations for compressible isentropic flow:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0. \end{cases} \tag{1.1}$$

with possibly degenerate viscosity coefficient $\mu(\rho) \geq 0$. Throughout the paper, we will assume that the pressure $P(\rho)$ verifies a γ type law $P(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma > 1$ and that the viscosity coefficient can be written under the form $\mu(\rho) = \mu\rho^\alpha$ with $\mu > 0$ and $\alpha \geq 0$. A large amount of literature is dedicated to the study of the compressible Navier-Stokes equations with constant viscosity case, however physically the viscosity of a gas depends on the temperature and on the density (in the isentropic case). Let us mention the case of the Chapman-Enskog viscosity law (see [5]) or the case of monoatomic gas

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($\gamma = \frac{5}{3}$) where $\mu(\rho) = \rho^{\frac{1}{3}}$. More generally, the viscosity coefficient $\mu(\rho)$ is expected to vanish as a power of the density ρ on the vacuum. We emphasize that the case $\alpha = 1$ corresponds to the so called viscous shallow water system. This system with friction has been derived by Gerbeau and Perthame in [10] from the Navier-Stokes system with a free moving boundary in the shallow water regime at the first order (it corresponds to a small shallowness parameter). This derivation relies on the hydrostatic approximation where the authors follow the role of viscosity and friction on the bottom.

Following [20, 18, 3, 16, 15] we can rewrite the previous system by using the so called effective velocity $v = u + \partial_x \varphi(\rho)$ with $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$:

$$\begin{cases} \partial_t \rho - \partial_x(\dfrac{\mu(\rho)}{\rho}\partial_x \rho) + \partial_x(\rho v) = 0, \\ \rho \partial_t v + \rho u \partial_x v + a\gamma \dfrac{\rho^{\gamma+1}}{\mu(\rho)} v = a\gamma \dfrac{\rho^{\gamma+1}}{\mu(\rho)} u. \end{cases} \tag{1.2}$$

It is important to note that this change of unknown is true for any viscosity coefficients (at the opposite in dimension $N \geq 2$ such change of unknown involves an algebraic relation between the coefficients μ and λ which precludes in particular the case of constant viscosity coefficients (see [20, 3] for more details)).
The questions addressed here are motivated by the fact that discontinuous solutions are fundamental both in the physical theory of non-equilibrium thermodynamics as well as in the mathematical study of inviscid models for compressible flow. It is natural, therefore, to express a rigorous theory which accommodates these discontinuities or shocks in the theory of hyperbolic systems. In order to obtain such results, a key point is to work with initial data with minimal regularity assumption as much as it is possible to do. The results of this paper effectively gives a first direction to the existence of global weak solution with initial density admitting shocks. More precisely we should deal with initial density ρ_0 in $BV(\mathbb{R})$ with $0 < c \leq \rho_0 \leq M < +\infty$ and initial momentum $\rho_0 u_0, \rho_0 v_0$ in $\mathcal{M}(\mathbb{R})$ the set of finite measure (including in particular Dirac masses). If $\mu_1 \in \mathcal{M}(\mathbb{R})$, the total variation of μ_1 is defined by:

$$\|\mu_1\|_{\mathcal{M}(\mathbb{R})} = \sup\{\int_{\mathbb{R}} \varphi d\mu_1 \ / \ \varphi \in C_0(\mathbb{R}),\ \|\varphi\|_{L^\infty(\mathbb{R})} \leq 1 \ \}$$

where $C_0(\mathbb{R})$ is the set of all real-valued functions on $\mathbb{R}$ vanishing at infinity. In addition we say that ρ_0 is in $BV(\mathbb{R})$ if ρ_0 is measurable, locally integrable, continuous on the left and such that the derivative in the sense of the distribution is a finite measure. An other way to describe the set $BV(\mathbb{R})$ is to consider the function ρ_0 with finite total variation with:

$$TV(\rho_0) = \sup\{\sum_{j=1}^{N} |\rho_0(x_j) - \rho_0(x_{j+1})| \ N \in \mathbb{N}^*,\ -\infty < x_0 < \cdots < x_N < +\infty\}.$$

The set $BV(\mathbb{R})$ of these functions is a Banach space when we equip it with the norm $TV(\rho_0) + |\rho_0(y)|$ with $y \in \mathbb{R}$.
We will observe that if in addition we assume that v_0 and $\rho_0 - \bar{\rho}$ belong to $L^2(\mathbb{R})$ then

the density is instantaneously regularizing inasmuch as the density becomes continuous. It is interesting to compare this kind of phenomenon with what is known on the behavior of the solutions of hyperbolic systems (typically the compressible isentropic Euler system which corresponds to $\mu(\rho) = 0$). In particular when the system is strictly hyperbolic (for a $N \times N$ system with $x \in \mathbb{R}$) and genuinely non linear there exists a global weak solution with small initial data in $(BV(\mathbb{R}))^N$ (see the famous paper of Glimm [12] and [6] for some extension in the case of a 2×2 system where the smallness assumption is weaken due to the existence of Riemann invariants). Concerning the uniqueness of such solutions, we refer to the works of Bressan et al (see [4] for a excellent review on all these results using in particular another way than Glimm to construct the global weak solution, the so-called front tracking methods which generate a contractive L^1 semi-group). It is important to note that for such hyperbolic systems there is a priori no regularizing effects on the unknown in the sense that the solution admits shocks all along the time (however a solution with a L^∞ initial data can become $BV(\mathbb{R})$ instantaneously, we refer to the scalar case with strictly convex flux which was in particular studied by Oleinik [36]). This explains why in particular the $BV(\mathbb{R})$ space is suitable for the study of the one dimensional case since this space contains functions with shocks. The simpler case is the solution of the Riemann problem due to Lax [28]. In particular it is known that for regular initial data, there is creation of shocks in finite time (except if we can use a characteristic methods, typically in the case of 2×2 systems where we have Riemann invariants). It is then remarkable to observe that in the case of compressible Navier Stokes systems the density can be instantaneously regularized even if the density seems to be governed by a transport equation in (1.1).

Before stating our results we would like to recall briefly the existing results on the existence of global weak solution in one dimension for compressible Navier-Stokes system. We would like to emphasize that all these results have been achieved in a framework with finite energy initial data, indeed systematically u_0 belongs in particular to $L^2(\mathbb{R})$. It will be not the case of our result since $\rho_0 u_0$ is in our case only a finite measure or a vortex. The existence of global solution for initial data which are finite measure is an important problem in mathematics and has been solved for other equations. We recall in particular that for the incompressible Navier-Stokes equations it exists a theory on the existence of global strong solution in dimension $N = 2$ for initial vorticity which belongs to the set of finite measure (see for example [11, 9]). It exists also a theory for the existence of global weal solution for the incompressible Euler equations in dimension $N = 2$ when the initial vorticity is a measure (see [7, 31]).

In the case of compressible Navier-Stokes equations in one dimension, the existence of global weak solutions was first obtained by Kazhikhov and Shelukin [26] for smooth enough data close to the equilibrium (in particular the initial density ρ_0 is bounded away from zero) with constant viscosity coefficient. The case of initial density admitting shocks has been treated by Hoff [22] where the initial density belongs to $BV(\mathbb{R})$, in addition the author needs smallness assumption on the initial data. In [23], Hoff proved the existence of global weak energy solution for constant viscosity coefficients provided that ρ_0 is only bounded in L^∞ norm and is far away from zero. In this work there is no smallness restriction on the initial data. In passing let us mention that the existence of global weak solution for constant viscosity coefficients in any dimension $N \geq 2$ has been proved for the first time by Lions in [30] and the result has been later refined by Feireisl et al [8].

The existence of global strong solution in one dimension with large initial data for initial density far away from the vacuum has been proved for the first time by Kanel [25] (see also [21]) in the case of constant viscosity coefficients.
The study of the compressible Navier Stokes equations with degenerate viscosity coefficients is more recent and has been in particular motivated by the introduction of a new entropy (see [2]) which provides new regularity estimate on the gradient of the density. In [24] Jiu and Xin proved the existence of global weak energy solution in one dimension when $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > \frac{1}{2}$ and with large initial energy data. In [33], Mellet and Vasseur showed the existence of global strong solution when $0 \leq \alpha < \frac{1}{2}$ for large initial data provided that the initial data $(\rho_0 - \bar{\rho}, u_0)$ are in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with $0 < c \leq \rho_0 \leq M < +\infty$. In [18] we extends this result to the case $\frac{1}{2} < \alpha \leq 1$ including in particular the case of the so called viscous shallow water system ($\alpha = 1$, see [10]).
In passing we would like to give few words on the existence of global weak solution for degenerate viscosity coefficients when N the dimension verifies $N \geq 2$. Mellet and Vasseur proved in [32] the stability of the global weak solution for compressible Navier-Stokes equation with viscosity coefficient verifying the so called BD entropy (see [2]) in dimension $N = 2, 3$ (we refer also to [2, 3] when we add friction terms). Let us mention in particular that the case $\mu(\rho) = \mu\rho$ with $\mu > 0$ and $\lambda(\rho) = 0$ verifies the algebraic relation related to the new entropy discovered in [2], it corresponds here to the so called viscous shallow water system. For $N = 2, 3$ the important problem of the existence of global weak solutions has been recently resolved independently by Vasseur and Yu [39, 40] and Li and Xin in [29].

2 Main result

We are going now to state the main results of this paper. Before we would like to recall a very interesting result due to Hoff which can indicate that when the effective velocity v is not sufficiently regular then there is no regularizing effects on the density inasmuch as the density does not become instantaneously continuous.

Theorem 2.1 (Hoff [23]) *Then the initial value problem (1.1) has a global weak solution (ρ, u) for which*

$$\begin{cases} \rho - \bar{\rho},\ \rho u \in C([0, +\infty), H^{-1}(\mathbb{R})), \\ u \in C((0, +\infty), L^2(\mathbb{R})), \\ u(t, \cdot), \mu\partial_x u(t, \cdot) - P(\rho(t, \cdot)) + P(\bar{\rho}) \in H^1(\mathbb{R}),\ t > 0, \\ \partial_t u(t, \cdot),\ \dot{u}(t, \cdot) \in L^2(\mathbb{R}),\ t > 0, \end{cases} \tag{2.3}$$

where $\dot{u}$ is the convective derivative $\dot{u} = \partial_t u + u\partial_x u$. Additionally, given $T > 0$, there is a positive constant $C(T)$ depending on T, μ, P and on upper bounds for $\|\rho_0 - \bar{\rho}\|_{L^2(\mathbb{R})}$, $\|u_0\|_{L^2(\mathbb{R})}$, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$ such that if $\sigma(t) = \min(1, t)$, then:

$$C(T)^{-1} \leq \rho(t, \cdot) \leq C(T) \ \ a.e, \tag{2.4}$$

$$\begin{aligned} &\sup_{0 \leq t \leq T} [\|\rho(t, \cdot) - \bar{\rho}\|_{L^2(\mathbb{R})} + \|u(t)\|_{L^2(\mathbb{R})} + \sigma(t)^{\frac{1}{2}} \\ &+ \sigma(t)\big(\|\dot{u}(t)\|_{L^2} + \|\mu\partial_x u(t, \cdot) - P(\rho)(t, \cdot) + P(\bar{\rho})\|_{L^2(\mathbb{R})}\big) \leq C(t), \end{aligned} \tag{2.5}$$

$$\int_0^T \big(\|\partial_x u(s,\cdot)\|_{L^2}^2 + \sigma(s)\|\dot{u}(s,\cdot)\|_{L^2}^2 + \sigma(s)\|\mu\partial_x u(s,\cdot) - P(\rho)(s,\cdot) + P(\bar{\rho})\|_{L^2(\mathbb{R})} + \sigma^2(s)\|\partial_x \dot{u}(s,\cdot)\|_{L^2}^2]ds \leq C(T), \tag{2.6}$$

and for $0 < \tau < T$

$$\sigma(\tau)^{\frac{1}{4}}\|u\|_{L^\infty(\mathbb{R}\times|\tau,T])} + \sigma(\tau)^{\frac{1}{2}}\langle u\rangle^{\frac{1}{2},\frac{1}{4}}_{[\tau,T]\times\mathbb{R}} \leq C(T). \tag{2.7}$$

where $\langle u\rangle^{\frac{1}{2},\frac{1}{4}}_{[\tau,T]\times\mathbb{R}}$ *is the usaul Hölder norm*

$$\sup\{\frac{|u(t,x)-u(s,y)|}{|x-y|^{\frac{1}{2}}+|t-s|^{\frac{1}{4}}}, x,y\in\mathbb{R},\ t,s\in[\tau,T],\ (t,x)\neq(s,y)\}.$$

Remark 1 *In the case of the constant viscosity coefficient* $\mu > 0$*, the effective initial velocity has the form* $v_0 = u_0 - \mu\partial_x(\frac{1}{\rho_0})$ *and under the assumptions of the theorem 2 we have* $v_0 \in L^2(\mathbb{R}) + W^{-1,\infty}(\mathbb{R})$*. From (1.2), we observe that the effective velocity* v *verifies a damped transport equation which prevents a priori any regularizing effects. It implies that* $\partial_x(\frac{1}{\rho})$ *is at the best in* $L^\infty(W^{-1,\infty} + L^2)$*, in other words it is not sufficient to show via Sobolev embedding that the density* ρ *becomes continuous. In the next theorems, it will be different since* v_0 *will be in* $L^2(\mathbb{R})$*.*

Remark 2 *In the previous theorem Hoff proved that the velocity* $\partial_x u - P(\rho) + P(1)$ *becomes regular, however to be able to prove that* $P(\rho)$ *becomes regular it is necessary to obtain additional information on* $\partial_x u$*. It explains also why we have no regularizing effects. The coupling with the effective velocity* v *is strongly since it provides an information on* $\partial_x(\frac{1}{\rho})$*.*

We define now the initial momentum which formally verifies $m_0 = \rho_0 u_0$ (m_0 has to be considered as an unknown). We define now the second momentum $m_0^1 = m_0 + \partial_x\varphi_1(\rho_0)$ (ρ_0 corresponds to the initial density and we have $\varphi_1(\rho_0) = \frac{\mu}{\alpha}\rho_0^\alpha$).
Let us give a definition of a global weak solution for initial data verifying when $\bar{\rho} > 0$:

$$\begin{cases} \rho_0 \geq 0, m_0 = 0 \ \text{ on } \ \{x\in\mathbb{R};\ \rho_0(x) = 0\}, \\ 0 < c \leq \rho_0 \leq M < +\infty \\ (\rho_0 - \bar{\rho}) \in L^2(\mathbb{R}),\ \varphi_1(\rho_0) \in BV(\mathbb{R}) \\ m_0^1 \in \mathcal{M}(\mathbb{R}), \dfrac{m_0^1}{\rho_0} \in L^2(\mathbb{R}). \end{cases} \tag{2.8}$$

In the sequel we will define m, m^1, m^2 as follows:

$$m = \rho u,\ m^1 = \rho u + \partial_x\varphi_1(\rho) \ \text{ and } \ m^2 = \sqrt{\rho}u + \partial_x\varphi_2(\rho),$$

with $\varphi_2(\rho) = \frac{\mu}{\alpha-\frac{1}{2}}\partial_x\rho^{\alpha-\frac{1}{2}}$ with $\alpha \neq \frac{1}{2}$. We refer to [30] for the definition of the Orlicz space $L_2^\gamma(\mathbb{R})$.

Definition 2.1 *A pair* (ρ, u) *is said to be a weak solution to (1.1) provided that:*

- $\rho \geq 0$ *a. e and for* $\bar{\rho} > 0$ *and any* $T > 0$, $1 \leq p < +\infty$ *:*

$$\begin{cases} \rho - \bar{\rho} \in L^\infty([0,T], L^\gamma_2(\mathbb{R})), \\ \rho \in C([0,+\infty[, W^{-\epsilon,p}_{loc}(\mathbb{R})),\ \rho \in L^\infty([0,T], L^\infty(\mathbb{R})) \\ \rho u \in L^\infty((0,T), L^1(\mathbb{R})), \sqrt{\rho}u \in L^{2+\epsilon}_{loc}([0,T]\times\mathbb{R}) \\ \partial_x \rho^{\frac{1}{2}(\gamma+\alpha-1)} \in L^2([0,T], L^2(\mathbb{R})), \partial_x(\rho^{\alpha-\frac{1}{2}}) \in L^2(([0,T], L^2(\mathbb{R})) \\ m^2 \in L^\infty([0,T], L^2(\mathbb{R})) \\ \partial_x \rho^{\alpha-\frac{1}{2}} \in L^\infty([1,+\infty[, L^2(\mathbb{R})) \\ m^1 \in L^\infty((0,T), \mathcal{M}(\mathbb{R})). \end{cases}$$

- $(\rho, \sqrt{\rho}u)$ *satisfies in the sense of distributions:*

$$\begin{cases} \partial_t \rho + \partial_x(\sqrt{\rho}\sqrt{\rho}u) = 0 \\ \rho(0,x) = \rho_0(x). \end{cases}$$

- *For any* $\varphi \in C^\infty_0([0,T[\times\mathbb{R})$ *and any* $T > 0$*:*

$$\begin{aligned} &\int_{\mathbb{R}} m_0(x)\varphi(0,x)dx + \int_0^T \int_{\mathbb{R}} [\sqrt{\rho}(\sqrt{\rho}u)\partial_t\varphi + ((\sqrt{\rho}u)^2 + \rho^\gamma)\partial_x\varphi]dtdx \\ &+ \langle \mu(\rho)\partial_x u, \partial_x\varphi\rangle = 0, \end{aligned} \tag{2.9}$$

where the diffusion term makes sense when written as:

$$\langle \mu(\rho)\partial_x u, \partial_x\varphi\rangle = \int_0^T \int_{\mathbb{R}} \partial_x(\varphi_2(\rho))\sqrt{\rho}u\partial_x\varphi dxdt + \int_0^T \int_{\mathbb{R}} \frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u)\partial_{xx}\varphi dxdt, \tag{2.10}$$

with $\varphi_2'(\rho) = \frac{\mu'(\rho)}{\sqrt{\rho}}$.

We obtain the main following theorems.

Theorem 2.2 *Assume that* $\mu(\rho) = \mu\rho^\alpha$ *with* $\mu, \alpha > 0$, $\alpha \neq \frac{1}{2}$ *and* $P(\rho) = a\rho^\gamma$ *with* $\gamma > 1$, $a > 1$, $\gamma \geq \alpha$ *and* $\gamma \geq 2\alpha - 1$ *if* $\alpha > \frac{1}{2}$*. The initial data satisfy for* $\bar{\rho} > 0$*:*

$$\begin{cases} \varphi_1(\rho_0) \in BV(\mathbb{R}),\ (\rho_0 - \bar{\rho}) \in L^\gamma_2(\mathbb{R}), \\ 0 < c \leq \rho_0(x) \leq C < +\infty \quad \textit{for any } x \in \mathbb{R}, \\ v_0 \in L^2(\mathbb{R}),\ \rho_0 v_0 \in \mathcal{M}(\mathbb{R}). \end{cases} \tag{2.11}$$

The momentum m_0 *is defined as follows:*

$$m_0 = \rho_0 v_0 - \partial_x\varphi_1(\rho_0),\ m_0 \in \mathcal{M}(\mathbb{R}). \tag{2.12}$$

In addition theres exists $\epsilon_0 > 0$ *such that if:*

$$\|\partial_x\varphi_1(\rho_0)\|_{\mathcal{M}(\mathbb{R})} + \|m_0\|_{\mathcal{M}(\mathbb{R})} \leq \epsilon_0, \tag{2.13}$$

then there exists a global weak solution $(\rho, \sqrt{\rho}u)$ *for the system (1.1) verifying the definition 2.1. In addition there exists* $T_\beta > 0$ *and* $C > 0$ *such that we have:*

$$\begin{cases} \|(\rho, \dfrac{1}{\rho})\|_{L^\infty([0,T_\beta],L^\infty(\mathbb{R})} \leq C, \\ \|u\|_{L^{p(s)-\epsilon}([0,T_\beta],H^s(\mathbb{R}))} \leq C \end{cases} \tag{2.14}$$

with $0 < s < 1$ *and* $p(s) = \frac{6}{1+2s}$ *and* $\epsilon > 0$ *sufficiently small such that* $p(s) - \epsilon \geq 1$. *We have in addition for a any* $t > 0$, *there exists* $C > 0$ *sufficiently large and and* $C(t) > 0$ *depending on* t *such that:*

$$\begin{cases} \|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C, \\ \|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t), \\ \|\sqrt{\rho}u\|_{L^\infty((0,t),L^2(\mathbb{R}))} \leq C(1 + \dfrac{1}{\sqrt{t}}) \\ \|m^2\|_{L^\infty((0,t),L^2(\mathbb{R}))} \leq C \\ \|\rho - \bar{\rho}\|_{L^\infty((0,t),L^2(\mathbb{R}))} \leq C \\ \|\partial_x(\rho^{\frac{1}{2}(\gamma+\alpha-1)})\|_{L^2((0,t),L^2(\mathbb{R}))} \leq C \\ \|\partial_x \rho^{\alpha-\frac{1}{2}}\|_{L^2([0,T_\beta],L^2(\mathbb{R}))+L^\infty([T_\beta,+\infty[,L^2(\mathbb{R}))} \leq C. \end{cases} \tag{2.15}$$

If $0 < \alpha < \frac{1}{2}$ *we have in addition for* $C > 0$*:*

$$\begin{cases} \|\dfrac{1}{\rho}\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))} \leq C \\ \|\partial_x u\|_{L^2([T_\beta,+\infty[,L^2(\mathbb{R}))} \leq C. \end{cases} \tag{2.16}$$

Furthermore we have for any $T > 0$*:*

$$\begin{cases} m \in C^w([0,T], \mathcal{M}(\mathbb{R})) \\ m^1 \in C^w([0,T], \mathcal{M}(\mathbb{R})), \end{cases} \tag{2.17}$$

with $m = \sqrt{\rho}\sqrt{\rho}u$ *and* $m^1 = \partial_x \varphi_1(\rho) + m$.

Remark 3 *The main interest of this result is to observe the two following points, first we can deal with initial momentum* m_0 *which are vortex. This means that we can take* m_0 *in* $\mathcal{M}(\mathbb{R})$. *The second point is that* $\varphi_1(\rho_0)$ *is in* $BV(\mathbb{R})$ *and by composition theorem* ρ_0 *is also in* $BV(\mathbb{R})$.*that* $\varphi_1(\rho_0)$ *is in* $BV(\mathbb{R})$. *This implies obviously that* ρ_0 *can admits initially some shocks.*
We can now observe that these shocks at the initial time $t = 0$ *are instantaneously regularizing inasmuch as we verify that* $\rho(t,\cdot) \in C(\mathbb{R})$ *for any* $0 < t \leq T_\beta$. *Indeed we observe that:*

$$\partial_x \varphi_1(\rho(t,\cdot)) = \rho u(t,\cdot) + \rho v(t,\cdot).$$

Since $\rho u(t,\cdot)$ *and* $\rho v(t,\cdot)$ *are in* $L^2(\mathbb{R})$ *for* $t > 0$ *and* $\varphi_1(\rho(t,\cdot)) - \varphi_1(\bar{\rho})$ *belongs to* $L^2(\mathbb{R})$ *for* $t \in [0, T_\beta]$, *we deduce by Sobolev embedding that* $\varphi_1(\rho)(t,\cdot) - \varphi_1(\bar{\rho})$ *is continuous and then* $\rho(t,\cdot)$ *is also continuous for any* $t > 0$ *(indeed we know that* $(\rho, \frac{1}{\rho})$ *belongs to*

$L^\infty([0,T_\beta],L^\infty(\mathbb{R}))$).
We can try to explain this phenomena, in fact v verify only a damped transport equation. In particular v has no regularizing effect but the velocity u admits regularizing effects. The fact that v is sufficiently regular and u has regularizing effects allows to prove that $\partial_x\varphi_1(\rho)$ is instantaneously bounded in $L^2(\mathbb{R})$. We can precise what we mean by sufficiently regular. In fact if $\rho_0 v_0$ is at least in $L^1(\mathbb{R})$, we expect that for $\rho_0 u_0 \in \mathcal{M}(\mathbb{R})$ we should observe regularizing effects on the density ρ. Indeed heuristically $\partial_x\varphi_1(\rho(t,\cdot))$ for $t\in[0,T_\beta]$ would be in $L^1(\mathbb{R})+L^2(\mathbb{R})$, and we know that $W^{1,1}(\mathbb{R})+H^1(\mathbb{R})$ is embedded in $C^0(\mathbb{R})$.
In other word the regularizing effects on the density ρ depends on the regularity of the coupling between the density and the velocity u which is expressed by the unknown $\rho_0 v_0$.

Remark 4 *The condition $0<c\leq\rho_0\leq C<+\infty$ is in fact deduced from (2.13) and the fact that $\rho_0-\bar\rho$ is in $L^2(\mathbb{R})\cap BV(\mathbb{R})$.*

Remark 5 *From (2.20), we give a sense to the initial momentum m_0. It is however not clear if we can define properly u_0. It would be natural now to write u_0 as $u_0=\frac{1}{\rho_0}m_0$ with $m_0\in\mathcal{M}(\mathbb{R})$. However since $\frac{1}{\rho_0}$ is a priori not in $C(\mathbb{R})$ (the space of bounded continuous function), the product $\frac{1}{\rho_0}m_0$ is not well defined. It is the same if we assume in addition that $(\rho_0-\bar\rho)\in\dot B^1_{1,\infty}(\mathbb{R})$ and $m_0\in\dot B^0_{1,\infty}(\mathbb{R})$ (we refer to [1] for the definition of the Besov space and on the notion of paraproduct law). In this case $\frac{1}{\rho_0}m_0$ can not be defined using classical paraproduct law, in fact the remainder $R(m_0,\frac{1}{\rho_0}-\frac{1}{\bar\rho})$ is not well defined.*
However if we consider $u_0=v_0+\partial_x\varphi(\rho_0)$ this unknown is well defined since v_0 belongs to $L^2(\mathbb{R})$ and $\partial_x\varphi(\rho_0)$ is in $\mathcal{M}(\mathbb{R})$. Indeed since $\varphi_1(\rho_0)$ is in $BV(\mathbb{R})$ and since we have $0<c\leq\rho_0\leq C<+\infty$, we deduce using the definition of the total variation and the mean value theorem that $\varphi(\rho_0)$ is in $BV(\mathbb{R})$.
In our case ρ_0 and m_0 have to be considered as the initial data. It is also remarkable that instantaneously u is well defined since on $[0,T_\beta]$, $u(t,\cdot)$ is in $L^1(\mathbb{R})$.

Remark 6 *Numerous works dedicated to the incompressible and compressible Navier-Stokes equations concerns the existence of global strong solution with small initial data in the case where the initial data (ρ_0,u_0) have critical regularity. By critical, we mean that the system (1.1) is solved in functional spaces with norm invariant by the changes of scales which leave (1.1) invariant. In the case of barotropic fluids, we can observe that the transformations:*

$$(\rho(t,x),u(t,x))\longrightarrow(\rho(l^2t,lx),lu(l^2t,lx)),\quad l\in\mathbb{R},\tag{2.18}$$

have that property, provided that the pressure term has been changed accordingly. Roughly speaking we expect that such spaces are optimal in term of regularity on the initial data in order to prove the well-posedness of the system (1.1). We can easily observe in particular that $\dot B^{\frac{1}{p}}_{p,r}(\mathbb{R})\times B^{\frac{1}{p}-1}_{p,r}(\mathbb{R})$ with $p\in[1,+\infty]$, $r\in[1,+\infty]$ is a good candidate since its norm remains invariant by the transformation (2.18).
In this sense the previous result is a theorem dealing with critical initial data since $BV(\mathbb{R})\times\mathcal{M}(\mathbb{R})$ is critical for (2.18) if we consider initial data (ρ_0,m_0). In our case, we need to impose the smallness assumption (2.13), it would be interesting to get a similar result without smallness assumption even if we must work with initial data slightly more

regular.
It is natural then to try to adapt the results of [1, 13] to the case of the dimension $N = 1$, in other words it consists to prove the existence of strong solution in finite time with initial data verifying:

$$0 < c \leq \rho_0 \leq M < +\infty,\ \rho_0 - \bar{\rho} \in \dot{B}^1_{1,1}(\mathbb{R}),\ u_0 \in \dot{B}^0_{1,1}(\mathbb{R}).$$

Unfortunately it seems delicate to prove this type of results, indeed if we apply the method develop in [1, 13] it is necessary to give a sense to the term $(\mu(\rho) - \mu(\bar{\rho}))\partial_{xx}u$ with $u \in \widetilde{L}^1([0,T], \dot{B}^2_{1,1}(\mathbb{R}))$, $(\rho - \bar{\rho}) \in \widetilde{L}^\infty([0,T], \dot{B}^1_{1,1}(\mathbb{R}))$ and $\rho \geq \frac{c}{2}$ on $[0,T]$ with $T < +\infty$. It implies that $\partial_{xx}u$ is in $\widetilde{L}^1([0,T], \dot{B}^0_{1,1}(\mathbb{R}))$. Unfortunately the product $(\mu(\rho)-\mu(\bar{\rho}))\partial_{xx}u$ is a priori not defined, indeed it we apply the classical paraproduct law, we can observe that $R((\mu(\rho) - \mu(\bar{\rho})), \partial_{xx}u)$ is not defined in dimension $N = 1$. This term is critical in the sense that we have $s_1 + s_2 + N\inf(0, \frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}) = 0$ with $s_1 = 0$, $s_2 = 1$, $N = 1$, $p = 1$, $p_1 = 1$ and $p_2 = 1$.
It is then not obvious to prove the existence of strong solution in critical Besov spaces in one domension, it would be however quite easy if we assume the initial data slightly subcritical $(\rho_0 - \bar{\rho}, u_0) \in (\dot{B}^1_{1,1}(\mathbb{R}) \cap \dot{B}^{1+\epsilon}_{1,1}(\mathbb{R})) \times (\dot{B}^0_{1,1}(\mathbb{R}) \cap \dot{B}^{\epsilon}_{1,1}(\mathbb{R}))$ with $\epsilon > 0$.
In this sense our result deals with critical initial data even if we have no result of uniqueness.

Remark 7 *This Theorem is a result of global weak solution with infinite energy space, this is due to the fact that $\sqrt{\rho_0}u_0$ does not belong to $L^2(\mathbb{R})$. To the best of our knowledge, this is the first result of this type for compressible Navier Stokes equations.*

We obtain now the following two corollaries. We refer to [1] for the definition of homogeneous Besov space.

Corollary 1 *Assume that $\mu(\rho) = \mu\rho^\alpha$ with $\mu, \alpha > 0$ and $P(\rho) = a\rho^\gamma$ with $\gamma > 1$, $a > 1$ and $\gamma \geq \alpha$. The initial data satisfy for $\bar{\rho} > 0$:*

$$\begin{cases} \varphi_1(\rho_0) \in BV(\mathbb{R}),\ (\rho_0 - \bar{\rho}) \in L^\gamma_2(\mathbb{R}) \\ 0 < c \leq \rho_0(x) \leq C < +\infty \quad \text{for any } x \in \mathbb{R} \\ u_0 \in L^2(\mathbb{R}),\ m_0 \in \mathcal{M}(\mathbb{R}). \end{cases} \tag{2.19}$$

The momentum m^1_0 is defined as follows:

$$m^1_0 = \rho_0 u_0 + \partial_x \varphi_1(\rho_0),\ m^1_0 \in \mathcal{M}(\mathbb{R}). \tag{2.20}$$

In addition theres exists $\epsilon_0 > 0$ such that if:

$$\|\partial_x \varphi_1(\rho_0)\|_{\mathcal{M}(\mathbb{R})} + \|m_0\|_{\mathcal{M}(\mathbb{R})} \leq \epsilon_0, \tag{2.21}$$

then there exists $T > 0$ and a weak solution (ρ, u) for the system (1.1) verifying the

definition 2.1 on the time interval $[0,T]$. *We have in addition:*

$$\begin{cases} u \in L^\infty([0,T_\beta], L^2(\mathbb{R})) \\ \partial_x u \in L^2([0,T_\beta], L^2(\mathbb{R})) \\ \rho, \dfrac{1}{\rho} \in L^\infty([0,T_\beta], L^\infty(\mathbb{R})) \\ \partial_x \rho \in L^\infty([0,T_\beta], \mathcal{M}(\mathbb{R})) \\ (\rho - \bar{\rho}) \in L^\infty([0,T_\beta], L^\gamma_2(\mathbb{R})) \\ \rho \in C([0,T_\beta], L^p_{loc}(\mathbb{R})) \\ \rho u \in C([0,T_\beta], \mathcal{M}(\mathbb{R})^*) \\ m^1 \in C([0,T_\beta], \mathcal{M}(\mathbb{R})^*). \end{cases} \tag{2.22}$$

Remark 8 *The previous corollary is a generalization of [22] for general viscosity coefficients (in [22] the author deals with constant viscosity coefficients). In the previous theorem* u_0 *is well defined and we can deal with initial density admitting shocks since* ρ_0 *is in* $BV(\mathbb{R})$.

Let us deal now with the case of constant viscosity coefficients, we obtain the following theorem.

Theorem 2.3 *Assume that* $\mu(\rho) = \mu$ *with* $\mu > 0$ *and* $P(\rho) = a\rho^\gamma$ *with* $\gamma > 1$, $a > 1$ *and* $\gamma \geq \alpha$. *The initial data satisfy for* $\bar{\rho} > 0$*:*

$$\begin{cases} m_0^1 \in \mathcal{M}(\mathbb{R}),\ \varphi_1(\rho_0) \in BV(\mathbb{R}), v_0 \in L^2(\mathbb{R}) \\ 0 < c \leq \rho_0(x) \leq C < +\infty \quad \textit{for any } x \in \mathbb{R} \\ \rho_0 - \bar{\rho} \in L^\gamma_2(\mathbb{R}). \end{cases} \tag{2.23}$$

The momentum m_0 *is defined as follows:*

$$m_0 = \rho_0 v_0 - \partial_x \varphi_1(\rho_0),\ m_0 \in \mathcal{M}(\mathbb{R}). \tag{2.24}$$

Then there exists a global weak solution (ρ, u) *for the system (1.1) verifying the definition 2.1, in addition for any* $t > 0$ *we have:*

$$\begin{cases} \|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} \leq C_1(t) \\ \|(\rho(t,\cdot), \dfrac{1}{\rho}(t,\cdot)\|_{L^\infty} \leq C_1(t) \\ \|\sqrt{\rho} u(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\dfrac{1}{\sqrt{t}} + 1) \\ \|\rho(t,\cdot) - \bar{\rho}\|_{L^\gamma_2(\mathbb{R})} \leq C \\ \|\partial_x \rho\|_{L^2_t(L^2(\mathbb{R}))} \leq C C_1(t) \\ \|m^2\|_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))} \leq C. \end{cases} \tag{2.25}$$

In addition u *belongs to* $L^{p(s)-\epsilon}_{loc}(H^s(\mathbb{R}))$ *with* $0 \leq s < 1$ *and* $p(s) = \frac{3}{1+2s}$ *with* $\epsilon > 0$ *sufficiently small such that* $p(s) - \epsilon \geq 1$. *We have also* $\rho \in C([0,+\infty[, W^{-\epsilon,p}_{loc}(\mathbb{R}))$ *and* $\rho u \in C([0,+\infty[, \mathcal{M}(\mathbb{R})^*)$.

Remark 9 *Compared with the Theorem 2.2, we have no smallness assumption on the initial data.*

Remark 10 *As in the Theorem 2.2, we can observe regularizing effects on the density. In [23], the initial density verifies only* $\rho_0 - \bar{\rho} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $\frac{1}{\rho_0} \in L^\infty(\mathbb{R})$ *(in particular the assumptions on the initial data are weaken as our Theorem). In particular the Theorem of [23] allows to deal with initial shocks on the density and a priori there is no reasons to observe regularizing effects on the density. One of the reasons is probably that* $\rho_0 v_0$ *is not sufficiently regular, indeed this terms is a priori only in* $W^{-1,\infty}(\mathbb{R})$.

We recall now some classical lemma of compactness in particular the classical lemma of Aubin-Lions.

Lemma 1 *Let* X_0, X, X_1 *Banach spaces. Assume that* X_0 *is compactly embedded in* X *and* X *is continuously embedded in* X_1. *Let* $1 \leq p, q \leq +\infty$. *We set for* $T > 0$*:*

$$W_T = \{u \in L^p([0,T], X_0),\ \frac{d}{dt}u \in L^q([0,T], X_1)\}.$$

Then if:

- $p < +\infty$ *then the embedding of* W *into* $L^p([0,T], X)$ *is compact.*
- $p = +\infty$ *and* $q > 1$, *then the embedding of* W *into* $C([0,T], X)$ *is compact.*

We refer to [37] for the following Lemma which is a consequence of the Helly's theorem.

Lemma 2 *We consider a sequence* $(\rho_n)_{n\in\mathbb{N}}$ *of functions defined on* $[0,T]\times\mathbb{R}$ *with values in* $\mathbb{R}$ *satisfying the following hypotheses:*

1. *There exists* $M > 0$ *such that* $TV(\rho_n(t,\cdot)) \leq M$ *for all* $n \in \mathbb{N}$.
2. $|\rho_n(t,\cdot)| \leq M$ *for all* $n \in \mathbb{N}$.
3. *There exists a sequence* $(\epsilon_n)_{n\in\mathbb{N}}$ *which converges to* 0^+, *such that:*

$$\int_{\mathbb{R}} |\rho_n(t,x) - \rho_n(s,x)|dx \leq \epsilon_n + M|t-s|^\beta$$

for all $n \in \mathbb{N}$, *all* $s, t \in [0,T]$ *and* $\beta > 0$.

Then this sequence is relatively compact in $L^1_{loc}((0,T)\times\mathbb{R})$.

Lemma 3 *Let* K *a compact subset of* $\mathbb{R}$ *and* v^ϵ *a sequel such that:*

- v^ϵ *is uniformly bounded in* $L^{p+\alpha}(K)$ *with* $\alpha > 0$ *and* $p \geq 1$,
- v^ϵ *converges almost everywhere to* v,

then v^ϵ *strongly converges to* v *in* $L^p(K)$ *with* $v \in L^{p+\alpha}(K)$.

Proof: First by the Fatou lemma v is in $L^{p+\alpha}(K)$. Next we have for any $M > 0$:

$$\int_K |v^\epsilon - v|^p dx \leq \int_{K\cap\{|v^\epsilon - v|\leq M\}} |v^\epsilon - v|^p dx + \int_{K\cap\{|v^\epsilon - v|\geq M\}} |v^\epsilon - v|^p dx. \tag{2.26}$$

We are dealing with the second member of the right hand side, by Hölder inequality and Tchebychev lemma we have for a $C > 0$:

$$\int_{K\cap\{|v^\epsilon - v|\geq M\}} |v^\epsilon - v|^p dx \leq (\int_K |v^\epsilon - v|^p \frac{|v^\epsilon - v|^\alpha}{M^\alpha} dx) \quad \leq \frac{C}{M^\alpha}. \tag{2.27}$$

In particular we have shown the strong convergence of v^ϵ to v, indeed from the inequality (2.26) it suffices to use the Lebesgue theorem for the first term on the right hand side and the estimate (2.27) with M going to $+\infty$. □

Section 3 deals with the proof of the theorem 2.2. In the section 4 we prove the corollaries 1. We show the theorem 2.3 which consider constant viscosity coefficients in the section 5.

3 Proof of theorem 2.2

We start by constructing a sequence (ρ_n, u_n) of global strong solutions of the system (1.1) provided that the initial data is sufficiently regular, in addition we change slightly the viscosity coefficient. More precisely (ρ_n, u_n) are solutions of the following systems:

$$\begin{cases} \partial_t \rho_n + \partial_x(\rho_n u_n) = 0, \\ \partial_t(\rho_n u_n) + \partial_x(\rho_n u_n^2) - \partial_x(\mu_n(\rho_n)\partial_x u_n) + \partial_x P(\rho_n) = 0, \end{cases} \tag{3.28}$$

with $\mu_n(\rho_n) = \frac{1}{n}\rho_n^\theta + \mu\rho_n^\alpha$ with $\theta \in (0, \frac{1}{2})$ (we will fix θ later). In addition we assume that:

$$\begin{cases} m_n(0,\cdot) = e^{\frac{1}{n}\Delta}(m_0) \\ \varphi_1(\rho_n)(0,\cdot) = e^{\frac{1}{n}\Delta}(\varphi_1(\rho_0)) \\ m_n^1(0,\cdot) = e^{\frac{1}{n}\Delta}(m_0 + \partial_x\varphi_1(\rho_0)) = e^{\frac{1}{n}\Delta} m_0^1, \end{cases}$$

with $\varphi_1(\rho) = \frac{\mu}{\alpha}\rho^\alpha$ if $\alpha \neq 0$ and $\varphi_1(\rho) = \mu \ln \rho$ if $\alpha = 0$. In the Theorem 2.2, we consider only the case $\alpha > 0$. We verify easily since $\rho_0 - \bar{\rho}$ is in $L_2^\gamma(\mathbb{R})$ and we have $0 < c \leq \rho_0 \leq C < +\infty$ that $\rho_0 - \bar{\rho} \in L^2(\mathbb{R})$. In addition by composition theorem we deduce that $\varphi_1(\rho_0) - \varphi_1(\bar{\rho})$ belongs to $L^2(\mathbb{R})$.

It implies from (2.19) that there exists $C_{n,s}$ depending on n and $s>0$ such that:

$$\begin{cases} \|m_n(0,\cdot)\|_{L^1(\mathbb{R})} \leq \|m_0^1\|_{\mathcal{M}(\mathbb{R})} + \|\varphi_1(\rho_0)\|_{BV(\mathbb{R})} \\ \|m_n^1(0,\cdot)\|_{L^1(\mathbb{R})} \leq \|m_0^1\|_{\mathcal{M}(\mathbb{R})} \\ \|\partial_x\varphi_1(\rho_n)(0,\cdot)\|_{L^1(\mathbb{R})} \leq \|\varphi_1(\rho_0)\|_{BV(\mathbb{R})} \\ 0 < \min(\varphi_1(c),\varphi_1(C)) \leq \varphi_1(\rho_n)(0,\cdot) \leq \max(\varphi_1(C),\varphi_1(c)) < +\infty \\ 0 < c \leq \rho_n(0,\cdot) \leq C < +\infty \\ \|\varphi_1(\rho_n(0,\cdot)) - \varphi_1(\bar{\rho})\|_{H^s(\mathbb{R})} \leq C_{n,s}\|\varphi_1(\rho_0) - \varphi_1(\bar{\rho})\|_{L^2(\mathbb{R})} \\ \|\rho_n(0,\cdot) - \bar{\rho}\|_{H^s(\mathbb{R})} \leq C_{n,s}\|\rho_0 - \bar{\rho}\|_{L^2(\mathbb{R})} \\ \|\rho_n(0,\cdot) - \bar{\rho}\|_{L^2(\mathbb{R})} \leq C(\|\rho_0\|_{L^\infty(\mathbb{R})}, \|\dfrac{1}{\rho_0}\|_{L^\infty(\mathbb{R})})\|\varphi_1(\rho_0) - \varphi_1(\bar{\rho})\|_{L^2(\mathbb{R})} \\ \|m_n^1(0,\cdot)\|_{L^2(\mathbb{R})} \leq \|m_0^1\|_{L^2(\mathbb{R})} \\ \|m_n^1(0,\cdot)\|_{H^s(\mathbb{R})} \leq C_{n,s}\|m_0^1\|_{L^2(\mathbb{R})} \\ \|m_n(0,\cdot)\|_{H^s(\mathbb{R})} \leq C_{n,s}\|m_0^1\|_{L^2(\mathbb{R})} + C_{n,s+1}\|\varphi_1(\rho_0) - \varphi_1(\bar{\rho})\|_{L^2(\mathbb{R})}. \end{cases} \tag{3.29}$$

We have previously used composition theorem to pass from the H^s norm on $\varphi_1(\rho_n(0,\cdot)) - \varphi_1(\bar{\rho})$ to the H^s norm on $\rho_n(0,\cdot) - \bar{\rho}$. Let us mention that we can now define $u_n(0,\cdot)$ and $v_n(0,\cdot)$, indeed if we write $u_n(0,\cdot) = \frac{1}{\rho_n(0,\cdot)} m_n(0,\cdot)$ and $v_n(0,\cdot) = \frac{1}{\rho_n(0,\cdot)} m_n^1(0,\cdot)$, these terms have a sense since they belong to $L^1(\mathbb{R})$.
In particular we deduce from (3.29) and using product in Sobolev space that we have for $C_{n,s}>0$ depending on n and s with $s>0$:

$$\begin{cases} \|\dfrac{1}{\rho_n(0,\cdot)} - \dfrac{1}{\bar{\rho}}\|_{H^s(\mathbb{R})} \leq C_{n,s} \\ \|u_n(0,\cdot)\|_{H^s(\mathbb{R})} \leq C_{n,s} \end{cases} \tag{3.30}$$

From [33], we know that there exists a global strong solution for the system (3.28) since $(\rho_n(0,\cdot) - \bar{\rho}, u_n(0,\cdot))$ belongs to $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with $0 < c \leq \rho_n(0,\cdot) \leq C < +\infty$. In addition we have used the fact that $\mu_n(\rho) \geq \frac{1}{n}\rho^\theta$ with $\theta \in [0, \frac{1}{2})$.

We are now going to prove uniform estimates in n on the solutions (ρ^n, u^n), in a second time we will prove that $(\rho^n, u^n)_{n\in\mathbb{N}}$ converges up to a subsequence to a global weak solution solution (ρ, u) of (1.1) with (ρ_0, u_0) verifying the condition of theorem 2.2. We can mention that the solution (ρ_n, u_n) here is classical in the sense that (ρ_n, u_n) is in $C^\infty([0,+\infty)\times\mathbb{R})$. This is due to the fact that the H^s regularity is preserved all along the time. From (2.13) and (3.29) we deduce that there exists $C>0$ such that:

$$\|\rho_n u_n(0,\cdot)\|_{L^1(\mathbb{R})} + \|\rho_n v_n(0,\cdot)\|_{L^1(\mathbb{R})} \leq C\epsilon_0. \tag{3.31}$$

3.1 Uniform estimates on (ρ_n, u_n, v_n)

Estimates of $\rho_n u_n$ and $\rho_n v_n$ in $L^\infty_T(L^1(\mathbb{R}))$ for any $T > 0$

For simplicity of the notations, we forget in the sequel the subscript n of $(\rho_n, u_n, v_n, \mu_n(\rho_n))$. From (1.2), we recall that we have:

$$\begin{cases} \rho\partial_t u + \rho u\partial_x u - \partial_x(\mu(\rho)\partial_x u) + \dfrac{P'(\rho)\rho^2}{\mu(\rho)}v = \dfrac{P'(\rho)\rho^2}{\mu(\rho)}u, \\ \rho\partial_t v + \rho u\partial_x v + \dfrac{P'(\rho)\rho^2}{\mu(\rho)}v = \dfrac{P'(\rho)\rho^2}{\mu(\rho)}u. \end{cases} \tag{3.32}$$

We set $j_k(s) = \sqrt{s^2 + \frac{1}{k}} - \sqrt{\frac{1}{k}}$ and we have $j_k'(s) = \frac{s}{\sqrt{s^2+\frac{1}{k}}}$, $j_k''(s) = \frac{1}{k(s^2+\frac{1}{k})^{\frac{3}{2}}}$. We are going now to multiply the first and the second equation respectively by $j_k'(u)$ and $j_k'(v)$. Since the solution (ρ, u, v) is classical and is in particular in $C^2([0, +\infty) \times \mathbb{R})$ we have:

$$\begin{cases} \rho\partial_t j_k(v) + \rho u\partial_x j_k(v) + \dfrac{P'(\rho)\rho^2}{\mu(\rho)}v j_k'(v) = \dfrac{P'(\rho)\rho^2}{\mu(\rho)}u j_k'(v) \\ \rho\partial_t j_k(u) + \rho u\partial_x j_k(u) - \partial_x(\mu(\rho)\partial_x j_k(u)) + \mu(\rho)j_k''(u)|\partial_x u|^2 \\ \qquad\qquad = -\dfrac{P'(\rho)\rho^2}{\mu(\rho)}v j_k'(u) + \dfrac{P'(\rho)\rho^2}{\mu(\rho)}u j_k'(u). \end{cases} \tag{3.33}$$

We integrate now on $(0, T) \times \mathbb{R}$ and we obtain:

$$\int_{\mathbb{R}} \rho(T,x) j_k(v)(T,x)dx + \int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(s,x)\frac{v^2(s,x)}{\sqrt{v^2(s,x)+\frac{1}{k}}}dsdx$$
$$= \int_{\mathbb{R}} \rho_0(x) j_k(v_0)(x)dx + \int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(s,x)\frac{v(s,x)u(s,x)}{\sqrt{v^2(s,x)+\frac{1}{k}}}dsdx.$$

$$\int_{\mathbb{R}} \rho(T,x) j_k(u)(T,x)dx + \int_0^T \int_{\mathbb{R}} \mu(\rho)j_k''(u)|\partial_x u|^2(s,x)dsdx$$
$$= \int_{\mathbb{R}} \rho_0(x) j_k(u_0)(x)dx - \int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(s,x)\frac{v(s,x)u(s,x)}{\sqrt{u^2(s,x)+\frac{1}{k}}}dsdx$$
$$+ \int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(s,x)\frac{u^2(s,x)}{\sqrt{u^2(s,x)+\frac{1}{k}}}dsdx.$$

We deduce that we have:

$$\int_{\mathbb{R}} \rho(T,x) j_k(v)(T,x)dx + \int_{\mathbb{R}} \rho(T,x) j_k(u)(T,x)dx \leq$$
$$\int_{\mathbb{R}} \rho_0(x)|u_0(x)|dx + \int_{\mathbb{R}} \rho_0(x)|v_0(x)|dx + 3\int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(|v(s,x)| + |u(s,x)|)dsdx. \tag{3.34}$$

We pass to the limit when $k \to +\infty$ and using the Fatou lemma we obtain:

$$\int_{\mathbb{R}} \rho(T,x)|v|(T,x)dx + \int_{\mathbb{R}} \rho(T,x)|u|(T,x)dx \leq$$
$$\int_{\mathbb{R}} \rho_0(x)|u_0(x)|dx + \int_{\mathbb{R}} \rho_0(x)|v_0(x)|dx + 3\int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\rho^2}{\mu(\rho)}(|v(s,x)| + |u(s,x)|)dsdx. \tag{3.35}$$

From the Gronwall lemma we deduce that for any $T > 0$:

$$\begin{aligned} &\|\rho v(T,\cdot)\|_{L^1(\mathbb{R})} + \|\rho u(T,\cdot)\|_{L^1(\mathbb{R})} \\ &\qquad \leq (\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})}) e^{3\int_0^T \|\frac{P'(\rho)\rho}{\mu(\rho)}(s,\cdot)\|_{L^\infty} ds}. \end{aligned} \tag{3.36}$$

At this level it is important to point out that we have:

$$\frac{P'(\rho)\rho}{\mu(\rho)} \leq \frac{a\gamma}{\mu}\rho^{\gamma-\alpha},$$

and we are going to use the fact that $\gamma - \alpha \geq 0$ in order to control the L^∞ norm of $\frac{P'(\rho)\rho}{\mu(\rho)}$ in terms of the L^∞ norm of the density ρ.

BD Entropy

Multiplying the momentum equation (1.2) and integrating over $(0,T)\times\mathbb{R}$, we obtain the following entropy:

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} (\rho|v|^2(T,x) + (\Pi(\rho) - \Pi(\bar\rho))(T,x)dx + \int_0^T \int_{\mathbb{R}} \frac{P'(\rho)\mu(\rho)}{\rho^2}(s,x)|\partial_x \rho|^2(s,x)dsdx \\ \leq \frac{1}{2}\int_{\mathbb{R}} (\rho_0|v_0|^2(x) + (\Pi(\rho_0) - \Pi(\bar\rho))(x)dx, \end{aligned} \tag{3.37}$$

with:

$$\Pi(s) = s(\int_{\bar\rho}^s \frac{P(z)}{z^2}dz - \frac{P(\bar\rho)}{\bar\rho}).$$

It gives when $\gamma + \alpha - 3 \neq 0$:

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} (\rho|v|^2(T,x) + (\Pi(\rho) - \Pi(\bar\rho))(T,x)dx + \frac{4a\gamma\mu}{(\gamma+\alpha-1)^2}\int_0^T \int_{\mathbb{R}} |\partial_x \rho^{\frac{1}{2}(\gamma+\alpha-1)})|^2(s,x)dsdx \\ \leq \frac{1}{2}\int_{\mathbb{R}} (\rho_0|v_0|^2(x) + (\Pi(\rho_0) - \Pi(\bar\rho))(x)dx, \end{aligned} \tag{3.38}$$

and we know that $\gamma + \alpha - 1 > 0$ since $\alpha \geq 0$ and $\gamma > 1$.

L^∞ estimate on the density ρ in finite time

From (3.36) we deduce that $\partial_x \varphi_1(\rho) = \rho v - \rho u$ is bounded in $L^\infty_T(L^1(\mathbb{R}))$ for any $T > 0$ with $\varphi_1'(\rho) = \frac{\mu(\rho)}{\rho}$ (it gives $\varphi_1(\rho) = \frac{\mu}{\alpha}\rho^\alpha$ since $\alpha > 0$). Since the density $\rho - \bar\rho$ and m is

bounded in $C([0,T],H^s(\mathbb{R}))$ for any $s>0$, we deduce from the mass equation that ρ is uniformly continuous. Furthermore from (3.37) we deduce that for any $t\geq 0$ we have:

$$\lim_{|x|\to+\infty}\varphi_1(\rho)(t,x)=\varphi_1(\bar{\rho}). \tag{3.39}$$

Now using the fact that the space $W^{1,1}(\mathbb{R})$ is embedded in $L^\infty(\mathbb{R})$ we deduce from (3.36) that for any $t>0$:

$$\begin{aligned}\|\varphi_1(\rho(t,\cdot))\|_{L^\infty}&\leq \varphi_1(\bar{\rho})+\|\partial_x\varphi_1(\rho(t,\cdot))\|_{L^1(\mathbb{R})}\\ &\leq \varphi_1(\bar{\rho})+(\|\rho_0v_0\|_{L^1(\mathbb{R})}+\|\rho_0u_0\|_{L^1(\mathbb{R})})e^{3\int_0^t\|\frac{P'(\rho)\rho}{\mu(\rho)}(s,\cdot)\|_{L^\infty}ds}.\end{aligned} \tag{3.40}$$

The previous inequality shows that we can prove L^∞ estimates on the density ρ in finite time by using bootstrap arguments. As previously, we assume that $\gamma-\alpha\geq 0$ and we have for $C>0$, $\alpha>0$ and any $t>0$:

$$\|\rho\|^\alpha_{L^\infty_t(L^\infty(\mathbb{R}))}\leq C\bar{\rho}^\alpha+C(\|\rho_0v_0\|_{L^1(\mathbb{R})}+\|\rho_0u_0\|_{L^1(\mathbb{R})})e^{3Ct\|\rho\|^{\gamma-\alpha}_{L^\infty_t(L^\infty(\mathbb{R}))}}. \tag{3.41}$$

And it yields that for $C'>0$ and $\alpha>0$:

$$\|\rho\|_{L^\infty_t(L^\infty(\mathbb{R}))}\leq C'\bar{\rho}+C'(\|\rho_0v_0\|_{L^1(\mathbb{R})}+\|\rho_0u_0\|_{L^1(\mathbb{R})})^{\frac{1}{\alpha}}\,e^{\frac{3C'}{\alpha}t\|\rho\|^{\gamma-\alpha}_{L^\infty_t(L^\infty(\mathbb{R}))}}. \tag{3.42}$$

Let us prove now that the density ρ is uniformly bounded in n in L^∞ norm on an time interval $[0,T^*]$ with $T^*>0$ independent on n. More precisely we define by:

$$T_n=\sup\{t\in(0,+\infty),\|\rho^n(t,\cdot)\|_{L^\infty}\leq\sup(2\|\rho_0\|_{L^\infty},C'\bar{\rho}+2C'(2\|\rho_0v_0\|_{\mathcal{M}(\mathbb{R})}+\|\partial_x\varphi_1(\rho_0)\|_{\mathcal{M}(\mathbb{R})})^{\frac{1}{\alpha}})\},$$

with $M=\sup(2\|\rho_0\|_{L^\infty},C'\bar{\rho}+2C'(2\|\rho_0v_0\|_{\mathcal{M}(\mathbb{R})}+\|\partial_x\varphi_1(\rho_0)\|_{\mathcal{M}(\mathbb{R})})$.
We observe that $T_n>0$ since ρ_n belongs to $C([0,+\infty[,L^\infty(\mathbb{R}))$ for any $n\in\mathbb{N}$ and since $\|\rho_0^n\|_{L^\infty}<M$. Let us define now T_* such that:

$$e^{\frac{3C'}{\alpha}T_*M^{\gamma-\alpha}}=\frac{3}{2}\quad\text{and}\quad T_*=\frac{\alpha\ln(\frac{3}{2})}{3C'M^{\gamma-\alpha}}.$$

From the definition of T_n and from (3.42), (3.29) we deduce that for any $n\in\mathbb{N}$ we have:

$$T_n\geq T_*>0.$$

It implies that ρ_n is uniformly bounded in n in $L^\infty((0,T_*),L^\infty(\mathbb{R}))$ with T_* independent on n. In other words we have for any $n\in\mathbb{N}^*$:

$$\|\rho_n\|_{L^\infty([0,T_*],L^\infty(\mathbb{R}))}\leq M. \tag{3.43}$$

In the sequel we will prove in fact that ρ_n is also uniformly bounded in n in $L^\infty((0,\infty),L^\infty(\mathbb{R}))$. Now if we combine (3.43), (3.29) and (3.36) we deduce that there exists $C>0$ independent on n such that:

$$\begin{aligned}&\|\rho_nu_n\|_{L^\infty([0,T_*],L^1(\mathbb{R}))}+\|\rho_nv_n\|_{L^\infty([0,T_*],L^1(\mathbb{R}))}\\ &\qquad\leq(2\|\rho_0v_0\|_{\mathcal{M}(\mathbb{R})}+\|\partial_x\varphi_1(\rho_0)\|_{\mathcal{M}(\mathbb{R})})e^{CT_*M^{\gamma-\alpha}}.\end{aligned} \tag{3.44}$$

Control of the L^∞ norm of $\frac{1}{\rho}$ in finite time

In the sequel we omit again the subscript n except when we want to emphasize on the fact that we have an uniform estimate in n. Now we are interested in estimating the L^∞ norm of $\frac{1}{\rho}$ on a finite time T_β independent on n. As in (3.40) we have for any $t \in [0, T_*]$: and for $C > 0$ large enough:

$$\begin{aligned} \|\varphi_1(\rho(t,\cdot)) - \varphi_1(\bar{\rho})\|_{L^\infty} &\leq \|\partial_x \varphi_1(\rho(t,\cdot))\|_{L^1(\mathbb{R})} \\ &\leq (\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})}) e^{tCM^{\gamma-\alpha}}. \end{aligned} \tag{3.45}$$

Now since we have $\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})} \leq C\epsilon_0$ with $C > 0$ independent on n and with ϵ_0 sufficiently small, we deduce that for $T_\beta \leq T_*$ such that:

$$e^{T_\beta C M^{\gamma-\alpha}} \leq 2, \tag{3.46}$$

we obtain that for any $t \in [0, T_\beta]$ and any $x \in \mathbb{R}$:

$$\varphi_1(\rho(t,x)) \geq \varphi_1(\bar{\rho}) - 2C\epsilon_0. \tag{3.47}$$

It implies that $\frac{1}{\rho}$ belongs to $L^\infty([0,T_\beta], L^\infty(\mathbb{R}))$ with $T_\beta > 0$ independent on n for $\epsilon_0 > 0$ small enough.

Gain of regularity on the velocity u_n

From (3.37), we deduce that for any $T > 0$ we have uniformly in n:

$$\begin{aligned} &\partial_x \rho^{\frac{1}{2}(\gamma+\alpha-1)} \in L^2_T(L^2(\mathbb{R})) \\ &\sqrt{\rho} v \in L^\infty_T(L^2(\mathbb{R})). \end{aligned} \tag{3.48}$$

Now from the definition of the effective velocity v we have:

$$\begin{aligned} \sqrt{\rho} u = \sqrt{\rho} v - \frac{\mu(\rho)}{\rho^{\frac{3}{2}}} \partial_x \rho &= \sqrt{\rho} v - \frac{2\mu}{\gamma+\alpha-1} \frac{\rho^\alpha}{\rho^{\frac{1}{2}(\gamma+\alpha)}} \partial_x \rho^{\frac{1}{2}(\gamma+\alpha-1)} \\ &= \sqrt{\rho} v - \frac{2\mu}{\gamma+\alpha-1} \rho^{\frac{1}{2}(\alpha-\gamma)} \partial_x \rho^{\frac{1}{2}(\gamma+\alpha-1)} \end{aligned} \tag{3.49}$$

From (3.48), (3.47) and since $\alpha - \gamma \leq 0$, we deduce that $\sqrt{\rho} u$ is uniformly bounded in $L^2((0,T_\beta) \times \mathbb{R})$, it gives with $T_\beta > 0$ independent on n,$C > 0$ depending on ϵ_0 and the initial data (ρ_0, v_0) that for any $n \in \mathbb{N}^*$:

$$\|\sqrt{\rho_n} u_n\|_{L^2((0,T_\beta)\times\mathbb{R})} \leq C. \tag{3.50}$$

Let us prove now some additional regularizing effects independent on n on the velocity u. It will be important in order to pass to the limit when n goes to $+\infty$. Let us multiply the momentum equation of (1.1) by su and integrate over $(0,t) \times \mathbb{R}$ with $t \in (0, T_\beta)$, we have then:

$$\begin{aligned} &\frac{1}{2} t \int_{\mathbb{R}} \rho |u|^2(t,x) dx - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \rho |u|^2(s,x) ds dx + \int_0^t \int_{\mathbb{R}} \mu(\rho) (\partial_x u)^2(s,x) s \, ds dx \\ &+ \int_0^t \int_{\mathbb{R}} s \partial_x P(\rho) \, u dx = 0 \end{aligned}$$

We recall now that we have:

$$\partial_t(t(\Pi(\rho)-\Pi(\bar\rho))+t\partial_x(\Pi(\rho)u)+tP(\rho)\partial_x u-(\Pi(\rho)-\Pi(\bar\rho))=0.$$

We deduce then that we have:

$$\begin{aligned}\frac{1}{2}t\int_{\mathbb{R}}\rho|u|^2(t,x)dx+\int_0^t\int_{\mathbb{R}}\mu(\rho)s(\partial_x u)^2(s,x)dsdx+t\int_{\mathbb{R}}(\Pi(\rho)(t,x)-\Pi(\bar\rho))dx\\ =\int_0^t\int_{\mathbb{R}}(\Pi(\rho)-\Pi(\bar\rho))(s,x)dsdx+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\rho|u|^2(s,x)dsdx.\end{aligned}\tag{3.51}$$

We deduce from (3.37), (3.47) and (3.50) that u verify uniformly on $[0,T_\beta]$:

$$\sqrt{t}u\in L^\infty([0,T_\beta],L^2(\mathbb{R}))\quad\text{and}\quad\sqrt{t}\partial_x u\in L^2([0,T_\beta],L^2(\mathbb{R})).\tag{3.52}$$

We can obtain additional informations on u by using interpolation estimates. Indeed from (3.44) we know that ρu is uniformly bounded in $L^\infty([0,T_\beta],L^1(\mathbb{R}))$. We recall that $\frac{1}{\rho}$ is uniformly bounded on $[0,T_\beta]$ then we deduce that u is uniformly bounded in $L^\infty([0,T_\beta],L^1(\mathbb{R}))$ and then is uniformly bounded in $L^\infty([0,T_\beta],H^{-\frac{1}{2}}(\mathbb{R}))$ by Sobolev embedding. It is important to mention that the bound depend only on ϵ_0 and $\|\rho_0\|_{L^\infty(\mathbb{R})}$. By interpolation, we have for $0<s<1$ and any $t\in(0,T_\beta]$:

$$t^{\frac{1}{6}(1+2s)}\|u(t,\cdot)\|_{H^s}\leq\|u(t,\cdot)\|^{\frac{2}{3}(1-s)}_{H^{-\frac{1}{2}}(\mathbb{R})}(\sqrt{t}\|u(t,\cdot)\|_{H^1(\mathbb{R})})^{\frac{1}{3}(1+2s)}.\tag{3.53}$$

From (3.47), (3.52) and (3.53) we deduce that u is uniformly bounded in $L^{p(s)-\epsilon}([0,T_\alpha],H^s(\mathbb{R}))$ with $s\in(0,1)$, $p(s)=\frac{3}{1+2s}$ and $\epsilon>0$ sufficiently small such that $p(s)-\epsilon\geq 1$.
We have finally obtained that for any $t\in[0,T_\beta]$, we have for $C>0$ independent on n:

$$\begin{cases}\|\rho_n(t,\cdot)-\bar\rho\|_{L^2(\mathbb{R})}\leq C\\ \|v_n(t,\cdot)\|_{L^2(\mathbb{R})}\leq C,\ \|v_n(t,\cdot)\|_{L^1(\mathbb{R})}\leq C\\ \|\sqrt{s}\partial_x u_n\|_{L^2([0,T_\beta],L^2(\mathbb{R}))}\leq C\\ \|u_n(t,\cdot)\|_{L^2(\mathbb{R})}\leq C(1+\frac{1}{\sqrt{t}}),\ \|u_n(t,\cdot)\|_{L^1(\mathbb{R})}\leq C\\ \|\partial_x\varphi(\rho_n)\|_{L^2(\mathbb{R})}\leq C(1+\frac{1}{\sqrt{t}})\\ \|u_n\|_{L^{p(s)-\epsilon}([0,t],H^s(\mathbb{R}))}\leq C\\ \|(\frac{1}{\rho_n},\rho_n)(t,\cdot)\|_{L^\infty(\mathbb{R})}\leq C.\end{cases}\tag{3.54}$$

We deduce in particular that choosing $s_1(\epsilon)\in(0,1)$ such that $\frac{1}{2}-s_1(\epsilon)=\frac{1}{p(s_1(\epsilon))-\epsilon}$ gives that $4\epsilon s_1(\epsilon)^2-10s_1(\epsilon)+(1-\epsilon)=0$. It implies that $s_1(\epsilon)=\frac{5}{4\epsilon}(1-\sqrt{1-\frac{4}{25}\epsilon(1-\epsilon)})$. We observe in fact that $s_1(\epsilon)\to_{\epsilon\to 0}\frac{1}{10}$. We observe in particular that $2<p(s_1(\epsilon))-\epsilon<3$ and we have from (3.54) the following uniform bound:

$$\|u_n\|_{L^{p(s_1(\epsilon))-\epsilon}([0,T_\beta]\times\mathbb{R})}\leq C,\tag{3.55}$$

with $C>0$ independent on n.

Estimate in long time of the L^∞ norm of the density ρ when $\alpha > \frac{1}{2}$

We know that $(\rho_n, u_n, v_n)_{n\in\mathbb{N}}$ verify the classical energy estimate on $(0,+\infty)\times\mathbb{R}$ (indeed the initial data belongs to the energy space, in particular $u_n(0,\cdot) \in L^2(\mathbb{R})$), however the energy estimate depends on the initial data and is not uniform in n (indeed the initial velocity u_0 of the Theorem 2.2 is not in $L^2(\mathbb{R})$). However for $t \geq t_1 > 0$ the energy estimate will be uniform in n. Indeed let us multiply the momentum equation of (1.1) by u and integrating over $[t_1, t_2]$ with $t_1 \in (0, T_\beta]$ and $t_2 > t_1$, it gives then:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}(\rho|u|^2(t_2,x) + (\Pi(\rho)-\Pi(\bar\rho))(t_2,x)dx + \int_{t_1}^{t_2}\int_{\mathbb{R}}\mu(\rho(s,x))(\partial_x u(s,x))^2 dsdx \\ \leq \frac{1}{2}\int_{\mathbb{R}}(\rho(t_1,x)|u(t_1,x)|^2(x) + (\Pi(\rho(t_1,x))-\Pi(\bar\rho))dx.\end{aligned} \tag{3.56}$$

From (3.51), (3.54) and (3.37) we deduce that there exists $C > 0$ depending only on the initial data such that:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}(\rho|u|^2(t_2,x) + (\Pi(\rho)-\Pi(\bar\rho))(t_2,x)dx + \int_{t_1}^{t_2}\int_{\mathbb{R}}\mu(\rho(s,x))(\partial_x u(s,x))^2 dsdx \\ \leq C(1+\frac{1}{t_1}).\end{aligned} \tag{3.57}$$

Combining (3.37) and (3.57) we deduce that for any $t > 0$ we have for $C > 0$ independent on n:

$$\begin{cases} \|\sqrt{\rho_n}v_n(t,\cdot)\|_{L^2(\mathbb{R})} \leq C \\ \|\sqrt{\rho_n}u_n(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(1+\frac{1}{\sqrt{t}}) \\ \|(\sqrt{s}1_{\{s\leq T_\beta\}} + \sqrt{\mu(\rho_n)}1_{\{s\geq T_\beta\}})\partial_x u_n\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|\partial_x \rho_n^{\alpha-\frac{1}{2}}(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(1+\frac{1}{\sqrt{t}}) \\ \|\partial_x \rho_n^{\frac{1}{2}(\gamma+\alpha-1)}\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|\rho_n(t,\cdot)-\bar\rho\|_{L^\gamma_2(\mathbb{R})} \leq C. \end{cases} \tag{3.58}$$

Lemma 4 *When $\alpha > \frac{1}{2}$ and $\gamma \geq 2\alpha - 1$, for any $t \geq 0$ there exists $C > 0$ independent on n such that:*

$$\|\rho_n(t,\cdot)\|_{L^\infty} \leq C. \tag{3.59}$$

Proof: We omit again in the sequel the subscript n. We know that for $t_1 \in (0, T_\beta]$, we have $C > 0$ independent on n such that:

$$\|\rho\|_{L^\infty([0,t_1],L^\infty(\mathbb{R}))} \leq C. \tag{3.60}$$

Following the lemma 3.7 in [24], it remains now only to prove that there exists $C > 0$ independent on n such that for all $t \geq t_1 > 0$, we have:

$$\|\rho(t,\cdot)\|_{L^\infty} \leq C. \tag{3.61}$$

Since ρ is regular (we recall that we forget the subscript n) it yields for any $t \geq t_1$:

$$
\begin{aligned}
&(\rho^{\alpha-\frac{1}{2}}(t,x) - \bar{\rho}^{\alpha-\frac{1}{2}})^{2\beta}(t,x) = \int_{-\infty}^{x} \partial_y((\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}})^{2\beta})(t,y)dy \\
&\leq 2\beta\Big(\int_{-\infty}^{x} |\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}}|^{2(2\beta-1)}dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{x} |\partial_y\rho^{\alpha-\frac{1}{2}}(t,y)|^2dy\Big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.62}
$$

From (3.58) we know that $\int_{-\infty}^{x} |\partial_y\rho^{\alpha-\frac{1}{2}}(t,y)|^2dy$ is uniformly bounded in n for $t \geq t_1 > 0$. We have now to estimate the following term:

$$
\begin{aligned}
&\int_{-\infty}^{x} |\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}}|^{2(2\beta-1)}dy = \int_{-\infty}^{x} |\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}}|^{2(2\beta-1)}[1_{\{|\rho-\bar{\rho}|\leq\frac{\bar{\rho}}{2}\}} + 1_{\{|\rho-\bar{\rho}|>\frac{\bar{\rho}}{2}\}}]dy \\
&= I_1(t) + I_2(t).
\end{aligned}
$$

We have now for $C, C_1 > 0$ sufficiently large and taking $\beta = 1$ for any $t \geq t_1$ and using (3.58):

$$
\begin{aligned}
I_1(t) &= \int_{-\infty}^{x} |\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}}|^{2(2\beta-1)}1_{\{|\rho-\bar{\rho}|\leq\frac{\bar{\rho}}{2}\}}dy \\
&\leq C\int_{-\infty}^{x} |\rho(t,y) - \bar{\rho}|^2 1_{\{|\rho-\bar{\rho}|\leq\frac{\bar{\rho}}{2}\}}dy \\
&\leq C_1.
\end{aligned}
\tag{3.63}
$$

For $\beta = 1$ we have now when $t \geq t_1$ and $C, C_1 > 0$ sufficiently large using (3.58) and the fact that $\gamma \geq 2\alpha - 1$:

$$
\begin{aligned}
I_2(t) &= \int_{-\infty}^{x} |\rho^{\alpha-\frac{1}{2}}(t,y) - \bar{\rho}^{\alpha-\frac{1}{2}}|^{2(2\beta-1)}1_{\{|\rho-\bar{\rho}|>\frac{\bar{\rho}}{2}\}}dy \\
&\leq C\int_{-\infty}^{x} |\rho(t,y) - \bar{\rho}|^{2(2\beta-1)(\alpha-\frac{1}{2})}1_{\{|\rho-\bar{\rho}|>\frac{\bar{\rho}}{2}\}}dy \\
&\leq C\int_{-\infty}^{x} |\rho(t,y) - \bar{\rho}|^{2(\alpha-\frac{1}{2})}1_{\{|\rho-\bar{\rho}|>\frac{\bar{\rho}}{2}\}}dy.
\end{aligned}
\tag{3.64}
$$

If $2\alpha - 1 \leq \gamma$ there exists $C_1 > 0$ large enough such that:

$$
I_2(t) \leq C_1\int_{-\infty}^{x} |\rho(t,y) - \bar{\rho}|^{\gamma}1_{\{|\rho-\bar{\rho}|>\frac{\bar{\rho}}{2}\}}dy. \tag{3.65}
$$

From (3.60), (3.62), (3.63) and (3.64) we conclude the proof of (3.59). ■

We deduce using the estimate (3.36) that for any $t > 0$ and any $n \in \mathbb{N}^*$ we have for $C(t) > 0$ depending only on t:

$$
\|\rho_n u_n(t,\cdot)\|_{L^1(\mathbb{R})} + \|\rho_n v_n(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t). \tag{3.66}
$$

Estimate in long time of the L^∞ norm of ρ and $\frac{1}{\rho}$ when $\alpha < \frac{1}{2}$

We recall that we have now:

$$
\partial_t u + u\partial_x u - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x u) - \partial_x\varphi(\rho)\,\partial_x u + \frac{a\gamma}{\gamma-1}\partial_x(\rho^{\gamma-1}) = 0.
$$

We deduce that we have for $t \in]t_1, T_\beta]$ with $0 < t_1 < T_\beta$:

$$\partial_t((t-t_1)\partial_x u) - \partial_x u + \partial_x(u\,(t-t_1)\partial_x u) - \partial_x\big(\frac{\mu(\rho)}{\rho}\partial_x((t-t_1)\partial_x u)\big)$$
$$- \partial_x\big((\partial_x\varphi(\rho) + \partial_x(\frac{\mu(\rho)}{\rho}))\,(t-t_1)\partial_x u\big) + \frac{a\gamma}{\gamma-1}\partial_{xx}(\rho^{\gamma-1}) = 0.$$

We set now $v = (t-t_1)\partial_x u$ and we have for $t \in]t_1, T_\beta]$:

$$\partial_t v - \partial_x u + \partial_x(u\,v) - \partial_x(\frac{\mu(\rho)}{\rho}\partial_x v) - \partial_x(\frac{\mu'(\rho)}{\rho}\partial_x\rho\, v) + \frac{a\gamma}{\gamma-1}\partial_{xx}((t-t_1)\rho^{\gamma-1}) = 0$$

Multiplying by v and integrating over $(t_1, t)\times\mathbb{R}$, we obtain:

$$\frac{1}{2}\int_{\mathbb{R}}|v|^2(t,x)dx + \int_{t_1}^t\int_{\mathbb{R}}\frac{\mu(\rho)}{\rho}(s,x)|\partial_x v(s,x)|^2 dsdx - \int_{t_1}^t\int_{\mathbb{R}}\partial_x u(s,x)v(s,x)dsdx$$
$$- \int_{t_1}^t\int_{\mathbb{R}}u(s,x)v(s,x)\partial_x v(s,x)dsdx + \int_{t_1}^t\int_{\mathbb{R}}\frac{\mu'(\rho)}{\rho}\partial_x\rho(s,x)v(s,x)\partial_x v(s,x)dsdx$$
$$- \frac{a\gamma}{\gamma-1}\int_{t_1}^t\int_{\mathbb{R}}(s-t_1)\partial_x\rho^{\gamma-1}(s,x)\partial_x v(s,x)dsdx = 0.$$

Using (3.54) we obtain then on (t_1, t) with $t \in]t_1, T_\beta]$, $c > 0$ and $C > 0$ independent on n:

$$\frac{1}{2}\int_{\mathbb{R}}|v|^2(t,x)dx + \mu c\int_{t_1}^t\int_{\mathbb{R}}|\partial_x v(s,x)|^2 dsdx \le \|\partial_x u\|_{L^2((t_1,t),L^2(\mathbb{R}))}\sqrt{t}\|v\|_{L^\infty((t_1,t),L^2(\mathbb{R}))}$$
$$+ \|\frac{\mu'(\rho)}{\rho}\partial_x\rho\|_{L^\infty((t_1,t),L^2(\mathbb{R}))}\|\partial_x v\|_{L^2((t_1,t),L^2(\mathbb{R}))}\|v\|_{L^2((t_1,t),L^\infty(\mathbb{R}))}$$
$$+ \|u\|_{L^\infty((t_1,t),L^2(\mathbb{R}))}\|\partial_x v\|_{L^2((t_1,t),L^2(\mathbb{R}))}\|v\|_{L^2((t_1,t),L^\infty(\mathbb{R}))}$$
$$+ Ct\|\partial_x\rho^{\gamma-1}\|_{L^2((t_1,t),L^2(\mathbb{R}))}\|\partial_x v\|_{L^2((t_1,t),L^2(\mathbb{R}))}.$$

Next by Gagliardo-Niremberg inequality:

$$\frac{1}{2}\int_{\mathbb{R}}|v|^2(t,x)dx + \mu c\int_{t_1}^t\int_{\mathbb{R}}|\partial_x v(s,x)|^2 dsdx \le$$
$$\|\partial_x u\|_{L^2((t_1,t),L^2(\mathbb{R}))}\sqrt{t}\|v\|_{L^\infty((t_1,t),L^2(\mathbb{R}))} + Ct\|\partial_x\rho^{\gamma-1}\|_{L^2((t_1,t),L^2(\mathbb{R}))}\|\partial_x v\|_{L^2((t_1,t),L^2(\mathbb{R}))}$$
$$+ C(\|\frac{\mu'(\rho)}{\rho}\partial_x\rho\|_{L^\infty((t_1,t),L^2(\mathbb{R}))} + \|u\|_{L^\infty((t_1,t),L^2(\mathbb{R}))})\|\partial_x v\|^{\frac{3}{2}}_{L^2((t_1,t),L^2(\mathbb{R}))}\|v\|^{\frac{1}{2}}_{L^2((t_1,t),L^2(\mathbb{R}))}.$$

Now using Young inequality we deduce that for $C_1 > 0$ large enough:

$$\frac{1}{4}\|v\|^2_{L^\infty((t_1,t),L^2(\mathbb{R}))} + \frac{\mu c}{2}\|\partial_x v\|^2_{L^2((t_1,t),L^2(\mathbb{R}))}$$
$$\le C_1(\|\partial_x u\|^2_{L^2((t_1,t),L^2(\mathbb{R}))}t + t^2\|\partial_x\rho^{\gamma-1}\|^2_{L^2((t_1,t),L^2(\mathbb{R}))}$$
$$+ (\|\frac{\mu'(\rho)}{\rho}\partial_x\rho\|_{L^\infty((t_1,t),L^2(\mathbb{R}))} + \|u\|_{L^\infty((t_1,t),L^2(\mathbb{R}))})^4\|v\|^2_{L^2((t_1,t),L^2(\mathbb{R}))}\big).$$

From (3.54) and (3.58) we deduce that for $t \in (t_1, T_\beta]$, we have for $C_1 > 0$ large enough:

$$\frac{1}{4}\|v\|^2_{L^\infty((t_1,t),L^2(\mathbb{R}))} + \frac{\mu c}{2}\|\partial_x v\|_{L^2((t_1,t),L^2(\mathbb{R}))} \leq C(t,t_1). \tag{3.67}$$

We have showed that for $t_1 \in]0, T_\beta]$ there exists $C(t,t_1) > 0$ independent on n and depending only on the initial data (ρ_0, v_0, u_0) such that for $t \in]t_1, T_\beta]$ and any $n \in \mathbb{N}^*$ we have:

$$\|(t-t_1)\partial_x u_n(t,\cdot)\|_{L^2} \leq C(t,t_1) < +\infty. \tag{3.68}$$

Remark 11 *It is important to mention that the previous estimate is true for any $\alpha > 0$.*

From (3.68) and (3.54) we have obtained that for $t_2 \in]t_1, T_\beta]$ with $0 < t_1 < T_\beta$, we have for $C > 0$ independent on n and depending only on the initial data (ρ_0, u_0, v_0):

$$\begin{cases} \|\partial_x u_n(t_2,\cdot)\|_{L^2(\mathbb{R})} \leq C \\ \|u_n(t_2,\cdot)\|_{L^2(\mathbb{R})} \leq C \\ \|\rho_n(t_2,\cdot) - \bar{\rho}\|_{H^1(\mathbb{R})} \leq C \\ \|(\rho_n, \dfrac{1}{\rho_n})(t_2,\cdot)\|_{L^\infty(\mathbb{R})} \leq C. \end{cases} \tag{3.69}$$

From the theorem of Mellet and Vasseur in [33], we know that there exists a global strong solution (ρ_μ, u_μ) with initial data $(\rho_n(t_2,\cdot), u_n(t_2,\cdot))$. In addition it is proved in [33] that for $t \geq 0$ there exists $C(t) > 0$ depending only on t, $\|u_n(t_2,\cdot)\|_{H^1(\mathbb{R})}$, $\|\rho_n(t_2,\cdot) - \bar{\rho}\|_{H^1(\mathbb{R})}$ and $\|(\rho_n, \frac{1}{\rho_n})(t_2,\cdot)\|_{L^\infty(\mathbb{R})}$ such that for any $t \geq 0$ we have:

$$\|(\frac{1}{\rho_\mu}, \rho_\mu)(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(t). \tag{3.70}$$

We deduce when $0 < \alpha < \frac{1}{2}$ that the regular solution (ρ_n, u_n) verify for any $t \geq t_1$:

$$(\rho_n(t,\cdot), u_n(t,\cdot)) = (\rho_\mu(t-t_1,\cdot), u_\mu(t-t_1,\cdot)), \tag{3.71}$$

This is a direct consequence of the uniqueness of the solution (ρ_μ, u_μ), indeed (ρ_n, u_n) is sufficiently regular such that the solution is in the class of uniqueness of (ρ_μ, u_μ). It implies from (3.72) that there exists $C(t)$ depnding only $\|u_n(t_2,\cdot)\|_{H^1(\mathbb{R})}$, $\|\rho_n(t_2,\cdot) - \bar{\rho}\|_{H^1(\mathbb{R})}$ and $\|(\rho_n, \frac{1}{\rho_n})(t_2,\cdot)\|_{L^\infty(\mathbb{R})}$ (it is important to mention that we have seen that these norms are independent on n, in particular $C(t)$ depends only on t and (ρ_0, u_0, v_0)) such that for any $t \geq 0$ we have:

$$\|(\frac{1}{\rho_n}, \rho_n)(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(t). \tag{3.72}$$

From (3.36) it yields that for any $t > 0$ there exists $C(t) > 0$ independent on n such that:

$$\|\rho_n u_n(t,\cdot)\|_{L^1(\mathbb{R})} + \|\rho_n v_n(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t). \tag{3.73}$$

Gain of integrability on the velocity u in long time when $\alpha > \frac{1}{2}$

Multiplying the momentum equation (1.1) by $u|u|^p$ (with $p = 2$) and integrating over $(t_1, t) \times \mathbb{R}$ with $t_1 \in (0, T_\beta]$ and $t > t_1$, we have then:

$$\begin{aligned} &\frac{1}{p+2}\int_{\mathbb{R}} \rho(t,x)|u|^{p+2}(t,x)dx + (p+1)\int_{t_1}^{t}\int_{\mathbb{R}} \mu(\rho(s,x))|\partial_x u(s,x)|^2|u(s,x)|^p dsdx \\ &+ \int_{t_1}^{t}\int_{\mathbb{R}} \partial_x P(\rho)(s,x)\, u|u|^p(s,x)dsdx = \frac{1}{p+2}\int_{\mathbb{R}} \rho(t_1,x)|u|^{p+2}(t_1,x)dx. \end{aligned} \tag{3.74}$$

From (3.68) and (3.54), and using interpolation we know that there exists $C(t_1) > 0$ independent on n such that:

$$|\int_{\mathbb{R}} \rho(t_1,x)|u|^{p+2}(t_1,x)dx| \leq C(t_1). \tag{3.75}$$

It remains now simply to estimate the term coming from the pressure. We have then using Young inequality for $\epsilon > 0$ and $C_\epsilon > 0$ large enough with $p = 2$:

$$\begin{aligned} &|\int_{t_1}^{t}\int_{\mathbb{R}} \partial_x P(\rho)(s,x)\, u|u|^p(s,x)dsdx| \\ &\leq a(p+1)|\int_{t_1}^{t}\int_{\mathbb{R}} \rho^\gamma(s,x)\,\partial_x u(s,x)|u|^p(s,x)dsdx| \\ &\leq \epsilon\int_{t_1}^{t}\int_{\mathbb{R}} \mu(\rho(s,x))|\partial_x u(s,x)|^2|u(s,x)|^2 dsdx + C_\epsilon\int_{t_1}^{t}\int_{\mathbb{R}} \frac{\rho^{2\gamma-1}(s,x)}{\mu(\rho(s,x))}\rho(s,x)|u|^2(s,x)dsdx. \end{aligned} \tag{3.76}$$

Since we have $2\gamma - 1 \geq \alpha$, we deduce from (3.54) and (3.59) that we have:

$$\begin{aligned} &|\int_{t_1}^{t}\int_{\mathbb{R}} \partial_x P(\rho)(s,x)\, u|u|^p(s,x)dsdx| \\ &\leq \epsilon\int_{t_1}^{t}\int_{\mathbb{R}} \mu(\rho(s,x))|\partial_x u(s,x)|^2|u(s,x)|^2 dsdx \\ &\qquad + C_\epsilon\|\rho\|^{2\gamma-1-\alpha}_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\int_{t_1}^{t} ds\|\sqrt{\rho}u\|^2_{L^\infty([t_1,+\infty),L^2(\mathbb{R}))}. \end{aligned} \tag{3.77}$$

From (3.58), (3.77), (3.75) and (3.74) we deduce that for any $n \in \mathbb{N}^*$ and for any $t_1 \in]0, T_\beta]$ we have:

$$\|\rho_n^{\frac{1}{4}} u_n\|_{L^\infty([t_1,t],L^4(\mathbb{R}))} \leq C(t_1,t), \tag{3.78}$$

with $C(t_1, t)$ depending only on t, t_1 but independent on n.

Compactness when $0 < \alpha < \frac{1}{2}$

We wish now to prove that the sequence $(\rho_n, u_n)_{n\in\mathbb{N}}$ converges up to a subsequence to a global weak solution (ρ, u) of the system (1.1). Using (3.58), (3.54), (3.72) and (3.73))

we have seen that there exists $C>0, C(t)>0$ independent on n (with $C(t)$ depending only on t) such that we have for any $t>0$:

$$\begin{cases} \|\sqrt{\rho_n}v_n(t,\cdot)\|_{L^2(\mathbb{R})} \leq C \\ \|\sqrt{\rho_n}u_n(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(1+\frac{1}{\sqrt{t}}) \\ \|(\sqrt{s}1_{\{s\leq T_\beta\}}+\sqrt{\mu(\rho_n)}1_{\{s\geq T_\beta\}})\partial_x u_n\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|\partial_x \rho_n^{\alpha-\frac{1}{2}}(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(1+\frac{1}{\sqrt{t}}) \\ \|\partial_x \rho_n^{\frac{1}{2}(\gamma+\alpha-1)}\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|u_n\|_{L^{p(s)-\epsilon}([0,T_\beta],H^s(\mathbb{R}))} \leq C \\ \|\rho_n(t,\cdot)-\bar{\rho}\|_{L^\gamma_2(\mathbb{R})} \leq C \\ \|(\frac{1}{\rho_n},\rho_n)(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(t) \\ \|\rho_n u_n(t,\cdot)\|_{L^1(\mathbb{R})}+\|\rho_n v_n(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t). \end{cases} \tag{3.79}$$

with $s\in(0,1)$, $p(s)=\frac{3}{1+2s}$ and $\epsilon>0$ sufficiently small such that $p(s)-\epsilon\geq 1$. From (3.79), we deduce that there exists $C(T)>0$ independent on n such that for any $T>0$ we have:

$$\|\rho_n-\bar{\rho}\|_{L^2((0,T),H^1(\mathbb{R}))} \leq C(T). \tag{3.80}$$

In addition we have $\partial_t(\rho_n-\bar{\rho})=-\partial_x(\rho_n u_n)$, then from (3.79) $\partial_t(\rho_n-\bar{\rho})$ is uniformly bounded in $L^1((0,T),H^{-1}(\mathbb{R}))$. From the Lemma 1, the diagonal process and compact Sobolev embedding, it implies that up to a subsequence ρ_n converges strongly to ρ in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R})$. We deduce that up to a subsequence ρ_n converges almost everywhere to ρ on $]0,+\infty[\times\mathbb{R}$. From Fatou lemma and (3.79), we obtain that there exists $C(t)>$ depending only on $t>0$ such that:

$$\begin{cases} \|\rho(t,\cdot)-\bar{\rho}\|_{L^\gamma_2(\mathbb{R})} \leq C \\ \|(\frac{1}{\rho},\rho)(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(t). \end{cases} \tag{3.81}$$

Next from (3.79), we deduce that u_n is uniformly bounded in $L^{p(s)-\epsilon}([0,T_\beta],H^s(\mathbb{R}))$ and u_n is uniformly bounded in $L^2_{loc}([T_\beta,+\infty),H^1(\mathbb{R}))$. Using classical paraproduct laws (see[1]) we obtain that for $t\in]0,T_\beta]$ and $0<s<\frac{1}{2}$:

$$\begin{aligned} &\|T_{(\rho_n(t,\cdot)-\bar{\rho})}u_n(t,\cdot)\|_{H^s(\mathbb{R})} \leq \|\rho_n(t,\cdot)-\bar{\rho}\|_{L^\infty(\mathbb{R})}\|u_n(t,\cdot)\|_{H^s(\mathbb{R})} \\ &\|T_{u_n(t,\cdot)}(\rho_n(t,\cdot)-\bar{\rho})\|_{H^{\frac{1}{2}+s}(\mathbb{R})} \leq \|\rho_n(t,\cdot)-\bar{\rho}\|_{H^1(\mathbb{R})}\|u_n(t,\cdot)\|_{H^s(\mathbb{R})} \\ &\|R(u_n(t,\cdot),(\rho_n(t,\cdot)-\bar{\rho}))\|_{H^{\frac{1}{2}+s}(\mathbb{R})} \leq \|\rho_n(t,\cdot)-\bar{\rho}\|_{H^1(\mathbb{R})}\|u_n(t,\cdot)\|_{H^s(\mathbb{R})} \end{aligned}$$

We deduce from (3.79) and (3.80) that $\rho_n u_n$ is uniformly bounded in $L^{q(s)}([0,T_\beta],H^s(\mathbb{R}))+L^{p(s)-\epsilon}([0,T_\beta],H^s(\mathbb{R}))$ with $0<s<\frac{1-\epsilon}{2(2+\epsilon)}$ and $\frac{1}{q(s)}=\frac{1}{2}+\frac{1}{p(s)-\epsilon}$ ($p(s)=\frac{3}{1+2s}$ and $\epsilon>0$ sufficiently small such that $p(s)-\epsilon\geq 1$). In particular since $p(s)-\epsilon\geq q(s)\geq 1$, we have proved that $\rho_n u_n$ is uniformly bounded in $L^{q(s)}([0,T_\beta],H^s(\mathbb{R}))$.

From (3.79), we observe that u_n and $(\rho_n - \bar{\rho})$ are uniformly bounded in $L^2([T_\beta, +\infty[, H^1(\mathbb{R}))$, by product of Sobolev space we deduce that $\rho_n u_n$ is bounded in $L^1_{loc}([T_\beta, +\infty[, H^1(\mathbb{R}))$. We have finally proved that $\rho_n u_n$ is bounded in $L^1_{loc}(\mathbb{R}^+, H^s(\mathbb{R}))$ for $0 < s < \frac{1-\epsilon}{2(2+\epsilon)}$ with $\epsilon > 0$ sufficiently small.
We recall now that we have:

$$\partial_t(\rho_n u_n) = -\partial_x(\rho_n u_n^2) - \partial_{xx}(\mu_n(\rho_n)u_n) + \partial_x(u_n \partial_x(\mu_n(\rho_n))) + \partial_x(a(\rho_n^\gamma - \bar{\rho}^\gamma)).$$

Let us deal with the term $\partial_x(u_n \partial_x(\mu_n(\rho_n)))$, from (3.79) we know that $\partial_x(\mu_n(\rho_n))$ is uniformly bounded in $L^2(\mathbb{R}^+, L^2(\mathbb{R}))$ and u_n is uniformly bounded in $L^2_{loc}(\mathbb{R}^+, L^2(\mathbb{R}))$, we deduce that $\partial_x(u_n \partial_x(\mu_n(\rho_n)))$ is uniformly bounded in $L^1_{loc}(\mathbb{R}^+, W^{-1,1}(\mathbb{R}))$. We proceed similarly for the other terms and we deduce by Sobolev embedding that $\partial_t(\rho_n u_n)$ is uniformly bounded in $L^1_{loc}(\mathbb{R}^+, H^{-s_1}(\mathbb{R}))$ with s_1 sufficiently large.
Since we have seen that $\rho_n u_n$ is bounded in $L^1_{loc}(\mathbb{R}^+, H^s(\mathbb{R}))$ for $0 < s < \frac{1-\epsilon}{2(2+\epsilon)}$ with $\epsilon > 0$ sufficiently small, using the lemma 1 we show that $\rho_n u_n$ converges strongly up to a subsequence to m in $L^1_{loc}(\mathbb{R}^+, L^2_{loc}(\mathbb{R}))$. It implies in particular that up to a subsequence $\rho_n u_n$ converges almost everywhere to m in $]0, +\infty[\times\mathbb{R}$. We define now u as $u = \frac{m}{\rho}$ which is well defined since $\frac{1}{\rho}$ belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R})$.
From Fatou lemma, we have for any $t > 0$:

$$\begin{cases} \|\sqrt{\rho} u(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(1 + \dfrac{1}{\sqrt{t}}) \\ \|m(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t). \end{cases} \tag{3.82}$$

From (3.79) we observe that u_n is uniformly bounded in $L^{p(s)-\epsilon}([0, T_\beta], H^s(\mathbb{R}))$ and is also uniformly bounded in $L^\infty([T_\beta, +\infty[, L^2(\mathbb{R})) \cap L^2([T_\beta, +\infty[, \dot{H}^1(\mathbb{R}))$. By Gagliardo-Niremberg we deduce that u_n is uniformly bounded in $L^4([T_\beta, +\infty[, L^\infty(\mathbb{R}))$. By interpolation it implies that u_n is uniformly bounded in $L^6([T_\beta, +\infty[, L^6(\mathbb{R}))$. Now taking $s = \epsilon_1 > 0$ sufficiently small and using the fact that u_n is uniformly bounded in $L^{p(s)-\epsilon}([0, T_\beta], H^s(\mathbb{R}))$, we deduce by Sobolev embedding that u_n is uniformly bounded in $L^{2+\epsilon_2}([0, T_\beta], L^{p_2}(\mathbb{R}))$ with $\epsilon_2 > 0$ sufficiently small and $p_2 > 2$. It implies in particular that there exists $C > 0$ independent on n and $\epsilon_2 > 0$ sufficiently small such that:

$$\|u_n\|_{L^{2+\epsilon_2}_{loc}(\mathbb{R}^+ \times \mathbb{R})} \leq C. \tag{3.83}$$

Using the Lemma 3, Lemma 1, (3.79), (3.83) and the fact that up to a subsequence $(\rho_n, u_n, m_n)_{n\in\mathbb{N}}$ converges almost everywhere to (ρ, u, m), we deduce that up to a subsequence:

- $(\rho_n, \frac{1}{\rho_n})_{n\in\mathbb{N}}$ converges strongly to $(\rho, \frac{1}{\rho})$ in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R})$ for any $1 \leq p < +\infty$.
- For any $T > 0$, $(\rho_n)_{n\in\mathbb{N}}$ converges strongly to ρ in $C([0, T], W^{-\epsilon,p}_{loc}(\mathbb{R}))$ for any $1 \leq p < +\infty$ and $\epsilon > 0$.
- u_n, $\sqrt{\rho_n} u_n$, m_n converges strongly in $L^{2+\epsilon_2}_{loc}(\mathbb{R}^+ \times \mathbb{R})$ for $\epsilon_2 > 0$ sufficiently small.

In particular since $\sqrt{\rho_n} v_n$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$, we deduce that up to a subsequence $(\sqrt{\rho_n} v_n)_{n\in\mathbb{N}}$ converges weakly in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ to m_3. Since $\sqrt{\rho_n} v_n =$

$\sqrt{\rho_n}u_n + \partial_x\varphi_2(\rho_n)$, we have seen using the previous informations that $(\sqrt{\rho_n}v_n)_{n\in\mathbb{N}}$ converges in the sense of the distributions to $\sqrt{\rho}u + \partial_x\varphi_2(\rho) = m^2$. It implies that $m_3 = m^2$. Now using the Fatou lemma for weak convergence we deduce that there exists $C > 0$ such that:

$$\|m2\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))} \leq C. \tag{3.84}$$

Since $(\rho_n, u_n)_{n\in\mathbb{N}}$ is a regular solution, we have for any test function $\varphi \in C_0^\infty([0,+\infty[\times\mathbb{R})$ after integration by parts:

$$\int_0^{+\infty}\int_{\mathbb{R}} \big(\rho_n u_n(s,x)\partial_s\varphi(s,x) + \rho_n u_n^2(s,x)\partial_x\varphi(s,x) + \mu_n(\rho_n)\, u_n(s,x)\partial_{xx}\varphi(s,x) \\ + \partial_x(\mu_n(\rho_n))\, u_n(s,x)\partial_x\varphi(s,x) + \rho_n^\gamma(s,x)\partial_x\varphi(s,x)\big)ds\,dx + \int_{\mathbb{R}} \rho_n u_n(0,x)\varphi(0,x)dx = 0. \tag{3.85}$$

From (3.79), we know that $\partial_x(\mu_n(\rho_n))$ is uniformly bounded in $L^2(\mathbb{R}^+, L^2(\mathbb{R}))$, then up to a subsequence $\partial_x(\mu_n(\rho_n))$ converges weakly in $L^2(\mathbb{R}^+, L^2(\mathbb{R}))$ to v. Since $\mu_n(\rho_n)$ converges to $\mu(\rho)$ in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R})$ for any $1 \leq p < +\infty$ using dominated convergence, we deduce that $v = \partial_x\mu(\rho)$ in the sense of the distributions. Now since we have seen that $u_n\partial_x\varphi$ converges strongly in $L^2(\mathbb{R}^+, L^2(\mathbb{R}))$ up to a subsequence to $u\partial_x\varphi$ we deduce that up to a subsequence we have:

$$\lim_{n\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}} \partial_x(\mu_n(\rho_n))\, u_n(s,x)\partial_x\varphi(s,x)dxds = \int_0^{+\infty}\int_{\mathbb{R}} \partial_x(\mu(\rho))\, u(s,x)\partial_x\varphi(s,x)dxds \tag{3.86}$$

Proceeding similarly for the other terms we have proved that (ρ, u) is a global weak solution in the sense of the definition 2.1. It is important in particular to observe that $m_n(0,\cdot)$ converges to m_0 in the sense of the measure and we have then:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} \rho_n u_n(0,x)\varphi(0,x)dx = \int_{\mathbb{R}} m_0(x)\varphi(0,x)dx. \tag{3.87}$$

We proceed also similarly for the mass equation. From (3.79) $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ and converges weakly * to v in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$. From the Fatou properties for the weak convergence we deduce that for any $t > 0$:

$$\begin{cases} \|v\|_{L^\infty([0,t],L^2(\mathbb{R}))} \leq C \\ \|(\sqrt{s}1_{\{s\leq T_\beta\}} + 1_{\{s\geq T_\beta\}})\partial_x u\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|\partial_x\rho^{\alpha-\frac{1}{2}}\|_{L^2([0,T_\beta],L^2(\mathbb{R}))+L^\infty([T_\beta,+\infty[,L^2(\mathbb{R}))} \leq C \\ \|\partial_x\rho^{\frac{1}{2}(\gamma+\alpha-1)}\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|u\|_{L^{p(s)-\epsilon}([0,T_\beta],H^s(\mathbb{R}))} \leq C. \end{cases} \tag{3.88}$$

Compactness when $\alpha \geq 1$

This case is more delicate essentially because $\frac{1}{\rho_n}$ is a priori not uniformly bounded in $L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}))$. We only know that there exists $T_\beta > 0$ independent on n such that $\frac{1}{\rho_n}$ is uniformly bounded in $L^\infty([0,T_\beta], L^\infty(\mathbb{R}))$. In particular we have the same estimates on (ρ_n, u_n, v_n) and (ρ, u, v) on the time interval $[0, T_\beta]$ ((ρ, u, v) are defined in a similar

way).
The main difficulty is to prove that ρ_n and a certain momentum $m_n = \rho_n^{\beta_1} u_n$ converges almost everywhere up to a subsequence to ρ and m.

Convergence almost everywhere of $(\rho_n)_{n\in\mathbb{N}}$

From (3.58) and (3.59), we have for any $\varphi \in C_0^\infty(\mathbb{R})$ with supp$\varphi \subset]-M, M[$, there exists $C(t_1) > 0$, $C > 0$ independent on n such that for $t_1 \in]0, T_\beta]$:

$$\begin{aligned}\|\partial_x(\varphi\rho_n^\alpha)\|_{L^\infty([t_1,+\infty[,L^1(\mathbb{R}))} &\leq C\|\varphi\sqrt{\rho_n}\|_{L^\infty([t_1,+\infty[,L^2(\mathbb{R}))}\|\partial_x\rho_n^{\alpha-\frac{1}{2}}\|_{L^\infty([t_1,+\infty[,L^2(\mathbb{R}))} \\ &\quad + \|\rho_n^\alpha\partial_x\varphi\|_{L^\infty([t_1,+\infty[,L^1(\mathbb{R}))} \\ &\leq (\|\varphi\|_{L^\infty(\mathbb{R})} + \|\partial_x\varphi\|_{L^\infty(\mathbb{R})})C(M,t_1).\end{aligned} \tag{3.89}$$

with $C(M, t_1)$ depending only on t_1, M. Next we have:

$$\partial_t(\varphi\rho_n^\alpha) = (1-\alpha)\varphi\rho_n^\alpha\partial_x u_n - \partial_x(\varphi\rho_n^\alpha u_n) + \partial_x\varphi\,\rho_n^\alpha u_n.$$

From (3.59) and (3.58), we deduce that $\varphi\rho_n^\alpha\partial_x u_n$ is uniformly bounded in $L^2([t_1,+\infty[, L^2(\mathbb{R}))$, $\partial_x(\varphi\rho_n^\alpha u_n)$ is uniformly bounded (since $\alpha > \frac{1}{2}$) in $L^\infty([t_1,+\infty[, W^{-1,2}(\mathbb{R})))$, $\partial_x\varphi\,\rho_n^\alpha u_n$ is uniformly bounded in $L^\infty([t_1,+\infty[, L^2(\mathbb{R})))$. It implies that $\partial_t(\varphi\rho_n^\alpha)$ is uniformly bounded in $L^2_{loc}([t_1,+\infty[, W^{-1,2}(\mathbb{R})))$. From the Lemma 1, we deduce that ρ_n^α converges strongly up to a subsequence in $C([t_1, T], L^{\beta_1}_{loc}(\mathbb{R})))$ for any $1 \leq \beta_1 < +\infty$ and $T > t_1$. Up to a subsequence it implies that ρ_n^α converges to ρ^α almost everywhere on $[t_1,+\infty[\times\mathbb{R}$. In particular up to a subsequence ρ_n converges almost everywhere to ρ on $[t_1,+\infty[\times\mathbb{R}$.
Now using the same arguments as the proof for $\alpha < \frac{1}{2}$ on $]0, t_1]$ with $t_1 \in]0, T_\beta]$, we obtain that ρ_n converges almost everywhere on $]0, t_1] \times \mathbb{R}$ to ρ up to a subsequence. Indeed we can use similar ideas since we know from (3.54) that $(\rho_n, \frac{1}{\rho_n})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0, T_\beta], L^\infty(\mathbb{R}))$. We have finally proved that ρ_n converges almost everywhere to ρ on $]0,+\infty[\times\mathbb{R}$. From (3.59) we deduce that for any $t > 0$ we have for $C > 0$:

$$\|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C. \tag{3.90}$$

In addition it implies using the dominated convergence and (3.59) that ρ_n^γ converges up to a subsequence to ρ^γ in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$.

Convergence almost everywhere of $(\rho_n^\alpha u_n)_{n\in\mathbb{N}}$ when $\alpha \geq 1$

We set $m_n = \rho_n^\alpha u_n$, we deduce from (3.58) and (3.59)that there exists $C(t_1) > 0$ such that for $t_1 \in]0, T_\beta]$ and $t > t_1$:

$$\begin{aligned}&\|\partial_x(\rho_n^\alpha u_n)\|_{L^2([t_1,t],L^2(\mathbb{R}))+L^\infty([t_1,t],L^1(\mathbb{R}))} \leq C\big(\|\rho_n^{\frac{\alpha}{2}}\partial_x u_n\|_{L^2([t_1,t],L^2(\mathbb{R}))}\|\rho_n^{\frac{\alpha}{2}}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))} \\ &+ \|\partial_x(\rho_n^{\alpha-\frac{1}{2}})\|_{L^\infty([t_1,t],L^2(\mathbb{R}))}\|\sqrt{\rho_n}u_n\|_{L^\infty([t_1,t],L^2(\mathbb{R}))}\big) \\ &\leq C(t_1),\end{aligned} \tag{3.91}$$

and:

$$\begin{aligned}\|\rho_n^\alpha u_n\|_{L^\infty([t_1,t],L^2(\mathbb{R}))} &\leq \|\sqrt{\rho_n}u_n\|_{L^\infty([t_1,t],L^2(\mathbb{R}))}\|\rho_n^{\alpha-\frac{1}{2}}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))}\\ &\leq C(t_1).\end{aligned} \tag{3.92}$$

Next we have:

$$\begin{aligned}&\partial_t(\rho_n^\alpha u_n) = \rho_n^{\alpha-1}\partial_t(\rho_n u_n) + \rho_n u_n \partial_t \rho_n^{\alpha-1}\\ &= \partial_x(\mu_n(\rho_n)\rho_n^{\alpha-1}\partial_x u_n) - \partial_x(\rho_n^{\alpha-1})\mu_n(\rho_n)\partial_x u_n - \partial_x(\rho_n^\alpha u_n^2) + \partial_x(\rho_n^{\alpha-1})\rho_n u_n^2\\ &- \frac{a\gamma}{\gamma+\alpha-1}\partial_x\rho_n^{\gamma+\alpha-1} - (\alpha-1)\rho_n^\alpha u_n\partial_x u_n - \rho_n u_n^2\partial_x(\rho_n^{\alpha-1})\\ &= \partial_x(\mu_n(\rho_n)\rho_n^{\frac{\alpha}{2}-1}\rho_n^{\frac{\alpha}{2}}\partial_x u_n) - (\alpha-1)\frac{\mu_n(\rho_n)}{\rho_n^{\frac{3}{2}}}\partial_x(\rho_n)\sqrt{\mu_n(\rho_n)}\partial_x u_n\frac{\rho_n^{\alpha-\frac{1}{2}}}{\sqrt{\mu_n(\rho_n)}} - \partial_x(\rho_n u_n^2\rho_n^{\alpha-1})\\ &- (\alpha-1)\rho_n^{\frac{\alpha}{2}}u_n\rho_n^{\frac{\alpha}{2}}\partial_x u_n - \frac{a\gamma}{\gamma+\alpha-1}\partial_x\rho_n^{\gamma+\alpha-1}.\end{aligned}$$

Here since $\alpha \geq 1$ it is important to point out that from (3.59), $\frac{\rho_n^{\alpha-\frac{1}{2}}}{\sqrt{\mu_n(\rho_n)}}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}))$. From (3.58) and (3.59) it yields that for $C>0, C(t_1)>0$ independent on n:

$$\begin{aligned}&\|\partial_t(\rho_n^\alpha u_n)\|_{L^2([t_1,t],H^{-1}(\mathbb{R}))+L^\infty([t_1,t],W^{-1,1}(\mathbb{R}))+L^2([t_1,t],L^1(\mathbb{R})+L^\infty([t_1,t],W^{-1,\infty}(\mathbb{R}))}\\ &\leq C\big(\|\mu_n(\rho_n)\rho_n^{\frac{\alpha}{2}-1}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))}\|\rho_n^{\frac{\alpha}{2}}\partial_x u_n\|_{L^2([t_1,t],L^2(\mathbb{R}))}\\ &+ \|\frac{\mu_n(\rho_n)}{\rho_n^{\frac{3}{2}}}\partial_x(\rho_n)\|_{L^\infty([t_1,t],L^2(\mathbb{R}))}\|\sqrt{\mu_n(\rho_n)}\partial_x u_n\|_{L^2([t_1,t],L^2(\mathbb{R}))}\|\frac{\rho_n^{\alpha-\frac{1}{2}}}{\sqrt{\mu_n(\rho_n)}}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))}\\ &+ \|\rho_n u_n^2\|_{L^\infty([t_1,t],L^1(\mathbb{R}))}\|\rho_n^{\alpha-1}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))} + \|\rho_n^{\gamma+\alpha-1}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))}\\ &+ \|\rho_n^{\frac{1}{2}}u_n\|_{L^\infty([t_1,t],L^2(\mathbb{R}))}\|\rho_n^{\frac{1}{2}(\alpha-1)}\|_{L^\infty([t_1,t],L^\infty(\mathbb{R}))}\|\rho_n^{\frac{\alpha}{2}}\partial_x u_n\|_{L^2([t_1,t],L^2(\mathbb{R}))}\\ &\leq C(t_1).\end{aligned} \tag{3.93}$$

We have seen that $(\rho_n^\alpha u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^2_{loc}([t_1,+\infty[,W^{1,1}_{loc}(\mathbb{R}))$ and $(\partial_t(\rho_n^\alpha u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^2_{loc}([t_1,+\infty[,W^{-s,\infty}_{loc}(\mathbb{R}))$ for $s>0$ large enough by Sobolev embedding.
From (3.91), (3.92), (3.93), the diagonal process and Lemma 1, we deduce that $(\rho_n^\alpha u_n)_{n\in\mathbb{N}}$ converges strongly up to a subsequence to m_α in $L^2_{loc}([t_1,+\infty[\times\mathbb{R})$. In addition up to a subsequence, it implies that $\rho_n^\alpha u_n$ converges almost everywhere on $[t_1,+\infty[\times\mathbb{R}$ to m_α.

Note that we can already define the velocity u with $u(t,x)=\frac{m_\alpha(t,x)}{\rho^\alpha(t,x)}$ on the set $\{(t,x)\in\mathbb{R}^+\times\mathbb{R};\ \rho(t,x)>0\}$. Let us verify now that $m_\alpha=0$ almost everywhere on the set $\{(t,x)\in\mathbb{R}^+\times\mathbb{R};\ \rho(t,x)=0\}$. Indeed we have for any $t>0$ and using the Fatou Lemma and (3.58):

$$\begin{aligned}\int_{\{\rho(t,x)\leq 1\}}\liminf_{n\to+\infty}\frac{(\rho_n^\alpha u_n(t,x))^2}{\rho_n^{2\alpha-1}(t,x)}dx &\leq \liminf_{n\to+\infty}\int_{\mathbb{R}}\rho_n|u_n|^2(t,x)dx\\ &\leq C(1+\frac{1}{t}),\end{aligned}$$

with $C > 0$ independent on n. It implies that $m_\alpha = 0$ almost everywhere on the set $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R};\ \rho(t,x) = 0\}$. We have then:

- $m_\alpha(t,x) = 0$ almost everywhere on $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R};\ \rho(t,x) = 0\}$.
- $u(t,x) = \frac{m_\alpha(t,x)}{\rho^\alpha(t,x)}$ on $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R};\ \rho(t,x) > 0\}$ and $u(t,x) = 0$ on $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R};\ \rho(t,x) = 0\}$.

We can observe that u is not uniquely defined on $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R};\ \rho(t,x) = 0\}$. Now from (3.78) and the Fatou lemma we deduce that there exists $C(t,t_1) > 0$ such that for any $t \geq t_1$ we have:

$$\|\rho^{\frac{1}{4}} u\|_{L^\infty([t_1,t],L^4(\mathbb{R}))} \leq C(t_1,t). \tag{3.94}$$

Similarly from (3.66) we have for any $t > 0$:

$$\|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(t). \tag{3.95}$$

We are going now to prove the strong L^2_{loc} convergence of $\sqrt{\rho_n} u_n$.

Lemma 5 *We have for any $T > 0$ and any compact K of $\mathbb{R}$:*

$$\int_0^T \int_K |\sqrt{\rho_n}(t,x) u_n(t,x) - \sqrt{\rho}(t,x) u(t,x)|^2 dt dx \to_{n \to +\infty} 0. \tag{3.96}$$

This limit is true up to a subsequence.

Proof: Following the proof of the case $\alpha < \frac{1}{2}$ (since $(\rho_n, \frac{1}{\rho_n})_{n \in \mathbb{N}}$ is uniformly bounded on $[0, T_\beta]$), we know that up to a subsequence we have for any compact K:

$$\int_0^{T_\beta} \int_K |\sqrt{\rho_n}(t,x) u_n(t,x) - \sqrt{\rho}(t,x) u(t,x)|^2 dt dx \to_{n \to +\infty} 0. \tag{3.97}$$

Let us deal now with the case $T > T_\beta$. We have seen that $\sqrt{\rho_n} u_n$ converges almost everywhere to $\sqrt{\rho} u$ on the set $\{(t,x) \in [t_1, +\infty[\times \mathbb{R};\ \rho(t,x) > 0\}$ with $t_1 \in]0, T_\beta]$. It implies that we have for any $T > 0$ using dominated convergence, (3.59) and taking $M > 0$ with $K_1 = K \cap \{(t,x) \in [t_1, T] \times \mathbb{R};\ \rho(t,x) > 0\}$:

$$\int_{t_1}^T \int_{K_1} |\sqrt{\rho_n}(t,x) u_n(t,x) 1_{\{|u_n|(t,x) \leq M\}} - \sqrt{\rho}(t,x) u(t,x) 1_{\{|u|(t,x) \leq M\}}|^2 dt dx \to_{n \to +\infty} 0. \tag{3.98}$$

We have now using (3.78), (3.94) and (3.59) that there exists $C > 0$ independent on n such that:

$$\begin{aligned}
&\int_{t_1}^T \int_{K \cap \{(t,x) \in [t_1,T] \times \mathbb{R};\ \rho(t,x) > 0\}} |\sqrt{\rho_n}(t,x) u_n(t,x)|^2 1_{\{|u_n|(t,x) > M\}} dt dx \\
&\leq \frac{C}{M^2} \int_{t_1}^T \int_{K \cap \{(t,x) \in [t_1,T] \times \mathbb{R};\ \rho(t,x) > 0\}} \rho_n |u_n|^4(t,x) dt dx \\
&\leq \frac{C(T,t_1)}{M^2} \to_{M \to +\infty} 0,
\end{aligned} \tag{3.99}$$

similarly we have:

$$\begin{aligned}&\int_{t_1}^{T}\int_{K\cap\{(t,x)\in[t_1,T]\times\mathbb{R};\ \rho(t,x)>0\}}|\sqrt{\rho_n}(t,x)u(t,x)|^2 1_{\{|u|(t,x)>M\}}dtdx\\&\leq\frac{C(T,t_1)}{M^2}\rightarrow_{M\rightarrow+\infty}0.\end{aligned}\tag{3.100}$$

On the set $\{(t,x)\in[t_1,+\infty[\times\mathbb{R};\ \rho(t,x)=0\}$, we have for any $M>0$:

$$\sqrt{\rho_n}|u_n(t,x)|1_{\{|u_n|(t,x)\leq M\}}\leq M\sqrt{\rho_n}(t,x)\rightarrow_{n\rightarrow+\infty}0=\sqrt{\rho}|u|(t,x).$$

From convergence dominated and Tchebytchev lemma we deduce that:

$$\int_{t_1}^{T}\int_{K\cap\{(t,x)\in[t_1,T]\times\mathbb{R};\ \rho(t,x)=0\}}|\sqrt{\rho_n}(t,x)u_n(t,x)|^2dtdx\rightarrow_{n\rightarrow+\infty}0.\tag{3.101}$$

From (3.97), (3.98), (3.99), (3.100), (3.101) and the fact that $\sqrt{\rho}u=0$ on $\{(t,x)\in[t_1,+\infty[\times\mathbb{R},\ \rho(t,x)=0\}$ we deduce (3.96). □

Let us prove now that the sequel $(\rho_n u_n)_{n\in\mathbb{N}}$ converges in the sense of the distribution to ρu. Indeed we have for any $T>0$, any K a compact of $\mathbb{R}$ and $C(T,K)>0$:

$$\begin{aligned}\|\rho_n u_n-\rho u\|_{L^1([0,T],L^1(K))}\leq C(T,K)\big(&\|\sqrt{\rho_n}\|_{L^\infty([0,T],L^\infty(K))}\|\sqrt{\rho_n}u_n-\sqrt{\rho}u\|_{L^2([0,T],L^2(K))}\\&+\|\sqrt{\rho_n}-\sqrt{\rho}\|_{L^2([0,T],L^2(K))}\|\sqrt{\rho}u\|_{L^2([0,T],L^2(K))}\big).\end{aligned}\tag{3.102}$$

From (3.96), (3.59), convergence dominated and the fact that $\sqrt{\rho_n}$ converges almost everywhere to $\sqrt{\rho}$ we deduce that $\rho_n u_n$ converges to ρu in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R})$. Now from (3.59), (3.78), (3.94), (3.55) and by interpolation we deduce that $\rho_n u_n$ converges to ρu in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R})$.

Let us deal now with the term of viscosity, we have then for any $\varphi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R})$:

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}\mu_n(\rho_n)\partial_x u_n(t,x)\partial_x\varphi(t,x)dxdt=-\mu\int_0^\infty\int_{\mathbb{R}}\rho_n^{\alpha-\frac{1}{2}}\sqrt{\rho_n}u_n(t,x)\partial_{xx}\varphi(t,x)dxdt\\&-\frac{1}{n}\int_0^\infty\int_{\mathbb{R}}\rho_n^\theta u_n(t,x)\partial_{xx}\varphi(t,x)dxdt-\int_0^\infty\int_{\mathbb{R}}\partial_x(\mu_n(\rho_n))(t,x)u_n(t,x)\partial_x\varphi(t,x)dxdt.\end{aligned}$$

We have only to consider the case of the interval $[t_1,+\infty[$ with $t_1\in]0,T_\beta]$ since the procedure follows the proof of the case $\alpha<\frac{1}{2}$ on $[0,T_\beta]$. We know from (3.58) and (3.78) that we have for any $t\in[t_1,+\infty[$:

$$\begin{cases}\|\partial_x\rho_n^{\alpha-\frac{1}{2}}(t,\cdot)\|_{L^2(\mathbb{R})}+\dfrac{1}{n}\|\rho_n^{\theta-\frac{3}{2}}\partial_x\rho_n(t,\cdot)\|_{L^2(\mathbb{R})}+\dfrac{1}{\sqrt{n}}\|\rho_n^{\frac{\alpha+\theta-3}{2}}\partial_x\rho_n(t,\cdot)\|_{L^2(\mathbb{R})}\leq C(t_1)\\\|\rho_n^{\frac{1}{4}}u_n(t,\cdot)\|_{L^4(\mathbb{R})}\leq C(t,t_1).\end{cases}\tag{3.103}$$

We have now:

$$u_n\partial_x(\mu_n(\rho_n))=\frac{\mu\alpha}{\alpha-\frac{1}{2}}\sqrt{\rho_n}u_n\partial_x\rho_n^{\alpha-\frac{1}{2}}+\frac{\theta}{n}\rho_n^{\frac{1}{4}}u_n\,\rho_n^{\theta-\frac{5}{4}}\partial_x\rho_n.$$

From (3.103), we have setting $\theta_1 \in (0,1)$ such that $\theta - \frac{5}{4} = \theta_1(\alpha - \frac{3}{2}) + (1-\theta_1)(\theta - \frac{3}{2})$ (this is true because we have $\alpha \geq 1$ and $\theta \leq \frac{1}{2}$) and for any $t \geq t_1$:

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_n^{2\theta-5}(t,x)|\partial_x \rho_n(t,x)|^2 dx \\
&\leq (\int_{\mathbb{R}} \rho_n^{2(\theta-\frac{3}{2})}(t,x)|\partial_x \rho_n(t,x)|^2 dx)^{(1-\theta_1)} (\int_{\mathbb{R}} \rho_n^{2(\alpha-\frac{3}{2})}(t,x)|\partial_x \rho_n(t,x)|^2 dx)^{\theta_1} \\
&\leq n^{2(1-\theta_1)} C(t_1).
\end{aligned} \tag{3.104}$$

Using Hölder's inequality, (3.103) and (3.104), we deduce that we have:

$$\begin{aligned}
&|\frac{\theta}{n}\int_{t_1}^{+\infty}\int_{\mathbb{R}} \rho_n^{\frac{1}{4}} u_n \, \rho_n^{\theta-\frac{5}{4}} \partial_x \rho_n(t,x)\partial_x\varphi(t,x)dtdx| \\
&\leq \frac{\theta}{n}\|\rho_n^{\frac{1}{4}} u_n |\partial_x\varphi|^{\frac{1}{2}}\|_{L^\infty([t_1,+\infty[,L^4(\mathbb{R}))} \|\rho_n^{\theta-\frac{5}{4}}\partial_x\rho_n |\partial_x\varphi|^{\frac{1}{2}}\|_{L^1([t_1,+\infty[,L^{\frac{4}{3}}(\mathbb{R}))} \\
&\leq C(\text{supp}\varphi)(\|\partial_x\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+1)\frac{\theta}{n}\|\rho_n^{\frac{1}{4}} u_n\|_{L^\infty([t_1,+\infty[,L^4(\mathbb{R}))} \\
&\qquad\qquad \times \|\rho_n^{\theta-\frac{5}{4}}\partial_x\rho_n\|_{L^\infty([t_1,+\infty[,L^2(\mathbb{R}))} \\
&\leq C(\text{supp}\varphi)C(t_1)(\|\partial_x\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+1)\frac{\theta}{n^{\theta_1}}.
\end{aligned} \tag{3.105}$$

We deduce then that:

$$\frac{\theta}{n}\int_{t_1}^{+\infty}\int_{\mathbb{R}} \rho_n^{\frac{1}{4}} u_n \, \rho_n^{\theta-\frac{5}{4}} \partial_x \rho_n(t,x)\partial_x\varphi(t,x)dtdx \to_{n\to 0} 0. \tag{3.106}$$

It is easy now to verify that:

$$\frac{\theta}{n}\int_{0}^{t_1}\int_{\mathbb{R}} \rho_n^{\frac{1}{4}} u_n \, \rho_n^{\theta-\frac{5}{4}} \partial_x \rho_n(t,x)\partial_x\varphi(t,x)dtdx \to_{n\to 0} 0. \tag{3.107}$$

We have finally proved that:

$$\frac{\theta}{n}\int_{0}^{+\infty}\int_{\mathbb{R}} \rho_n^{\frac{1}{4}} u_n \, \rho_n^{\theta-\frac{5}{4}} \partial_x \rho_n(t,x)\partial_x\varphi(t,x)dtdx \to_{n\to 0} 0. \tag{3.108}$$

Similarly taking $\theta = \frac{1}{4}$, we obtain from (3.103):

$$\frac{1}{n}\int_{0}^{\infty}\int_{\mathbb{R}} \rho_n^{\theta} u_n(t,x)\partial_{xx}\varphi(t,x)dxdt \to_{n\to 0} 0. \tag{3.109}$$

We can prove easily now that up to a subsequence:

$$\begin{aligned}
&-\mu\int_0^\infty\int_{\mathbb{R}} \rho_n^{\alpha-\frac{1}{2}}\sqrt{\rho_n}u_n(t,x)\partial_{xx}\varphi(t,x)dxdt - \int_0^\infty\int_{\mathbb{R}}\frac{\mu\alpha}{\alpha-\frac{1}{2}}\sqrt{\rho_n}u_n\partial_x\rho_n^{\alpha-\frac{1}{2}}\partial_x\varphi(t,x)dxdt \\
&\to_{n\to+\infty} -\mu\int_0^\infty\int_{\mathbb{R}} \rho^{\alpha-\frac{1}{2}}\sqrt{\rho}u(t,x)\partial_{xx}\varphi(t,x)dxdt - \int_0^\infty\int_{\mathbb{R}}\frac{\mu\alpha}{\alpha-\frac{1}{2}}\sqrt{\rho}u\partial_x\rho^{\alpha-\frac{1}{2}}\partial_x\varphi(t,x)dxdt
\end{aligned} \tag{3.110}$$

Indeed since $\partial_x \rho_n^{\alpha-\frac{1}{2}}$ is uniformly bounded in $L^2_{loc}([t_1,+\infty[\times\mathbb{R})$, it implies that up to a subsequence $\partial_x \rho_n^{\alpha-\frac{1}{2}}$ converges weakly in $L^2([t_1,+\infty[\times\mathbb{R})$ to v. Since $\rho_n^{\alpha-\frac{1}{2}}$ converges to $\rho^{\alpha-\frac{1}{2}}$ in $L^2_{loc}([t_1,+\infty[\times\mathbb{R})$ via the dominated convergence, we deduce that $\partial_x \rho_n^{\alpha-\frac{1}{2}}$ converges in the sense of the distribution to $v = \partial_x \rho^{\alpha-\frac{1}{2}}$. Now using the fact that $\sqrt{\rho_n}u_n$ converges strongly to $\sqrt{\rho}u$ in $L^2_{loc}([t_1,+\infty[\times\mathbb{R})$, we obtain the convergence of the second term on the left hand side of (3.110) (indeed on the time interval $[0,t_1]$ we proceed as in the previous section with $0 < t_1 < T_\beta$). We proceed similarly for the other term.
We have proved finally that (ρ_n, u_n) converges in the sense of the distribution to a global weak solution (ρ, u) of the system (1.1).
Finally we can prove the estimates (2.15) by using Fatou lemma in the framework of the weak convergence.

Compactness when $\frac{1}{2} < \alpha < 1$

We proceed as in the previous section excepted that it suffices to use the Lemma 1 for the unknown $m_n = \rho_n u_n$. Indeed we observe that we have from (3.58) and (3.59) with $t_1 \in]0, T_\beta]$:

$$\|\rho_n u_n\|_{L^\infty([t_1,+\infty[,L^2(\mathbb{R}))} \leq C(t_1), \tag{3.111}$$

and since:

$$\partial_t(\rho_n u_n) = -\partial_x(\rho_n u_n^2) + \partial_x(\mu_n(\rho_n)\partial_x u_n) - \partial_x(a\rho_n^\gamma),$$

we have then:

$$\|\rho_n u_n\|_{L^\infty([t_1,+\infty[,W^{-1,1}(\mathbb{R}))+L^2([t_1,+\infty[,H^{-1}(\mathbb{R}))+L^\infty([t_1,+\infty[,W^{-1,+\infty}(\mathbb{R}))} \leq C(t_1). \tag{3.112}$$

We can then apply the Lemma 1 and conclude as in the previous section.

Continuity in time of the momentum ρu and m^1

Proposition 3.1 *For any $T > 0$, m and ρv belongs to $C([0,T], \mathcal{M}(\mathbb{R})^*)$.*

Proof: VERIFIER CETTE PARTIE!
From (3.36), the sequence $(\rho_n u_n)_{n\in\mathbb{N}}$ is uniformly bounded in n on $[0,T]$ in $\mathcal{M}(\mathbb{R})$ when $\|\rho_0^n u_0^n\|_{L^1} \leq M$, indeed there exists $M(T) > 0$ such that for any $t \in [0,T]$ we have:

$$\|\rho_n u_n(t,\cdot)\|_{\mathcal{M}(\mathbb{R})} \leq M(T).$$

We are now going to prove that for any $\varphi \in C_0(\mathbb{R})$, $(\rho_n u_n(t,\cdot), \varphi)$ is uniformly continuous in n on $[0,T]$. Indeed we have using the momentum equation of (1.1), the Fubini theorem

when $t > s \geq 0$ and (3.58), (3.59), (3.54) and (3.78) that for any $\varphi \in C_0^\infty(\mathbb{R})$:

$$
\begin{aligned}
&|(\rho_n u_n(t,\cdot),\varphi) - (\rho_n u_n(s,\cdot),\varphi)| \\
&= |\int_{\mathbb{R}}\int_s^t [-\partial_x(\rho_n u_n^2)(\tau,x) + \partial_x(\mu_n(\rho_n)\partial_x u_n)(\tau,x) - (\partial_x P(\rho_n))(\tau,x)]\varphi(x)\,d\tau\,dx| \\
&= |\int_s^t \int_{\mathbb{R}} [\rho_n u_n^2(\tau,x) + \partial_x(\mu_n(\rho_n))u_n(\tau,x) + P(\rho_n)(\tau,x)]\partial_x\varphi(x)\,d\tau\,dx \\
&\qquad\qquad + \int_s^t\int_{\mathbb{R}} \mu_n(\rho_n(\tau,x))u_n(\tau,x)\partial_{xx}\varphi(x)d\tau dx| \\
&\leq \int_s^t \big(\|\rho_n\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\|1_{[0,T_\beta]}(\tau)u_n(\tau,\cdot)\|^2_{L^2(\mathbb{R})} + \|1_{t\geq T_\beta}(\tau)\sqrt{\rho_n}u_n(\tau,\cdot)\|^2_{L^2(\mathbb{R})}\big)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}d\tau \\
&+ \int_s^t (1_{[0,T_\beta]}(\tau)\sqrt{\tau}\|\sqrt{\rho_n}\partial_x\varphi_n(\rho_n)(\tau,\cdot)\|_{L^2}\|\sqrt{\rho_n}\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\frac{1}{\sqrt{\tau}}\|u_n(\tau,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x\varphi\|_{L^\infty(\mathbb{R})} \\
&+ 1_{\tau\geq T_\beta}(\tau)\|\sqrt{\rho_n}u_n\|_{L^2(\mathbb{R}))}\|\sqrt{\rho_n}\partial_x\varphi_n(\rho_n)\|_{L^2(\mathbb{R})}\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\big)d\tau \\
&+ \|P(\rho_n)\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\|\partial_x\varphi\|_{L^1(\mathbb{R})}(t-s) \\
&+ \int_s^t \big(1_{[0,T_\beta]}(\tau)\|\mu(\rho_n)\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\|u_n(\tau,\cdot)\|_{L^2(\mathbb{R})}\|\partial_{xx}\varphi\|_{L^2(\mathbb{R})} \\
&+ 1_{\tau\geq T_\beta}(\tau)\|\rho_n^{\frac{1}{4}}u_n(\tau,\cdot)\|_{L^4(\mathbb{R})}\|\partial_{xx}\varphi\|_{L^{\frac{4}{3}}(\mathbb{R})}\|\frac{\mu_n(\rho_n)}{\rho_n^{\frac{1}{4}}}\|_{L^\infty(\mathbb{R}^+,L^\infty(\mathbb{R}))}\big)d\tau \\
&\leq C(|t-s| + |t-s|^{\alpha_1})(1 + \|\partial_x\varphi\|_{L^\infty(\mathbb{R})} + \|\partial_x\varphi\|_{L^1(\mathbb{R})} + \|\partial_{xx}\varphi\|_{L^\infty(\mathbb{R})} + \|\partial_{xx}\varphi\|_{L^1(\mathbb{R})}),
\end{aligned}
$$

with $\alpha_1 > 0$. We note that we have used the fact that u_n is uniformly bounded in $L^{3-\epsilon}([0,T_\beta],L^2(\mathbb{R}))$ with $\epsilon > 0$ small enough.
By a density argument and the fact that $\rho_n u_n$ is uniformly bounded in $L^\infty([0,T],\mathcal{M}(\mathbb{R}))$ it implies that for $\varphi \in C_0(\mathbb{R})$, $(\rho_n u_n(t,\cdot),\varphi)$ is uniformly continuous in n on $[0,T]$. In other words the sequence $(\rho_n u_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,T]$ for the weak * topology on $\mathcal{M}(\mathbb{R})$ when $\|\rho_0^n u_0^n\|_{L^1} \leq M$.

Using the Ascoli theorem on the metric space $B(0,M(T))_{\mathcal{M}(\mathbb{R})}$ endowed with the weak * topology (this is true because $C_0(\mathbb{R})$ is separable), we deduce that up to a subsequence $\rho_n u_n(t,\cdot)$ converges uniformly to $m(t,\cdot)$ on $[0,T]$ in the weak * topology of $\mathcal{M}(\mathbb{R})$. In particular it implies that m belongs to $C([0,T],\mathcal{M}(\mathbb{R})*)$ for any $T > 0$. The proof is similar for m^2. □

4 Proof of the Corollary 1

We proceed as in the proof of the theorem 2.2, we use in particular the same regularizing process. We have then a sequence $(\rho_n,u_n)_{n\in\mathbb{N}}$ solution of the system (3.28) and we want to verify that up to a subsequence this sequence converges in the sense of the distributions to a weak solution of the system (1.1). Compared with the theorem 2.2, we have no estimate as (3.37) since v_0 does not belong to $L^2(\mathbb{R})$. However we have the classical energy estimate, indeed this is due to the fact that $u_n(0,\cdot)$ is uniformly bounded in $L^2(\mathbb{R})$. Multiplying the momentum equation of (3.28) and integrate over $(0,T)\times\mathbb{R}$

we have then:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\big(\rho_n(T,x)|u_n(T,x)|^2+(\Pi(\rho_n(T,x))-\Pi(\bar{\rho}))\big)dx\\
&+\int_0^T\int_{\mathbb{R}}\mu_n(\rho_n(s,x))(\partial_x u_n(s,x))^2dxds \qquad (4.113)\\
&\qquad\qquad\leq\frac{1}{2}\int_{\mathbb{R}}\big(\rho_n(0,x)|u_n(0,x)|^2+(\Pi(\rho_n(0,x))-\Pi(\bar{\rho}))\big)dx.
\end{aligned}$$

Now proceeding as in the proof of the theorem 2.2, we establish that there exists $T_\beta>0$, $C>0$ independent on n such that:

$$\|(\frac{1}{\rho_n},\rho_n)\|_{L^\infty([0,T_\beta],L^\infty(\mathbb{R}))}\leq C. \qquad (4.114)$$

From (3.36), we deduce that there exists $C>0$ independent on n such that:

$$\begin{aligned}
&\|\rho_n u_n\|_{L^\infty([0,T_\beta],L^1(\mathbb{R}))}+\|\rho_n v_n\|_{L^\infty([0,T_\beta],L^1(\mathbb{R}))}\\
&\qquad\leq(\|\rho_n u_n(0,\cdot)\|_{L^1(\mathbb{R})}+\|\rho_n v_n(0,\cdot)\|_{L^1(\mathbb{R})})e^{CT_\beta^{\gamma-\alpha}} \qquad (4.115)\\
&\qquad\leq\epsilon_0 e^{CT_\beta^{\gamma-\alpha}}.
\end{aligned}$$

It implies in particular from (4.114) and (4.115) that for any $t\in[0,T_\beta]$, there exists $C>0$ independent on n such that:

$$\|\rho_n^\alpha(t,\cdot)\|_{BV(\mathbb{R})}\leq C+\epsilon_0 e^{CT_\beta^{\gamma-\alpha}}. \qquad (4.116)$$

From (4.114), (4.116), composition theorem we deduce that for $C>0$ independent on n and any $t\in[0,T_\beta]$:

$$\|\rho_n(t,\cdot)\|_{BV(\mathbb{R})}\leq C(1+\epsilon_0 e^{CT_\beta^{\gamma-\alpha}}). \qquad (4.117)$$

We wish now to prove that ρ_n converges almost everywhere on $]0,T_\beta]\times\mathbb{R}$ up to a subsequence. We are going to use the lemma 2. To do this, we must prove each assumption of the lemma 2. First we have since ρ_n is regular on $[0,T_\beta]\times\mathbb{R}$ and for any increasing $-\infty<x_0<\cdots<x_N<+\infty$ and any $t\in[0,T_\beta]$ we have from (4.115) and (4.114) for $C>0$:

$$\begin{aligned}
\sum_{i=0}^{N-1}|\rho_n(t,x_{i+1})-\rho_n(t,x_i)|&\leq\sum_{i=0}^{N-1}|\int_{x_i}^{x_{i+1}}\partial_x\rho_n(t,x)dx|\\
&\leq C\epsilon_0 e^{CT_\beta^{\gamma-\alpha}}.
\end{aligned} \qquad (4.118)$$

From (4.114) , we have for any $t\in[0,T_\beta]$:

$$|\rho_n(t,0)|\leq C \qquad (4.119)$$

From the mass equation, we deduce now that for any compact $K \in \mathbb{R}$ we have from (4.113), (4.114), Sobolev embedding and for $C > 0$ independent on n:

$$\begin{aligned}
&\|\rho_n(t,\cdot) - \rho_n(s,\cdot)\|_{L^1(K)} \leq \int_s^t \|\partial_x(\rho_n u_n)(s',\cdot)\|_{L^1(K)} ds' \\
&\leq \int_s^t \big(\sqrt{|K|}\,\|\frac{\rho_n}{\sqrt{\mu_n(\rho_n)}}(t,\cdot)\|_{L^\infty(K)}\|\sqrt{\mu_n(\rho_n)}\partial_x u_n(t,\cdot)\|_{L^2(K)} \\
&+ \|\partial_x\varphi_1(\rho_n)(t,\cdot)\|_{L^1(K)}\|\frac{\rho_n}{\mu_n(\rho_n)}(t,\cdot)\|_{L^\infty(K)}\|u_n(t,\cdot)\|_{H^1(\mathbb{R})}\big) ds \\
&\leq C\sqrt{t-s}(1+\sqrt{|K|}).
\end{aligned} \tag{4.120}$$

From the Lemma 2 by localizing the argument (it suffices to consider the sequel $(\varphi\rho_n)_{n\in\mathbb{N}}$ with $\varphi \in C_0^\infty(\mathbb{R})$ and to use the diagonal process), we deduce that up to a subsequence $(\rho_n)_{n\in\mathbb{N}}$ converges strongly in $L^1_{loc}([0,T_\beta]\times\mathbb{R})$. It implies in particular that up to a subsequence $(\rho_n)_{n\in\mathbb{N}}$ converges almost everywhere to ρ on $[0,T_\beta]\times\mathbb{R}$. From (4.114) and the dominated convergence theorem we deduce that:

- ρ_n^γ, ρ_n^α converges strongly in $L^p_{loc}([0,T_\beta]\times\mathbb{R})$ for any $1 \leq p < +\infty$ to respectively ρ^γ and ρ^α.

From (4.113) and (4.114) we observe that u_n is uniformly bounded in $L^2([0,T_\beta], H^1(\mathbb{R}))$. Up to a subsequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to u in $L^2([0,T_\beta], H^1(\mathbb{R}))$. Since ρ_n^α converges strongly to ρ^α in $L^2_{loc}([0,T_\beta]\times\mathbb{R})$, $\mu_n(\rho_n)\partial_x u_n$ converges to $\mu(\rho)\partial_x u$ in the sense of the distribution on $[0,T_\beta]\times\mathbb{R}$.
Easily we can prove now that:

- $(\rho_n u_n)_{n\in\mathbb{N}}$ converges up to a subsequence to ρu in $\mathcal{D}'([0,T_\beta]\times\mathbb{R})$.
- $\rho_n u_n^2$ converges up to a subsequence to ρu^2 in $\mathcal{D}'([0,T_\beta]\times\mathbb{R})$.

Indeed it suffices to use the arguments developed in [30]. To finish since $\partial_x\rho_n$ is uniformly bounded in $L^\infty([0,T_\beta], L^1(\mathbb{R}))$ we deduce that up to a subsequence $\partial_x\rho_n$ converges * weakly to $\partial_x\rho$ in $L^\infty([0,T_\beta], \mathcal{M}(\mathbb{R}))$.
In addition we have the following estimates for $C > 0$ large enough by using the Fatou lemma for weak convergence:

$$\begin{cases}
\|u\|_{L^\infty([0,T_\beta],L^2(\mathbb{R}))} \leq C \\
\|\nabla u\|_{L^2([0,T_\beta],L^2(\mathbb{R}))} \leq C \\
\|(\rho, \frac{1}{\rho})\|_{L^\infty([0,T_\beta],L^\infty(\mathbb{R}))} \leq C \\
\|\partial_x\rho\|_{L^\infty([0,T_\beta],\mathcal{M}(\mathbb{R}))} \leq C \\
\|(\rho-\bar\rho)\|_{L^\infty([0,T_\beta],L^\gamma_2(\mathbb{R}))} \leq C.
\end{cases} \tag{4.121}$$

In addition from (4.113), (4.114), (4.115) and Sobolev embedding, we deduce that $\rho_n - \bar\rho$ is uniformly bounded in $L^\infty([0,T_\beta], H^{\frac{1}{2}}(\mathbb{R}))$. From the mass equation $\partial_t\rho_n$ is uniformly in $L^\infty([0,T_\beta], W^{-1,1}(\mathbb{R}))$. Using the Lemma 1, we deduce that $(\rho_n)_{n\in\mathbb{N}}$ converges strongly to ρ in $C([0,T_\beta], L^p_{loc}(\mathbb{R}))$ for $1 \leq p < +\infty$.
Following the same argument than in the previous section, we prove also that $(\rho_n u_n)_{n\in\mathbb{N}}$ converges to ρu in $C([0,T_\beta], \mathcal{M}(\mathbb{R})^*)$.

5 Proof of the theorem 2.3

Let (ρ_0, u_o, v_0) verifying the assumptions of the Theorem 2.3, we define now the regularizing initial data $(\rho_0^n = j_n * \rho_0, m_0^n = j_n * m_0)$ with L_n a regularizing kernel. From [25, 33] since $(\rho_0^n - \bar{\rho}, u_0^n)$ belongs to $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and we have $0 \leq c \leq \rho_0^n \leq M < +\infty$, we know that there exists a global strong solution (ρ_n, u_n). In addition since the regularity H^s with $s \geq 1$ is preserved this solution (ρ_n, u_n) is in $C^\infty(\mathbb{R}^+ \times \mathbb{R})$. We are going now to prove uniform estimates in n on the solution (ρ_n, u_n), for the simplicity in the sequel we forget the subscript n. When we will have sufficiently uniform estimates, we will verify that (ρ_n, u_n) converges up to a subsequence in a suitable functional space to a global weak solution (ρ, u) of (1.1).

Similarly we have the following entropy:

$$\frac{1}{2}\int_{\mathbb{R}} (\rho|v|^2(T,x) + (\Pi(\rho) - \Pi(\bar{\rho}))(T,x)dx + \frac{4a\gamma\mu}{(\gamma+\alpha-1)^2}\int_0^T\int_{\mathbb{R}} |\partial_x \rho^{\frac{1}{2}(\gamma-1)}|^2(s,x)dsdx \leq \frac{1}{2}\int_{\mathbb{R}} (\rho_0|v_0|^2(x) + (\Pi(\rho_0) - \Pi(\bar{\rho}))(x)dx, \tag{5.122}$$

L^∞ estimate of $\frac{1}{\rho}$ and ρ in L^∞ norm on a finite time intervall

In this case the only difference is the way that we are going to use to prove L^∞ estimate on ρ and $\frac{1}{\rho}$. We can show the estimate (3.36) in a similar way as the proof of the Theorem 2.2. We deduce as (3.40) that we have for $C > 0$ large enough independent on n and any $t > 0$:

$$\|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} + \|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} \leq C(\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})}) e^{\frac{C}{\mu}\int_0^t \|P'(\rho)\rho(s,\cdot)\|_{L^\infty} ds} \tag{5.123}$$

Since we have $\rho v = \rho u + \mu \partial_x \ln \rho$, we deduce in particular that for any $t > 0$ we have using (5.122):

$$\begin{aligned} \|\ln \rho(t,\cdot)\|_{L^\infty} &\leq C(|\ln(\bar{\rho})| + \|\partial_x \ln(\rho(t,\cdot))\|_{L^1(\mathbb{R})}) \\ &\leq C\big(|\ln(\bar{\rho})| + (\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})}) e^{\frac{C}{\mu}\int_0^t \|P'(\rho)\rho(s,\cdot)\|_{L^\infty} ds}\big). \end{aligned} \tag{5.124}$$

And it yields that for $C' > 0$ since

$$\|\ln \rho\|_{L^\infty_t(L^\infty(\mathbb{R}))} \leq C'\big(|\ln \bar{\rho}| + (\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})}) e^{C' t e^{\gamma \|\ln \rho\|_{L^\infty_t(L^\infty(\mathbb{R}))}}}\big). \tag{5.125}$$

Let us prove now that ρ and $\frac{1}{\rho}$ are bounded in L^∞ independently on any n and on an time interval $[0, T^*]$ with T^* independent on n. More precisely we define by:

$$T_n = \sup\{t \in (0, +\infty), \|\ln \rho^n(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq M_1.\},$$

with $M_1 = \sup(2\|\ln \rho_0\|_{L^\infty}, C'\big(|\ln \bar{\rho}| + 2(\|\rho_0 v_0\|_{L^1(\mathbb{R})} + \|\rho_0 u_0\|_{L^1(\mathbb{R})})\big))$. We observe that $T_n > 0$ since $\ln \rho_n$ belongs to $C([0, +\infty[, L^\infty(\mathbb{R}))$ for any $n \in \mathbb{N}$ (indeed this is due to the

regularity of the solution (ρ_n, u_n)) and since $\| \ln \rho_0^n \|_{L^\infty} < M_1$. Let us define now T_2 such that:

$$e^{C' T_2 e^{\gamma M_1}} = \frac{3}{2} \quad \text{and} \quad T_2 = \frac{\ln(\frac{3}{2})}{C' e^{\gamma M_1}}.$$

From the definition of T_n and from (5.125) we have for any $n \in \mathbb{N}$:

$$T_n \geq T_2 > 0.$$

We have then proved that $\ln \rho^n$ is uniformly bounded in n in $L^\infty((0, T_2), L^\infty(\mathbb{R}))$ with T_2 independent on n. In particular this gives:

$$\|(\rho, \frac{1}{\rho})\|_{L^\infty([0,T_2], L^\infty(\mathbb{R}))} \leq e^{M_1}. \tag{5.126}$$

Now proceeding as in the previous section we can show additional regularity information on the velocity u and ρ, we have in particular for any $t > 0$ there exists $C > 0$ independent on n such that:

$$\begin{aligned} &\|\sqrt{\rho_n} u_n(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(\frac{1}{\sqrt{t}} + 1) \\ &\|(\sqrt{s} 1_{\{s \leq 1\}} + 1_{\{s \geq 1\}}) \partial_x u_n\|_{L^2_t(L^2(\mathbb{R}))} \leq C. \end{aligned} \tag{5.127}$$

We recall in addition that from (5.122) we have also for any $t > 0$ and $C > 0$ independent on n:

$$\begin{cases} \|\sqrt{\rho_n} v_n(t, \cdot)\|_{L^2} \leq C \\ \|\Pi(\rho_n(t, \cdot)) - \Pi(\bar{\rho})\|_{L^1(\mathbb{R})} \leq C \\ \|\partial_x(\rho_n^{\frac{1}{2}(\gamma-1)})\|_{L^2_t(L^2(\mathbb{R}))} \leq C. \end{cases} \tag{5.128}$$

From (5.127), (5.128) and the definition of the effective velocity v, we deduce that for any $t > 0$ and $C > 0$ large enough:

$$\|\partial_x(\frac{1}{\sqrt{\rho_n}(t, \cdot)})\|_{L^2(\mathbb{R})} \leq C(\frac{1}{\sqrt{t}} + 1). \tag{5.129}$$

Control of the L^∞ norm for long time of ρ and $\frac{1}{\rho}$

We are going to follow the method develop by Hoff in [23]. First for any $t_1 < t_2 \leq T$ with $t_1 \in]0, T_2]$ and $x_0 \in \mathbb{R}$, then there exists x_1 and a constant $C(T) > 0$ depending only on T such that we have:

$$\begin{cases} |x_0 - x_1| \leq C(T) \\ C(T)^{-1} \leq \rho(t_2, x_1) \leq C(T). \end{cases} \tag{5.130}$$

We observe that there exists $C > 0$ such that:

$$\lim_{\rho \to 0} \inf(\Pi(\rho) - \Pi(\bar{\rho})) \geq C^{-1}.$$

We deduce in particular that there exists $\delta > 0$ such that:

$$\Pi(\rho) - \Pi(\bar{\rho}) \geq \frac{C^{-1}}{2}.$$

Then if for any $R>0$ we have:

$$\sup_{x\in[x_0-R,x_0+R]}\rho(t_2,x)<\delta,$$

then we have from the previous estimate and (5.122) (with $C_1>0$ depends on the initial data (ρ_0,u_0)):

$$C^{-1}R\leq\int_{\mathbb{R}}(\Pi(\rho(t_2,x)-\Pi(\bar{\rho}))dx\leq C_1.$$

In particular R can not goes to 0. We proceed similarly for the case $\rho(t_2,x_1)\geq C(T)$. We use now the characteristic method, we set for $t_1\in]0,T_2]$ and $t\in[t_1,t_2]$ with $t_2>t_1$:

$$\begin{cases}\dfrac{d}{dt}X_j(s)=u(t,X_j(s))\\ X_j(t_2)=x_j\in\mathbb{R},\ \ j=0,1.\end{cases}$$

We set $\Delta L(t)=\log\rho(t,X_1(t))-\log\rho(t,X_0(t))$. From (1.1) we deduce that:

$$\begin{aligned}\frac{d}{dt}\Delta L(t)&=-\int_{X_0(t)}^{X_1(t)}\partial_{xx}u(t,x)dx\\&=-\frac{1}{\mu}\int_{X_0(t)}^{X_1(t)}(\partial_t(\rho u)(t,x)+\partial_x(\rho u^2)(t,x)+\partial_xP(\rho)(t,x)dx\\&=-\frac{d}{dt}I(t)-\Delta P(t),\end{aligned}$$

where $I(t)=\int_{X_0(t)}^{X_1(t)}\rho u(t,x)dx$ and $\Delta P(t)=P(\rho(t,X_1(t)))-P(\rho(t,X_0(t)))$. We observe now as in [23] that $\alpha(t)=\frac{\Delta L(t)}{\Delta P(t)}$ is positive. We deduce then that we have the following ordinary differential equation:

$$\frac{d}{dt}\Delta L(t)+\alpha(t)\Delta L(t)=-\frac{d}{dt}I(t).$$

We deduce that we have for any $t\in[t_1,t_2]$:

$$|\Delta L(t)|\leq|\Delta L(t_1)|+|I(t_1)|+|I(t)|+\int_{t_1}^{t}e^{-\int_s^t\alpha(\tau)d\tau}\alpha(s)|I(s)|ds.\tag{5.131}$$

For the latter, we take $X_1>X_0$. We observe easily that for any $t\in[t_1,t_2]$ we have from (5.127) and for $C>0$ large enough:

$$\begin{aligned}&I(t)\leq\|\sqrt{\rho}u(t,\cdot)\|_{L^2}\|\sqrt{\rho}\|_{L^2([X_0(t),X_1(t)])}\\&\leq C(1+\frac{1}{\sqrt{t}})(1+\|\rho(t,\cdot)-\bar{\rho}\|_{L^\gamma_2})\\&\qquad\times\big((|X_0(t)-X_1(t)|^{\frac{1}{2}}+|X_0(t)-X_1(t)|^{1-\frac{1}{\gamma}}+|X_0(t)-X_1(t)|\big).\end{aligned}\tag{5.132}$$

It remains now only to estimate $|X_0(t)-X_1(t)|$, next we have:

$$\begin{aligned}\frac{d}{dt}(X_1-X_0)(t)&=\int_{X_0(t)}^{X_1(t)}\partial_xu(t,x)dx\\&\leq\frac{1}{2}(X_1(t)-X_0(t))+\frac{1}{2}\int_{X_0(t)}^{X_1(t)}(\partial_xu)^2(t,x)dx.\end{aligned}$$

Using (5.127) we get:

$$X_1(t) - X_0(t) \leq X_1(t_1) - X_0(t_1) + \int_{t_1}^{t} (X_1(s) - X_0(s))ds + \int_{t_1}^{t} \int_{X_0(s)}^{X_1(s)} (\partial_x u)^2(s,x)dxds$$
$$\leq |x_1 - x_0| + \int_{t_1}^{t} (X_1(s) - X_0(s))ds + \int_{t_1}^{t} \frac{1}{s} \int_{\mathbb{R}} s(\partial_x u)^2(s,x)dx$$
$$\leq |x_1 - x_0| + \int_{t_1}^{t} (X_1(s) - X_0(s))ds + \frac{C}{t_1}.$$

From Gronwall lemma and (5.130) we deduce that for $t_1 \leq t \leq t_2 \leq T$, we have:

$$|X_1(t) - X_0(t)| \leq C_1(T, t_1), \tag{5.133}$$

with $C_1(T, t_1)$ depending only on T and t_1. Combining (5.132) and (5.133), there exists a continuous fonction C such that for $t_1 \leq t \leq t_2 \leq T$, there exists $C_1(T, t_1)$ such that:

$$I(t) \leq C_1(T, t_1). \tag{5.134}$$

Now using (5.131) and (5.134) we get for any $t_1 \leq t \leq t_2 \leq T$:

$$|\Delta L(t)| \leq |\Delta L(t_1)| + 5C(T, t_1). \tag{5.135}$$

Now since $(\frac{1}{\rho}(t_1, \cdot), \rho(t_1, \cdot))$ is uniformly bounded in n in $L^\infty(\mathbb{R})$, we deduce that:

$$|\ln \rho(t_2, x_0) - \ln \rho(t_2, x_1)| \leq C(T, t_1). \tag{5.136}$$

From (5.130) and (5.136), we deduce that for any t_2 with $0 < t_1 \leq t_2 \leq T$ and any $x_0 \in \mathbb{R}$ there exists $C(T, t_1)$ such that:

$$C(t_1, T)^{-1} \leq \rho(t_2, x_0) \leq C(T, t_1).$$

It implies in particular that we have for C_1 an increasing continuous function and any $t \geq t_1$

$$\|(\rho(t, \cdot), \frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C_1(t, t_1). \tag{5.137}$$

If we summarize the previous estimates (5.127), (5.128), (5.129) and (5.137), we have proved that for any $t > 0$ there exists $C > 0$ and C_1 a continuous increasing function not depending on n such that:

$$\begin{cases} \|(\rho_n u_n(t, \cdot), \rho_n v_n(t, \cdot))\|_{L^1(\mathbb{R})} \leq C_1(t) \\ \|(\rho_n(t, \cdot), \frac{1}{\rho_n}(t, \cdot)\|_{L^\infty} \leq C_1(t) \\ \|\sqrt{\rho_n} u_n(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(\frac{1}{\sqrt{t}} + 1) \\ \|(\sqrt{s} 1_{\{s \leq 1\}} + 1_{\{s \geq 1\}})\partial_x u_n\|_{L^2_t(L^2(\mathbb{R}))} \leq C \\ \|\sqrt{\rho_n} v_n(t, \cdot)\|_{L^2} \leq C \\ \|\rho_n(t, \cdot) - \bar{\rho}\|_{L^\gamma_2(\mathbb{R})} \leq C \\ \|\partial_x \rho_n\|_{L^2_t(L^2(\mathbb{R}))} \leq C C_1(t) \\ \|\partial_x \rho_n(t, \cdot)\|_{L^2(\mathbb{R})} \leq C C_1(t)(\frac{1}{\sqrt{t}} + 1). \end{cases} \tag{5.138}$$

Compactness arguments

From (5.138) we deduce that ρ_n is uniformly bounded in $L^2_{loc}(H^1(\mathbb{R}))$. In addition from (5.138) and from the fact that $\partial_t \rho_n = -\partial_x(\rho_n u_n)$ we deduce that $\partial_t \rho_n$ is uniformly bounded in $L^\infty_{loc}(W^{-1,1}(\mathbb{R}))$ and is bounded uniformy in n in $L^{2-\epsilon}_{loc}(W^{-\frac{3}{2},2}(\mathbb{R}))$. Using the Aubin-Lions lemma 1, we deduce that the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges up to a subsequence in $L^2_{loc}(H^s_{loc}(\mathbb{R}))$ with $0 \leq s < 1$ to ρ. It implies in particular that up to a subsequence $(\rho_n)_{n\in\mathbb{N}}$ converges almost everywhere to ρ. From convergence dominated and (5.138) we show that ρ_n^γ converges to ρ^γ in $L^p_{loc}(\mathbb{R}\times\mathbb{R}^+)$ for any $1 \leq p < +\infty$. In addition from (5.138) for any $t > 0$, there exists $C(t) > 0$ such that we have:

$$\|(\frac{1}{\rho}(t,\cdot), \rho(t,\cdot))\|_{L^\infty(\mathbb{R})} \leq C(t). \tag{5.139}$$

Proceeding as for (3.53), we deduce that u_n is uniformly bounded in n in $L^{p(s)-\epsilon}_{loc}(H^s(\mathbb{R}))$ with $0 \leq s < 1$ and $p(s) = \frac{3}{1+2s}$ with $\epsilon > 0$ sufficiently small such that $p(s) - \epsilon \geq 1$. Now using paraproduct law we deduce that we have for any $t > 0$ and $0 \leq s \leq \frac{1}{2}$:

$$\begin{aligned} &\|(\rho_n - \bar{\rho})u_n(t,\cdot)\|_{H^s(\mathbb{R})} \leq \\ &\|u_n(t,\cdot)\|_{H^s(\mathbb{R})}\|\rho_n(t,\cdot) - \bar{\rho}\|_{L^\infty} + \|\rho_n(t,\cdot) - \bar{\rho}\|_{H^1(\mathbb{R})}\|u_n(t,\cdot)\|_{H^{-\frac{1}{2}}(\mathbb{R})} \end{aligned} \tag{5.140}$$

Combining the fact that at u_n is uniformly bounded in n in $L^{p(s)-\epsilon}_{loc}(H^s(\mathbb{R}))$ and (5.138) we deduce that $\rho_n u_n$ is uniformly bounded in $L^{q(s)}_{loc}(H^s(\mathbb{R}))$ with $q(s) = \min(2, p(s))$ and $s \in [0, \frac{1}{2}]$.
Now we recall that we have $\partial_t(\rho_n u_n) = -\partial_x(\rho_n u_n^2) + \mu\partial_{xx}u_n - \partial_x P(\rho_n)$, taking $s = 0$ we have seen that u_n^2 is bounded in $L^{\frac{3}{2}}_{loc}(L^1(\mathbb{R}))$ and ρ_n is uniformly bounded in $L^\infty_{loc}(L^\infty(\mathbb{R}))$ then $\rho_n u_n^2$ is uniformly bounded in $L^{\frac{3}{2}}_{loc}(L^1(\mathbb{R}))$. Proceeding similarly we observe by Sobolev embedding that $\partial_t(\rho_n u_n)$ is bounded uniformly in $L^1_{loc}(H^{-s_2}(\mathbb{R}))$ with $s_2 > 0$ large enough. Using the Aubin-Lions lemma 1 we deduce that $\rho_n u_n$ converges strongly up to a subsequence to m in in $L^{q(s)}_{loc}(H^s_{loc}(\mathbb{R}))$ with $q(s) = \min(2, p(s))$ and $s \in [0, \frac{1}{2}[$. By Sobolev embedding we deduce that $\rho_n u_n$ converges strongly to m in $L^2_{loc}(L^2_{loc}(\mathbb{R}))$. Now since $(\rho_n)_{n\in\mathbb{N}}$ and $(\frac{1}{\rho_n})_{n\in\mathbb{N}}$ converges respectively to ρ and $\frac{1}{\rho}$ in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R})$ for $1 \leq p < +\infty$ on every compact and from the fact that $\frac{1}{\rho_n}$ is uniformly bounded in $L^\infty_{loc}(L^\infty(\mathbb{R}))$ we deduce that u_n converges up to a subsequence strongly to $\frac{m}{\rho} = u$ in $L^{2-\epsilon}_{loc}(L^{2-\epsilon}_{loc}(\mathbb{R}))$ with $\epsilon > 0$ and $(u_n)_{n\in\mathbb{N}}$ converges up to a subsequence almost everywhere to u.
All these estimates are now sufficient to show that $(\rho_n, u_n)_{n\in\mathbb{N}}$ converges up to a subsequence in the sense of the distribution to a global weak solution (ρ, u).
In addition using Fatou lemma and weak convergence, we deduce from (5.138) that for

any $t > 0$ there exists $C > 0$ and $C_1(t) > 0$ such that:

$$\begin{cases} \|\rho u(t,\cdot)\|_{L^1(\mathbb{R})} \leq C_1(t) \\ \|(\rho(t,\cdot), \dfrac{1}{\rho}(t,\cdot)\|_{L^\infty} \leq C_1(t) \\ \|\sqrt{\rho} u(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\dfrac{1}{\sqrt{t}} + 1) \\ \|\rho(t,\cdot) - \bar{\rho}\|_{L^\gamma_2(\mathbb{R})} \leq C \\ \|\partial_x \rho\|_{L^2_t(L^2(\mathbb{R}))} \leq C C_1(t). \end{cases} \tag{5.141}$$

In addition u belongs to $L^{p(s)-\epsilon}_{loc}(H^s(\mathbb{R}))$ with $0 \leq s < 1$ and $p(s) = \frac{3}{1+2s}$ with $\epsilon > 0$ sufficiently small such that $p(s) - \epsilon \geq 1$. In addition as in the previous proof, we can show that ρ is in $C([0, +\infty[, W^{-\epsilon,p}_{loc}(\mathbb{R}))$ and ρu is in $C([0,T], \mathcal{M}(\mathbb{R})^*)$.
This achieves the proof of the Theorem 2.3.

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