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# Spectrum of a linear differential equation with constant coefficients

Tinhinane A. AZZOUZ

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## Abstract

In this paper we compute the spectrum, in the sense of Berkovich, of an ultrametric linear differential equation with constant coefficients, defined over an affinoid domain of the analytic affine line  $\mathbb{A}_k^{1,an}$ . We show that it is a finite union of either closed disks or topological closures of open disks and that it satisfies a continuity property.

## 1 Introduction

Differential equations constitute an important tool for the investigation of algebraic and analytic varieties, over the complex and the  $p$ -adic numbers. Notably, de Rham cohomology is one of the most powerful way to obtain algebraic and analytic informations. Besides, ultrametric phenomena appeared naturally studying formal Taylor solutions of the equation around singular and regular points.

As soon as the theory of ultrametric differential equations became a central topic of investigation around 1960, after the work of B. Dwork, P. Robba (et al.), the following interesting phenomena appeared. In the ultrametric setting, the solutions of a linear differential equation may fail to converge as expected, even if the coefficients of the equation are entire functions. For example, over the ground field  $\mathbb{Q}_p$  of  $p$ -adic numbers, the exponential power series  $\exp(T) = \sum_{n \geq 0} \frac{T^n}{n!}$  which is solution of the equation  $y' = y$  has radius of convergence equal to  $|p|^{\frac{1}{(p-1)}}$  even though the equation shows no singularities. However, the behaviour of the radius of convergence is well controlled, and its knowledge permits to obtain several informations about the equation. Namely it controls the *finite dimensionality* of the de Rham cohomology. For more details, we refer the reader to the recent work of J. Poineau and A. Pulita [Pul15], [PP15], [PP13a] and [PP13b].

The starting point of this paper is an interesting relation between this radius and the notion of *spectrum* (in the sense of Berkovich). Before discussing more in detail our results, we shall quickly explain this relation.

Consider a ring  $A$  together with a derivation  $d : A \rightarrow A$ . A *differential module* on  $(A, d)$  is a finite free  $A$ -module  $M$  together with a  $\mathbb{Z}$ -linear map

$$\nabla : M \rightarrow M$$

satisfying for all  $m \in M$  and all  $f \in A$  the relation  $\nabla(fm) = d(f)m + f\nabla(m)$ . If an isomorphism  $M \cong A^\nu$  is given, then  $\nabla$  coincides with an operator of the form

$$d + G : A^\nu \rightarrow A^\nu$$

where  $d$  acts on  $A^\nu$  component by component and  $G$  is a square matrix with coefficients in  $A$ .

If  $A$  is moreover a Banach algebra with respect to a given norm  $\|\cdot\|$  and  $A^\nu$  is endowed with the max norm, then we can endow the algebra of continuous  $\mathbb{Z}$ -linear endomorphisms with the operator norm, that we still denote  $\|\cdot\|$ . The spectral norm of  $\nabla$  is given by

$$\|\nabla\|_{Sp} = \lim_{n \rightarrow \infty} \|\nabla^n\|^{\frac{1}{n}}.$$

The link between the radius of convergence of  $\nabla$  and its spectrum is then the following: on the one hand, when  $\|\cdot\|$  is a Berkovich point of type other than 1 of the affine line and  $A$  is the field of this point, the spectral norm  $\|\nabla\|_{Sp}$  coincides with the inverse of the radius of convergence of  $M$  at  $\|\cdot\|$  multiplied by some constant (cf. [CD94, p. 676] or [Ked10, Definition 9.4.4]). On the other hand the spectral norm  $\|\nabla\|_{Sp}$  is also equal to the radius of the smallest disk centred at zero and containing the spectrum of  $\nabla$  in the sense of Berkovich (cf. [Ber90, Theorem 7.1.2]). We refer to the introduction of Section 3.2 for a more precise explanation.

The spectrum appears then as a new invariant of the connection  $\nabla$  generalizing and refining the radius of convergence. This has been our first motivation, however the study of the spectrum of an operator has its own interest and it deserves its own independent theory.

In this paper we focus on the computation of the spectrum of  $\nabla$  in the case where  $\nabla$  has constant coefficients, which means that there is a basis of  $M$  in which the matrix  $G$  has coefficients in the base field of constants (i.e. the field of elements killed by the derivations). We show that when we change the domain of definition of the equation the spectrum has a uniform behaviour: it is a finite union of either closed disks or topological closures (in the sense of Berkovich) of open disks.

Let  $k$  be an arbitrary field and  $E$  be a  $k$ -algebra with unit. Recall that classically, the spectrum  $\Sigma_f(E)$  of an element  $f$  of  $E$  (cf. [Bou07, §1.2. Definition 1]) is the set of elements  $\lambda$  of  $k$  such that  $f - \lambda.1_E$  is not invertible in  $E$ . In the case where  $k = \mathbb{C}$  and  $(E, \|\cdot\|)$  is a  $\mathbb{C}$ -Banach algebra, the spectrum of  $f$  satisfies the following properties:

- It is not empty and compact.
- The smallest disk centred at 0 and containing  $\Sigma_f(E)$  has radius equal to  $\|f\|_{Sp} = \lim_{n \rightarrow +\infty} \|f^n\|^{\frac{1}{n}}$ .
- The resolvent function  $R_f : \mathbb{C} \setminus \Sigma_f \rightarrow E$ ,  $\lambda \mapsto (f - \lambda.1_E)^{-1}$  is an analytic function with values in  $E$ .

Unfortunately, this may fail in the ultrametric case. In [Vis85] M. Vishik provides an example of operator with empty spectrum and with a resolvent which is only locally analytic. We illustrate this pathology with an example in our context, where connections with empty classical spectrum abound.

**Example 1.1.** Consider the field  $k = \mathbb{C}$  endowed with the trivial absolute value. Let  $A := \mathbb{C}((S))$  be the field of Laurent power series endowed with the  $S$ -adic absolute value given by  $|\sum_{n \geq n_0} a_n S^n| = r^{n_0}$ , if  $a_{n_0} \neq 0$ , where  $|S| = r < 1$  is a nonzero real number<sup>1</sup>. We consider a rank one *irregular* differential module over  $A = \mathbb{C}((S))$  defined by the operator  $\frac{d}{dS} + g : A \rightarrow A$ , where  $g \in A$  has  $S$ -adic valuation  $n_0$  that is less than or equal to  $-2$ . We consider the connection  $\nabla = \frac{d}{dS} + g$  as an element of the  $\mathbb{C}$ -Banach algebra  $E = \mathcal{L}_{\mathbb{C}}(\mathbb{C}((S)))$  of bounded  $\mathbb{C}$ -linear maps of  $\mathbb{C}((S))$  with respect to the usual operator norm:  $\|\varphi\| = \sup_{f \in A \setminus \{0\}} \frac{\|\varphi(f)\|}{\|f\|}$ , for all  $\varphi \in E$ . Then, the classical spectrum of the differential operator  $\frac{d}{dS} + g$  is empty. Indeed, since  $\|\frac{d}{dS}\| = \frac{1}{r}$  and  $|(g - a)^{-1}| \leq r^2$  for all  $a \in \mathbb{C}$ , then  $\|(g - a)^{-1} \circ (\frac{d}{dS})\| \leq r$  (resp.  $\|(\frac{d}{dS}) \circ (g - a)^{-1}\| \leq r$ ) and the series  $\sum_{n \geq 0} (-1)^n \cdot ((g - a)^{-1} \circ (\frac{d}{dS}))^n$  (resp.  $\sum_{n \geq 0} (-1)^n \cdot ((\frac{d}{dS}) \circ (g - a)^{-1})^n$ ) converges in  $\mathcal{L}_k(\mathbb{C}((S)))$ . Hence, for all  $a \in \mathbb{C}$ ,  $(\sum_{n \geq 0} (-1)^n \cdot ((g - a)^{-1} \circ (\frac{d}{dS}))^n) \circ (g - a)^{-1}$  (resp.  $(g - a)^{-1} \circ (\sum_{n \geq 0} (-1)^n \cdot ((\frac{d}{dS}) \circ (g - a)^{-1})^n)$ ) is a left (resp. right) inverse of  $\frac{d}{dS} + (g - a)$  in  $\mathcal{L}_k(\mathbb{C}((S)))$  and  $a$  does not belong to the spectrum.<sup>2</sup>

To deal with this issue V. Berkovich understood that it was better not to define the spectrum as a subset of the base field  $k$ , but as a subset of the analytic line  $\mathbb{A}_k^{1,an}$ , which is a bigger space<sup>3</sup>. His theory of analytic spaces (cf. [Ber90], [Ber93]) enjoys several good local topological properties such as

<sup>1</sup>In the language of Berkovich, this field can be naturally identified with the complete residual field of the point  $x_{0,r} \in \mathbb{A}_k^{1,an}$  (cf. section 2.2).

<sup>2</sup>Notice that it is relatively easy to show that any non trivial rank one connection over  $\mathbb{C}((S))$  is set theoretically bijective. This follows from the classical index theorem of B. Malgrange [Mal74]. However, the set theoretical inverse of the connection may not be automatically bounded. This is due to the fact that the base field  $\mathbb{C}$  is trivially valued and the Banach open mapping theorem does not hold in general. However, it is possible to prove that any such set theoretical inverse is bounded (cf. [Azz18]).

<sup>3</sup>This can be motivated by the fact that the resolvent is an analytic function on the complement of the spectrum.

compactness, arc connectedness ... In this setting Berkovich developed a spectral theory for ultrametric operators [Ber90, Chapter 7]. The definition of the spectrum given by Berkovich is the following: Let  $(k, |\cdot|)$  be complete field with respect to an ultrametric absolute value and let  $\mathbb{A}_k^{1,an}$  be the Berkovich affine line. For a point  $x \in \mathbb{A}_k^{1,an}$  we denote by  $\mathcal{H}(x)$  the associated complete residual field. We fix on  $\mathbb{A}_k^{1,an}$  a coordinate function  $T$ . The spectrum  $\Sigma_f(E)$  of an element  $f$  of a  $k$ -Banach algebra  $E$  is the set of points  $x \in \mathbb{A}_k^{1,an}$ , such that  $f \otimes 1 - 1 \otimes T(x)$  is not invertible in  $E \hat{\otimes}_k \mathcal{H}(x)$ . It can be proved (cf. [Ber90, Proposition 7.1.6]) that this is equivalent to say that there exists a complete valued field  $\Omega$  containing isometrically  $k$  and a constant  $c \in \Omega$  such that

- the image of  $c$  by the canonical projection  $\mathbb{A}_\Omega^{1,an} \rightarrow \mathbb{A}_k^{1,an}$  is  $x$ ;
- $f \otimes 1 - 1 \otimes c$  is not invertible as an element of  $E \hat{\otimes}_k \Omega$ .

In some sense  $\Omega = \mathcal{H}(x)$  and  $c = T(x)$  are the minimal possible choices. This spectrum is compact, non empty and it satisfies the properties listed above (cf. [Ber90, Theorem 7.12]). Notice that if  $f = \nabla$  is a connection and if  $E = \mathcal{L}_k(M)$  is the  $k$ -Banach algebra of continuous  $k$ -linear endomorphisms of  $M$ , then  $f \otimes 1 - 1 \otimes c$  is no more a connection as an element of  $E \hat{\otimes}_k \Omega$ . Indeed,  $\mathcal{L}_k(M) \hat{\otimes}_k \Omega$  does not coincide with  $\mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$ , unless  $\Omega$  is a finite extension. In particular, *no index theorem can be used* to test the set theoretical bijectivity of  $\nabla \otimes 1 - 1 \otimes c$ .

Coming back to the above example, using this definition it can be proved that the spectrum of  $\frac{d}{dS} + g$  is now reduced to the individual *non-rational* point  $x_{0,r^{n_0}} \in \mathbb{A}_\mathbb{C}^{1,an}$  (cf. [Azz18]). The dependence on  $r$  shows that the Berkovich spectrum *depends on the chosen absolute value on  $k$* ; whereas, instead, the classical spectrum is a completely algebraic notion.

Now, let  $(k, |\cdot|)$  be an ultrametric complete field. Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$  and  $x$  a point of type (2), (3) or (4). To avoid confusion we fix another coordinate function  $S$  over  $X$ . We set  $A = \mathcal{O}(X)$  or  $\mathcal{H}(x)$  and  $d = g(S) \frac{d}{dS}$  with  $g(S) \in A$ . We will distinguish the notion of differential equation in the sense of a differential polynomial  $P(d)$  acting on  $A$  from the notion of differential module  $(M, \nabla)$  over  $(A, d)$  associated to  $P(d)$  in a cyclic basis (cf. section 3.2). Indeed, it is not hard to prove that the spectrum of  $P(\frac{d}{dS}) = (\frac{d}{dS})^n + a_{n-1}(\frac{d}{dS})^{n-1} + \dots + a_0$  as an element of  $\mathcal{L}_k(A)$ , where  $a_i \in k$ , is given by the easy formula (cf. [Bou07, p. 2] and Lemma 2.25)

$$\Sigma_{P(\frac{d}{dS})}(\mathcal{L}_k(A)) = P(\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(A))). \quad (1)$$

This set is always either a closed disk or the topological closure of an open disk (cf. Lemma 2.25, Remark 4.16). On the other hand, surprisingly enough, this differs from the spectrum  $\Sigma_\nabla(\mathcal{L}_k(M))$  of the differential module  $(M, \nabla)$  associated to  $P(\frac{d}{dS})$  in a cyclic basis (i.e the spectrum of  $\nabla$  as an element of  $\mathcal{L}_k(M)$ ). Indeed, this last is a finite union of either closed disks or topological closures of open disks (cf. Theorem 1.2) centered on the roots of the (commutative) polynomial  $Q = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in k[X]$  associated to  $P$ .<sup>4</sup> In order to introduce our next result, we denote by  $\tilde{k}$  the residual field of  $k$  (cf. Section 2) and we set:

$$\omega = \begin{cases} |p|^{\frac{1}{p-1}} & \text{if } \text{char}(\tilde{k}) = p \\ 1 & \text{if } \text{char}(\tilde{k}) = 0 \end{cases}. \quad (2)$$

The main statement of this paper is then the following:

**Theorem 1.2.** *We suppose that  $k$  is algebraically closed. Let  $X$  be a connected affinoid domain of  $\mathbb{A}_k^{1,an}$  and  $x \in \mathbb{A}_k^{1,an}$  a point of type (2), (3) or (4). We set  $A = \mathcal{O}(X)$  or  $\mathcal{H}(x)$ . Let  $(M, \nabla)$  be a differential module over  $(A, \frac{d}{dS})$  such that there exists a basis for which the associated matrix  $G$  has constant entries (i.e.  $G \in \mathcal{M}_n(k)$ ). Let  $\{a_1, \dots, a_n\} \subset k$  be the set of eigenvalues of  $G$ . Then the behaviour of the spectrum  $\Sigma_\nabla(\mathcal{L}_k(M))$  of  $\nabla$  as an element of  $\mathcal{L}_k(M)$  is summarized in the following table:*

<sup>4</sup>Notice that  $Q$  is the Fourier transform of  $P$  and that its roots are related to the eigenvalues of the matrix  $G$  of the connection in the companion form (cf. Remark 4.15).

$A =$	$\mathcal{O}(X)$		$\mathcal{H}(x)$	
	$X = D^+(c, r)$	$X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$	$x$ of type (2) or (3)	$x$ of type (4)
$\text{char}(\tilde{k}) = p > 0$	$\bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r})$	$\bigcup_{i=1}^N D^+(a_i, \frac{\omega}{\min_j r_j})$	$\bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r(x)})$	
$\text{char}(\tilde{k}) = 0$	$\bigcup_{i=1}^N \overline{D^-(a_i, \frac{1}{r})}$	$\bigcup_{i=1}^N D^+(a_i, \frac{1}{\min_j r_j})$	$\bigcup_{i=1}^N D^+(a_i, \frac{1}{r(x)})$	$\bigcup_{i=1}^N \overline{D^-(a_i, \frac{1}{r(x)})}$

Using this result we can prove that when  $x$  varies over Berkovich a segment  $(x_1, x_2) \subset \mathbb{A}_k^{1, \text{an}}$ , the behaviour of the spectrum is left continuous, and continuous at the points of type (3) (cf. Section 5).

The paper is organized as follows. Section 2 is devoted to recalling the definitions and properties. It is divided into three parts: in the first one we recall some basic definitions and properties of  $k$ -Banach spaces, in the second we provide settings and notations related to the affine line  $\mathbb{A}_k^{1, \text{an}}$  and in the last one we recall the definition of the spectrum given by Berkovich and some properties.

In Section 3, we introduce the spectrum associated to a differential module, and show how it behaves under exact sequences. The main result of this section is the following:

**Proposition 1.3.** *We suppose that  $k$  is algebraically closed. Let  $A$  be as in Theorem 1.2, and let  $d = g(S)d/dS$ , with  $g \in A$ . Let  $(M, \nabla)$  be a differential module over  $(A, d)$  such that there exists a basis for which the associated matrix  $G$  has constant entries (i.e  $G \in \mathcal{M}_n(k)$ ). Then the spectrum of  $\nabla$  is  $\Sigma_{\nabla} = \bigcup_{i=1}^n (a_i + \Sigma_d(\mathcal{L}_k(A)))$ , where  $\{a_1, \dots, a_n\} \subset k$  is the multiset of the eigenvalues of  $G$ .*

In particular the spectrum *highly depends on the choice of the derivation  $d$* . In this paper we choose  $d = \frac{d}{dS}$ . This claim shows the importance of computing the spectrum of  $\Sigma_d(\mathcal{L}_k(A))$ . Therefore, in Section 4, a large part is devoted to the computation of the spectrum of  $\frac{d}{dS}$  acting on various rings. In the last part of this section, we state and prove the main result.

In the last section, we will explain, in the case of a differential equation with constant coefficients, that the spectrum associated to  $(M, \nabla)$  over  $(\mathcal{H}(x), d)$  satisfies a continuity property, when  $x$  varies over a segment  $(x_1, x_2) \subset \mathbb{A}_k^{1, \text{an}}$ .

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## 2 Definitions and notations

All rings are with unit element. We denote by  $\mathbb{R}$  the field of real numbers, and  $\mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$ . In all the paper  $(k, |\cdot|)$  will be a valued field of characteristic 0, complete with respect to an ultrametric absolute value  $|\cdot| : k \rightarrow \mathbb{R}_+$  (i.e. verifying  $|1| = 1$ ,  $|a \cdot b| = |a||b|$ ,  $|a + b| \leq \max(|a|, |b|)$  for all  $a, b \in k$  and  $|a| = 0$  if and only if  $a = 0$ ). We set  $|k| := \{r \in \mathbb{R}_+ | \exists a \in k \text{ such that } r = |a|\}$ ,  $k^\circ := \{a \in k | |a| \leq 1\}$ ,  $k^{\circ\circ} := \{a \in k | |a| < 1\}$  and  $\widehat{k} := k^\circ / k^{\circ\circ}$ . Let  $E(k)$  be the category whose objects are pairs  $(\Omega, |\cdot|)$ , where  $\Omega$  is a field extension of  $k$  complete with respect to  $|\cdot|$ , and whose morphisms are the isometric rings morphisms. For  $(\Omega, |\cdot|) \in E(k)$ , we set  $\Omega^{\text{alg}}$  to be an algebraic closure of  $\Omega$ , the absolute value extends uniquely to an absolute value defined on  $\Omega^{\text{alg}}$ . We denote by  $\widehat{\Omega^{\text{alg}}}$  the completion of  $\Omega^{\text{alg}}$  with respect to this absolute value.

### 2.1 Banach spaces

An *ultrametric norm* on a  $k$ -vector space  $M$  is a map  $\|\cdot\| : M \rightarrow \mathbb{R}_+$  verifying:

- $\|m\| = 0 \Leftrightarrow m = 0$ .

- $\forall m \in M, \forall \lambda \in k; \|\lambda m\| = |\lambda| \|m\|.$
- $\forall m, n \in M, \|m + n\| \leq \max(\|m\|, \|n\|).$

A normed  $k$ -vector space  $M$  is a  $k$ -vector space endowed with an ultrametric norm  $\|\cdot\|$ . If moreover  $M$  is complete with respect to this norm, we say that  $M$  is  *$k$ -Banach space*. A  $k$ -linear map  $\varphi : M \rightarrow N$  between two normed  $k$ -vector spaces is a *bounded  $k$ -linear map* satisfying the following condition:

$$\exists C \in \mathbb{R}_+, \forall m \in M; \|\varphi(m)\| \leq C \|m\|.$$

If  $C = 1$  we say that  $\varphi$  is a *contracting map*.

Let  $M$  and  $N$  be two ultrametric normed  $k$ -vector spaces. We endow the tensor product  $M \otimes_k N$  with the following norm:

$$\begin{aligned} \|\cdot\| : M \otimes_k N &\longrightarrow \mathbb{R}_+ \\ f &\longmapsto \inf \left\{ \max_i \|m_i\| \|n_i\| \mid f = \sum_i m_i \otimes n_i \right\}. \end{aligned} \quad (3)$$

The completion of  $M \otimes_k N$  for this norm will be denoted by  $M \hat{\otimes}_k N$ .

An ultrametric norm on a  $k$ -algebra  $A$  is an ultrametric norm  $\|\cdot\| : A \rightarrow \mathbb{R}_+$  on the  $k$ -vector space  $A$ , satisfying the additional properties:

- $\|1\| = 1$  if  $A$  has a unit element.
- $\forall m, n \in A; \|mn\| \leq \|m\| \|n\|.$

A normed  $k$ -algebra  $A$  is a  $k$ -algebra endowed with an ultrametric norm  $\|\cdot\|$ . If moreover  $A$  is complete with respect to this norm, we say that  $A$  is a  *$k$ -Banach algebra*.

We set  $\mathbf{Ban}_k$  the category whose objects are  $k$ -Banach vector spaces and arrows are bounded  $k$ -linear maps. An isomorphism in this category will be called bi-bounded isomorphism. We set  $\mathbf{BanAL}_k$  the category whose objects are  $k$ -Banach algebras and arrows are bounded morphisms of  $k$ -algebras.

**Definition 2.1.** Let  $A$  be a  $k$ -Banach algebra. The *spectral semi-norm* associated to the norm of  $A$  is the map:

$$\begin{aligned} \|\cdot\|_{Sp,A} : A &\longrightarrow \mathbb{R}_+ \\ f &\longmapsto \lim_{n \rightarrow +\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|f^n\|^{\frac{1}{n}}. \end{aligned} \quad (4)$$

The existence of the limit and its equality with the infimum are well known (cf. [Bou07, Définition 1.3.2]). If  $\|\cdot\|_{Sp,A} = \|\cdot\|$ , we say that  $A$  is a *uniform algebra*.

**Lemma 2.2.** Let  $M$  be a  $k$ -vector space and let  $\|\cdot\| : M \rightarrow \mathbb{R}_+$  be a map verifying the following properties:

- $\|m\| = 0 \Leftrightarrow m = 0.$
- $\forall m \in M, \forall \lambda \in k; \|\lambda m\| \leq |\lambda| \|m\|.$
- $\forall m, n \in M, \|m + n\| \leq \max(\|m\|, \|n\|).$

Then  $\|\cdot\|$  is an ultrametric norm on  $M$ .

*Proof.* See [BGR84, Section 2.1.1, Proposition 4]. □

**Lemma 2.3.** Let  $\Omega \in E(k)$  and let  $M$  be a  $k$ -Banach space. Then, the inclusion  $M \hookrightarrow M \hat{\otimes}_k \Omega$  is an isometry. In particular, for all  $v \in M$  and  $c \in \Omega$  we have  $\|v \otimes c\| = |c| \|v\|$ .

*Proof.* Since  $\Omega$  contains isometrically  $k$ , the morphism  $M \hat{\otimes}_k \Omega \rightarrow \Omega$  (resp.  $\Omega \rightarrow M \hat{\otimes}_k \Omega$ ) is an isometry (cf. [Poi13, Lemma 3.1]). We know that  $\|v \otimes c\| \leq |c| \|v\| = |c| \|v \otimes 1\|$  for all  $v \in M$  and  $c \in \Omega$ . Therefore, by Lemma 2.2 the tensor norm is a norm on  $M \hat{\otimes}_k \Omega$  as an  $\Omega$ -vector space. Consequently, we obtain  $\|v \otimes c\| = |c| \|v\|$  for all  $v \in M$  and  $c \in \Omega$ . □

**Proposition 2.4.** *Let  $M$  be a  $k$ -Banach space, and  $B$  be a uniform  $k$ -Banach algebra. Then in  $M \hat{\otimes}_k B$  we have:*

$$\forall m \in M, \forall f \in B, \|m \otimes f\| = \|m\| \|f\|.$$

*Proof.* Let  $\mathcal{M}(B)$  be the analytic spectrum of  $B$  (see [Ber90, Chapter 1]). For all  $x$  in  $\mathcal{M}(B)$  the canonical map  $B \rightarrow \mathcal{H}(x)$  is a contracting map, therefore the map  $M \hat{\otimes}_k B \rightarrow M \hat{\otimes}_k \mathcal{H}(x)$  is a contracting map too. Then, by Lemme 2.4 we have:

$$\forall x \in \mathcal{M}(B) : \forall m \in M, \forall f \in B, \|m \otimes f(x)\| = \|m\| |f(x)| \leq \|m \otimes f\| \leq \|m\| \|f\|$$

thus,

$$\forall m \in M, \forall f \in B, \|m\| \max_{x \in \mathcal{M}(B)} |f(x)| \leq \|m \otimes f\| \leq \|m\| \|f\|.$$

Since  $B$  is uniform, we have  $\max_{x \in \mathcal{M}(B)} |f(x)| = \|f\|_{sp} = \|f\|$  (cf. [Ber90, Theorem 1.3.1]), and  $\|m\| \|f\| \leq \|m \otimes f\| \leq \|m\| \|f\|$ . □

Let  $M$  and  $N$  be two  $k$ -Banach spaces. We denote by  $\mathcal{L}_k(M, N)$  the  $k$ -Banach algebra of bounded  $k$ -linear maps  $M \rightarrow N$  endowed with the operator norm:

$$\begin{aligned} \mathcal{L}_k(M, N) &\longrightarrow \mathbb{R}_+ \\ \varphi &\longmapsto \sup_{m \in M \setminus \{0\}} \frac{\|\varphi(m)\|}{\|m\|}. \end{aligned} \quad (5)$$

We set  $\mathcal{L}_k(M) := \mathcal{L}_k(M, M)$ .

**Lemma 2.5.** *Let  $\Omega \in E(k)$ . There exists an isometric  $k$ -linear map  $\mathcal{L}_k(M) \hookrightarrow \mathcal{L}_k(M \hat{\otimes}_k \Omega)$  which extends to an  $\Omega$ -linear contracting map  $\mathcal{L}_k(M) \hat{\otimes}_k \Omega \rightarrow \mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$ .*

*Proof.* Let  $\varphi \in \mathcal{L}_k(M)$ , we have the bilinear map:

$$\begin{aligned} \varphi \times 1 : M \times \Omega &\longrightarrow M \otimes_k \Omega \\ (x, a) &\longmapsto \varphi(x) \otimes a \end{aligned}$$

where  $M \times \Omega$  is endowed with the product topology and  $M \otimes \Omega$  with the topology induced by tensor norm (cf. (3)). The map  $\varphi \times 1$  is continuous see [Sch01, Chap. IV Lemma 17.1]. The diagram below:

$$\begin{array}{ccc} M \times \Omega & \xrightarrow{\varphi \times 1} & M \otimes_k \Omega \\ \otimes \downarrow & \nearrow \exists! \varphi \otimes 1 & \\ M \otimes_k \Omega & & \end{array}$$

is commutative, in addition  $\varphi \otimes 1$  is continuous see [Sch01, Chap. IV Lemma 17.1]. By the universal property of the completion of a metric space, the map  $\varphi \otimes 1$  extends to a continuous map  $\varphi \hat{\otimes} 1 : M \hat{\otimes}_k \Omega \rightarrow M \hat{\otimes}_k \Omega$ . So we obtain a  $k$ -linear map  $\mathcal{L}_k(M) \hookrightarrow \mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$ . We prove now that it is an isometry. Indeed, let  $m \in M$  and  $a \in \Omega$  then by Lemma 2.3:

$$\|\varphi \otimes 1(m \otimes a)\| = \|\varphi(m) \otimes a\| = \|\varphi(m)\| \|a\| \leq \|\varphi\| \|m\| \|a\| = \|\varphi\| \|m \otimes a\|.$$

Now, let  $x = \sum_i m_i \otimes a_i \in M \hat{\otimes}_k \Omega$ , then we have:

$$\|\varphi \hat{\otimes} 1(x)\| \leq \inf \{ \max_i (\|\varphi(m_i)\| \|a_i\|) \mid x = \sum_i m_i \otimes a_i \} \leq \|\varphi\| \inf \{ \max_i \|m_i \otimes a_i\| \mid x = \sum_i m_i \otimes a_i \}.$$

Consequently,

$$\|\varphi \hat{\otimes} 1\| \leq \|\varphi\|.$$



On other hand, since  $M \hookrightarrow M \hat{\otimes}_k \Omega$  is an isometry and  $\varphi \hat{\otimes} 1|_M = \varphi$ , we have  $\|\varphi\| \leq \|\varphi \hat{\otimes} 1\|$ . The map  $\mathcal{L}_k(M) \hookrightarrow \mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$  extends to a continuous  $\Omega$ -linear map  $\mathcal{L}_k(M) \hat{\otimes}_k \Omega \hookrightarrow \mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$ . We need to prove now that it is a contracting map. Let  $\psi = \sum_i \varphi_i \otimes a_i$  be an element of  $\mathcal{L}_k(M) \hat{\otimes}_k \Omega$ , its image in  $\mathcal{L}_\Omega(M \hat{\otimes}_k \Omega)$  is the element  $\sum_i a_i \varphi_i \hat{\otimes} 1$ . we have:

$$\left\| \sum_i a_i \varphi_i \hat{\otimes} 1 \right\| \leq \max_i \|a_i \varphi_i \hat{\otimes} 1\| = \max_i \|\varphi_i\| |a_i|.$$

Consequently,

$$\left\| \sum_i a_i \varphi_i \hat{\otimes} 1 \right\| \leq \inf \{ \max_i \|\varphi_i\| |a_i| \mid \psi = \sum_i \varphi_i \otimes a_i \} = \|\psi\|.$$

Hence we obtain the result.  $\square$

**Lemma 2.6.** *Let  $M$  be a  $k$ -Banach space and  $L$  be a finite extension of  $k$ . Then we have a bi-bounded isomorphism:*

$$\mathcal{L}_k(M) \hat{\otimes}_k L \simeq \mathcal{L}_L(M \hat{\otimes}_k L).$$

*Proof.* As  $L$  is a sequence of finite intermediary extensions generated by one element, by induction we can assume that  $L = k(\alpha)$ , where  $\alpha$  is an algebraic element over  $k$ . By Lemma 2.5 we have a morphism  $\mathcal{L}_k(M) \hat{\otimes}_k L \rightarrow \mathcal{L}_L(M \hat{\otimes}_k L)$ . As  $L = k(\alpha)$ , we have a  $k$ -isomorphism  $M \hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} M \otimes (\alpha^i \cdot k)$ . Let  $\psi \in \mathcal{L}_L(M \hat{\otimes}_k L)$ . The restriction  $\psi|_{M \otimes 1} : M \otimes 1 \rightarrow M \hat{\otimes}_k L$  is of the form  $m \otimes 1 \mapsto \sum_{i=0}^{n-1} \varphi_i(m) \otimes \alpha^i$ , where  $\varphi_i \in \mathcal{L}_k(M)$ . As  $\psi$  is  $L$ -linear, it is determined by the  $\varphi_i$ . This gives rise to an inverse  $L$ -linear map  $\mathcal{L}_L(M \hat{\otimes}_k L) \rightarrow \mathcal{L}_k(M) \hat{\otimes}_k L$ . In the case where  $k$  is not trivially valued, by the open mapping theorem (see [BGR84, Section 2.8 Theorem of Banach]) the last map is bounded. Otherwise, the extension  $L$  is trivially valued. Consequently, we have an isometric  $k$ -isomorphisms:  $L \simeq \bigoplus_{i=0}^{n-1} k$  equipped with the max norm. Therefore, we have  $\mathcal{L}_k(M) \hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} \mathcal{L}_k(M)$  and  $M \hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} M$  with respect to the max norm. Then we have:

$$\max_i \|\varphi_i\| \leq \|\psi|_{M \otimes 1}\| \leq \|\psi\|.$$

Hence, we obtain the result.  $\square$

## 2.2 Berkovich line

In this paper, we will consider  $k$ -analytic spaces in the sense of Berkovich (see [Ber90]). We denote by  $\mathbb{A}_k^{1,an}$  the affine analytic line over the ground field  $k$ , with coordinate  $T$ . We set  $k[T]$  to be the ring of polynomial with coefficients in  $k$  and  $k(T)$  its fractions field.

Recall that a point  $x \in \mathbb{A}_k^{1,an}$  corresponds to a multiplicative semi-norm  $|\cdot|_x$  on  $k[T]$  (i.e.  $|0|_x = 0$ ,  $|P + Q|_x \leq \max(|P|_x, |Q|_x)$  and  $|P \cdot Q|_x = |P|_x |Q|_x$  for all  $P, Q \in k[T]$ ), that its restriction coincides with the absolute value of  $k$ . The set  $\mathfrak{p}_x := \{P \in k[T] \mid |f|_x = 0\}$  is a prime ideal of  $k[T]$ . Therefore, the semi-norm extends to a multiplicative norm on the fraction field  $\text{Frac}(A/\mathfrak{p}_x)$ .

**Notation 2.7.** We denote by  $\mathcal{H}(x)$  the completion of  $\text{Frac}(A/\mathfrak{p}_x)$  with respect to  $|\cdot|_x$ , and by  $|\cdot|$  the absolute value on  $\mathcal{H}(x)$  induced by  $|\cdot|_x$ .

Let  $\Omega \in E(k)$  and  $c \in \Omega$ . For  $r \in \mathbb{R}_+^*$  we set

$$D_\Omega^+(c, r) = \{x \in \mathbb{A}_\Omega^{1,an} \mid |T(x) - c| \leq r\}$$

and

$$D_\Omega^-(c, r) = \{x \in \mathbb{A}_\Omega^{1,an} \mid |T(x) - c| < r\}$$

Denote by  $x_{c,r}$  the unique point in the Shilov boundary of  $D_\Omega^+(c, r)$ .

For  $r_1, r_2 \in \mathbb{R}_+$ , such that  $0 < r_1 \leq r_2$  we set



$$C_{\Omega}^{+}(c, r_1, r_2) = \{x \in \mathbb{A}_{\Omega}^{1,an} | r_1 \leq |T(x) - c| \leq r_2\}$$

and for  $r_1 < r_2$  we set:

$$C_{\Omega}^{-}(c, r_1, r_2) = \{x \in \mathbb{A}_{\Omega}^{1,an} | r_1 < |T(x) - c| < r_2\}$$

We may delete the index  $\Omega$  when it is obvious from the context.

**Hypothesis 2.8.** Until the end of this section we will suppose that  $k$  is algebraically closed.

Each affinoid domain  $X$  of  $\mathbb{A}_k^{1,an}$  is a finite union of connected affinoid domain of the form:

$$D^{+}(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^{-}(c_i, r_i) \quad (6)$$

Where  $c_0, \dots, c_{\mu} \in D^{+}(c_0, r_0) \cap k$  and  $0 < r_1, \dots, r_{\mu} \leq r_0$  (the case where  $\mu = 0$  is included).

Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ , we denote by  $\mathcal{O}(X)$  the  $k$ -Banach algebra of global sections of  $X$ . For a disk  $D^{+}(c, r)$  we have:

$$\mathcal{O}(D^{+}(c, r)) = \left\{ \sum_{i \in \mathbb{N}} a_i (T - c)^i \mid a_i \in k, \lim_{i \rightarrow +\infty} |a_i| r^i = 0 \right\}$$

equipped with the multiplicative norm:

$$\left\| \sum_{i \in \mathbb{N}} a_i (T - c)^i \right\| = \max_{i \in \mathbb{N}} |a_i| r^i$$

More generally, if  $X$  is of the form (6), then by the Mittag-Leffler decomposition [FP04, Proposition 2.2.6], we have:

$$\mathcal{O}(X) = \bigoplus_{i=1}^{\mu} \left\{ \sum_{j \in \mathbb{N}^*} \frac{a_{ij}}{(T - c_i)^j} \mid a_{ij} \in k, \lim_{j \rightarrow +\infty} |a_{ij}| r_i^{-j} = 0 \right\} \oplus \mathcal{O}(D^{+}(c_0, r_0)).$$

where  $\left\| \sum_{j \in \mathbb{N}^*} \frac{a_{ij}}{(T - c_i)^j} \right\| = \max_j |a_{ij}| r_i^{-j}$  and the sum above is equipped with the maximum norm.

For  $\Omega \in E(k)$ , we set  $X_{\Omega} = X \hat{\otimes}_k \Omega$ . We have a canonical projection of analytic spaces:

$$\pi_{\Omega/k} : X_{\Omega} \rightarrow X \quad (7)$$

**Definition 2.9.** Let  $x \in \mathbb{A}_k^{1,an}$ . we define the radius of  $x$  to be the value:

$$r_k(x) = \inf_{a \in k} |T(x) - a|.$$

We may delete  $k$  if it is obvious from the context.

We can describe the field  $\mathcal{H}(x)$ , where  $x$  is a point of  $\mathbb{A}_k^{1,an}$ , in a more explicit way. In the case where  $x$  is of type (1), we have  $\mathcal{H}(x) = k$ . If  $x$  is of type (3) of the form  $x = x_{c,r}$  where  $c \in k$  and  $r \notin |k|$ , then it is easy to see that  $\mathcal{H}(x) = \mathcal{O}(C^{+}(c, r, r))$ . But for the points of type (2) and (4), a description is not obvious, we have the following Propositions:

**Proposition 2.10** (Mittag-Leffler Decomposition). *Let  $x = x_{c,r}$  be a point of type (2) of  $\mathbb{A}_k^{1,an}$  ( $c \in k$  and  $r \in |k^*|$ ). We have the decomposition:*

$$\mathcal{H}(x) = E \oplus \mathcal{O}(D^{+}(c, r))$$

where  $E$  is the closure in  $\mathcal{H}(x)$  of the ring of rational fractions of  $k(T - c)$  whose poles are in  $D^{+}(c, r)$ . i.e. for  $\gamma \in k$  with  $|\gamma| = r$ :

$$E := \widehat{\bigoplus_{\alpha \in \bar{k}} \left\{ \sum_{i \in \mathbb{N}^*} \frac{a_{\alpha i}}{(T - c + \gamma \alpha)^i} \mid a_{\alpha i} \in k, \lim_{i \rightarrow +\infty} |a_{\alpha i}| r^{-i} = 0 \right\}}.$$

*Proof.* In the case where  $k$  is not trivially valued we refer to [Chr83, Theorem 2.1.6]. Otherwise, the only point of type (2) of  $\mathbb{A}_k^{1,an}$  is  $x_{0,1}$ , which corresponds to the trivial norm on  $k[T]$ . Therefore, we have  $\mathcal{H}(x) = k(T)$ .  $\square$

**Lemma 2.11.** *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (4). The field  $\mathcal{H}(x)$  is the completion of  $k[T]$  with respect to the norm  $|\cdot|_x$ .*

*Proof.* Recall that for a point  $x \in \mathbb{A}_k^{1,an}$  of type (4), the field  $\mathcal{H}(x)$  is the completion of  $k(T)$  with respect to  $|\cdot|_x$ . To prove that  $\mathcal{H}(x)$  is the completion of  $k[T]$ , it is enough to show that  $k[T]$  is dense in  $k(T)$  with respect to  $|\cdot|_x$ . For all  $a \in k$  it is then enough to show that there exists a sequence  $(P_i)_{i \in \mathbb{N}} \subset k[T]$  which converges to  $\frac{1}{T-a}$ . Let  $a \in k$ . Since  $x$  is of type (4), there exists  $c \in k$  such that  $|T-c|_x < |T-a|_x$ . Therefore we have  $|c-a|_x = |T-a|_x$  and we obtain:

$$\frac{1}{T-a} = \frac{1}{(T-c) + (c-a)} = \frac{1}{c-a} \sum_{i \in \mathbb{N}} \frac{(T-c)^i}{(a-c)^i}.$$

So we conclude.  $\square$

**Proposition 2.12** ([Chr83]). *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (4). Then there exists an isometric isomorphism  $\psi : \mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x))) \rightarrow \mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x)$  of  $\mathcal{H}(x)$ -Banach algebras.*

*Proof.* Note that, for any element  $f \in \mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x)$  with  $|f| \leq r_k(x)$ , we can define a morphism of  $\mathcal{H}(x)$ -Banach algebra  $\psi : \mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x))) \rightarrow \mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x)$ , that associates  $f$  to  $T - T(x)$ . To prove the statment we choose  $f = T(x) \otimes 1 - 1 \otimes T(x)$ . Indeed, for all  $a \in k$  we have  $T(x) \otimes 1 - 1 \otimes T(x) = (T(x) - a) \otimes 1 - 1 \otimes (T(x) - a)$ . Hence,

$$|T(x) \otimes 1 - 1 \otimes T(x)| \leq \inf_{a \in k} (\max(|(T(x) - a)|, |(T(x) - a)|)) = \inf_{a \in k} |T(x) - a| = r_k(x)$$

By construction  $\psi$  is a contracting  $\mathcal{H}(x)$ -linear map. In order to prove that it is an isometric isomorphism, we need to construct its inverse map and show that it is also a contracting map. For all  $a \in k$ ,  $|T(x) - a| > r_k(x)$ , hence  $T - a$  is invertible in  $\mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))$ . This means that  $k(T) \subset \mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))$  as  $k$ -vector space. As for all  $a \in k$  we have:

$$|T - a| = \max(r_k(x), |T(x) - a|) = |T(x) - a|$$

the restriction of the norm of  $\mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))$  to  $k(T)$  coincides with  $|\cdot|_x$ . Consequently, the closure of  $k(T)$  in  $\mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))$  is exactly  $\mathcal{H}(x)$ , which means that we have an isometric embedding  $\mathcal{H}(x) \hookrightarrow \mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))$  of  $k$ -algebras which associates  $T(x)$  to  $T$ . This map extends uniquely to a contracting morphism of  $\mathcal{H}(x)$ -algebras:

$$\begin{array}{ccc} \mathcal{H}(x) & \hookrightarrow & \mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x))) \\ \downarrow & \nearrow \exists! \varphi & \\ \mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x) & & \end{array}$$

Then we have  $\varphi(T(x) \otimes 1 - 1 \otimes T(x)) = T - T(x)$ . Since  $(T - T(x))$  (resp.  $T(x) \otimes 1 - 1 \otimes T(x)$ ) is a topological generator of the  $k$ -algebra  $D_{\mathcal{H}(x)}^+(T(x), r_k(x))$  (resp.  $\mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x)$ ) and both of  $\varphi \circ \psi$  and  $\psi \circ \varphi$  are bounded morphisms of  $k$ -Banach algebras, we have  $\varphi \circ \psi = Id_{\mathcal{O}(D_{\mathcal{H}(x)}^+(T(x), r_k(x)))}$  and  $\psi \circ \varphi = Id_{\mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x)}$ . Hence, we obtain the result.  $\square$

For any point  $x$  of  $\mathbb{A}_k^{1,an}$  and any extension  $\Omega \in E(k)$  the tensor norm on the algebra  $\mathcal{H}(x) \hat{\otimes}_k \Omega$  is multiplicative (see [Poi13, Corollary 3.14.]). We denote this norm by  $\sigma_{\Omega/k}(x)$ .

**Proposition 2.13.** *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (i), where  $i \in \{2, 3, 4\}$ . Let  $\Omega \in E(k)$  algebraically closed. If  $\Omega \notin E(\mathcal{H}(x))$ , then  $\pi_{\Omega/k}^{-1}\{x\} = \{\sigma_{\Omega/k}(x)\}$ .*

*Proof.* Recall that if  $\pi_{\Omega/k}^{-1}\{x\} \setminus \{\sigma_{\Omega/k}(x)\}$  is not empty, then it is a union of disjoint open disks (cf. [PP15, Theorem 2.2.9]). Therefore, it contains points of type (1) which implies that  $\Omega \in E(\mathcal{H}(x))$ . Hence, we obtain a contradiction.  $\square$

## 2.3 Berkovich's spectral theory

We recall here the definition of the sheaf of analytic functions with value in a  $k$ -Banach space over an analytic space and the definition of the spectrum given by V. Berkovich in [Ber90].

**Definition 2.14.** Let  $X$  be a  $k$ -affinoid space and  $B$  be a  $k$ -Banach space. We define the sheaf of analytic functions with values in  $B$  over  $X$  to be the sheaf:

$$\mathcal{O}_X(B)(U) = \varprojlim_{V \subset U} B \hat{\otimes}_k A_V$$

where  $U$  is an open subset of  $X$ ,  $V$  an affinoid domain and  $A_V$  the  $k$ -affinoid algebra associated to  $V$ .

As each  $k$ -analytic space is obtained by gluing  $k$ -affinoid spaces (see [Ber90], [Ber93]), we can extend the definition to  $k$ -analytic spaces. Let  $U$  be an open subset of  $X$ . Every element  $f \in \mathcal{O}_X(B)(U)$  induces a function:  $f : U \rightarrow \prod_{x \in U} B \hat{\otimes}_k \mathcal{H}(x)$ ,  $x \mapsto f(x)$ , where  $f(x)$  is the image of  $f$  by the map  $\mathcal{O}_X(B)(U) \rightarrow B \hat{\otimes}_k \mathcal{H}(x)$ . We will call an *analytic function over  $U$  with value in  $B$* , any function  $\psi : U \rightarrow \prod_{x \in U} B \hat{\otimes}_k \mathcal{H}(x)$  induced by an element  $f \in \mathcal{O}_X(B)(U)$ .

**Hypothesis 2.15.** Until the end of the paper, we will assume that all Banach algebras are with unit element.

**Definition 2.16.** Let  $E$  be  $k$ -Banach algebra and  $f \in E$ . The spectrum of  $f$  is the set  $\Sigma_{f,k}(E)$  of points  $x \in \mathbb{A}_k^{1,an}$  such that the element  $f \otimes 1 - 1 \otimes T(x)$  is not invertible in the  $k$ -Banach algebra  $E \hat{\otimes}_k \mathcal{H}(x)$ . The resolvent of  $f$  is the function:

$$\begin{aligned} R_f : \mathbb{A}_k^{1,an} \setminus \Sigma_{f,k}(E) &\longrightarrow \prod_{x \in \mathbb{A}_k^{1,an} \setminus \Sigma_{f,k}(E)} E \hat{\otimes}_k \mathcal{H}(x) \\ x &\longmapsto (f \otimes 1 - 1 \otimes T(x))^{-1} \end{aligned}$$

*Remark 2.17.* If there is no confusion we denote the spectrum of  $f$ , as an element of  $E$ , just by  $\Sigma_f$ .

**Theorem 2.18.** Let  $E$  be a Banach  $k$ -algebra and  $f \in E$ . Then:

1. The spectrum  $\Sigma_f$  is a non-empty compact subset of  $\mathbb{A}_k^{1,an}$ .
2. The radius of the smallest (inclusion) closed disk with center at zero which contains  $\Sigma_f$  is equal to  $\|f\|_{S_p}$ .
3. The resolvent  $R_f$  is an analytic function on  $\mathbb{A}_k^{1,an} \setminus \Sigma_f$  which is equal to zero at infinity.

*Proof.* See [Ber90, Theorem 7.1.2]. □

*Remark 2.19.* Let  $E$  be a  $k$ -Banach algebra and  $r \in \mathbb{R}_+^*$ . Note that we have

$$\mathcal{O}_{\mathbb{A}_k^{1,an}}(E)(D^+(a, r)) = E \hat{\otimes}_k \mathcal{O}(D^+(a, r)),$$

i.e any element of  $\mathcal{O}_{\mathbb{A}_k^{1,an}}(E)(D^+(a, r))$  has the form  $\sum_{i \in \mathbb{N}} f_i \otimes (T-a)^i$  with  $f_i \in E$ . Let  $\varphi = \sum_{i \in \mathbb{N}} f_i \otimes (T-a)^i$  be an element of  $\mathcal{O}_{\mathbb{A}_k^{1,an}}(E)(D^+(a, r))$ . Since  $\|f_i \otimes (T-a)^i\| = \|f_i\| \|(T-a)\|^i$  in  $\mathcal{O}_{\mathbb{A}_k^{1,an}}(E)(D^+(a, r))$  (cf. Proposition 2.4), the radius of convergence of  $\varphi$  with respect to  $T-a$  is equal to  $\liminf_{i \rightarrow +\infty} \|f_i\|^{-\frac{1}{i}}$ .

**Lemma 2.20.** We maintain the same assumption as in Theorem 2.18. If  $a \in (\mathbb{A}_k^{1,an} \setminus \Sigma_f) \cap k$ , then the biggest open disk centred in  $a$  contained in  $\mathbb{A}_k^{1,an} \setminus \Sigma_f$  has radius  $R = \|(f-a)^{-1}\|_{S_p}^{-1}$ .

*Proof.* Since  $\Sigma_f$  is compact and not empty, the biggest disk  $D^-(a, R) \subset \mathbb{A}_k^{1,an} \setminus \Sigma_f$  has finite positive radius  $R$ . In the neighbourhood of the point  $a$ , we have:

$$R_f = (f \otimes 1 - 1 \otimes T)^{-1} = ((f-a) \otimes 1 - 1 \otimes (T-a))^{-1} = ((f-a)^{-1} \otimes 1) \sum_{i \in \mathbb{N}} \frac{1}{(f-a)^i} \otimes (T-a)^i.$$

On the one hand the radius of convergence of the latter series with respect to  $(T-a)$  is equal to  $\|(f-a)^{-1}\|_{S_p}^{-1}$  (cf. Remark 2.19). On the other hand, the analyticity of  $R_f$  on  $D^-(a, R)$  implies that the radius of convergence with respect to  $T-a$  is equal to  $R$ . Hence, we have  $R = \|(f-a)^{-1}\|_{S_p}^{-1}$ . □

**Proposition 2.21.** *Let  $E$  be a commutative  $k$ -Banach algebra element, and  $f \in E$ . The spectrum  $\Sigma_f$  of  $f$  coincides with the image of the analytic spectrum  $\mathcal{M}(E)$  by the map induced by the ring morphism  $k[T] \rightarrow E$ ,  $T \mapsto f$ .*

*Proof.* See [Ber90, Proposition 7.1.4]. □

**Definition 2.22.** Let  $E$  be a  $k$ -Banach algebra and  $B$  a commutative  $k$ -subalgebra of  $E$ . We say that  $B$  is a maximal subalgebra of  $E$ , if for any subalgebra  $B'$  of  $E$  we have the following property:

$$(B \subset B' \subset E) \Leftrightarrow (B' = B).$$

*Remark 2.23.* A maximal subalgebra  $B$  is necessarily closed in  $E$ , hence a  $k$ -Banach algebra.

**Proposition 2.24.** *Let  $E$  be a  $k$ -Banach algebra. For any maximal commutative subalgebra  $B$  of  $E$ , we have:*

$$\forall f \in B; \quad \Sigma_f(B) = \Sigma_f(E).$$

*Proof.* See [Ber90, Proposition 7.2.4]. □

Let  $P(T) \in k[T]$ , let  $E$  be a Banach  $k$ -algebra and let  $f \in E$ . We set  $P(f)$  to be the image of  $P(T)$  by the morphism  $k[T] \rightarrow E$ ,  $T \mapsto f$ , and  $P : \mathbb{A}_k^{1,an} \rightarrow \mathbb{A}_k^{1,an}$  to be the analytic map associated to  $k[T] \rightarrow k[T]$ ,  $T \mapsto P(T)$ .

**Lemma 2.25.** *Let  $P(T) \in k[T]$ , let  $E$  be a Banach  $k$ -algebra and let  $f \in E$ . We have the equality of sets:*

$$\Sigma_{P(f)} = P(\Sigma_f)$$

*Proof.* Let  $B$  a maximal commutative  $k$ -subalgebra of  $E$  containing  $f$  (which exists by Zorn's Lemma). Then  $B$  contains also  $P(f)$ . By Proposition 2.24 we have  $\Sigma_f(E) = \Sigma_f(B)$  and  $\Sigma_{P(f)}(E) = \Sigma_f(B)$ . Let  ${}^*f : \mathcal{M}(B) \rightarrow \mathbb{A}_k^{1,an}$  (resp.  ${}^*P(f) : \mathcal{M}(B) \rightarrow \mathbb{A}_k^{1,an}$ ) be the map induced by  $k[T] \rightarrow E$ ,  $T \mapsto f$  (resp.  $T \mapsto P(f)$ ). By Proposition 2.21 we have  $\Sigma_f(B) = {}^*f(\mathcal{M}(B))$  and  $\Sigma_{P(f)}(B) = {}^*P(f)(\mathcal{M}(B))$ . Since  ${}^*P(f) = P \circ {}^*f$ , we obtain the equality. □

*Remark 2.26.* Note that, we can imitate the proof provided in [Bou07, p.2] to prove the statement of Lemma 2.25.

**Lemma 2.27.** *Let  $E$  and  $E'$  be two Banach  $k$ -algebras and  $\varphi : E \rightarrow E'$  be a bounded morphism of  $k$ -algebras. If  $f \in E$  then we have:*

$$\Sigma_{\varphi(f)}(E') \subset \Sigma_f(E).$$

*If moreover  $\varphi$  is a bi-bounded isomorphism then we have the equality.*

*Proof.* Consequence of the definition. □

Let  $M_1$  and  $M_2$  be two  $k$ -Banach spaces, let  $M = M_1 \oplus M_2$  endowed with the max norm (i.e.  $\forall m_1 \in M_1$  and  $\forall m_2 \in M_2$   $\|m_1 + m_2\| = \max(\|m_1\|, \|m_2\|)$ ). We set:

$$\mathcal{M}(M_1, M_2) = \left\{ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \mid L_1 \in \mathcal{L}_k(M_1), L_2 \in \mathcal{L}_k(M_2, M_1), L_3 \in \mathcal{L}_k(M_1, M_2), L_4 \in \mathcal{L}_k(M_2) \right\}$$

We define the multiplication in  $\mathcal{M}(M_1, M_2)$  as follows:

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} A_1 B_1 + A_2 B_3 & A_1 B_2 + A_2 B_4 \\ A_3 B_1 + A_4 B_3 & A_3 B_2 + A_4 B_4 \end{pmatrix}.$$

Then  $\mathcal{M}(M_1, M_2)$  endowed with the max norm is a  $k$ -Banach algebra.

**Lemma 2.28.** *We have a bi-bounded isomorphism of  $k$ -Banach algebras:*

$$\mathcal{L}_k(M) \simeq \mathcal{M}(M_1, M_2).$$

*Proof.* Let  $p_j$  be the projection of  $M$  onto  $M_j$  and  $i_j$  be the inclusion of  $M_j$  into  $M$ , where  $j \in \{1, 2\}$ . We define the following two  $k$ -linear maps:

$$\begin{aligned}\Psi_1 : \mathcal{L}_k(M) &\longrightarrow \mathcal{M}(M_1, M_2) \\ \varphi &\mapsto \begin{pmatrix} p_1 \varphi i_1 & p_1 \varphi i_2 \\ p_2 \varphi i_1 & p_2 \varphi i_2 \end{pmatrix} \\ \Psi_2 : \mathcal{M}(M_1, M_2) &\longrightarrow \mathcal{L}_k(M) \\ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} &\mapsto i_1 L_1 p_1 + i_1 L_2 p_2 + i_2 L_3 p_1 + i_2 L_4 p_2\end{aligned}$$

Since the projections and inclusions are bounded maps, then the maps  $\Psi_1$  and  $\Psi_2$  are bounded too. It is easy to show that  $\Psi_1 \circ \Psi_2 = id_{\mathcal{M}(M_1, M_2)}$  and  $\Psi_2 \circ \Psi_1 = id_{\mathcal{L}_k(M)}$ . Hence we have an isomorphism of  $k$ -Banach spaces.  $\square$

We will need the following lemma for the computation of the spectrum:

**Lemma 2.29.** *Let  $M_1$  and  $M_2$  be  $k$ -Banach spaces and let  $M = M_1 \oplus M_2$  endowed with the max norm. Let  $p_1, p_2$  be the respective projections associated to  $M_1$  and  $M_2$  and  $i_1, i_2$  be the respective inclusions. Let  $\varphi \in \mathcal{L}_k(M)$  and set  $\varphi_1 = p_1 \varphi i_1 \in \mathcal{L}_k(M_1)$  and  $\varphi_2 = p_2 \varphi i_2 \in \mathcal{L}_k(M_2)$ . If  $\varphi(M_1) \subset M_1$ , then we have:*

- i)  $\Sigma_{\varphi_i}(\mathcal{L}_k(M_i)) \subset \Sigma_{\varphi}(\mathcal{L}_k(M)) \cup \Sigma_{\varphi_j}(\mathcal{L}_k(M_j))$ , where  $i, j \in \{1, 2\}$  and  $i \neq j$ .
- ii)  $\Sigma_{\varphi}(\mathcal{L}_k(M)) \subset \Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$ . Furthermore, if  $\varphi(M_2) \subset M_2$ , then we have the equality.
- iii) If  $\Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cap \Sigma_{\varphi_2}(\mathcal{L}_k(M_2)) = \emptyset$ , then  $\Sigma_{\varphi}(\mathcal{L}_k(M)) = \Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$ .

*Proof.* By Lemma 2.28, we can represent the elements of  $\mathcal{L}_k(M)$  as follows:

$$\mathcal{L}_k(M) = \left\{ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \mid L_1 \in \mathcal{L}_k(M_1), L_2 \in \mathcal{L}_k(M_2, M_1), L_3 \in \mathcal{L}_k(M_1, M_2), L_4 \in \mathcal{L}_k(M_2) \right\}$$

and  $\varphi$  has the form  $\begin{pmatrix} \varphi_1 & L \\ 0 & \varphi_2 \end{pmatrix}$ , where  $L \in \mathcal{L}_k(M_2, M_1)$ .

Let  $x \in \mathbb{A}_k^{1, an}$ . We have an isomorphism of  $k$ -Banach algebras:

$$\mathcal{L}_k(M) \hat{\otimes}_k \mathcal{H}(x) = \left\{ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \mid \begin{array}{l} L_1 \in \mathcal{L}_k(M_1) \hat{\otimes}_k \mathcal{H}(x), L_2 \in \mathcal{L}_k(M_2, M_1) \hat{\otimes}_k \mathcal{H}(x), \\ L_3 \in \mathcal{L}_k(M_1, M_2) \hat{\otimes}_k \mathcal{H}(x), L_4 \in \mathcal{L}_k(M_2) \hat{\otimes}_k \mathcal{H}(x) \end{array} \right\}$$

Consequently,

$$\varphi \otimes 1 - 1 \otimes T(x) = \begin{pmatrix} \varphi_1 \otimes 1 - 1 \otimes T(x) & L \otimes 1 \\ 0 & \varphi_2 \otimes 1 - 1 \otimes T(x) \end{pmatrix}.$$

We first prove i). Let  $\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}$  be an invertible element of  $\mathcal{L}_k(M) \hat{\otimes}_k \mathcal{H}(x)$ . We claim that if, for  $i \in \{1, 2\}$ ,  $L_i$  is invertible in  $\mathcal{L}_k(M_i) \hat{\otimes}_k \mathcal{H}(x)$ , then so is  $L_j$ , where  $j \neq i$ . Indeed, let  $\begin{pmatrix} L'_1 & C' \\ B & L'_2 \end{pmatrix}$  such that we have:

$$\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} L'_1 & C' \\ B & L'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} L'_1 & C' \\ B & L'_2 \end{pmatrix} \begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we obtain:

$$\begin{cases} L_1 L'_1 + C B = 1 \\ L_1 C' + C L_2 = 0 \\ L_2 B = 0 \\ L_2 L'_2 = 1 \end{cases} \quad ; \quad \begin{cases} L'_1 L_1 = 1 \\ L'_1 C + C' L_2 = 0 \\ B L_1 = 0 \\ B C + L'_2 L_2 = 1 \end{cases}.$$

We deduce that  $L_1$  is left invertible and  $L_2$  is right invertible. If  $L_1$  is invertible, then  $B = 0$  which implies that  $L_2$  is left invertible, hence invertible. If  $L_2$  is invertible, then  $B = 0$  which implies that  $L_1$  is right invertible, hence invertible. Therefore, if  $\varphi \otimes 1 - 1 \otimes T(x)$  and  $\varphi_i \otimes 1 - 1 \otimes T(x)$  are invertible where  $i \in \{1, 2\}$ , then  $\varphi_j \otimes 1 - 1 \otimes T(x)$  is invertible for  $j \in \{1, 2\} \setminus \{i\}$ . We conclude that  $\Sigma_{\varphi_j} \subset \Sigma_{\varphi} \cup \Sigma_{\varphi_i}$  where  $i, j \in \{1, 2\}$  and  $i \neq j$ .

We now prove ii). If  $\varphi_1 \otimes 1 - 1 \otimes T(x)$  and  $\varphi_2 \otimes 1 - 1 \otimes T(x)$  are invertible, then  $\varphi \otimes 1 - 1 \otimes T(x)$  is invertible. This proves that  $\Sigma_{\varphi} \subset \Sigma_{\varphi_1} \cup \Sigma_{\varphi_2}$ . If  $\varphi(M_2) \subset M_2$ , then  $L = 0$  which implies that: if  $\varphi \otimes 1 - 1 \otimes T(x)$  is invertible, then  $\varphi_1 \otimes 1 - 1 \otimes T(x)$  and  $\varphi_2 \otimes 1 - 1 \otimes T(x)$  are invertible. Hence we have the equality.

We now prove iii). If  $\Sigma_{\varphi_1} \cap \Sigma_{\varphi_2} = \emptyset$ , then by above we have  $\Sigma_{\varphi_1} \subset \Sigma_{\varphi}$  and  $\Sigma_{\varphi_2} \subset \Sigma_{\varphi}$ . Therefore,  $\Sigma_{\varphi_1} \cup \Sigma_{\varphi_2} \subset \Sigma_{\varphi}$ . □

*Remark 2.30.* Set notations as in Lemma 2.29. In the proof above, we showed also: if  $\varphi \otimes 1 - 1 \otimes T(x)$  is invertible, then  $\varphi_1 \otimes 1 - 1 \otimes T(x)$  is left invertible and  $\varphi_2 \otimes 1 - 1 \otimes T(x)$  is right invertible.

### 3 Differential modules and spectrum

#### 3.1 Preliminaries

Recall that a differential  $k$ -algebra, denoted by  $(A, d)$ , is a commutative  $k$ -algebra  $A$  endowed with a  $k$ -linear derivation  $d : A \rightarrow A$ . A differential module  $(M, \nabla)$  over  $(A, d)$  is a finite free  $A$ -module  $M$  equipped with a  $k$ -linear map  $\nabla : M \rightarrow M$ , called connection of  $M$ , satisfying  $\nabla(fm) = df \cdot m + f \cdot \nabla(m)$  for all  $f \in A$  and  $m \in M$ . If we fix a basis of  $M$ , then we get an isomorphism of  $A$ -modules  $M \xrightarrow{\sim} A^n$ , and the operator  $\nabla$  is given in this basis by the rule:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad (8)$$

where  $G \in \mathcal{M}_n(A)$  is a matrix. Conversely the data of such a matrix defines a differential module structure on  $A^n$  by the rule (8).

A morphism between differential modules is a  $k$ -linear map  $M \rightarrow N$  commuting with connections. We set  $A\langle D \rangle = \bigoplus_{i \in \mathbb{N}} A \cdot D^i$  to be the ring of differential polynomials equipped with the non-commutative multiplication defined by the rule:  $D \cdot f = df + f \cdot D$  for all  $f \in A$ . Let  $P(D) = g_0 + \dots + g_{\nu-1} D^{\nu-1} + D^{\nu}$  be a monic differential polynomial. The quotient  $A\langle D \rangle / A\langle D \rangle \cdot P(D)$  is a finite free  $A$ -module of rank  $\nu$ . Equipped with the multiplication by  $D$ , it is a differential module over  $(A, d)$ . In the basis  $\{1, D, \dots, D^{\nu-1}\}$  the multiplication by  $D$  satisfies:

$$D \begin{pmatrix} f_1 \\ \vdots \\ f_{\nu} \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_{\nu} \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 & -g_0 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -g_{\nu-1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{\nu} \end{pmatrix}$$

**Theorem 3.1** (The cyclic vector theorem). *Let  $(A, d)$  be a  $k$ -differential field (i.e  $A$  is a field), with  $d \neq 0$ , and let  $(M, \nabla)$  be a differential module over  $(A, d)$  of rank  $n$ . Then there exists  $m \in M$  such that  $\{m, \nabla(m), \dots, \nabla^{n-1}(m)\}$  is a basis of  $M$ . In this case we say that  $m$  is cyclic vector.*

*Proof.* See [Ked10, Theorem 5.7.2.]. □

*Remark 3.2.* The last theorem means that there exists an isomorphism of differential modules between  $(M, \nabla)$  and  $(A\langle D \rangle / A\langle D \rangle \cdot P(D), D)$  for some monic differential polynomial  $P(D)$  of degree  $n$ .

**Lemma 3.3.** *Let  $L$ ,  $P$  and  $Q$  be differential polynomials, such that  $L = QP$ . Then we have an exact sequence of differential modules:*

$$0 \longrightarrow A\langle D \rangle / A\langle D \rangle Q \xrightarrow{i} A\langle D \rangle / A\langle D \rangle L \xrightarrow{p} A\langle D \rangle / A\langle D \rangle P \longrightarrow 0$$

where the maps  $i$  and  $p$  are defined as follows: for a differential polynomial  $R$ ,  $i(\bar{R}) = \overline{RP}$  and  $p(\bar{R}) = \bar{R}$ .

*Proof.* See [Chr83, Section 3.5.6].  $\square$

### 3.2 Spectrum associated to a differential module

**Hypothesis 3.4.** From now on  $A$  will be either a  $k$ -affinoid algebra associated to an affinoid domain of  $\mathbb{A}_k^{1,an}$  or  $\mathcal{H}(x)$  for some  $x \in \mathbb{A}_k^{1,an}$  not of type (1). Let  $d$  be a bounded derivation on  $A$ . It is of the form  $d = g(T) \frac{d}{dT}$  where  $g(T) \in A$ .

Let  $(M, \nabla)$  be a differential module over  $(A, d)$ . In order to associate to this differential module a spectrum we need to endow it with a structure of  $k$ -Banach space. For that, recall the following proposition:

**Proposition 3.5.** *There exists an equivalence of category between the category of finite Banach  $A$ -modules with bounded  $A$ -linear maps as morphisms and the category of finite  $A$ -modules with  $A$ -linear maps as morphisms.*

*Proof.* See [Ber90, Proposition 2.1.9].  $\square$

This means that we can endow  $M$  with a structure of finite Banach  $A$ -module isomorphic to  $A^n$  equipped with the maximum norm, and any other structure of finite Banach  $A$ -module on  $M$  is equivalent to the previous one. This induces a structure of Banach  $k$ -space on  $M$ . As  $\nabla$  satisfies the rule (8) and  $d \in \mathcal{L}_k(A)$ , we have  $\nabla \in \mathcal{L}_k(M)$ . The spectrum associated to  $(M, \nabla)$  is denoted by  $\Sigma_{\nabla, k}(\mathcal{L}_k(M))$ <sup>5</sup> (or just by  $\Sigma_{\nabla}$  if the dependence is obvious from the context).

Let  $\varphi : (M, \nabla) \rightarrow (N, \nabla')$  be a morphism of differential modules. If we endow  $M$  and  $N$  with a structure of  $k$ -Banach space (as above) then  $\varphi$  is automatically an admissible<sup>6</sup> bounded  $k$ -linear map (see [Ber90, Proposition 2.1.10]). In the case  $\varphi$  is an isomorphism, then it induces a bi-bounded  $k$ -linear isomorphism and according to Lemma 2.27 we have:

$$\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \Sigma_{\nabla', k}(\mathcal{L}_k(N)). \quad (9)$$

This prove the following proposition:

**Proposition 3.6.** *The spectrum of a connection is an invariant by isomorphisms of differential modules.*

**Proposition 3.7.** *Let  $0 \rightarrow (M_1, \nabla_1) \rightarrow (M, \nabla) \rightarrow (M_2, \nabla_2) \rightarrow 0$  be an exact sequence of differential modules over  $(A, d)$ .*

*Then we have:  $\Sigma_{\nabla}(\mathcal{L}_k(M)) \subset \Sigma_{\nabla_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\nabla_2}(\mathcal{L}_k(M_2))$ , with equality if  $\Sigma_{\nabla_1}(\mathcal{L}_k(M_1)) \cap \Sigma_{\nabla_2}(\mathcal{L}_k(M_2)) = \emptyset$ .*

*Proof.* As  $M_1$ ,  $M_2$  and  $M$  are free  $A$ -modules, the sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

splits. Hence, we have  $M = M_1 \oplus M_2$  where  $f$  is the inclusion of  $M_1$  into  $M$  and  $g$  is the projection of  $M$  onto  $M_2$ . Let  $p_1$  be the projection of  $M$  onto  $M_1$  and  $i_2$  be the inclusion of  $M_2$  into  $M$ . As both  $f$  and  $g$  are morphisms of differential modules, we have  $\nabla(M_1) \subset M_1$ ,  $\nabla_1 = p_1 \nabla f$  and  $\nabla_2 = g \nabla i_2$ . By Lemma 2.29 and Remark 2.30 we obtain the result.  $\square$

<sup>5</sup>Note that, since all the structures of finite Banach  $A$ -module on  $M$  are equivalent, the spectrum does not depend on the choice of such structure.

<sup>6</sup> Which means that:  $M / \text{Ker } \varphi$  endowed with quotient the topology is isomorphic as  $k$ -Banach space to  $\text{Im } \varphi$ .



*Remark 3.8.* We maintain the assumption of Lemma 3.7. If in addition we have an other exact sequence of the form:

$$0 \rightarrow (M_2, \nabla_2) \rightarrow (M, \nabla) \rightarrow (M_1, \nabla_1) \rightarrow 0,$$

then the equality holds, this is a consequence of Remark 2.30. Indeed, If  $\nabla \otimes 1 - 1 \otimes T(x)$  is invertible, then the first exact sequence shows that  $\nabla_1 \otimes 1 - 1 \otimes T(x)$  is left invertible and  $\nabla_2 \otimes 1 - 1 \otimes T(x)$  is right invertible, the second exact sequence to  $\nabla_2 \otimes 1 - 1 \otimes T(x)$  is left invertible and  $\nabla_1 \otimes 1 - 1 \otimes T(x)$  is right invertible. Therefore, both of  $\nabla_1 \otimes 1 - 1 \otimes T(x)$  and  $\nabla_2 \otimes 1 - 1 \otimes T(x)$  are invertible. Hence, we obtain  $\Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} \subset \Sigma_{\nabla}$ .

*Remark 3.9.* If moreover we have  $M = M_1 \oplus M_2$  as differential modules then we have  $\Sigma_{\nabla} = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2}$ .

*Remark 3.10.* Set notation as in Proposition 3.7. We suppose that  $A = \mathcal{H}(x)$  for some point  $x \in \mathbb{A}_k^{1,an}$  not of type (1). For the spectral semi-norm it is know (see [Ked10, Lemma 6.2.8]) that we have:

$$\|\nabla\|_{Sp} = \max\{\|\nabla_1\|_{Sp}, \|\nabla_2\|_{Sp}\}.$$

We say that a differential module  $(M, \nabla)$  over  $(A, d)$  of rank  $n$  is trivial if it is isomorphic to  $(A^n, d)$  as a differential module.

**Lemma 3.11.** *Let  $(M, \nabla)$  be a differential module over a differential field  $(K, d)$  of rank  $n$ . If the  $k$ -vector space  $\text{Ker } \nabla$  has dimension equal to  $n$ , then  $(M, \nabla)$  is a trivial differential module.*

*Proof.* See [Chr83, Proposition 3.5.3]. □

**Corollary 3.12.** *We suppose that  $A = \mathcal{H}(x)$  for some  $x \in \mathbb{A}_k^{1,an}$  not of type (1). Let  $(M, \nabla)$  be a differential module over  $(A, d)$ . If  $\dim(\text{Ker } \nabla) = n$ , then  $\Sigma_{\nabla}(\mathcal{L}_k(M)) = \Sigma_d(\mathcal{L}_k(A))$ .*

*Proof.* By Lemma 3.11 there exists  $\{e_1, \dots, e_n\}$  a basis of  $M$  as an  $A$ -module for which  $\nabla$  satisfies the rule:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix}.$$

By induction and Remark 3.9 we obtain the result. □

**Lemma 3.13.** *We suppose that  $k$  is algebraically closed. Let  $(M, \nabla)$  be a differential module over  $(A, d)$  such that  $G \in \mathcal{M}_n(k)$  (cf. (8)) and  $\{a_1, \dots, a_N\}$  is the set of the eigenvalues of  $G$ . Then we have an isomorphisme of differential modules:*

$$(M, \nabla) \simeq \left( \bigoplus_{1 \leq i \leq N} \bigoplus_{1 \leq j \leq N_i} A\langle D \rangle / (D - a_i)^{n_{i,j}}, D \right)$$

where the  $n_{i,j}$  are positive integers such that  $\sum_{j=1}^{N_i} n_{i,j}$  is the multiplicity of  $a_i$  and  $\sum_{i,j} n_{i,j} = n$ .

*Proof.* Consequence of the Jordan reduction. □

**Lemma 3.14.** *Let  $(M, \nabla)$  be the differential module over  $(A, d)$  associated to the differential polynomial  $(D - a)^n$ , where  $a \in k$ . The spectrum of  $\nabla$  is  $\Sigma_{\nabla}(\mathcal{L}_k(M)) = a + \Sigma_d(\mathcal{L}_k(A))$  (the image of  $\Sigma_d$  by the polynomial  $T + a$ ).*

*Proof.* By Lemma 3.3, we have the exact sequences:

$$0 \rightarrow (A\langle D \rangle / (D - a)^{n-1}, D) \rightarrow (A\langle D \rangle / (D - a)^n, D) \rightarrow (A\langle D \rangle / (D - a), D) \rightarrow 0$$

and

$$0 \rightarrow (A\langle D \rangle / (D - a), D) \rightarrow (A\langle D \rangle / (D - a)^n, D) \rightarrow (A\langle D \rangle / (D - a)^{n-1}, D) \rightarrow 0$$

By induction and Remark 3.8, we have  $\Sigma_D = \Sigma_{d+a}$ . By Lemma 2.25, we obtain  $\Sigma_D = a + \Sigma_d$ . □

**Proposition 3.15.** *We suppose that  $k$  is algebraically closed. Let  $(M, \nabla)$  be a differential module over  $(A, d)$  such that:*

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

with  $G \in \mathcal{M}_n(k)$ . The spectrum of  $\nabla$  is  $\Sigma_\nabla = \bigcup_{i=1}^N (a_i + \Sigma_d)$ , where  $\{a_1, \dots, a_N\}$  are the eigenvalues of  $G$ .

*Proof.* Using the decomposition of Lemma 3.13, 3.14 and Remark 3.9, we obtain the result.  $\square$

**Remark 3.16.** This claim shows that the spectrum of a connection depends highly on the choice of the derivation  $d$ .

**Notation 3.17.** From now on we will fix  $S$  to be the coordinate function of the analytic domain where the linear differential equation is defined and  $T$  to be the coordinate function on  $\mathbb{A}_k^{1,an}$  (for the computation of the spectrum).

**Lemma 3.18.** *We assume that  $k$  is algebraically closed. Let  $\omega$  be the real positive number introduced in (2).*

- *Let  $X$  be a connected affinoid domain as in (6) and set  $r = \min_{0 \leq i \leq \mu} r_i$ . The operator norm of  $(\frac{d}{dS})^n$  as an element of  $\mathcal{L}_k(\mathcal{O}(X))$  satisfies:*

$$\|(\frac{d}{dS})^n\|_{\mathcal{L}_k(\mathcal{O}(X))} = \frac{|n!|}{r^n}, \quad \|\frac{d}{dS}\|_{Sp, \mathcal{L}_k(\mathcal{O}(X))} = \frac{\omega}{r}.$$

- *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (2), (3) or (4). The operator norm of  $(\frac{d}{dS})^n$  as an element of  $\mathcal{L}_k(\mathcal{H}(x))$  satisfies:*

$$\|(\frac{d}{dS})^n\|_{\mathcal{L}_k(\mathcal{H}(x))} = \frac{|n!|}{r(x)^n}, \quad \|\frac{d}{dS}\|_{Sp, \mathcal{L}_k(\mathcal{H}(x))} = \frac{\omega}{r(x)}.$$

*Proof.* See [Pul15, Lemma 4.4.1].  $\square$

**Remark 3.19.** We maintain the assumption that  $k$  is algebraically closed. Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ . Then  $X = \bigcup_{i=1}^\mu X_i$ , where  $X_i$  are connected affinoid domains and  $X_i \cap X_j = \emptyset$  for  $i \neq j$ . We have  $\mathcal{O}(X) = \bigoplus_{i=1}^\mu \mathcal{O}(X_i)$ . As  $\frac{d}{dS}$  stabilises each Banach space of the direct sum, we have:

$$\|\frac{d}{dS}\|_{Sp, \mathcal{L}_k(\mathcal{O}(X))} = \max_{0 \leq i \leq \mu} \|\frac{d}{dS}\|_{Sp, \mathcal{L}_k(\mathcal{O}(X_i))}.$$

Let  $\Omega \in E(k)$  and let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ . Let  $d = f(S)\frac{d}{dS}$  be a derivation defined on  $\mathcal{O}(X)$ . We can extend it to a derivation  $d_\Omega = f(S)\frac{d}{dS}$  defined on  $\mathcal{O}(X_\Omega)$ . The derivation  $d_\Omega$  is the image of  $d \otimes 1$  by the morphism  $\mathcal{L}_k(\mathcal{O}(X)) \hat{\otimes}_k \Omega \rightarrow \mathcal{L}_\Omega(\mathcal{O}(X_\Omega))$  defined in Lemma 2.5.

**Lemma 3.20.** *Let  $\pi_{\Omega/k} : X_\Omega \rightarrow X$  be the canonical projection. We have:*

$$\pi_{\Omega/k}(\Sigma_{d_\Omega, \Omega}(\mathcal{L}_\Omega(\mathcal{O}(X_\Omega))) \subset \Sigma_{d, k}(\mathcal{L}_k(\mathcal{O}(X))).$$

*Proof.* By Lemma 2.5 and 2.27 we have  $\Sigma_{d_\Omega, \Omega}(\mathcal{L}_\Omega(\mathcal{O}(X_\Omega))) \subset \Sigma_{d \otimes 1, \Omega}(\mathcal{L}_k(\mathcal{O}(X)) \hat{\otimes}_k \Omega)$ . Since  $\Sigma_{d \otimes 1, \Omega}(\mathcal{L}_k(\mathcal{O}(X)) \hat{\otimes}_k \Omega) = \pi_{\Omega/k}^{-1}(\Sigma_{d, k}(\mathcal{L}_k(\mathcal{O}(X))))$  (see [Ber90, Proposition 7.1.6]), we obtain the result.  $\square$

## 4 Main result

This section is divided in two parts. The first one is for the computation of the spectrum of  $\frac{d}{dS}$ , the second to state and prove the main result which is the computation of the spectrum associated to a linear differential equation with constant coefficients.

**Hypothesis 4.1.** In this section we will suppose that  $k$  is algebraically closed.

### 4.1 The spectrum of $\frac{d}{dS}$ defined on several domains

Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$  and  $x \in \mathbb{A}_k^{1,an}$  be a point of type (2), (3) or (4). In this part we compute the spectrum of  $\frac{d}{dS}$  as a derivation of  $A = \mathcal{O}(X)$  or  $\mathcal{H}(x)$  as an element of  $\mathcal{L}_k(A)$ . We treat the case of positive residual characteristic separately. We will also distinguish the case where  $X$  is a closed disk from the case where it is a connected affinoid subdomain, and the case where  $x$  is point of type (4) from the others.

#### 4.1.1 The case of positive residual characteristic

We suppose that  $\text{char}(\tilde{k}) = p > 0$ . In this case  $\omega = |p|^{\frac{1}{p-1}}$ .

**Proposition 4.2.** *The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(D^+(c, r)))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(D^+(c, r)))) = D^+(0, \frac{\omega}{r}).$$

*Proof.* We set  $A = \mathcal{O}(D^+(c, r))$  and  $d = \frac{d}{dS}$ . We prove firstly this claim for a field  $k$  that is spherically complete and satisfies  $|k| = \mathbb{R}_+$ . By Lemma 3.18 the spectral norm of  $d$  is equal to  $\|d\|_{Sp} = \frac{\omega}{r}$ . By Theorem 2.18 we have  $\Sigma_d \subset D^+(0, \frac{\omega}{r})$ . We prove now that  $D^+(0, \frac{\omega}{r}) \subset \Sigma_d$ .

Let  $x \in D^+(0, \frac{\omega}{r}) \cap k$ . Then

$$d \otimes 1 - 1 \otimes T(x) = (d - a) \otimes 1$$

where  $T(x) = a \in k$ . The element  $d \otimes 1 - 1 \otimes T(x)$  is invertible in  $\mathcal{L}_k(A) \hat{\otimes} \mathcal{H}(x)$  if and only if  $d - a$  is invertible in  $\mathcal{L}_k(A)$  [Ber90, Lemma 7.1.7].

If  $|a| < \frac{\omega}{r}$ , then  $\exp(a(S - c)) = \sum_{n \in \mathbb{N}} (\frac{a^n}{n!})(S - c)^n$  exists and it is an element of  $A$ . Hence,  $\exp(a(S - c)) \in \ker(d - a)$ , in particular  $d - a$  is not invertible. Consequently,  $D^-(0, \frac{\omega}{r}) \cap k \subset \Sigma_d$ .

Now we suppose that  $|a| = \frac{\omega}{r}$ . We prove that  $d - a$  is not surjective. Let  $g(S) = \sum_{n \in \mathbb{N}} b_n(S - c)^n \in A$ . If there exists  $f(S) = \sum_{n \in \mathbb{N}} a_n(S - c)^n \in A$  such that  $(d - a)f = g$ , then for each  $n \in \mathbb{N}$  we have:

$$a_n = \frac{(\sum_{i=0}^{n-1} i! b_i a^{n-1-i}) + a^n a_0}{n!}. \quad (10)$$

We now construct a series  $g \in A$  such that its antecedent  $f$  does not converge on the closed disk  $D^+(c, r)$ . Let  $\alpha, \beta \in k$ , such that  $|\alpha| = r$  and  $|\beta| = |p|^{1/2}$ . For  $n \in \mathbb{N}$  we set:

$$b_n = \begin{cases} \frac{\beta^l}{\alpha^{p^l-1}} & \text{if } n = p^l - 1 \text{ with } l \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Then,  $|b_n| r^n$  is either 0 or  $|p|^{\log_p(n+1)}$  and  $g \in A$ . If we suppose that there exists  $f \in A$  such that  $(d - a)f = g$  then we have:

$$\forall l \in \mathbb{N}; \quad a_{p^l} = \frac{a^{p^l-1}}{(p^l)!} \left[ \sum_{j=0}^l \frac{(p^j - 1)! \beta^j}{a^{p^j-1} \alpha^{p^j-1}} + a a_0 \right].$$

As  $|p^l| = \omega^{p^l-1}$  (cf. [DGS94, p. 51]) we have:

$$|a_{p^l}| = \frac{1}{r^{p^l-1}} \left| \sum_{j=0}^l \frac{(p^j - 1)! \beta^j}{a^{p^j-1} \alpha^{p^j-1}} + a a_0 \right|.$$

Since  $|\frac{(p^j-1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}}| = |p|^{-j/2}$ , we have:

$$|\sum_{j=0}^l \frac{(p^j-1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}}| = \max_{0 \leq j \leq l} |p|^{-j/2} = |p|^{-l/2},$$

therefore  $|a_{p^l}|r^{p^l} \xrightarrow{k \rightarrow +\infty} +\infty$ , which proves that the power series  $f$  is not in  $A$  and this is a contradiction. Hence,  $D^+(0, \frac{\omega}{r}) \cap k \subset \Sigma_d$ . As the points of type (1) are dense in  $D^+(0, \frac{\omega}{r})$  and  $\Sigma_d$  is compact, we deduce that  $D^+(0, \frac{\omega}{r}) \subset \Sigma_d$ .

Let us now consider an arbitrary field  $k$ . Let  $\Omega \in E(k)$  algebraically closed, spherically complete such that  $|\Omega| = \mathbb{R}_+$ . We denote  $A_\Omega = \mathcal{O}(X_\Omega)$  and  $d_\Omega = \frac{d}{dS}$  the derivation on  $A_\Omega$ . From above, we have  $\Sigma_{d_\Omega} = D_\Omega^+(0, \frac{\omega}{r})$ , then  $\pi_{\Omega/k}(\Sigma_{d_\Omega}) = D^+(0, \frac{\omega}{r})$ . By Lemma 3.20 we have  $D^+(0, \frac{\omega}{r}) = \pi_{\Omega/k}(\Sigma_{d_\Omega}) \subset \Sigma_d$ . As  $\|d\|_{S_p} = \frac{\omega}{r}$ , we obtain  $\Sigma_d = D^+(0, \frac{\omega}{r})$ .  $\square$

*Remark 4.3.* The statement holds even if the field  $k$  is not algebraically closed. Indeed, we did not use this assumption.

**Proposition 4.4.** *Let  $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^\mu D^-(c_i, r_i)$  be a connected affinoid domain of  $\mathbb{A}_k^{1,an}$  different from the closed disk. The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(X))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \frac{\omega}{\min_{0 \leq i \leq \mu} r_i}).$$

*Proof.* We set  $A = \mathcal{O}(X)$  and  $d = \frac{d}{dS}$ . The spectral norm of  $d$  is equal to  $\|d\|_{S_p} = \frac{\omega}{\min_{0 \leq i \leq \mu} r_i}$  (cf. Lemma 3.18), which implies  $\Sigma_d \subset D^+(0, \frac{\omega}{\min_{0 \leq i \leq \mu} r_i})$ . Now, let  $x \in D^+(0, \frac{\omega}{\min_{0 \leq i \leq \mu} r_i})$ . We set  $A_{\mathcal{H}(x)} = \mathcal{O}(X_{\mathcal{H}(x)})$  and  $d_{\mathcal{H}(x)} = \frac{d}{dS} : A_{\mathcal{H}(x)} \rightarrow A_{\mathcal{H}(x)}$ . From Lemma 2.5 we have the bounded morphism:

$$\mathcal{L}_k(A) \hat{\otimes}_k \mathcal{H}(x) \rightarrow \mathcal{L}_{\mathcal{H}(x)}(A_{\mathcal{H}(x)}).$$

The image of  $d$  by this morphism is the derivation  $d_{\mathcal{H}(x)}$ . By the Mittag-Leffler decomposition [FP04, Proposition 2.2.6], we have:

$$\mathcal{O}(X_{\mathcal{H}(x)}) = \bigoplus_{i=1}^n \{ \sum_{j \in \mathbb{N}^*} \frac{a_{ij}}{(S - c_i)^j} \mid a_{ij} \in \mathcal{H}(x), \lim_{j \rightarrow +\infty} |a_{ij}|r_i^{-j} = 0 \} \oplus \mathcal{O}(D_{\mathcal{H}(x)}^+(c_0, r_0)).$$

Each Banach space of the direct sum above is stable under  $d_{\mathcal{H}(x)}$ .

We set  $F_i = \{ \sum_{j \in \mathbb{N}^*} \frac{a_{ij}}{(S - c_i)^j} \mid a_{ij} \in \mathcal{H}(x), \lim_{j \rightarrow +\infty} |a_{ij}|r_i^{-j} = 0 \}$ , and  $d_i = d_{\mathcal{H}(x)}|_{F_i}$ . By Lemma 2.29,  $\Sigma_{d_{\mathcal{H}(x)}} = \bigcup \Sigma_{d_i}$ . Let  $i_0 > 0$  be the index such that  $r_{i_0} = \min_{0 \leq i \leq \mu} r_i$ . We will prove that  $d_{i_0} - T(x)$  is not surjective. Indeed, let  $g(S) = \sum_{n \in \mathbb{N}^*} \frac{b_n}{(S - c_{i_0})^n} \in F_{i_0}$ , if there exists  $f(S) = \sum_{n \in \mathbb{N}^*} \frac{a_n}{(S - c_{i_0})^n} \in F_{i_0}$  such that  $(d_{i_0} - T(x))f(S) = g(S)$ , then for each  $n \in \mathbb{N}^*$  we have:

$$a_n = \frac{(n-1)!}{(-T(x))^n} \sum_{i=1}^n \frac{(-T(x))^{i-1}}{(i-1)!} b_i.$$

We choose  $g(S) = \frac{1}{S - c_{i_0}}$ , in this case  $a_n = \frac{(n-1)!}{(-T(x))^n}$  and  $|a_n| = \frac{|(n-1)!|}{|T(x)|^n}$ . As  $|T(x)| \leq \frac{\omega}{r_{i_0}}$ , the sequence  $|a_n|r_{i_0}^{-n}$  diverges. We obtain contradiction since  $f \in F_{i_0}$ . Hence,  $d_{\mathcal{H}(x)} - T(x)$  is not invertible, which implies that  $d \otimes 1 - 1 \otimes T(x)$  is not invertible and we obtain the result.  $\square$

**Corollary 4.5.** *Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ . The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(X))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \|\frac{d}{dS}\|_{S_p}).$$

*Proof.* In this case we may write  $X = \bigcup_{i=1}^u X_i$ , where the  $X_i$  are connected affinoid domain of  $\mathbb{A}_k^{1,an}$  such that  $X_i \cap X_j = \emptyset$  for  $i \neq j$ . We have:

$$\mathcal{O}(X) = \bigoplus_{i=1}^u \mathcal{O}(X_i).$$

Each Banach space of the direct sum above is stable under  $d$ , we denote by  $d_i$  the restriction of  $d$  to  $\mathcal{O}(X_i)$ . We have  $\|d\|_{S_p} = \max_{1 \leq i \leq u} d_i$  (cf. Remark 3.19). By Lemma 2.29 and Proposition 4.4 we have  $\Sigma_d = \bigcup_{i=1}^u D^+(0, \|d_i\|_{S_p}) = D^+(0, \max_i \|d_i\|_{S_p})$ . Hence, we obtain the result.  $\square$

**Proposition 4.6.** *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (2), (3) or (4). The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{H}(x))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{H}(x))) = D^+(0, \frac{\omega}{r(x)}).$$

Where  $r(x)$  is the value defined in Definition 2.9.

*Proof.* We set  $d = \frac{d}{dS}$ . We distinguish three cases:

- **$x$  is point a of type (2):** Let  $c \in k$  such that  $x = x_{c,r(x)}$ . By Proposition 2.10, we have:

$$\mathcal{H}(x) = E \oplus \mathcal{O}(D^+(c, r(x))).$$

As both  $E$  and  $\mathcal{O}(D^+(c, r(x)))$  are stable under  $d$ , by Lemma 2.29 we have  $\Sigma_d = \Sigma_{d|_E} \cup \Sigma_{d|_{\mathcal{O}(D^+(c, r(x)))}}$ . By Proposition 4.2  $\Sigma_{d|_{\mathcal{O}(D^+(c, r(x)))}} = D^+(0, \frac{\omega}{r(x)})$ . Since  $\|d\|_{S_p} = \frac{\omega}{r(x)}$  (cf. Lemma 3.18), then  $\Sigma_d = D^+(0, \frac{\omega}{r(x)})$ .

- **$x$  is point a of type (3):** Let  $c \in k$  such that  $x = x_{c,r(x)}$ . In this case  $\mathcal{H}(x) = \mathcal{O}(D^+(c, r(x), r(x)))$ . By Proposition 4.4 we obtain the result.
- **$x$  is point a of type (4):** By Proposition 2.12 we have  $\mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x) \simeq \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x)))$ . From Lemma 2.5 we have the bounded morphism:

$$\mathcal{L}_k(\mathcal{H}(x)) \hat{\otimes}_k \mathcal{H}(x) \rightarrow \mathcal{L}_{\mathcal{H}(x)}(\mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x)))).$$

The image of  $d \otimes 1$  by this morphism is the derivation  $d_{\mathcal{H}(x)} = \frac{d}{dS} : \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x))) \rightarrow \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x)))$ .

From Proposition 4.2 we have  $\Sigma_{d_{\mathcal{H}(x)}} = D_{\mathcal{H}(x)}^+(0, \frac{\omega}{r(x)})$ , then  $\pi_{\mathcal{H}(x)/k}(\Sigma_{d_{\mathcal{H}(x)}}) = D^+(0, \frac{\omega}{r(x)})$ . By Lemma 3.20 we have  $D^+(0, \frac{\omega}{r(x)}) \subset \Sigma_d$ . Since  $\|d\|_{S_p} = \frac{\omega}{r(x)}$  (cf. Lemma 3.18), we obtain  $\Sigma_d = D^+(0, \frac{\omega}{r(x)})$  (cf. Theorem 2.18).  $\square$

#### 4.1.2 The case of residual characteristic zero

We suppose that  $\text{char}(\tilde{k}) = 0$ .

**Proposition 4.7.** *The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(D^+(c, r)))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(D^+(c, r)))) = \overline{D^-(0, \frac{1}{r})}$$

(the topological closure of  $D^-(0, \frac{1}{r})$ ).

*Proof.* We set  $A = \mathcal{O}(D^+(c, r))$  and  $d = \frac{d}{dS}$ . The spectral norm of  $d$  is equal to  $\|d\|_{Sp} = \frac{1}{r}$  (cf. Lemma 3.18), which implies that  $\Sigma_d \subset D^+(0, \frac{1}{r})$  (cf. Theorem 2.18). For  $a \in D^-(0, \frac{1}{r}) \cap k$ ,  $d - a$  is not injective, therefore  $a \in \Sigma_d$ . Let  $x \in D^-(0, \frac{1}{r}) \setminus k$ .

We set  $A_{\mathcal{H}(x)} = A \hat{\otimes}_k \mathcal{H}(x) = \mathcal{O}(D^+_{\mathcal{H}(x)}(c, r))$  and  $d_{\mathcal{H}(x)} = \frac{d}{dS} : A_{\mathcal{H}(x)} \rightarrow A_{\mathcal{H}(x)}$ . From Lemma 2.5 we have the bounded morphism:

$$\mathcal{L}_k(A) \hat{\otimes}_k \mathcal{H}(x) \rightarrow \mathcal{L}_{\mathcal{H}(x)}(A_{\mathcal{H}(x)})$$

The derivation  $d_{\mathcal{H}(x)}$  is the image of  $d \otimes 1$  by this morphism. As  $|T(x)(S - c)| < 1$ ,  $f = \exp(T(x)(S - c))$  exists and it is an element of  $A_{\mathcal{H}(x)}$ . As  $f \in \text{Ker}(d_{\mathcal{H}(x)} - T(x))$ ,  $d_{\mathcal{H}(x)} - T(x)$  is not invertible. Therefore  $d \otimes 1 - 1 \otimes T(x)$  is not invertible which is equivalent to saying that  $x \in \Sigma_d$ . By compactness of the spectrum we have  $\overline{D^-(0, \frac{1}{r})} \subset \Sigma_d$ . In order to end the proof, we need to prove firstly the statement for the case where  $k$  is trivially valued.

• **Trivially valued case:** We need to distinguish two cases:

- $r \neq 1$  : In this case we have  $D^+(0, \frac{1}{r}) = \overline{D^-(0, \frac{1}{r})}$ , hence  $\Sigma_d = \overline{D^-(0, \frac{1}{r})}$ .
- $r = 1$  : In this case we have  $\mathcal{O}(D^+(c, 1)) = k[S - c]$  equipped with the trivial valuation, and  $\mathcal{L}_k(k[S - c])$  is the  $k$ -algebra of all  $k$ -linear maps equipped with the trivial norm (i.e.  $\|\varphi\| = 1$  for all  $\varphi \in \mathcal{L}_k(k[S - c]) \setminus \{0\}$ ). Let  $a \in k \setminus \{0\}$ . Since the power series  $\exp(a(S - c)) = \sum_{n \in \mathbb{N}} \frac{a^n}{n!} (S - c)^n$  does not converge in  $\mathcal{O}(D^+(c, 1))$ , the operator  $d - a : k[S - c] \rightarrow k[S - c]$  is injective. It is also surjective. Indeed, let  $g(S) = \sum_{n=0}^m b_n (S - c)^n \in \mathcal{O}(D^+(c, 1))$ . The polynomial  $f(S) = \sum_{n=0}^m a_n (S - c)^n \in \mathcal{O}(D^+(c, 1))$  that its coefficients verifies

$$a_n = \frac{-a^{n-1}}{n!} \sum_{i=n}^m i! b_i a^{-i}.$$

for all  $0 \leq n \leq m$ , satisfies  $(d - a)f = g$ . Hence,  $d - a$  is invertible in  $\mathcal{L}_k(k[S - c])$ . Since the norm is trivial on  $\mathcal{L}_k(k[S - c])$ , we have  $\|(d - a)^{-1}\|_{Sp}^{-1} = 1$ . Therefore, by Lemma 2.20, for all  $x \in D^-(a, 1)$  the element  $d \otimes 1 - 1 \otimes T(x)$  is invertible. Consequently, for all  $a \in k \setminus \{0\}$  the disk  $D^-(a, 1)$  is not meeting the spectrum  $\Sigma_d$ . This means that  $\Sigma_d$  is contained in  $D^+(0, 1) \setminus \bigcup_{a \in k \setminus \{0\}} D^-(a, 1) = [0, x_{0,1}]$ . Since  $[0, x_{0,1}] = \overline{D^-(0, 1)}$  we have  $\Sigma_d = \overline{D^-(0, 1)}$ .

• **Non-trivially valued case:** We need to distinguish two cases:

- $r \notin |k^*|$  : In this case we have  $D^+(0, \frac{1}{r}) = \overline{D^-(0, \frac{1}{r})}$ , hence  $\Sigma_d = \overline{D^-(0, \frac{1}{r})}$ .
- $r \in |k^*|$  : We can reduce our case to  $r = 1$ . Indeed, there exists an isomorphism of  $k$ -Banach algebras

$$\mathcal{O}(D^+(c, r)) \rightarrow \mathcal{O}(D^+(c, 1)),$$

that associates to  $S - c$  the element  $\alpha(S - c)$ , with  $\alpha \in k$  and  $|\alpha| = r$ . This induces an isomorphism of  $k$ -Banach algebras

$$\mathcal{L}_k(\mathcal{O}(D^+(c, r))) \rightarrow \mathcal{L}_k(\mathcal{O}(D^+(c, 1))),$$

which associates to  $d : \mathcal{O}(D^+(c, r)) \rightarrow \mathcal{O}(D^+(c, r))$  the derivation  $\frac{1}{\alpha} \cdot \frac{d}{dS} : \mathcal{O}(D^+(c, 1)) \rightarrow \mathcal{O}(D^+(c, 1))$ . By Lemmas 2.27 and 2.25 we obtain

$$\Sigma_d(\mathcal{L}_k(\mathcal{O}(D^+(c, r)))) = \frac{1}{\alpha} \Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(D^+(c, 1)))).$$

We now suppose that  $r = 1$ . Let  $k'$  be a maximal (for the order given by the inclusion) trivially valued field included in  $k$  (which exists by Zorn's Lemma). As  $k$  is algebraically closed then so is  $k'$ . The complete residual field of  $x_{0,1} \in \mathbb{A}_{k'}^{1,an}$  is  $\mathcal{H}(x_{0,1}) = k'(S)$  endowed with the trivial valuation, the maximality of  $k'$  implies that  $\mathcal{H}(x_{0,1})$  can not be included in  $k$ , therefore  $k \not\subset E(\mathcal{H}(x_{0,1}))$ . By Proposition 2.13, we obtain  $\pi_{k/k'}^{-1}(x_{0,1}) = \{x_{0,1}\}$ . We

set  $d' = \frac{d}{dS}$  as an element of  $\mathcal{L}_{k'}(\mathcal{O}(D_{k'}^+(c, 1)))$ . We know by [Ber90, Proposition 7.1.6] that the spectrum of  $d' \otimes 1$ , as an element of  $\mathcal{L}_{k'}(\mathcal{O}(D_{k'}^+(c, 1))) \hat{\otimes}_{k'} k$ , satisfies  $\Sigma_{d' \otimes 1} = \pi_{k/k'}^{-1}(\Sigma_{d'})$ , by the result above we have  $\pi_{k/k'}^{-1}(\Sigma_{d'}) = \overline{D^-(0, 1)}$ . From Lemma 2.5 we have a bounded morphism  $\mathcal{L}_{k'}(\mathcal{O}(D_{k'}^+(c, 1))) \hat{\otimes}_{k'} k \rightarrow \mathcal{L}_k(\mathcal{O}(D^+(c, 1)))$ , the image of  $d' \otimes 1$  by this morphism is  $d$ . Therefore,  $\Sigma_d \subset \pi_{k/k'}^{-1}(\Sigma_{d'}) = \overline{D^-(0, 1)}$ . Then we obtain the result.  $\square$

**Proposition 4.8.** *Let  $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$  be a connected affinoid domain of  $\mathbb{A}_k^{1,an}$  different from the closed disk. The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(X))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \frac{1}{\min_{0 \leq i \leq \mu} r_i}).$$

*Proof.* Here  $\omega = 1$  (cf. (2)). The proof is the same as in Proposition 4.4.  $\square$

**Corollary 4.9.** *Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$  which does not contain a closed disk as a connected component. The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{O}(X))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \|\frac{d}{dS}\|_{S_P}).$$

*Remark 4.10.* Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ . Then  $X = Y \cup D$  with  $Y \cap D = \emptyset$ , where  $Y$  is an affinoid domain as in the corollary above and  $D$  is a disjoint union of disks. We set  $d_Y = \frac{d}{dS}|_{\mathcal{O}(Y)}$  and  $d_D = \frac{d}{dS}|_{\mathcal{O}(D)}$ . If  $\|d_Y\|_{S_P} \geq \|d_D\|_{S_P}$  then  $\Sigma_{\frac{d}{dS}} = D^+(0, \|d_Y\|_{S_P})$ . Otherwise,  $\Sigma_{\frac{d}{dS}} = \overline{D^-(0, \|d_D\|_{S_P})}$ .

**Proposition 4.11.** *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (2), (3). The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{H}(x))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{H}(x))) = D^+(0, \frac{1}{r(x)}).$$

Where  $r(x)$  is the value defined in Definition 2.9.

*Proof.* We set  $d = \frac{d}{dS}$ . We distinguish two cases:

- **$x$  is point of type (2):** Let  $c \in k$  such that  $x = x_{c, r(x)}$ . By Proposition 2.10 we have

$$\mathcal{H}(x) = F \oplus \mathcal{O}(C^+(c, r(x), r(x)))$$

where,

$$F := \widehat{\bigoplus_{\alpha \in \tilde{k} \setminus \{0\}} \left\{ \sum_{i \in \mathbb{N}^*} \frac{a_{\alpha i}}{(T - c + \gamma \alpha)^i} \mid a_{\alpha i} \in k, \lim_{i \rightarrow +\infty} |a_{\alpha i}| r^{-i} = 0 \right\}}.$$

We use the same arguments as in Proposition 4.6.

- **$x$  is point of type (3):** Let  $c \in k$  such that  $x = x_{c, r(x)}$ . In this case  $\mathcal{H}(x) = \mathcal{O}(C^+(c, r(x), r(x)))$ , by Proposition 4.8 we conclude.  $\square$

**Proposition 4.12.** *Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (4). The spectrum of  $\frac{d}{dS}$  as an element of  $\mathcal{L}_k(\mathcal{H}(x))$  is:*

$$\Sigma_{\frac{d}{dS}}(\mathcal{L}_k(\mathcal{H}(x))) = \overline{D^-(0, \frac{1}{r(x)})}.$$

Where  $r(x)$  is the value defined in Definition 2.9.



*Proof.* We set  $d = \frac{d}{dS}$ . By Proposition 2.12 we have  $\mathcal{H}(x) \hat{\otimes}_k \mathcal{H}(x) \simeq \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x)))$ . From Lemma 2.5 we have the bounded morphism:

$$\mathcal{L}_k(\mathcal{H}(x)) \hat{\otimes}_k \mathcal{H}(x) \rightarrow \mathcal{L}_{\mathcal{H}(x)}(\mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x))))$$

which associates to  $d \otimes 1$  the derivation  $d_{\mathcal{H}(x)} = \frac{d}{dS} : \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x))) \rightarrow \mathcal{O}(D_{\mathcal{H}(x)}^+(S(x), r(x)))$ . From Proposition 4.7 we have  $\Sigma_{d_{\mathcal{H}(x)}} = \overline{D_{\mathcal{H}(x)}^-(0, \frac{1}{r(x)})}$ , hence  $\pi_{\mathcal{H}(x)/k}(\Sigma_{d_{\mathcal{H}(x)}}) = \overline{D^-(0, \frac{1}{r(x)})}$ . By Lemma 3.20 we have  $\overline{D^-(0, \frac{1}{r(x)})} \subset \Sigma_d$ . From now on we set  $r = r(x)$ . Since  $\|d\|_{Sp} = \frac{1}{r}$  (cf. Lemma 3.19) and  $\Sigma_d \subset D^+(0, \|d\|_{Sp})$  (cf. Theorem 2.18), in order to prove the statement it is enough to show that for all  $a \in k$  such that  $|a| = \frac{1}{r}$ , we have  $D^-(a, \frac{1}{r}) \subset \mathbb{A}_k^{1,an} \setminus \Sigma_d$ . Let  $a \in k$  such that  $|a| = \frac{1}{r}$ . The restriction of  $d - a$  to the normed  $k$ -algebra  $k[S]$  is a bijective bounded map  $d - a : k[S] \rightarrow k[S]$  with respect to the restriction of  $|\cdot|_x$ . We set  $\varphi = (d - a)|_{k[S]}$ . As  $\mathcal{H}(x)$  is the completion of  $k[S]$  with respect to  $|\cdot|_x$  (cf. Lemma 2.11), to prove that it extends to an isomorphism, it is enough to show that  $\varphi^{-1} : k[S] \rightarrow k[S]$  is a bounded  $k$ -linear map. A family of closed disks  $\{D^+(c_l, r_l)\}_{l \in I}$  is called embedded if the set of index  $I$  is endowed with total order  $\leq$  and for  $i \leq j$  we have  $D^+(c_i, r_i) \subset D^+(c_j, r_j)$ . Since  $x$  is a point of type (4), then there exists a family of embedded disks  $\{D^+(c_l, r_l)\}_{l \in I}$  such that  $\bigcap_{l \in I} D^+(c_l, r_l) = \{x\}$ . If we consider  $d - a$  as an element of  $\mathcal{L}_k(\mathcal{O}(D^+(c_l, r_l)))$ , then it is invertible (cf. Proposition 4.7) and its restriction to  $k[S]$  coincides with  $\varphi$  as  $k$ -linear map. Let  $f(S) = \sum_{i \in \mathbb{N}} a_i (S - c_l)^i$  and  $g(S) = \sum_{i \in \mathbb{N}} b_i (S - c_l)^i$  be two elements of  $\mathcal{O}(D^+(c_l, r_l))$  such that  $(d - a)f = g$ . Using the same induction to obtain the equation (10) we obtain: for all  $n \in \mathbb{N}$

$$a_n = \frac{-a^{n-1}}{n!} \sum_{i \geq n} i! b_i a^{-i}. \quad (11)$$

Hence,

$$|a_n| = \frac{1}{r^{n-1}} \left| \sum_{i \geq n} i! b_i a^{-i} \right| \leq \frac{1}{r^{n-1}} \max_{i \geq n} |b_i| r^i \leq r \max_{i \geq n} |b_i| r^{i-n} \leq r \max_{i \geq n} |b_i| r_l^{i-n} \leq \frac{r}{r_l^n} \max_{i \geq n} |b_i| r_l^i \quad (12)$$

Therefore,

$$|a_n| r_l^n \leq r \max_{i \geq n} |b_i| r_l^i. \quad (13)$$

Consequently,

$$|f|_{x_{c_l, r_l}} \leq r |g|_{x_{c_l, r_l}}$$

In the special case where  $f$  and  $g$  are in  $k[S]$ , then  $f = \varphi^{-1}(g)$  and we have for all  $l \in \mathbb{N}$ :

$$|\varphi^{-1}(g)|_{x_{c_l, r_l}} \leq r |g|_{x_{c_l, r_l}}.$$

Hence,

$$|\varphi^{-1}(g)|_x = \inf_{l \in \mathbb{N}} |\varphi^{-1}(g)|_{x_{c_l, r_l}} \leq \inf_{l \in \mathbb{N}} r |g|_{x_{c_l, r_l}} = r |g|_x.$$

This means that  $\varphi^{-1}$  is bounded, hence  $d - a$  is invertible in  $\mathcal{L}_k(\mathcal{H}(x))$  and  $\|(d - a)^{-1}\| \leq r$ , hence  $\|(d - a)^{-1}\|_{Sp} \leq r$ . Since  $\|(d - a)^{-1}\|_{Sp}^{-1}$  is the radius of the biggest disk centred in  $a$  contained in  $\mathbb{A}_k^{1,an} \setminus \Sigma_d$  (cf. Lemma 2.20), we obtain  $D^-(a, \frac{1}{r}) \subset \mathbb{A}_k^{1,an} \setminus \Sigma_d$ .  $\square$

## 4.2 Spectrum of a linear differential equation with constant coefficients

Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$  and  $x \in X$  a point of type (2), (3) or (4). We set here  $A = \mathcal{O}(X)$  or  $\mathcal{H}(x)$  and  $d = \frac{d}{dS}$ . Recall that a linear differential equation with constant coefficients is a differential module  $(M, \nabla)$  over  $(A, d)$  associated to a differential polynomial  $P(D) = g_0 + g_1 D + \dots + g_{\nu-1} D^{\nu-1} + D^{\nu}$  with  $g_i \in k$ , or in an equivalent way there exists a basis for which the matrix  $G$  of the rule (8) has constant coefficients (i.e  $G \in \mathcal{M}_{\nu}(k)$ ). Here we compute the spectrum of  $\nabla$  as an element of  $\mathcal{L}_k(M)$  (cf. Section 3.2).

**Theorem 4.13.** Let  $X$  be a connected affinoid domain of  $\mathbb{A}_k^{1,an}$ . We set here  $A = \mathcal{O}(X)$ . Let  $(M, \nabla)$  be a defferential module over  $(A, d)$  such that the matrix  $G$  of the rule (8) has constant entries (i.e.  $G \in \mathcal{M}_\nu(k)$ ), and let  $\{a_1, \dots, a_N\}$  be the set of eigenvalues of  $G$ . Then we have:

- If  $X = D^+(c_0, r_0)$ ,

$$\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r_0}) & \text{If } \text{char}(\tilde{k}) > 0 \\ \bigcup_{i=1}^N \overline{D^-(a_i, \frac{1}{r_0})} & \text{If } \text{char}(\tilde{k}) = 0 \end{cases}.$$

- If  $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^\mu D^-(c_i, r_i)$  with  $\mu \geq 1$ ,

$$\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{\min_{0 \leq i \leq \mu} r_i}).$$

Where  $\omega$  is the positive real number introduced in (2).

*Proof.* By Propositions 4.2, 4.4, 4.7, 4.8 and 3.15 we obtain the result.  $\square$

**Theorem 4.14.** Let  $x \in \mathbb{A}_k^{1,an}$  be a point of type (2), (3) or (4). We set here  $A = \mathcal{H}(x)$ . Let  $(M, \nabla)$  be a defferential module over  $(A, d)$  such taht the matrix  $G$  of the rule (8) has constant entries (i.e.  $G \in \mathcal{M}_\nu(k)$ ), and let  $\{a_1, \dots, a_N\}$  be the set of eigenvalues of  $G$ . Then we have:

- If  $x$  is a point of type (2) or (3),

$$\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r(x)}).$$

- If  $x$  is a point of type (4),

$$\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r(x)}) & \text{If } \text{char}(\tilde{k}) > 0 \\ \bigcup_{i=1}^N \overline{D^-(a_i, \frac{1}{r(x)})} & \text{If } \text{char}(\tilde{k}) = 0 \end{cases}.$$

Where  $\omega$  is the positive real number introduced in (2).

*Proof.* By Propositions 4.6, 4.11, 4.12 and 3.15 we obtain the result.  $\square$

*Remark 4.15.* Notice that since the spectrum of  $\nabla$  is independant of the choice of the basis, if  $G'$  is another associated matrix to the differential module  $(M, \nabla)$  with constant entries. Then the set of eigenvalues  $\{a'_1, \dots, a'_{N'}\}$  of  $G'$  can not be arbitrary, namely it must satisfy: for each  $a'_i$  there exists  $a_j$  such that  $a'_i$  belongs to the connected componente of the spectrum containing  $a_j$ .

*Remark 4.16.* As mentioned in the introduction, if we consider the differential polynomial  $P(d)$  as an element of  $\mathcal{L}_k(A)$ , then its spectrum  $\Sigma_{P(d)} = P(\Sigma_d)$  (cf. Lemma 2.25) which is in general diffrent from the spectrum of the associated connexion.

## 5 Variation of the spectrum

In this section, we will discuss about the behaviour of the spectrum of  $(M, \nabla)$  over  $(\mathcal{H}(x), d)$ , when we vary  $x$  over  $[x_1, x_2] \subset \mathbb{A}_k^{1,an}$ , where  $x_1$  and  $x_2$  are points of type (2), (3) or (4). For that we need to define a topology over  $K(\mathbb{A}_k^{1,an})$  the set of nonempty compact subsets of  $\mathbb{A}_k^{1,an}$ . Note that, in the case  $\mathbb{A}_k^{1,an}$  is metrisable, we can endow  $K(\mathbb{A}_k^{1,an})$  with a metric called Hausdorff metric. However, in general  $\mathbb{A}_k^{1,an}$  is not metrisable. Indeed, it is metrizable if and only if  $\tilde{k}$  is countable. In the first part of the section, we will introduce the topology on  $K(\mathcal{T})$  (set of nonempty compact subset of a Hausdorff topological space  $\mathcal{T}$ ), that coincides with the topology induced by the Hausdorff metric in the metrizable case. In the second, we will prove that the variation of the spectrum of a differential equation with constant coefficients is left continuous.

### 5.1 The topology on $K(\mathcal{T})$

Let  $\mathcal{T}$  be a Hausdorff topological space, we will denote by  $K(\mathcal{T})$  the set of nonempty compact subset of  $\mathcal{T}$ . Recall that, in the case where  $\mathcal{T}$  is metrizable. For an associated metric  $\mathcal{M}$ , the respective Hausdorff metric  $\mathcal{M}_H$  defined on  $K(\mathcal{T})$  is given as follows. Let  $\Sigma, \Sigma' \in K(\mathcal{T})$

$$\mathcal{M}_H(\Sigma, \Sigma') = \max\left\{\sup_{\beta \in \Sigma'} \inf_{\alpha \in \Sigma} \mathcal{M}(\alpha, \beta), \sup_{\alpha \in \Sigma} \inf_{\beta \in \Sigma'} \mathcal{M}(\alpha, \beta)\right\}. \quad (14)$$

We introduce below a topology on  $K(\mathcal{T})$  for an arbitrary Hausdorff topological space  $\mathcal{T}$ , that coincides with the topology induced by the Hausdorff metric in the metrizable case.

**The topology on  $K(\mathcal{T})$ :** Let  $U$  be an open of  $\mathcal{T}$  and  $\{U_i\}_{i \in I}$  be a finite open cover of  $U$ . We set:

$$(U, \{U_i\}_{i \in I}) = \{\Sigma \in K(\mathcal{T}) \mid \Sigma \subset U, \Sigma \cap U_i \neq \emptyset \forall i\}. \quad (15)$$

The family of sets of this form is stable under finite intersection. Indeed, we have:

$$(U, \{U_i\}_{i \in I}) \cap (V, \{V_j\}_{j \in J}) = (U \cap V, \{U_i \cap V\}_{i \in I} \cup \{V_j \cap U\}_{j \in J}).$$

We endow  $K(\mathcal{T})$  with the topology generated by this family of sets.

**Lemma 5.1.** *The topological space  $K(\mathcal{T})$  is Hausdorff.*

*Proof.* Let  $\Sigma$  and  $\Sigma'$  be two compact subsets of  $\mathcal{T}$  such that  $\Sigma \neq \Sigma'$ . We may assume that  $\Sigma' \not\subset \Sigma$ . Let  $x \in \Sigma \setminus \Sigma'$ . Since  $\mathcal{T}$  is a Hausdorff space, there exists an open neighbourhood  $U_x$  of  $x$  and an open neighbourhood  $U'$  of  $\Sigma'$ , such that  $U_x \cap U' = \emptyset$ . Let  $U$  be an open neighbourhood of  $\Sigma$  such that  $U_x \subset U$ . Then the open set  $(U, \{U, U_x\})$  (resp.  $(U', \{U'\})$ ) is a neighbourhood of  $\Sigma$  (resp.  $\Sigma'$ ) in  $K(\mathcal{T})$  such that  $(U, \{U, U_x\}) \cap (U', \{U'\}) = \emptyset$ . □

**Lemma 5.2.** *Assume that  $\mathcal{T}$  is metrizable. The topology on  $K(\mathcal{T})$  coincides with the topology induced by the Hausdorff metric.*

*Proof.* Let  $\mathcal{M}$  be a metric associated to  $\mathcal{T}$ . For  $x \in \mathcal{T}$  and  $r \in \mathbb{R}_+^*$  we set

$$B_{\mathcal{M}}(x, r) := \{y \in \mathcal{T} \mid \mathcal{M}(x, y) < r\}.$$

For  $\Sigma \in K(\mathcal{T})$  and  $r \in \mathbb{R}_+^*$ , we set  $B_{\mathcal{M}_H}(\Sigma, r) := \{\Sigma' \in K(\mathcal{T}) \mid \mathcal{M}_H(\Sigma, \Sigma') < r\}$ . Let  $\Sigma \in K(\mathcal{T})$ . To prove the statement, we need to show that for all  $r \in \mathbb{R}_+^*$  there exists an open neighbourhood  $(U, \{U_i\}_{i \in I})$  of  $\Sigma$  such that  $(U, \{U_i\}_{i \in I}) \subset B_{\mathcal{M}_H}(\Sigma, r)$ , and for each open neighbourhood  $(U, U_i)_{i \in I}$  of  $\Sigma$ , there exists  $r \in \mathbb{R}_+^*$  such that  $B_{\mathcal{M}_H}(\Sigma, r) \subset (U, \{U_i\}_{i \in I})$ .

Let  $r \in \mathbb{R}_+^*$ . Let  $\{c_1, \dots, c_m\} \subset \Sigma$  such that  $\Sigma \subset \bigcup_{i=1}^m B_{\mathcal{M}}(c_i, \frac{r}{3})$ , we set  $U = \bigcup_{i=1}^m B_{\mathcal{M}}(c_i, \frac{r}{3})$ . We claim that  $(U, \{B_{\mathcal{M}}(c_i, \frac{r}{3})\}_{i=1}^m) \subset B_{\mathcal{M}_H}(\Sigma, r)$ . Indeed, let  $\Sigma' \in (U, \{B_{\mathcal{M}}(c_i, \frac{r}{3})\}_{i=1}^m)$ . As  $\Sigma' \in U$ , for all  $y \in \Sigma'$  we have  $\min_{1 \leq i \leq m} (\mathcal{M}(y, c_i)) < \frac{r}{3}$ . Therefore,

$$\sup_{y \in \Sigma'} \inf_{x \in \Sigma} \mathcal{M}(y, x) \leq \frac{r}{3} < r.$$

Since for each  $c_i$  there exists  $y \in \Sigma'$  such that  $\mathcal{M}(c_i, y) < \frac{r}{3}$ , for each  $x \in \Sigma$  there exists  $y \in \Sigma'$  such that  $\mathcal{M}(x, y) < \frac{2r}{3}$ . Indeed, there exists  $c_i$  such that  $x \in B(c_i, \frac{r}{3})$ , therefore we have

$$\mathcal{M}(x, y) \leq \mathcal{M}(c_i, y) + \mathcal{M}(c_i, x) < \frac{2r}{3}.$$

This implies

$$\sup_{x \in \Sigma} \inf_{y \in \Sigma'} \mathcal{M}(x, y) \leq \frac{2r}{3} < r.$$

Consequently,  $\mathcal{M}_H(\Sigma, \Sigma') < r$ .

Now let  $(U, \{U_i\}_{i \in I})$  be an open neighbourhood of  $\Sigma$ . Let  $\alpha = \inf_{y \in \mathcal{T} \setminus U} \inf_{x \in \Sigma} \mathcal{M}(x, y)$ , since  $\Sigma \subsetneq U$  we have  $\alpha \neq 0$ . For each  $1 \leq i \leq m$ , let  $c_i \in \Sigma \cap U_i$ . There exists  $0 < \beta < \alpha$  such that for all  $0 < r < \beta$  we have  $B_{\mathcal{M}}(c_i, r) \subset U_i$  for each  $1 \leq i \leq m$ .

We claim that  $B_{\mathcal{M}_H}(\Sigma, r) \subset (U, \{U_i\}_{i=1}^m)$ . Indeed, let  $\Sigma' \in B_{\mathcal{M}_H}(\Sigma, r)$  this means that:

$$\sup_{y \in \Sigma'} \inf_{x \in \Sigma} \mathcal{M}(x, y) < r; \quad \sup_{x \in \Sigma} \inf_{y \in \Sigma'} \mathcal{M}(x, y) < r.$$

The first inequality implies  $\Sigma' \subset U$ . The second implies that for each  $c_i$  there exists  $y \in \Sigma'$  such that  $\mathcal{M}(c_i, y) < r$ . Hence,  $\Sigma' \cap U_i = \emptyset$  for each  $i$ .  $\square$

## 5.2 Variation of the spectrum

Let  $X$  be an affinoid domain of  $\mathbb{A}_k^{1,an}$ . Let  $(M, \nabla)$  be a differential module over  $(\mathcal{O}(X), \frac{d}{dS})$  such that there exists a basis for which the associated matrix  $G$  has constant entries. For a point  $x \in X$  not of type (1), the differential module  $(M, \nabla)$  extends to a differential module  $(M_x, \nabla_x)$  over  $(\mathcal{H}(x), \frac{d}{dS})$ . In the corresponding basis of  $(M_x, \nabla_x)$  the associated matrix is  $G$ .

**Theorem 5.3.** *Suppose that  $k$  is algebraically closed. Let  $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$  be a connected affinoid domain and  $x \in X$  be a point of type (2), (3) or (4). Let  $(M, \nabla)$  be a differential module over  $(\mathcal{O}(X), \frac{d}{dS})$  such that there exists a basis for which the corresponding matrix  $G$  has constant entries. We set:*

$$\begin{aligned} \Psi : [x, x_{c_0, r_0}] &\longrightarrow K(\mathbb{A}_k^{1,an}) \\ y &\longmapsto \Sigma_{\nabla_y}(\mathcal{L}_k(M_y)) \end{aligned}.$$

Then we have:

- for each  $y \in [x, x_{c_0, r_0}]$ , the restriction of  $\Psi$  to  $[x, y]$  is continuous at  $y$ .
- If  $y \in [x, x_{c_0, r_0}]$  is a point of type (3), then  $\Psi$  is continuous at  $y$ .
- If  $\text{char}(\tilde{k}) = 0$  and  $y \in [x, x_{c_0, r_0}]$  is a point of type (4), then  $\Psi$  is continuous at  $y$ .

*Proof.* Let  $\{a_1, \dots, a_N\} \subset k$  be the set of eigenvalues of  $G$ . We identify  $[x, x_{c_0, r_0}]$  with the interval  $[r(x), r_0]$  by the map  $y \mapsto r(y)$  (cf. Definition 2.9). Let  $y \in [x, x_{c_0, r_0}]$ . We set  $\Sigma_y = \Sigma_{\nabla_y}(\mathcal{L}_k(M_y))$ . By Theorem 4.14, for all  $y' \in (x, x_{c_0, r_0}]$  ( $y'$  can not be a point of type (4)) we have  $\Sigma_{y'} = \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r(y')})$ . Let  $(U, \{U_i\}_{i \in I})$  be an open neighbourhood of  $\Sigma_y$ . Since  $\Sigma_y$  is a finite union of closed disks, we may assume that  $U$  is a finite union of open disks. Let  $R$  be the smallest radius of those disks.

- If  $y = x$  then it is obvious that the restriction of  $\Psi$  to  $[x, y] = \{y\}$  is continuous at  $y$ . Now, we assume that  $y \neq x$ . Let  $x_R \in (x, y)$  be the point with radius  $r(x_R) = \max(r(x), \frac{\omega}{R})$ . Then for each  $y' \in (x_R, y]$  we have  $\frac{\omega}{r(y')} < R$ . Hence,  $\Sigma_{y'} \subset U$  for each  $y' \in (x_R, y]$ . Since  $\frac{\omega}{r(y)} \leq \frac{\omega}{r(y')}$  for each  $y' \in (x_R, y]$ , we have  $\Sigma_y \subset \Sigma_{y'}$ . Therefore  $\Sigma_{y'} \cap U_i \neq \emptyset$  for each  $y' \in (x_R, y]$ . Then we obtain  $(x_R, y] \subset \Psi^{-1}((U, \{U_i\}_{i \in I}))$ .
- Let  $y \in [x, x_{c_0, r_0}]$  be point of type (3). Since  $k$  is algebraically closed,  $\omega \in |k|$ . Therefore,  $\frac{\omega}{r(y)} \notin |k|$  and we have  $\Sigma_y = \bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r(y)}) = \bigcup_{i=1}^N (D^-(a_i, \frac{\omega}{r(y)}) \cup \{x_{a_i, \frac{\omega}{r(y)}}\})$ . In order to prove the continuity at  $y$  it is enough to prove that the restriction  $\Psi$  to  $[y, x_{c_0, r_0}]$  is continuous at  $y$ . Let

$y' \in [y, x_{c_0, r_0}]$ . Since  $\frac{\omega}{r(y')} \leq \frac{\omega}{r(y)}$ , we have  $\Sigma_{y'} \subset \Sigma_y \subset U$ . Now we need to show that there exists  $x' \in (y, x_{c_0, r_0}]$  such that for all  $y' \in [y, x']$  we have  $\Sigma_{y'} \cap U_i \neq \emptyset$ . Let  $\alpha_1, \dots, \alpha_m \in \Sigma_y$  such that  $\alpha_i \in \Sigma_y \cap U_i$  for each  $i$ . Since  $\Sigma_y$  is an affinoid domain we can assume that the  $\alpha_i$  are either of type (1) or (3)<sup>7</sup>. For each  $\alpha_i$  there exists  $a_{j_i}$  such that  $\alpha_i \in D^-(a_{j_i}, \frac{\omega}{r(y)}) \cup \{x_{a_{j_i}, \frac{\omega}{r(y)}}\} \subset \Sigma_y$ .

If  $\alpha_i$  is point of type (1), then  $\alpha_i \in D^-(a_{j_i}, \frac{\omega}{r(y)})$ . Since the open disks form a basis of neighbourhoods for the points of type (1), there exists  $D^-(\alpha_i, L_i) \subset D^-(a_{j_i}, \frac{\omega}{r(y)}) \cap U_i$ . We set  $R_i = \max(|b_i - a_{j_i}|, L_i)$ . Let  $x_i \in [y, x_{c_0, r_0}]$  with  $r(x_i) = \min(r(x_{c_0, r_0}), \frac{\omega}{R_i})$ . Then for all  $y' \in (y, x_i)$  we have  $R_i < \frac{\omega}{r(y')} < \frac{\omega}{r(y)}$ . This implies  $D^-(\alpha_i, R_i) \subset D^-(a_{j_i}, \frac{\omega}{r(y')}) \subset \Sigma_{y'}$  for each  $y' \in [y, x_i]$ .

Suppose now that  $\alpha_i = x_{b_i, L_i}$  is point of type (3). If  $\alpha_i \neq x_{a_{j_i}, \frac{\omega}{r(y)}}$ , we set  $R_i = \max(|a_{j_i} - b_i|, L_i)$ . Let  $x_i \in (y, x_{c_0, r_0}]$  with  $r(x_i) = \min(r(x_{c_0, r_0}), \frac{\omega}{R_i})$ . Since  $\frac{\omega}{r(y')} > R_i$  for each  $y' \in [y, x_i]$ , we have  $\alpha_i \in D^-(a_{j_i}, \frac{\omega}{r(y')}) \subset \Sigma_{y'}$ . If  $\alpha_i = x_{a_{j_i}, \frac{\omega}{r(y)}}$ , since it is a point of type (3), the open annulus  $C^-(a_{j_i}, L_i^1, L_i^2)$  form a basis of neighbourhoods of  $\alpha_i$ . Therefore, there exists  $C^-(a_{j_i}, L_i^1, L_i^2) \subset U_i$  containing  $\alpha_i$ . This implies that there exists  $C^+(a_{j_i}, R_i, \frac{\omega}{r(y)}) \subset D^+(a_{j_i}, \frac{\omega}{r(y)}) \cap U_i$ . Let  $x_i \in [y, x_{c_0, r_0}]$  with  $r(x_i) = \min(r(x_{c_0, r_0}), \frac{\omega}{R_i})$ . Then for all  $y' \in (y, x_i)$  we have  $R_i < \frac{\omega}{r(y')} < \frac{\omega}{r(y)}$ . Therefore  $C^+(a_{j_i}, R_i, \frac{\omega}{r(y)}) \cap D^+(a_{j_i}, \frac{\omega}{r(y')}) = C^+(a_{j_i}, R_i, \frac{\omega}{r(y')})$ . Then we obtain  $\Sigma_{y'} \cap U_i \neq \emptyset$  for each  $y' \in [y, x_i]$ . Hence, for all  $y' \in \bigcap_{i=1}^m [y, x_i] = [y, x_{i_0}]$  we have  $\Sigma_{y'} \cap U_i \neq \emptyset$ .

- We assume that  $\text{char}(\tilde{k}) = 0$ . Let  $y \in [x, x_{c_0, r_0}]$  be a point of type (4). Since  $\Sigma_y = \bigcup_{i=1}^N (D^-(a_i, \frac{\omega}{r(y)}) \cup \{x_{a_i, \frac{\omega}{r(y)}}\})$  (cf. Theorem 4.14), using the same arguments above we obtain the result. □

*Remark 5.4.* Set notations as in the proof of Theorem 5.3. In the case where  $y \in [x, x_{c_0, r_0}]$  is a point of type (2), then the map  $\Psi$  is never continuous at  $y$ . Indeed, since  $y$  is of type (2), there exists  $b \in (D^+(a_i, \frac{\omega}{r(y)}) \setminus D^-(a_i, \frac{\omega}{r(y)})) \cap k$ . As for each  $y' \in (y, x_{c_0, r_0})$  we have  $\Sigma_{y'} \cap D^-(b, \frac{\omega}{r(y)}) = \emptyset$ ,  $\Sigma_{y'} \not\subset (U \cup D^-(b, \frac{\omega}{r(y)}), \{U_i\}_{i \in I} \cup \{D^-(b, \frac{\omega}{r(y)})\})$  for all  $y' \in (y, x_{c_0, r_0})$ .

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<sup>7</sup>Note that, in the case where  $k$  is not trivially valued, we may assume that the  $\alpha_i$  are of type (1).

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