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# Does Klinkenberg's Law Survive Upscaling?

J. CHASTANET, P. ROYER and J.-L. AURIAULT

*Laboratoire 'Sols, Solides, Structures' (3S), UJF, INPG, CNRS, BP 53, 38041 Grenoble Cedex, France. e-mails: juliette.chastanet@hmg.inpg.fr, pascale.royer@hmg.inpg.fr, jean-louis.auriault@hmg.inpg.fr*

**Abstract.** This work deals with the large-scale mathematical modelling of flow of gas at low pressure in porous media. At the pore scale, this type of flow is characterised by a wall-slip effect, which at the sample scale entails a dependence of permeability upon gas pressure. This latter property is described by Klinkenberg's law. The goal of the present work is to examine the robustness of this law, by determining whether it is still verified on a large-scale: upscaling is thus applied, starting with Klinkenberg's law at the local scale. A Klinkenberg's flow of gas in a two-constituent composite porous medium is considered, and the constituents are firstly assumed to be homogeneous. The cases of low and of high permeability contrast are successively examined. Upscaling is performed using the homogenisation method of multiple scale expansions. In both cases, the large-scale permeability tensor differs from its liquid counterpart. Except in the particular case of equal Klinkenberg factors, Klinkenberg's law is not verified at low permeability contrast. At high permeability contrast, the large-scale gas permeability verifies Klinkenberg's law. The case of heterogeneous constituents is then examined. It is shown that the large-scale permeability differs from its liquid counterpart, but it does not verify Klinkenberg's law.

**Key words:** gas flow, composite porous medium, Klinkenberg's law, gas permeability, upscaling, homogenisation.

## 1. Introduction

Darcy's law may break down for flow of gases at low pressures through porous media: Klinkenberg's effect may occur which can have significant impact on gas flow behaviours, especially in low permeable media. Klinkenberg (1941) observed that at low or near atmospheric pressure, the gas permeability  $k_g$  of a porous sample is greater than its liquid permeability  $k_l$ . This phenomenon is called Klinkenberg's effect and is described by the so-called Klinkenberg's equation

$$k_g = k_l \left( 1 + \frac{b}{\bar{p}} \right), \quad (1.1)$$

where  $b$  is the Klinkenberg factor, which depends on the rock properties, as well as, to a lesser degree, on the nature of the gas used, and  $\bar{p}$  is the arithmetic mean of the inlet and outlet pressures of the sample. Therefore, Klinkenberg's equation (1.1) describes a non-local flow behaviour. This law characterises the fact that

Klinkenberg's effect becomes less significant at higher pressures, and it follows that

$$\lim_{\bar{p} \rightarrow \infty} k_g = k_1. \quad (1.2)$$

Physically, Klinkenberg's effect is significant in any situation where the mean free path of a gas molecule  $\lambda$ , that is, the mean distance that it travels before it collides with another molecule (Cercignani, 1988, 1990)

$$\lambda = \frac{\mu \sqrt{\pi RT/2M}}{p}, \quad (1.3)$$

is comparable with the pore dimension  $l_p$ . This leads to a non-negligible Knudsen number, the ratio of molecular mean free path to pore size

$$Kn = \frac{\lambda}{l_p} \approx 1. \quad (1.4)$$

Klinkenberg's effect is thus expected to occur at low pressure, as  $\lambda$  is inversely proportional to  $p$ , and in very fine-grained porous media. In Darcy's flow regime and at high pressure ( $\lambda \ll l_p$ ), the boundary condition at the fluid/solid interface at the pore scale is that the fluid molecules are static and therefore that the fluid adheres to the pore wall. At low pressure and if the mean free path  $\lambda$  approaches the pore size, then significant molecular collisions are with the pore wall rather than with other gas molecules. Thus, the fluid/solid boundary molecules are no longer static and a wall-slip flow occurs: the local flow velocity is a non-vanishing velocity on the pore walls. Gas molecules move on the pore surface and contribute to their overall flux. This slippage increases the flow rate over that predicted by Darcy's law. Since the vast majority of permeability measurements are done with near atmospheric gas flow, the deviations from Darcy's law may thus be important and, consequently, the permeability may be overestimated.

Modelling and predicting the occurrence of Klinkenberg's effect is crucial to any area which involves gas flow through porous media: petroleum engineering (Jones, 1972; Sampath and Keighin, 1982), geothermal reservoir engineering (Wu *et al.*, 1998; Persoff and Hulen, 2001) contaminant transport and remediation (Reda, 1987), civil engineering (McVay and Rish, 1995), and material engineering for the forming of fibrous materials (Starr, 1995; Marschall and Milos, 1998), or for testing analysis of ceramics (Glass and Green, 1999), or pneumatics (Baehr and Hult, 1991). In Reda (1897), Baehr and Hult (1991) and Persoff and Hulen (2001), experimental evidence is given Klinkenberg's effect could not be neglected in low permeability media. In comparison with the considerable amount of studies on theory and applications of isothermal flow of gases through porous media, very few studies are explicitly concerned with Klinkenberg's effect. In Wu *et al.* (1998), a set of analytical solutions for steady-state and transient gas flows in porous media with Klinkenberg's effect is developed. The coupling between wall-slip and inertial

effects has been investigated in Skjetne and Gudmundsson (1993) and Skjetne (1995). Most of the works on Klinkenberg's effect are actually concerned with the determination of the Klinkenberg factor  $b$  and its correlation with the properties of the porous medium. Jones (1972) showed that  $b$  decreases with increasing permeability. A general analytical expression for  $b$  was derived in Skjetne and Gudmundsson (1993), by averaging the local properties of the porous medium

$$b = \frac{\phi T^* R_h}{k} \mu \sqrt{\frac{\pi R T}{2M}},$$

where  $\phi$  is the porosity,  $T^*$  is the tortuosity,  $R_h$  represents the hydraulic radius,  $\mu$  is the fluid viscosity,  $R$  is the gas constant,  $T$  denotes the temperature and  $M$  is the molecular weight. The parameter  $b$  may take a large range of values (see Table I) from  $10^{-2}$  for media of great permeability (Klinkenberg, 1941), to 18 for low permeability media (Wu *et al.*, 1998). In Skjetne and Auriault (1999), Klinkenberg's law is rigorously derived by homogenisation; the fluid flow equations and the wall-slip condition

$$\mathbf{v} = -c\lambda \mathbf{t} \cdot \nabla \mathbf{v} \cdot \mathbf{n} \mathbf{t} \quad \text{on } \Gamma,$$

(in which  $c$  is a constant,  $\mathbf{n}$  and  $\mathbf{t}$  are unit normal and tangential vectors to  $\Gamma$ ) are considered at the pore-scale. Homogenisation of this local description leads to a tensorial form of (1.1) in which the term  $1/\bar{p}$  is replaced by  $1/p$ :

$$\tilde{K}_g = \tilde{K}_l \left( \tilde{I} + \frac{\beta}{p} \tilde{H} \right), \quad (1.5)$$

where  $\tilde{K}_l$  (the liquid permeability) and  $\tilde{H}$  are second-rank and positive tensors and

$$\beta = \frac{c\mu\sqrt{\pi R T/2M}}{l_p}.$$

Table I. Typical values of the Klinkenberg factor  $b$

Author(s)	Material	Gas	$k$ ( $10^{-5} \text{ m}^2$ )	$b$ (MPa)	$p$ (MPa)
Klinkenberg (1941)	Glass filter	Air	2.36	0.68	[0.02, 9]
	Rock	Air	23.6	0.16	[0.01, 5.1]
	Rock	Air	170	0.075	[0.02, 0.17]
Liang <i>et al.</i> (2001)	Quartzite	Air	0.05	1.11	[2, 12]
Wu <i>et al.</i> (1998)	Rock	Nitrogen	$1.61 \times 10^{-5}$	18.8	[1, 2]
McVay and Rish (1995)	Cement	Nitrogen	$8.04 \times 10^{-2}$	0.81	[2.5, 10]
	Cement	Vapor	$8.04 \times 10^{-2}$	0.77	[2.5, 10]
Marschall and Milos (1998)	Fibrous tile	Air	$21.5 \times 10^3$	$3.9 \times 10^{-2}$	$[0.5 \times 10^{-2}, 0.7]$

Equation (1.5) shows that any component of  $\tilde{K}_g$  satisfies Klinkenberg's law. An important feature of law (1.5) is that the term  $1/\bar{p}$  of (1.1) is replaced by  $1/p$ : the gas permeability is therefore independent of the boundary conditions and therefore Equation (1.5) describes a local flow behaviour.

The aim of the present work is to examine whether the structure of Klinkenberg's law does survive the upscaling procedure, that is, if it is still satisfied on a large-scale when a Klinkenberg's flow in a two porous constituent medium is considered at the local scale. The upscaling technique is the homogenisation method of multiple scale expansions introduced by Bensoussan *et al.* (1978) and Sanchez-Palencia (1980). We apply the formulation of the method suggested in Auriault (1991), which uses the dimensionless numbers that arise from the physical description at the local scale. Section 2 is devoted to a brief introduction to the method. Then, in Section 3 we consider a Klinkenberg's flow of gas in a two-constituent porous composite. The local scale under consideration is the composite scale: it consists of two interconnected porous constituents in which Klinkenberg's law is supposed to be valid. We analyse the local description and define the range of dimensionless numbers. The order-of-magnitude analysis of the parameters highlights the influence of the gas permeability ratio. Section 4 is devoted to the case of homogeneous constituents. Constituent permeabilities are first considered to be of same order: the obtained large-scale gas permeability tensor strongly differs from its liquid counterpart, but except in the particular case of equal Klinkenberg factors, the  $1/p$ -form of Klinkenberg's law is not satisfied. Then, we consider the case of high permeability contrast: the large-scale description is a strongly non-linear model with memory effects, of whom we extract a linearised version. We demonstrate that in this case Klinkenberg's law is always valid. All the above mentioned results are examined on the simple geometry of a bilaminated composite. Finally, in Section 5 the case of heterogeneous constituents is considered, for which it is shown that the large-scale tensor of gas permeability differs from its liquid counterpart but does not verify Klinkenberg's law.

## 2. Homogenisation Method

The essence of homogenisation techniques is to determine an equivalent macroscopic behaviour by upscaling the local description. The fundamental assumption behind any homogenisation method is that the scales must be separate. For a two-scale medium, this condition is expressed as  $l \ll L$ , where  $l$  and  $L$  are the characteristic lengths at the heterogeneity scale and at the macroscopic scale, respectively (Figure 1). This definition conjures up a purely geometric separation of scales, but this fundamental condition must also be satisfied by the physical process considered. As an example, for fluid flow in porous media, the heterogeneity

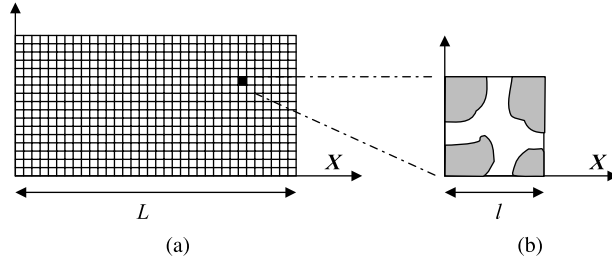


Figure 1. Two-scale medium: (a) macroscopic sample; (b) periodic cell.

characteristic length  $l$  must be small compared to both the length related to the pressure gradient

$$L_e = O\left(\frac{|p|}{|\nabla p|}\right),$$

and to the geometric length (i.e., the sample size)  $L_g$ . The macroscopic scale length is therefore defined as

$$L = \text{Min}(L_e, L_g).$$

We use the homogenisation method for periodic structures (also called method of multiple scale expansions) introduced by Bensoussan *et al.* (1978) and Sanchez-Palencia (1980). When dealing with a two-scale medium, the key parameter of the method is the scale ratio

$$\varepsilon = \frac{l}{L} \ll 1. \quad (2.1)$$

With this homogenisation method, the medium is also assumed to be periodic (Figure 1). This assumption is actually not a restriction; it allows derivation of the macroscopic model without any assumption on the form of the macroscopic equations. In this study, we use the formulation of the method suggested in Auriault (1991), with which the problem is tackled in a more physical manner. This formulation uses the dimensionless numbers that arise from the local description. These dimensionless numbers must be estimated with respect to the scale ratio  $\varepsilon$ . Using the two characteristic lengths, two dimensionless space variables are defined

$$\mathbf{y} = \frac{\mathbf{X}}{l}, \quad \mathbf{x} = \frac{\mathbf{X}}{L}, \quad (2.2)$$

where  $\mathbf{X}$  is the physical space variable. If the condition of separation of scales is satisfied, then  $\mathbf{y}$  and  $\mathbf{x}$  appear as two independent space variables:  $\mathbf{y}$  is the microscopic variable and describes the heterogeneity scale, whereas  $\mathbf{x}$  is the macroscopic space variable. As a consequence, any physical variable of the problem  $\varphi$  (e.g., pressure, velocity, permeability. . .) is *a priori* function of  $\mathbf{y}$  and  $\mathbf{x}$

$$\varphi = \varphi(\mathbf{y}, \mathbf{x}). \quad (2.3)$$

The homogenisation method of multiple scale expansions is based on the fundamental premise that if the scales are well separated, then all physical variables can be looked for in the form of asymptotic expansions in powers of  $\varepsilon$

$$\varphi = \varphi^{(0)}(\mathbf{y}, \mathbf{x}) + \varepsilon \varphi^{(1)}(\mathbf{y}, \mathbf{x}) + \dots, \quad (2.4)$$

in which the functions  $\varphi^i$  are  $\mathbf{y}$ -periodic.

The method consists of incorporating the variable expansions (2.4) in the dimensionless form of the local description. Solving the boundary-value problems arising at the successive orders of  $\varepsilon$  leads to the macroscopic description.

### 3. Klinkenberg's Flow in a Composite Porous Medium

#### 3.1. LOCAL DESCRIPTION

We consider a periodic composite which consists of two interconnected rigid porous constituents. Both constituents are saturated by gas at low pressure. Let denote by  $\Omega$  the period of characteristic size  $O(l)$ . Within the period both constituents occupy domains  $\Omega_1$  and  $\Omega_2$ , respectively, and their common interface is denoted by  $\Gamma$  (Figure 2). For simplicity, both constituents are assumed to be isotropic and their intrinsic permeabilities, that is, their permeabilities to liquid are denoted by  $k_1$  and  $k_2$ , respectively. The whole structure is subject to a macroscopic pressure gradient, which gives rise to an isothermal Klinkenberg's flow. The scale length of the period  $\Omega$ ,  $l$ , is assumed to be small compared to the macroscopic scale length  $L$

$$\varepsilon = \frac{l}{L} \ll 1. \quad (3.1)$$

In each constituent, gas flow is described by Klinkenberg's law

$$\mathbf{v}_\alpha = -K_\alpha \nabla p_\alpha \quad \text{in } \Omega_\alpha \quad (\alpha = 1, 2), \quad (3.2)$$

in which  $K_\alpha$  is the gas permeability of constituent  $\alpha$  and verifies Klinkenberg's law

$$K_\alpha = k_\alpha \left( 1 + \frac{b_\alpha}{p_\alpha} \right) > 0 \quad (\alpha = 1, 2). \quad (3.3)$$

In the above equations,  $p_\alpha$ ,  $\mathbf{v}_\alpha$ ,  $k_\alpha$  and  $b_\alpha$  denote the gas pressure, the gas velocity, the intrinsic liquid permeability and the Klinkenberg factor of constituent  $\alpha$ , respectively.

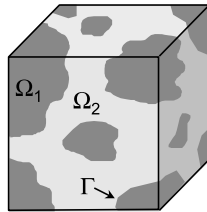


Figure 2. Periodic cell of the composite.

We consider the following linear isothermal equation of state for the gas

$$\rho_\alpha = Ap_\alpha \quad \text{in } \Omega_\alpha \quad (\alpha = 1, 2), \quad (3.4)$$

where  $\rho_\alpha$  denotes the fluid density and  $A$  is a constant.

Gas flow over the periodic cell  $\Omega$  is thus governed by the following set of equations

$$\phi_\alpha \frac{\partial p_\alpha}{\partial t} - \nabla \cdot (p_\alpha K_\alpha \nabla p_\alpha) = 0 \quad \text{in } \Omega_\alpha \quad (\alpha = 1, 2), \quad (3.5)$$

$$[pK\nabla p]_\Gamma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (3.6)$$

$$p_1 = p_2 \quad \text{on } \Gamma, \quad (3.7)$$

where  $\phi_\alpha$  is the porosity of constituent  $\alpha$ ,  $\mathbf{n}$  denotes a unit normal vector to  $\Gamma$ ,  $[\varphi]_\Gamma$  represents the discontinuity of  $\varphi$  over  $\Gamma$  and where the gas permeability  $K_\alpha$  is defined by (3.3). Due to both the fluid compressibility and the dependence of  $K_\alpha$  upon gas pressure, the above set of equations turns out to be strongly non-linear.

### 3.2. NORMALISATION AND ORDER-OF-MAGNITUDE ANALYSIS

The purpose of this section is to define the set of non-dimensional numbers that characterise the local description (3.5)–(3.7), and then to estimate them with respect to powers of the scale ratio  $\varepsilon$  (3.1). From Equation (3.5) we can define

$$Q_1 = \frac{|\phi_1(\partial p_1/\partial t)|}{|\nabla \cdot (p_1 K_1 \nabla p_1)|}, \quad (3.8)$$

and

$$Q_2 = \frac{|\phi_2(\partial p_2/\partial t)|}{|\nabla \cdot (p_2 K_2 \nabla p_2)|} = O(Q_1) \times O\left(\frac{\phi_2}{\phi_1}\right) \times O\left(\frac{p_1}{p_2}\right) \times O\left(\frac{K_1}{K_2}\right). \quad (3.9)$$

Now, from Equation (3.6) arises

$$A = \frac{|(p_1 K_1 \nabla p_1) \cdot \mathbf{n}|}{|(p_2 K_2 \nabla p_2) \cdot \mathbf{n}|} = O\left(\frac{K_1}{K_2}\right) \times O\left(\frac{(p_1)^2}{(p_2)^2}\right). \quad (3.10)$$

For estimating these non-dimensional numbers, let arbitrarily consider  $l$  as the reference characteristic length. When using  $l$  as the reference length, the estimations of  $Q_1$ ,  $Q_2$  and  $A$  are denoted by  $Q_{1l}$ ,  $Q_{2l}$  and  $A_l$ . By assuming for simplicity that

$$\left| \frac{\phi_1}{\phi_2} \right| = O(1), \quad (3.11)$$



and by considering boundary condition (3.7) which gives:

$$\left| \frac{p_1}{p_2} \right| = O(1), \quad (3.12)$$

we thus get

$$Q_{2_1} = O(Q_{1_1}) \times O\left(\frac{K_1}{K_2}\right), \quad (3.13)$$

$$A_1 = O\left(\frac{K_1}{K_2}\right). \quad (3.14)$$

Now, the orders of magnitude of  $Q_1$  and  $Q_2$  are subject to restrictive conditions: there exist orders of magnitude of  $Q_\alpha$  for which homogenisation cannot be applied. Moreover, the macroscopic transient regime is closely linked to the flow regime in the constituent of greater permeability. Assuming that  $\Omega_1$  be the constituent of greater permeability ( $K_1 \geq K_2$ ), gas flow can macroscopically be described by means of an equivalent continuous macroscopic model (i.e., homogenisation can be applied), if on a macroscopic point of view and in an order-of-magnitude sense, the transient term is lower than or equal to the mass-balance term in  $\Omega_1$

$$\left| \phi_1 \frac{\partial p_1}{\partial t} \right|_L \leq O(|\nabla \cdot (K_1 p_1 \nabla p_1)|_L). \quad (3.15)$$

This condition, which is a classical result from homogenisation theory (Auriault *et al.*, 1990), actually means that the existence of an equivalent macroscopic description requires a sufficiently large characteristic flow time  $T_c$ . Condition (3.15), which is expressed with respect to  $L$ , can also be written

$$Q_{1_L} = O\left(\frac{\phi_1 L^2}{p_1 K_1 T_c}\right) \leq O(1). \quad (3.16)$$

It can be shown that the order

$$Q_{1_L} = O(1) \quad (3.17)$$

gives rise to a transient regime at the macroscopic scale, while the order

$$Q_{1_L} < O(1) \quad (3.18)$$

leads to a steady-state macroscopic flow regime. We consider the case (3.17) of greater interest, which when using  $l$  for estimating  $Q_1$  corresponds to

$$Q_{1_l} = O(\varepsilon^2). \quad (3.19)$$

According to (3.13), it follows that

$$Q_{2_1} = O(\varepsilon^2) \times O\left(\frac{K_1}{K_2}\right). \quad (3.20)$$

The only remaining parameter is the order of magnitude of the permeability ratio

$$\kappa = O\left(\frac{K_1}{K_2}\right),$$

and we thus have to consider the following orders of magnitude

$$Q_{1_1} = O(\varepsilon^2), \quad Q_{2_1} = O(\varepsilon^2 \kappa), \quad A_1 = O(\kappa), \quad B_1 = O(1). \quad (3.21)$$

There are two orders of magnitude of interest for  $\kappa$

- Low permeability contrast:  $\kappa = O(1)$ ;
- High permeability contrast:  $\kappa = O(\varepsilon^{-2})$ . (3.22)

The order of magnitude  $\kappa = O(\varepsilon^{-2})$  gives rise to macroscopic memory effects. This is a classical result of homogenisation theory applied to composite media which has first been demonstrated in Auriault (1983). We do not consider the case  $\kappa = O(\varepsilon^{-1})$ , as it would lead to a particular subcase of the model obtained for  $\kappa = O(1)$  (Auriault, 1983).

### 3.3. DIMENSIONLESS LOCAL DESCRIPTION

According to the above estimations and considerations, the dimensionless writing of the local description (3.5)–(3.7) is therefore the following, in which all quantities are now non-dimensional quantities (for simplicity and clarity we have kept, for the dimensionless quantities, the same notations as for their physical counterparts)

$$\varepsilon^2 \phi_1 \frac{\partial p_1}{\partial t} - \nabla \cdot (p_1 K_1 \nabla p_1) = 0 \quad \text{in } \Omega_1, \quad (3.23)$$

$$\varepsilon^2 \kappa \phi_2 \frac{\partial p_2}{\partial t} - \nabla \cdot (p_2 K_2 \nabla p_2) = 0 \quad \text{in } \Omega_2, \quad (3.24)$$

$$(p_1 K_1 \nabla p_1) \cdot \mathbf{n} = \kappa^{-1} (p_2 K_2 \nabla p_2) \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (3.25)$$

$$p_1 = p_2 \quad \text{on } \Gamma, \quad (3.26)$$

in which

$$K_\alpha = k_\alpha \left(1 + \frac{b_\alpha}{p_\alpha}\right) \quad (\alpha = 1, 2). \quad (3.27)$$

We consider the dimensionless space variables  $\mathbf{y}$  and  $\mathbf{x}$  defined by (2.2); the dimensionless gradient operator is therefore given by

$$\nabla = \nabla_{\mathbf{y}} + \varepsilon \nabla_{\mathbf{x}}, \quad (3.28)$$

where  $\nabla_{\mathbf{y}}$  and  $\nabla_{\mathbf{x}}$  are the gradient operators with respect to  $\mathbf{y}$  and  $\mathbf{x}$ , respectively.

As physical variables of the problem, the pressure fields  $p_\alpha$  and the gas permeabilities  $K_\alpha$  are functions of both space variables

$$p_\alpha = p_\alpha(\mathbf{y}, \mathbf{x}, t), \quad K_\alpha = K_\alpha(\mathbf{y}, \mathbf{x}, t).$$

Two situations may at that stage be distinguished:

- Homogeneous constituents:  $k_\alpha$  and  $b_\alpha$  are constants;
- Heterogeneous constituents:  $k_\alpha = k_\alpha(\mathbf{y}, \mathbf{x})$ ,  $b_\alpha = b_\alpha(\mathbf{y}, \mathbf{x})$ .

For applying the homogenisation procedure, the physical variables are looked for in the form of asymptotic expansions in power of  $\varepsilon$ . Thus, the pressure fields  $p_\alpha$  and the gas permeabilities  $K_\alpha$  are written as follows:

$$p_\alpha(\mathbf{y}, \mathbf{x}, t) = p_\alpha^{(0)}(\mathbf{y}, \mathbf{x}, t) + \varepsilon p_\alpha^{(1)}(\mathbf{y}, \mathbf{x}, t) + \dots, \quad (3.29)$$

$$K_\alpha(\mathbf{y}, \mathbf{x}, t) = K_\alpha^{(0)}(\mathbf{y}, \mathbf{x}, t) + \varepsilon K_\alpha^{(1)}(\mathbf{y}, \mathbf{x}, t) + \dots, \quad (3.30)$$

in which the functions  $p_\alpha^{(i)}$  and  $K_\alpha^{(i)}$  are  $\mathbf{y}$ -periodic.

When considering heterogeneous constituents, the intrinsic permeabilities  $k_\alpha$  and the Klinkenberg factors  $b_\alpha$  must also be looked for in the form of asymptotic expansions in power of  $\varepsilon$ :

$$k_\alpha(\mathbf{y}, \mathbf{x}) = k_\alpha^{(0)}(\mathbf{y}, \mathbf{x}) + \varepsilon k_\alpha^{(1)}(\mathbf{y}, \mathbf{x}) + \dots, \quad (3.31)$$

$$b_\alpha(\mathbf{y}, \mathbf{x}) = b_\alpha^{(0)}(\mathbf{y}, \mathbf{x}) + \varepsilon b_\alpha^{(1)}(\mathbf{y}, \mathbf{x}) + \dots, \quad (3.32)$$

in which the functions  $k_\alpha^{(i)}$  and  $b_\alpha^{(i)}$  are  $\mathbf{y}$ -periodic.

The method consists then of incorporating the above expansions and expression (3.28) for the dimensionless gradient operator in the dimensionless local description (3.23)–(3.26) and in Equation (3.27) which defines the gas permeabilities  $K_\alpha$ . Then, identification at the successive orders of  $\varepsilon$  allows construction of appropriate boundary-value problems. Solving the boundary-value problems leads to the macroscopic behaviour. Application of this procedure to the two cases defined by (3.22) and when both constituents are homogeneous is the purpose of the next section. The case of heterogeneous constituents is then considered in Section 5.

## 4. Homogeneous Constituents

### 4.1. LOW PERMEABILITY CONTRAST

#### 4.1.1. Derivation of the Large-Scale Model

When  $\kappa = K_1/K_2 = O(1)$ , the dimensionless local description is written as

$$\varepsilon^2 \phi_\alpha \frac{\partial p_\alpha}{\partial t} - \nabla \cdot (p_\alpha K_\alpha \nabla p_\alpha) = 0 \quad \text{in } \Omega_\alpha \quad (\alpha = 1, 2), \quad (4.1)$$

$$[p K \nabla p]_\Gamma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (4.2)$$

$$p_1 = p_2 \text{ on } \Gamma. \quad (4.3)$$

From the first-order boundary-value problem, we obtain

$$p_1^{(0)} = p_2^{(0)} = p^{(0)}(\mathbf{x}, t), \quad (4.4)$$

and

$$K_1^{(0)} = K_1^{(0)}(\mathbf{x}, t), \quad K_2^{(0)} = K_2^{(0)}(\mathbf{x}, t), \quad (4.5)$$

where

$$K_\alpha^{(0)} = k_\alpha \left( 1 + \frac{b_\alpha}{p_\alpha^{(0)}} \right). \quad (4.6)$$

At the next order, the set (4.1)–(4.3) leads to a boundary-value problem of unknown  $p_\alpha^{(1)}$ , whose solution is given by

$$p^{(1)} = \boldsymbol{\tau}_g(\mathbf{y}, p^{(0)}) \cdot \nabla_x p^{(0)} + \bar{p}^{(1)}(\mathbf{x}, t), \quad (4.7)$$

where  $\bar{p}^{(1)}$  is an arbitrary function independent of  $\mathbf{y}$ . The vector  $\boldsymbol{\tau}_g$  is  $\mathbf{y}$ -periodic, continuous over  $\Gamma$ , average to zero for uniqueness

$$\langle \boldsymbol{\tau}_g \rangle = \frac{1}{|\Omega|} \int_{\Omega} \boldsymbol{\tau}_g \, d\Omega = 0,$$

and is the solution to the well-posed following linear boundary-value problem

$$\frac{\partial}{\partial y_i} \left( \frac{\partial \tau_{g\alpha j}}{\partial y_i} + \delta_{ij} \right) = 0 \quad \text{in } \Omega_\alpha, \quad (4.8)$$

$$K_1^{(0)} \left( \frac{\partial \tau_{g1j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i = K_2^{(0)} \left( \frac{\partial \tau_{g2j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i \quad \text{on } \Gamma, \quad (4.9)$$

$$\tau_{g1_i} = \tau_{g2_i} \quad \text{on } \Gamma. \quad (4.10)$$

We note that  $\boldsymbol{\tau}_g$  depends upon the first-order pressure field  $p^{(0)}$ .

Then, we average over  $\Omega$  the third-order boundary-value problem, to obtain

$$\langle \phi \rangle \frac{\partial p^{(0)}}{\partial t} - \nabla_x \cdot \langle p^{(0)} K^{(0)} (\nabla_y p^{(1)} + \nabla_x p^{(0)}) \rangle_{\Omega} = 0, \quad (4.11)$$

in which  $\langle \phi \rangle = \eta_1 \phi_1 + \eta_2 \phi_2$  denotes the composite bulk porosity and  $\eta_\alpha = |\Omega_\alpha|/|\Omega|$  represents the volume fraction of constituent  $\alpha$ . Symbol  $\langle \cdot \rangle_{\Omega}$  denotes the average over  $\Omega$  and is defined by

$$\langle \Psi \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \Psi \, d\Omega = \frac{1}{|\Omega|} \int_{\Omega_1} \Psi_1 \, d\Omega + \frac{1}{|\Omega|} \int_{\Omega_2} \Psi_2 \, d\Omega.$$

Using the expression (4.7) obtained for  $p^{(1)}$ , Equation (4.11) can be rewritten as

$$\langle \phi \rangle \frac{\partial p^{(0)}}{\partial t} - \nabla_x \cdot \left( p^{(0)} \tilde{\mathcal{K}}_g^* \nabla_x p^{(0)} \right) = 0, \quad (4.12)$$

in which  $\tilde{\mathcal{K}}_g^*$  is the tensor of effective gas permeability and is defined by:

$$\mathcal{K}_{g_{ij}}^* = \frac{1}{|\Omega|} \int_{\Omega} K^{(0)} \left( \frac{\partial \tau_{g_j}}{\partial y_i} + I_{ij} \right) d\Omega. \quad (4.13)$$

Equation (4.12) describes the large-scale gas flow behaviour. In its writing, this macroscopic model is identical to that obtained in the classical case of gas flow in a porous composite. The only specific aspect of this model is with respect to the effective gas permeability tensor,  $\tilde{\mathcal{K}}_g^*$ , which we examine below. For details on the derivation of the macroscopic model, the reader is referred to Appendix A.

#### 4.1.2. Effective Gas Permeability

The tensor of effective gas permeability is thus given by

$$\mathcal{K}_{g_{ij}}^* = \frac{1}{|\Omega|} \int_{\Omega} k \left( 1 + \frac{b}{p^{(0)}} \right) \left( \frac{\partial \tau_{g_j}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (4.14)$$

where  $\tau_g$  is the solution to the boundary-value problem (4.8)–(4.10).

It can easily be shown that  $\tilde{\mathcal{K}}_g^*$  is a positive symmetrical second-rank tensor.

Let now examine the corresponding tensor of effective liquid permeability  $\tilde{\mathcal{K}}_l^*$ , which is defined by

$$\mathcal{K}_{l_{ij}}^* = \frac{1}{|\Omega|} \int_{\Omega} k \left( \frac{\partial \tau_{l_j}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (4.15)$$

in which  $\tau_l$  has the same properties as  $\tau_g$  ( $\mathbf{y}$ -periodic, continuous over  $\Gamma$  and average to zero), but is the solution to the following boundary-value problem

$$\frac{\partial}{\partial y_i} \left( \frac{\partial \tau_{l_j}}{\partial y_i} + \delta_{ij} \right) = 0 \quad \text{in } \Omega_{\alpha}, \quad (4.16)$$

$$k_1 \left( \frac{\partial \tau_{l_j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i = k_2 \left( \frac{\partial \tau_{l_j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i \quad \text{on } \Gamma, \quad (4.17)$$

$$\tau_{l_i} = \tau_{l_i} \quad \text{on } \Gamma. \quad (4.18)$$

Comparing boundary-value problem (4.8)–(4.10) to the set (4.16)–(4.18) that define  $\tau_g$  and  $\tau_l$ , respectively, it follows that

$$\tau_g = \tau_l \quad \text{when } b_1 = b_2, \quad \tau_g \neq \tau_l \quad \text{otherwise.}$$

The tensor  $\tilde{\mathcal{K}}_g^*$  can be rewritten as

$$\tilde{\mathcal{K}}_g^* = \tilde{G} \left( \tilde{I} + \frac{1}{p^{(0)}} \tilde{J} \right), \quad (4.19)$$

where

$$\begin{aligned} \tilde{G} &= \tilde{\mathcal{K}}_1^*, \quad \tilde{J} \text{ independent of } p^{(0)}, \quad \text{when } b_1 = b_2, \\ \tilde{G} &\neq \tilde{\mathcal{K}}_1^*, \quad \tilde{J} = \tilde{J}(p^{(0)}), \quad \text{otherwise.} \end{aligned}$$

Therefore, Klinkenberg's law is only verified in the particular case of equal Klinkenberg factors:  $b_1 = b_2$ . When  $b_1 \neq b_2$ , even though the effective gas permeability differs from the effective liquid permeability, the tensor does not verify Klinkenberg's law; it can only be satisfied for isolated components (under specific geometric and flow conditions) for which

$$\frac{\partial \tau_{1i}}{\partial y_j} = \frac{\partial \tau_{g^i}}{\partial y_j} = 0.$$

#### 4.1.3. Illustration: Bilaminated Composite

In order to illustrate the above results, we consider a periodic bilaminated porous composite. The composite is  $l$ -periodic in the  $y_1$ -direction and of infinity period in both the  $y_2$  and  $y_3$  directions. The periodic cell, which is sketched in Figure 3, consists of two homogeneous porous layers  $\Omega_1$  and  $\Omega_2$  of thickness  $\eta l$  and  $(1 - \eta)l$ , respectively, where  $\eta$  denotes the volume fraction of layer  $\Omega_1$ . Each layer is assumed to be homogeneous and isotropic, of liquid permeability  $k_\alpha$  ( $\alpha = 1, 2$ ) and of gas permeability

$$K_\alpha = k_\alpha \left( 1 + \frac{b_\alpha}{p} \right) \quad (\alpha = 1, 2). \quad (4.20)$$

The components of the effective liquid and gas tensors ( $\tilde{\mathcal{K}}_1^*$  and  $\tilde{\mathcal{K}}_g^*$ ) are the arithmetic mean and the geometric mean, for the flows parallel and perpendicular to the layers, respectively:

$$\mathcal{K}_{1_{12}}^* = \mathcal{K}_{1_{13}}^* = \mathcal{K}_{1_{23}}^* = 0, \quad (4.21)$$

$$\mathcal{K}_{1_{22}}^* = \mathcal{K}_{1_{33}}^* = \eta k_1 + (1 - \eta)k_2 = \mathcal{K}_{1_{\parallel}}^*, \quad (4.22)$$

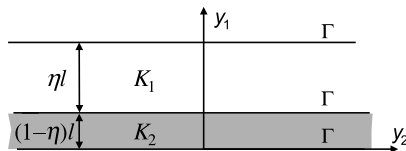


Figure 3. Periodic cell of the bilaminated composite.

$$\mathcal{K}_{111}^* = \frac{k_1 k_2}{(1-\eta)k_1 + \eta k_2} = \mathcal{K}_{1\perp}^*, \quad (4.23)$$

$$\mathcal{K}_{g12}^* = \mathcal{K}_{g13}^* = \mathcal{K}_{g23}^* = 0, \quad (4.24)$$

$$\mathcal{K}_{g22}^* = \mathcal{K}_{g33}^* = \eta K_1 + (1-\eta)K_2 = \mathcal{K}_{g\parallel}^*, \quad (4.25)$$

$$\mathcal{K}_{g11}^* = \frac{K_1 K_2}{(1-\eta)K_1 + \eta K_2} = \mathcal{K}_{g\perp}^*. \quad (4.26)$$

Introducing expressions (4.20) into (4.25) and (4.26) yields

$$\begin{aligned} \mathcal{K}_{g11}^* &= \mathcal{K}_{111}^* \frac{(1+b_1/p)(1+b_2/p)}{1+B/p}, & \mathcal{K}_{g22}^* &= \mathcal{K}_{122}^* \left(1 + \frac{B}{p}\right), \\ \mathcal{K}_{g33}^* &= \mathcal{K}_{133}^* \left(1 + \frac{B}{p}\right), \end{aligned} \quad (4.27)$$

where

$$B = \frac{(1-\eta)k_2 b_2 + \eta k_1 b_1}{(1-\eta)k_2 + \eta k_1}.$$

Therefore,  $\mathcal{K}_{g22}^*$  and  $\mathcal{K}_{g33}^*$  verify Klinkenberg's law whereas  $\mathcal{K}_{g11}^*$  does not, which shows that for the flow in a bilaminated composite, Klinkenberg's law survives upscaling for a flow parallel to the layers but it does not for a flow perpendicular to the layers. As expected, when  $b_1 = b_2 = b$ ,  $\mathcal{K}_{g11}^*$  verifies Klinkenberg's law:

$$\mathcal{K}_{g11}^* = \mathcal{K}_{111}^* \left(1 + \frac{b}{p}\right).$$

## 4.2. HIGH PERMEABILITY CONTRAST

### 4.2.1. Derivation of the Large-Scale Model

When  $\kappa = K_1/K_2 = O(\varepsilon^{-2})$ , the local description (3.23)–(3.26) becomes

$$\varepsilon^2 \phi_1 \frac{\partial p_1}{\partial t} - \nabla \cdot (p_1 K_1 \nabla p_1) = 0 \quad \text{in } \Omega_1, \quad (4.28)$$

$$\phi_2 \frac{\partial p_2}{\partial t} - \nabla \cdot (p_2 K_2 \nabla p_2) = 0 \quad \text{in } \Omega_2, \quad (4.29)$$

$$(p_1 K_1 \nabla p_1) \cdot \mathbf{n} = \varepsilon^2 (p_2 K_2 \nabla p_2) \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (4.30)$$

$$p_1 = p_2 \quad \text{on } \Gamma \quad (4.31)$$

Considering Equations (4.28) and (4.30) at the first order, we obtain a boundary-value problem defined over  $\Omega_1$ , from which we get

$$p_1^{(0)} = p_1^{(0)}(\mathbf{x}, t), \quad K_1^{(0)} = K_1^{(0)}(\mathbf{x}, t), \quad (4.32)$$

and where

$$K_1^{(0)} = k_1 \left( 1 + \frac{b_1}{p_1^{(0)}} \right). \quad (4.33)$$

At the second order, Equations (4.28) and (4.30) constitute a well-posed linear boundary-value problem of unknown  $p_1^{(1)}$ , and whose solution can be written as

$$p_1^{(1)} = \boldsymbol{\chi}(\mathbf{y}, p_1^{(0)}) \cdot \nabla p_1^{(0)} + \bar{p}_1^{(1)}(\mathbf{x}), \quad (4.34)$$

where  $\bar{p}_1^{(1)}$  is an arbitrary function and  $\boldsymbol{\chi}$  is  $\mathbf{y}$ -periodic, satisfies the condition

$$\frac{1}{|\Omega_1|} \int_{\Omega_1} \boldsymbol{\chi} \, d\Omega = 0,$$

and is the solution to the following boundary-value problem

$$\frac{\partial}{\partial y_i} \left( \frac{\partial \chi_j}{\partial y_i} + \delta_{ij} \right) = 0 \quad \text{in } \Omega_1, \quad (4.35)$$

$$\left( \frac{\partial \chi_j}{\partial y_i} + \delta_{ij} \right) n_i = 0 \quad \text{on } \Gamma. \quad (4.36)$$

We now consider Equations (4.29) and (4.31) at the first order

$$\phi_2 \frac{\partial p_2^{(0)}}{\partial t} - \nabla_{\mathbf{y}} \cdot \left( p_2^{(0)} K_2^{(0)} \nabla_{\mathbf{y}} p_2^{(0)} \right) = 0 \quad \text{in } \Omega_2, \quad (4.37)$$

$$p_2^{(0)} = p_1^{(0)}(\mathbf{x}, t) \quad \text{on } \Gamma, \quad (4.38)$$

where  $p_2^{(0)}$  and  $K_2^{(0)}$  are  $\mathbf{y}$ -periodic and where

$$K_2^{(0)} = k_2 \left( 1 + \frac{b_2}{p_2^{(0)}} \right). \quad (4.39)$$

The solution  $p_2^{(0)}$  to the above boundary-value problem can be written as:

$$p_2^{(0)} = \mathcal{F} \left( p_1^{(0)}(\mathbf{x}, t), \mathbf{y}, t \right), \quad (4.40)$$

in which  $\mathcal{F}$  is a non-linear functional with memory effects (Auriault, 1983). Due to the strong non-linearity of the problem, there is no explicit expression for defining  $\mathcal{F}$ .



We then average over  $\Omega_1$  the third-order of Equation (4.28); After using the divergence theorem and the third-order of boundary condition (4.30), the  $\Omega_2$ -average of (4.37) is then required to obtain

$$\eta_1 \phi_1 \frac{\partial p_1^{(0)}}{\partial t} + \phi_2 \frac{\partial \langle p_2^{(0)} \rangle_{\Omega_2}}{\partial t} - \nabla_x \cdot \left\langle p_1^{(0)} K_1^{(0)} \left( \nabla_y p_1^{(1)} + \nabla_x p_1^{(0)} \right) \right\rangle_{\Omega_1} = 0, \quad (4.41)$$

in which  $\eta_1 = |\Omega_1|/|\Omega|$  denotes the volume fraction of domain  $\Omega_1$  and where the  $\Omega_\alpha$ -average ( $\alpha = 1, 2$ ) is defined by

$$\langle \cdot \rangle_{\Omega_\alpha} = \frac{1}{|\Omega|} \int_{\Omega_\alpha} \cdot \, d\Omega.$$

Using the expression obtained for  $p_1^{(1)}$ , Equation (4.41) can be rewritten as

$$\eta_1 \phi_1 \frac{\partial p_1^{(0)}}{\partial t} + \phi_2 \frac{\partial \langle \mathcal{F} \rangle_{\Omega_2}}{\partial t} - \nabla_x \cdot \left( p_1^{(0)} \tilde{\mathcal{K}}_g^{**} \nabla_x p_1^{(0)} \right) = 0, \quad (4.42)$$

that describes the large-scale gas flow behaviour, and in which the tensor of effective gas permeability  $\tilde{\mathcal{K}}_g^{**}$  is defined by

$$\mathcal{K}_{g_{ij}}^{**} = K_1^{(0)} \frac{1}{|\Omega|} \int_{\Omega_1} \left( \frac{\partial \chi_j}{\partial y_i} + I_{ij} \right) \, d\Omega. \quad (4.43)$$

This large-scale description is a strongly non-linear model with memory effects (characterised by the term  $\partial \langle \mathcal{F} \rangle_{\Omega_2} / \partial t$ ), of whom a linearised version is extracted below. The fact that a high contrast of properties leads to memory effects is a well-known result of the homogenisation theory (Auriault, 1983) for thermal conduction in composites. In the absence of Klinkenberg's effect, the mathematical model (4.42) would have the same form: as in Section 4.1, Klinkenberg's effect entails only a modification of the effective tensor of permeability, which we examine in paragraph 4.2.3. Details on the above derivation are given in Appendix B.1.

#### 4.2.2. Linearised Model

In order to get more insight into expression (4.40) and its consequences on the macroscopic flow behaviour, we consider a low pressure level

$$\frac{b_2}{p_2^{(0)}} \gg 1,$$

which enables us to derive a linearised form of model (4.42). In effect,  $K_2^{(0)}$ , which is defined by (4.39), can now be approximate by

$$K_2^{(0)} \simeq k_2 \frac{b_2}{p_2^{(0)}}. \quad (4.44)$$

It can be shown (see Appendix B.2 for details) that, in Fourier space, the macroscopic flow behaviour is described by

$$[\langle\phi\rangle - \eta_2\phi_2 \langle\beta\rangle] i\omega p_1^{(0)} - \nabla_x \cdot \left( p_1^{(0)} \tilde{\mathcal{K}}^{**} \nabla_x p_1^{(0)} \right) = 0, \quad (4.45)$$

in which  $\langle\phi\rangle$  is the bulk porosity, and where

$$\beta(\mathbf{x}, \mathbf{y}, \omega) = \frac{p_2^{(0)} - p_1^{(0)}}{p_1^{(0)}}, \quad \langle\beta\rangle = \frac{1}{|\Omega_2|} \int_{\Omega_2} \beta(\mathbf{x}, \mathbf{y}, \omega) d\Omega.$$

When written in time space, Equation (4.45) becomes

$$\langle\phi\rangle \frac{\partial p_1^{(0)}}{\partial t} - \eta_2\phi_2 \int_{-\infty}^t \hat{B}(t - \tau) \frac{\partial^2 p_1^{(0)}}{\partial t^2} d\tau - \nabla_x \cdot \left( p_1^{(0)} \tilde{\mathcal{K}}_g^{**} \nabla_x p_1^{(0)} \right) = 0, \quad (4.46)$$

where  $\hat{B}$  denotes the inverse Fourier transform of  $\langle\beta\rangle / i\omega$ .

Equation (4.46) is a linearised form of the macroscopic gas flow behaviour (4.42). The convolution product in Equation (4.46) characterises the presence of memory effects.

#### 4.2.3. Effective Gas Permeability

The tensors of effective gas and liquid permeabilities read, respectively

$$\mathcal{K}_{g_{ij}}^{**} = k_1 \left( 1 + \frac{b_1}{p^{(0)}} \right) \frac{1}{|\Omega_1|} \int_{\Omega_1} \left( \frac{\partial \chi_j}{\partial y_i} + I_{ij} \right) d\Omega, \quad (4.47)$$

and

$$\mathcal{K}_{l_{ij}}^{**} = k_1 \frac{1}{|\Omega_1|} \int_{\Omega_1} \left( \frac{\partial \chi_j}{\partial y_i} + I_{ij} \right) d\Omega, \quad (4.48)$$

where  $\chi$  is the solution to the boundary-value problem (4.35)–(4.36). It follows that

$$\mathcal{K}_{g_{ij}}^{**} = \mathcal{K}_{l_{ij}}^{**} \left( 1 + \frac{b_1}{p^{(0)}} \right). \quad (4.49)$$

Klinkenberg's law is therefore always verified at large-scale in the case of high permeability contrast.

#### 4.2.4. Illustration: Bilaminated Composite

Let now examine the above results on the bilaminated composite described in Section 4.1.3 (Figure 3). The components of the effective tensors  $\tilde{\mathcal{K}}_1^{**}$  and  $\tilde{\mathcal{K}}_g^{**}$  are the following

$$\mathcal{K}_{1_{11}}^{**} = 0, \quad (4.50)$$

$$\mathcal{K}_{122}^{**} = \mathcal{K}_{133}^{**} = \eta k_1 = \mathcal{K}_{1\parallel}^{**}, \quad (4.51)$$

$$\mathcal{K}_{112}^{**} = \mathcal{K}_{113}^{**} = \mathcal{K}_{123}^{**} = 0, \quad (4.52)$$

$$\mathcal{K}_{g11}^{**} = 0, \quad (4.53)$$

$$\mathcal{K}_{g22}^{**} = \mathcal{K}_{g33}^{**} = \eta K_1 = \mathcal{K}_{g\parallel}^{**}, \quad (4.54)$$

$$\mathcal{K}_{g12}^{**} = \mathcal{K}_{g13}^{**} = \mathcal{K}_{g23}^{**} = 0. \quad (4.55)$$

It follows that

$$\mathcal{K}_{g11}^{**} = \mathcal{K}_{111}^{**} = 0, \quad (4.56)$$

$$\mathcal{K}_{g22}^{**} = \mathcal{K}_{122}^{**} \left( 1 + \frac{b_1}{p} \right), \quad (4.57)$$

$$\mathcal{K}_{g33}^{**} = \mathcal{K}_{133}^{**} \left( 1 + \frac{b_1}{p} \right), \quad (4.58)$$

which confirms that Klinkenberg's law is valid.

Let us now examine on this particular geometry the linearised model derived in Section 4.2.2. The coefficient  $\langle \beta \rangle$  of the model expressed in Fourier space (4.45) and the memory function  $\hat{B}$  of the time-space model (4.46) can easily be calculated (Auriault, 1983)

$$\langle \beta \rangle = (1 - \eta) \left( 1 - \frac{\tanh(\gamma \sqrt{i})}{\gamma \sqrt{i}} \right) \quad \text{with } \gamma = \sqrt{\frac{\omega \phi_2 (1 - \eta) l}{k_2 b_2}} \frac{1}{2}. \quad (4.59)$$

$$\hat{B}(t) = 8(1 - \eta) \sum_{j=0}^{\infty} \frac{\exp(-(2j+1)^2 \pi^2 \tau / 4)}{(2j+1)^2 \pi^2}, \quad \tau = \frac{4k_2 b_2 t}{\phi_2 (1 - \eta)^2 l^2}. \quad (4.60)$$

## 5. Heterogeneous Constituents

The above results can easily be extended to the case of heterogeneous constituents, for which we have to account for expansions (3.31) and (3.32). Introducing the dependence of  $k_\alpha$  and  $b_\alpha$  on the space variables does not change the writing of the large-scale models (4.12) and (4.42), obtained for low and for high permeability contrast, respectively. The large-scale gas flow behaviours are anyhow modified, due to the consequences on the definitions of the effective tensors of gas permeability.

### 5.1. LOW PERMEABILITY CONTRAST

The effective tensor of gas permeability becomes

$$\mathcal{K}_{gij}^* = \frac{1}{|\Omega|} \int_{\Omega} K^{(0)}(\mathbf{y}, \mathbf{x}) \left( \frac{\partial \tau_{gj}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (5.1)$$

where

$$K_{\alpha}^{(0)}(\mathbf{y}, \mathbf{x}) = k_{\alpha}^{(0)}(\mathbf{y}, \mathbf{x}) \left( 1 + \frac{b_{\alpha}^{(0)}(\mathbf{y}, \mathbf{x})}{p^{(0)}(\mathbf{x})} \right), \quad (5.2)$$

and in which the vector  $\boldsymbol{\tau}_g$  is  $\mathbf{y}$ -periodic, average to zero and is the solution to the following boundary-value problem, for  $\gamma = K^{(0)}(\mathbf{y}, \mathbf{x})$ :

$$\frac{\partial}{\partial y_i} \left[ \gamma_{\alpha} \left( \frac{\partial \tau_{\alpha j}}{\partial y_i} + \delta_{ij} \right) \right] = 0 \quad \text{in } \Omega_{\alpha}, \quad (5.3)$$

$$\gamma_1 \left( \frac{\partial \tau_{1j}}{\partial y_i} + \delta_{ij} \right) n_i = \gamma_2 \left( \frac{\partial \tau_{2j}}{\partial y_i} + \delta_{ij} \right) n_i \quad \text{on } \Gamma, \quad (5.4)$$

$$\tau_{1i} = \tau_{2i} \quad \text{on } \Gamma. \quad (5.5)$$

As for the effective tensor of liquid permeability, it is now defined by

$$\mathcal{K}_{lij}^* = \frac{1}{|\Omega|} \int_{\Omega} k^{(0)}(\mathbf{y}, \mathbf{x}) \left( \frac{\partial \tau_{lj}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (5.6)$$

where the vector  $\boldsymbol{\tau}_l$  is  $\mathbf{y}$ -periodic, average to zero and is the solution to (5.3)–(5.5) for  $\gamma = k^{(0)}(\mathbf{y}, \mathbf{x})$ . It follows that  $\boldsymbol{\tau}_g \neq \boldsymbol{\tau}_l$ , and the tensor  $\tilde{\mathcal{K}}_g^*$  can be written as

$$\tilde{\mathcal{K}}_g^* = \tilde{G} \left( \tilde{I} + \frac{1}{p^{(0)}} \tilde{J} \right), \quad \tilde{G} \neq \tilde{\mathcal{K}}_l, \quad \tilde{J} = \tilde{J}(p^{(0)}). \quad (5.7)$$

Therefore, Klinkenberg's law is never verified in that case.

### 5.2. HIGH PERMEABILITY CONTRAST

The effective tensor of gas permeability is now defined by

$$\mathcal{K}_{gij}^{**} = \frac{1}{|\Omega|} \int_{\Omega_1} K_1^{(0)} \left( \frac{\partial \chi_{gj}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (5.8)$$

where

$$K_1^{(0)}(\mathbf{y}, \mathbf{x}) = k_1^{(0)}(\mathbf{y}, \mathbf{x}) \left( 1 + \frac{b_1^{(0)}(\mathbf{y}, \mathbf{x})}{p^{(0)}(\mathbf{x})} \right), \quad (5.9)$$

and in which the vector  $\boldsymbol{\chi}_g$  is  $\mathbf{y}$ -periodic, average to zero and is the solution to the following boundary-value problem, for  $\gamma = K_1^{(0)}(\mathbf{y}, \mathbf{x})$ :

$$\frac{\partial}{\partial y_i} \left[ \gamma \left( \frac{\partial \chi_j}{\partial y_i} + \delta_{ij} \right) \right] = 0 \quad \text{in } \Omega_1, \quad (5.10)$$

$$\gamma \left( \frac{\partial \chi_j}{\partial y_i} + \delta_{ij} \right) n_i = 0 \quad \text{on } \Gamma. \quad (5.11)$$

The effective tensor of liquid permeability is now given by

$$\mathcal{K}_{ij}^{**} = \frac{1}{|\Omega|} \int_{\Omega_1} k_1^{(0)}(\mathbf{y}, \mathbf{x}) \left( \frac{\partial \chi_{lj}}{\partial y_i} + I_{ij} \right) d\Omega, \quad (5.12)$$

where the vector  $\boldsymbol{\chi}_1$  is  $\mathbf{y}$ -periodic, average to zero and is the solution to (5.10)–(5.11) for  $\gamma = k_1^{(0)}(\mathbf{y}, \mathbf{x})$ .

It turns out that  $\boldsymbol{\chi}_g \neq \boldsymbol{\chi}_1$ , and  $\tilde{K}_g^{**}$  can therefore be written in the form (5.7). Klinkenberg's law is thus not verified. Note that if constituent  $\Omega_2$  is heterogeneous while constituent  $\Omega_1$  is homogeneous, the conclusions of Section 4.2 remain valid: Klinkenberg's law is verified.

## 6. Conclusions

The objective of this work was to determine whether Klinkenberg's law survives upscaling, that is, if the large-scale tensor of effective gas permeability could be expressed as

$$\tilde{\mathcal{K}}_g = \tilde{\mathcal{K}}_1 \left( \tilde{I} + \frac{\beta}{p} \tilde{H} \right),$$

where  $\tilde{\mathcal{K}}_1$  is the effective liquid permeability. Using the homogenisation method of multiple scale expansions, we have derived the large-scale mathematical models and the effective tensors of gas permeability of a Klinkenberg's gas flow in a two-constituent porous composite. By distinguishing the cases of low and of high permeability contrast we have derived two kinds of large-scale models. In their writing, these models are identical to those obtained in the absence of Klinkenberg's effect. The only specific aspect of these models is concerned with the effective tensors of gas permeability which differ from their liquid counterparts. We have shown that Klinkenberg's law survives upscaling only in the two following cases:

1. At low permeability contrast, when both constituents are homogeneous and both Klinkenberg factors are equal;
2. At high permeability contrast when the constituent of greater permeability is homogeneous.

In any other situation, Klinkenberg's law is not verified and the large-scale effective tensor of gas permeability can be expressed as follows:

$$\tilde{\mathcal{K}}_g = \tilde{G} \left( \tilde{I} + \frac{\tilde{J}}{p} \right), \quad \tilde{G} \neq \tilde{K}_1, \quad \tilde{J} = \tilde{J}(p).$$

### Appendix A. Homogeneous Constituents and Low Permeability Contrast: Derivation of the Large-Scale Model (Section 4.1)

The dimensionless local description is written as

$$\varepsilon^2 \phi_\alpha \frac{\partial p_\alpha}{\partial t} - \nabla \cdot (p_\alpha K_\alpha \nabla p_\alpha) = 0 \quad \text{in } \Omega_\alpha \quad (\alpha = 1, 2), \quad (\text{A.1})$$

$$[p K \nabla p]_\Gamma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (\text{A.2})$$

$$p_1 = p_2 \quad \text{on } \Gamma. \quad (\text{A.3})$$

The first-order boundary-value problem reads

$$\nabla_y \cdot (p_\alpha^{(0)} K_\alpha^{(0)} \nabla_y p_\alpha^{(0)}) = 0 \quad \text{in } \Omega_\alpha, \quad (\text{A.4})$$

$$[p^{(0)} K^{(0)} \nabla_y p^{(0)}]_\Gamma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (\text{A.5})$$

$$p_1^{(0)} = p_2^{(0)} \quad \text{on } \Gamma, \quad (\text{A.6})$$

in which  $p_\alpha^{(0)}$  and  $K_\alpha^{(0)}$  are  $\mathbf{y}$ -periodic and where

$$K_\alpha^{(0)} = k_\alpha \left( 1 + \frac{b_\alpha}{p_\alpha^{(0)}} \right). \quad (\text{A.7})$$

This is a well-posed non-linear problem of unknowns  $p_\alpha^{(0)}$  and  $K_\alpha^{(0)}$ .

After using the divergence theorem, boundary-condition (A.5) and periodicity, the  $\Omega$ -average of Equation (A.4) gives

$$\int_\Omega K_\alpha^{(0)} \nabla_y (p_\alpha^{(0)})^2 d\Omega = 0.$$

As a result of the positivity of  $K_\alpha^{(0)}$ , we get

$$\nabla_y (p_\alpha^{(0)})^2 = 0,$$

According to boundary-condition (A.6) we get

$$p_1^{(0)} = p_2^{(0)} = p^{(0)}(\mathbf{x}, t), \quad (\text{A.8})$$

and from Equation (A.7) we deduce

$$K_1^{(0)} = K_1^{(0)}(\mathbf{x}, t), \quad K_2^{(0)} = K_2^{(0)}(\mathbf{x}, t). \quad (\text{A.9})$$

At the next order, the set (A.1)–(A.3) gives the following linear boundary-value problem of unknown  $p_\alpha^{(1)}$

$$\nabla_{\mathbf{y}} \cdot [K_\alpha^{(0)}(\nabla_{\mathbf{y}} p_\alpha^{(1)} + \nabla_{\mathbf{x}} p_\alpha^{(0)})] = 0 \quad \text{in } \Omega_\alpha, \quad (\text{A.10})$$

$$[K^{(0)}(\nabla_{\mathbf{y}} p^{(1)} + \nabla_{\mathbf{x}} p^{(0)})]_{\Gamma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (\text{A.11})$$

$$p_1^{(1)} = p_2^{(1)} \quad \text{on } \Gamma, \quad (\text{A.12})$$

in which  $p_\alpha^{(1)}$  is  $\mathbf{y}$ -periodic.

Let  $\mathcal{V}(\Omega)$  be the Hilbert space of regular functions  $\theta$  defined and continuous over  $\Omega$ , that are  $\mathbf{y}$ -periodic and that satisfy the condition

$$\langle \theta \rangle = \frac{1}{|\Omega|} \int_{\Omega} \theta \, d\Omega = 0. \quad (\text{A.13})$$

Furthermore, this Hilbert space is equipped with the following inner product

$$(\theta_1, \theta_2)_{\mathcal{V}(\Omega)} = \int_{\Omega} K^{(0)} \nabla_{\mathbf{y}} \theta_1 \nabla_{\mathbf{y}} \theta_2 \, d\Omega.$$

Note that the additional condition (A.13) is introduced in order to provide the product  $(\theta_1, \theta_2)_{\mathcal{V}(\Omega)}$  with the required properties of an inner product. The following variational formulation is thus equivalent, modulo an added constant, to the boundary-value problem (A.10)–(A.12):

$$\forall \theta \in \mathcal{V}(\Omega), \quad (p^{(1)}, \theta)_{\mathcal{V}(\Omega)} = - \int_{\Omega} K^{(0)} \nabla_{\mathbf{x}} p^{(0)} \nabla_{\mathbf{y}} \theta \, d\Omega. \quad (\text{A.14})$$

Existence and uniqueness of the solution to (A.14) are proved by Lax–Milgram lemma. This solution is given by

$$p^{(1)} = \boldsymbol{\tau}_{\mathbf{g}}(\mathbf{y}, p^{(0)}) \cdot \nabla_{\mathbf{x}} p^{(0)} + \bar{p}^{(1)}(\mathbf{x}, t), \quad (\text{A.15})$$

where  $\bar{p}^{(1)}$  is an arbitrary function independent of  $\mathbf{y}$ . The vector  $\boldsymbol{\tau}_{\mathbf{g}}$  is  $\mathbf{y}$ -periodic, continuous over  $\Gamma$ , average to zero for uniqueness

$$\langle \boldsymbol{\tau}_{\mathbf{g}} \rangle = \frac{1}{|\Omega|} \int_{\Omega} \boldsymbol{\tau}_{\mathbf{g}} \, d\Omega = 0,$$

and is the solution to the well-posed following boundary-value problem

$$\frac{\partial}{\partial y_i} \left( \frac{\partial \tau_{\mathbf{g}\alpha_j}}{\partial y_i} + \delta_{ij} \right) = 0 \quad \text{in } \Omega_\alpha, \quad (\text{A.16})$$

$$K_1^{(0)} \left( \frac{\partial \tau_{g1j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i = K_2^{(0)} \left( \frac{\partial \tau_{g2j}}{\partial y_i} + \delta_{ij} \right) \cdot n_i \quad \text{on } \Gamma, \quad (\text{A.17})$$

$$\tau_{g1_i} = \tau_{g2_i} \quad \text{on } \Gamma. \quad (\text{A.18})$$

In other words,  $\tau_{g_i}$  is the particular solution to the boundary-value problem defined by Equations (A.10)–(A.12) when  $\partial p^{(0)}/\partial x_j \vec{e}_j = \vec{e}_i$ .

At the third-order, boundary-value problem (A.10)–(A.12) reads

$$\begin{aligned} \phi_\alpha \frac{\partial p^{(0)}}{\partial t} - \nabla_x \cdot [p^{(0)} K_\alpha^{(0)} (\nabla_y p_\alpha^{(1)} + \nabla_x p^{(0)})] - \\ - \nabla_y \cdot [p^{(0)} K_\alpha^{(0)} (\nabla_y p_\alpha^{(2)} + \nabla_x p_\alpha^{(1)})] - \\ - \nabla_y \cdot [(p^{(0)} K_\alpha^{(1)} + p_\alpha^{(1)} K_\alpha^{(0)}) (\nabla_y p_\alpha^{(1)} + \nabla_x p^{(0)})] = 0 \quad \text{in } \Omega_\alpha, \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} [p^{(0)} K_\alpha^{(0)} (\nabla_y p_\alpha^{(2)} + \nabla_x p_\alpha^{(1)})]_\Gamma \cdot \mathbf{n} + \\ + [(p^{(0)} K_\alpha^{(1)} + p_\alpha^{(1)} K_\alpha^{(0)}) (\nabla_y p_\alpha^{(1)} + \nabla_x p^{(0)})]_\Gamma \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \end{aligned} \quad (\text{A.20})$$

$$p_1^{(2)} = p_2^{(2)} \quad \text{on } \Gamma, \quad (\text{A.21})$$

where  $p_1^{(2)}$  and  $p_2^{(2)}$  are  $\mathbf{y}$ -periodic. Then, we average (A.19) over  $\Omega$ .

After using the divergence theorem and boundary condition (A.20), we obtain

$$\langle \phi \rangle \frac{\partial p^{(0)}}{\partial t} - \nabla_x \cdot \langle p^{(0)} K^{(0)} (\nabla_y p^{(1)} + \nabla_x p^{(0)}) \rangle_\Omega = 0. \quad (\text{A.22})$$

Using the expression (A.15) obtained for  $p^{(1)}$ , Equation (A.22) can be rewritten as

$$\langle \phi \rangle \frac{\partial p^{(0)}}{\partial t} - \nabla_x \cdot (p^{(0)} \tilde{\mathcal{K}}_g^* \nabla_x p^{(0)}) = 0, \quad (\text{A.23})$$

in which  $\tilde{\mathcal{K}}_g^*$  is the tensor of effective permeability and is defined by:

$$\mathcal{K}_{g_{ij}}^* = \frac{1}{|\Omega|} \int_\Omega K^{(0)} \left( \frac{\partial \tau_{g_j}}{\partial y_i} + I_{ij} \right) d\Omega. \quad (\text{A.24})$$

## Appendix B. Homogeneous Constituents and High Permeability Contrast: Derivation of the Large-Scale Model (Section 4.2)

### B.1. GENERAL LARGE-SCALE MODEL (SECTION 4.2.1)

The local description is written as follows

$$\varepsilon^2 \phi_1 \frac{\partial p_1}{\partial t} - \nabla \cdot (p_1 K_1 \nabla p_1) = 0 \quad \text{in } \Omega_1, \quad (\text{B.1})$$



$$\phi_2 \frac{\partial p_2}{\partial t} - \nabla \cdot (p_2 K_2 \nabla p_2) = 0 \quad \text{in } \Omega_2, \quad (\text{B.2})$$

$$(p_1 K_1 \nabla p_1) \cdot \mathbf{n} = \varepsilon^2 (p_2 K_2 \nabla p_2) \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (\text{B.3})$$

$$p_1 = p_2 \quad \text{on } \Gamma \quad (\text{B.4})$$

Considering Equations (B.1) and (B.3) at the first order, we obtain the following non-linear boundary-value problem defined over  $\Omega_1$

$$\nabla_{\mathbf{y}} \cdot (p_1^{(0)} K_1^{(0)} \nabla_{\mathbf{y}} p_1^{(0)}) = 0 \quad \text{in } \Omega_1, \quad (\text{B.5})$$

$$(p_1^{(0)} K_1^{(0)} \nabla_{\mathbf{y}} p_1^{(0)}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (\text{B.6})$$

in which  $p_1^{(0)}$  and  $K_1^{(0)}$  are  $\mathbf{y}$ -periodic and where

$$K_1^{(0)} = k_1 \left( 1 + \frac{b_1}{p_1^{(0)}} \right). \quad (\text{B.7})$$

As a result of the positivity of  $K_1^{(0)}$  (the reasoning is identical to that conducted for the first-order problem in Appendix A), we get

$$p_1^{(0)} = p_1^{(0)}(\mathbf{x}, t), \quad K_1^{(0)} = K_1^{(0)}(\mathbf{x}, t). \quad (\text{B.8})$$

At the second order, Equations (B.1) and (B.3) read

$$\nabla_{\mathbf{y}} \cdot [K_1^{(0)} (\nabla_{\mathbf{y}} p_1^{(1)} + \nabla_{\mathbf{x}} p_1^{(0)})] = 0 \quad \text{in } \Omega_1, \quad (\text{B.9})$$

$$[K_1^{(0)} (\nabla_{\mathbf{y}} p_1^{(1)} + \nabla_{\mathbf{x}} p_1^{(0)})] \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (\text{B.10})$$

where  $p_1^{(1)}$  is  $\mathbf{y}$ -periodic. The equivalent variational formulation of the above linear boundary-value problem of unknown  $p_1^{(1)}$  is written as

$$\forall \theta \in \mathcal{V}(\Omega_1), \quad (p_1^{(1)}, \theta)_{\mathcal{V}(\Omega_1)} = - \int_{\Omega} K_1^{(0)} \nabla_{\mathbf{x}} p_1^{(0)} \nabla_{\mathbf{y}} \theta \, d\Omega, \quad (\text{B.11})$$

and its solution can be written as

$$p_1^{(1)} = \boldsymbol{\chi}(\mathbf{y}, p_1^{(0)}) \cdot \nabla p_1^{(0)} + \bar{p}_1^{(1)}(\mathbf{x}), \quad (\text{B.12})$$

where  $\bar{p}_1^{(1)}$  is an arbitrary function and  $\boldsymbol{\chi}$  is  $\mathbf{y}$ -periodic, satisfies the condition

$$\frac{1}{|\Omega_1|} \int_{\Omega_1} \boldsymbol{\chi} \, d\Omega = 0,$$

and is the solution to the following boundary-value problem

$$\frac{\partial}{\partial y_i} \left( \frac{\partial \chi_j}{\partial y_i} + \delta_{ij} \right) = 0 \quad \text{in } \Omega_1, \quad (\text{B.13})$$

$$\left(\frac{\partial \chi_j}{\partial y_i} + \delta_{ij}\right) n_i = 0 \quad \text{on } \Gamma. \quad (\text{B.14})$$

We now consider Equations (B.2) and (B.4) at the first order

$$\phi_2 \frac{\partial p_2^{(0)}}{\partial t} - \nabla_y \cdot (p_2^{(0)} K_2^{(0)} \nabla_y p_2^{(0)}) = 0 \quad \text{in } \Omega_2, \quad (\text{B.15})$$

$$p_2^{(0)} = p_1^{(0)}(\mathbf{x}, t) \quad \text{on } \Gamma, \quad (\text{B.16})$$

where  $p_2^{(0)}$  and  $K_2^{(0)}$  are  $\mathbf{y}$ -periodic and where

$$K_2^{(0)} = k_2 \left(1 + \frac{b_2}{p_2^{(0)}}\right). \quad (\text{B.17})$$

Assuming the existence of the solution  $p_2^{(0)}$  to the above boundary-value problem, it can be written as:

$$p_2^{(0)} = \mathcal{F}(p_1^{(0)}(\mathbf{x}, t), \mathbf{y}, t), \quad (\text{B.18})$$

in which  $\mathcal{F}$  is a non-linear functional with memory effects.

At the third order, Equations (B.1) and (B.3) read

$$\begin{aligned} \phi_1 \frac{\partial p_1^{(0)}}{\partial t} - \nabla_x \cdot [p_1^{(0)} K_1^{(0)} (\nabla_y p_1^{(1)} + \nabla_x p_1^{(0)})] - \\ - \nabla_y \cdot [p_1^{(0)} K_1^{(0)} (\nabla_y p_1^{(2)} + \nabla_x p_1^{(1)})] - \\ - \nabla_y \cdot [(p_1^{(0)} K_1^{(1)} + p_1^{(1)} K_1^{(0)}) (\nabla_y p_1^{(1)} + \nabla_x p_1^{(0)})] = 0 \quad \text{in } \Omega_1, \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} [p_1^{(0)} K_1^{(0)} (\nabla_y p_1^{(2)} + \nabla_x p_1^{(1)})] \cdot \mathbf{n} + \\ + [(p_1^{(0)} K_1^{(1)} + p_1^{(1)} K_1^{(0)}) (\nabla_y p_1^{(1)} + \nabla_x p_1^{(0)})] \cdot \mathbf{n} \\ = (p_2^{(0)} K_2^{(0)} \nabla_y p_2^{(0)}) \cdot \mathbf{n} \quad \text{on } \Gamma, \end{aligned} \quad (\text{B.20})$$

where  $p_1^{(2)}$  and  $K_1^{(1)}$  are  $\mathbf{y}$ -periodic.

We then average (B.19) over  $\Omega_1$ ; after using the divergence theorem and boundary condition (B.20), the  $\Omega_2$ -average of (B.15) is then required to obtain

$$\eta_1 \phi_1 \frac{\partial p_1^{(0)}}{\partial t} + \phi_2 \frac{\partial \langle p_2^{(0)} \rangle_{\Omega_2}}{\partial t} - \nabla_x \cdot \langle p_1^{(0)} K_1^{(0)} (\nabla_y p_1^{(1)} + \nabla_x p_1^{(0)}) \rangle_{\Omega_1} = 0, \quad (\text{B.21})$$

where according to (B.18) we have

$$\phi_2 \frac{\partial \langle p_2^{(0)} \rangle_{\Omega_2}}{\partial t} = \phi_2 \frac{1}{|\Omega|} \int_{\Omega_2} \frac{\partial p_2^{(0)}}{\partial t} d\Omega = \phi_2 \frac{1}{|\Omega|} \int_{\Omega_2} \frac{\partial \mathcal{F}(p_1^{(0)}, \mathbf{y}, t)}{\partial t} d\Omega.$$

Using the expression obtained for  $p_1^{(1)}$ , Equation (B.21) can be rewritten as

$$\eta_1 \phi_1 \frac{\partial p_1^{(0)}}{\partial t} + \phi_2 \frac{\partial \langle \mathcal{F} \rangle_{\Omega_2}}{\partial t} - \nabla_x \cdot \left( p_1^{(0)} \tilde{\mathcal{K}}_g^{**} \nabla_x p_1^{(0)} \right) = 0, \quad (\text{B.22})$$

in which the tensor of effective permeability  $\tilde{\mathcal{K}}^{**}$  is defined by

$$\mathcal{K}_{gij}^{**} = K_1^{(0)} \frac{1}{|\Omega_1|} \int_{\Omega_1} \left( \frac{\partial \chi_j}{\partial y_i} + I_{ij} \right) d\Omega \quad (\text{B.23})$$

## B.2. LINEARISED LARGE-SCALE MODEL (SECTION 4.2.2)

In order to get more insight into expression (B.18), we consider a low pressure level such that

$$\frac{b_2}{p_2^{(0)}} \gg 1.$$

As a consequence,  $K_2^{(0)}$ , defined by (B.17), can be approximate as follows

$$K_2^{(0)} \simeq k_2 \frac{b_2}{p_2^{(0)}}, \quad (\text{B.24})$$

and Equation (B.15) becomes linear

$$\phi_2 \frac{\partial p_2^{(0)}}{\partial t} - \nabla_y \cdot (k_2 b_2 \nabla_y p_2^{(0)}) = 0 \quad \text{in } \Omega_2. \quad (\text{B.25})$$

Now we define

$$W = p_1^{(0)}(\mathbf{x}, t) - p_2^{(0)} \quad (\text{B.26})$$

In Fourier space, the simplified form of boundary-value problem (B.15)–(B.16) with respect to  $W$  is written as

$$\phi_2 i\omega (p_1^{(0)} - W) + \nabla_y \cdot (k_2 b_2 \nabla_y W) = 0 \quad \text{in } \Omega_2, \quad (\text{B.27})$$

$$W = 0 \quad \text{on } \Gamma, \quad (\text{B.28})$$

where  $W$  is  $\mathbf{y}$ -periodic.

Let  $\mathcal{W}$  be the Hilbert space of regular, complex-valued functions  $\theta$  defined on  $\Omega_2$ ,  $\mathbf{y}$ -periodic, zero-valued over  $\Gamma$ , which is equipped with the following inner product

$$(\theta_1, \theta_2)_{\mathcal{W}(\Omega_2)} = \int_{\Omega_2} (k_2 b_2 \nabla_y \theta_1 \nabla_y \tilde{\theta}_2 + i\omega \phi_2 \theta_1 \tilde{\theta}_2) d\Omega,$$

where  $\tilde{\theta}_2$  denotes the complex conjugate of  $\theta_2$ . The equivalent variational formulation of (B.27)–(B.28) is written as

$$\forall \theta \in \mathcal{W}(\Omega_2), \quad (W, \theta)_{\mathcal{W}(\Omega_2)} = - \int_{\Omega_2} i\omega \phi_2 p_1^{(0)} \tilde{\theta} \, d\Omega. \quad (\text{B.29})$$

Existence and uniqueness of the solution stem from Lax–Milgram lemma.

This solution can be written as

$$W(\mathbf{x}, \mathbf{y}) = -\beta(\mathbf{x}, \mathbf{y}, \omega) p_1^{(0)}(\mathbf{x}), \quad (\text{B.30})$$

where scalar  $\beta$  is the pulsation dependent and complex-valued particular solution to Equation (B.29) when  $p_1^{(0)} = -1$ .

In Fourier space, the macroscopic description is thus written as

$$[\langle \phi \rangle - \eta_2 \phi_2 \langle \beta \rangle] i\omega p_1^{(0)} - \nabla_x \cdot (p_1^{(0)} \tilde{\mathcal{K}}_g^{**} \nabla_x p_1^{(0)}) = 0 \quad (\text{B.31})$$

When written in time space, Equation (B.31) becomes

$$\langle \phi \rangle \frac{\partial p_1^{(0)}}{\partial t} - \eta_2 \phi_2 \int_{-\infty}^t \hat{B}(t - \tau) \frac{\partial^2 p_1^{(0)}}{\partial \tau^2} \, d\tau - \nabla_x \cdot (p_1^{(0)} \tilde{\mathcal{K}}_g^{**} \nabla_x p_1^{(0)}) = 0,$$

where  $\hat{B}$  denotes the inverse Fourier transform of  $\langle \beta \rangle / i\omega$ .

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