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# Lipschitz stability for an inverse hyperbolic problem of determining two coefficients by a finite number of observations

L. Beilina <sup>\*</sup>      M. Cristofol <sup>†</sup>      S. Li <sup>‡</sup>      M. Yamamoto <sup>§</sup>

## Abstract

We consider an inverse problem of reconstructing two spatially varying coefficients in an acoustic equation of hyperbolic type using interior data of solutions with suitable choices of initial condition. Using a Carleman estimate, we prove Lipschitz stability estimates which ensures unique reconstruction of both coefficients. Our theoretical results are justified by numerical studies on the reconstruction of two unknown coefficients using noisy backscattered data.

## 1 Statement of the problem

### 1.1 Introduction

The main purpose of this paper is to study the inverse problem of determining simultaneously the function  $\rho(x)$  and the conductivity  $p(x)$  in the following:

$$\rho(x)\partial_t^2 u - \operatorname{div}(p(x)\nabla u) = 0 \tag{1.1}$$

from a finite number of boundary observations on the domain  $\Omega$  which is a bounded open subset of  $\mathbb{R}^n$ ,  $n \geq 1$ .

The reconstruction of two coefficients of the principal part of an operator with a finite number of observations is very challenging since we mix at least two difficulties, see [15] for the case of a principal matrix

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term in the divergence form, arising from anisotropic media, [25] for Lamé system and [6, 13, 38, 39, 40] for Maxwell system.

To the best knowledge of the authors, the present paper for the first time establishes the uniqueness and the Lipschitz stability (Theorem 1.1) in determining two important physical coefficients  $\rho$  and  $p$  in the conductivity equation with observed data coming from suitably chosen two initial values. Taking into consideration the degrees of freedom of unknown coefficients and observation data, we should note that the formulation with twice changed initial data is not over-determining. On the other hand, in [15], for determining  $\frac{n(n+1)}{2}$  coefficients, we need to choose initial displacement  $\frac{n(n+3)}{2}$ -times with spatial dimension  $n$ , which is over-determining. Furthermore, we use the uniqueness and the Lipschitz stability to perform numerical reconstruction with noisy observations to be more close to real-life applications.

Bukhgeim and Klibanov [19] created the methodology by Carleman estimate for proving the uniqueness in coefficient inverse problems. After [19] increasing number of works have been published on this topic. We refer to some of them, see [11, 12, 15, 16, 17], [26] - [28], [32] - [34], [37, 48]. In all these works except the recent works [5, 6], only theoretical studies are presented. From the other side, the existence of a stability theorems allows us to improve the results of the numerical reconstruction by choosing different regularization strategies in the minimization procedure. Particularly, we refer to Imanuvilov and Yamamoto [27] which established the Lipschitz stability for the coefficient inverse problem for a hyperbolic equation. Our argument in this paper is a simplification of [27] and Klibanov and Yamamoto [37].

To the authors knowledge, there exist few works which study numerical reconstruction based on the theoretical stability analysis for the inverse problem with finite and restricted measurements. Furthermore, the case of the reconstruction of the conductivity coefficient in the divergence form for the hyperbolic operator induces some numerical difficulties, see [3, 7, 10, 22] for details.

In numerical simulations of this paper we use similar optimization approach which was applied recently in works [3, 5, 6, 8, 10]. More precisely, we minimize the Tikhonov functional in order to reconstruct unknown spatially distributed wave speed and conductivity functions of the acoustic wave equation from transmitted or backscattered boundary measurements. For minimization of the Tikhonov functional we construct the associated Lagrangian and minimize it on the adaptive locally refined meshes using the domain decomposition finite element/finite difference method similar to one of [3]. Details of this method can be found in forthcoming publication. The adaptive optimization method is implemented efficiently in the software package WavES [47] in C++/PETSc [45].

Our numerical simulations show that we can accurately reconstruct location of both space-

dependent wave speed and conductivity functions already on a coarse non-refined mesh. The contrast of the conductivity function is also reconstructed correctly. However, the contrast of the wave speed function should be improved. In order to obtain better contrast, similarly with [2, 7, 8], we applied an adaptive finite element method, and refined the finite element mesh locally only in places, where the a posteriori error of the reconstructed coefficients was large. Our final results attained on a locally refined meshes show that an adaptive finite element method significantly improves reconstruction obtained on a coarse mesh.

The outline of this paper is as follows. In Section 2, we show a key Carleman estimate, in Section 3 we complete the proofs of Theorems 1.1 and 1.2. Finally, in Section 4 we present numerical simulations taking into account the theoretical observations required in Theorem 1.1 as an important guidance. Section 5 concludes the main results of this paper.

## 1.2 Settings and main results

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ . We consider an acoustic equation

$$\rho(x)\partial_t^2 u(x, t) - \operatorname{div}(p(x)\nabla u(x, t)) = 0, \quad x \in \Omega, 0 < t < T. \quad (1.2)$$

To (1.2) we attach the initial and boundary conditions:

$$u(x, 0) = a(x), \quad \partial_t u(x, 0) = 0, \quad x \in \Omega \quad (1.3)$$

and

$$u(x, t) = h(x, t), \quad (x, t) \in \partial\Omega \times (0, T). \quad (1.4)$$

By  $u(p, \rho, a, h)$  we denote a solution in  $W^{4,\infty}(\Omega \times (0, T))$  to the problem (1.2)-(1.4). For sufficiently smooth positive coefficients  $p$  and  $\rho$  and initial and boundary data, we can prove the unique existence of such a solution by a classical argument (e.g., Lions and Magenes [42]) but omit details. Functions  $p, \rho$  are assumed to be positive on  $\bar{\Omega}$  and are unknown in  $\Omega$ . They should be determined by extra data of solutions  $u$  in  $\Omega$ .

Throughout this paper, we set  $\partial_j = \frac{\partial}{\partial x_j}$ ,  $\partial_i \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$ ,  $\partial_t^2 = \frac{\partial^2}{\partial t^2}$ ,  $1 \leq i, j \leq n$ .

Let  $\omega \subset \Omega$  be a suitable subdomain of  $\Omega$  and  $T > 0$  be given. In this paper, we consider an inverse problem of determining coefficients  $p = p(x)$  and  $\rho = \rho(x)$  of the principal term, from the interior observations:

$$u(x, t), \quad x \in \omega, 0 < t < T.$$

In order to formulate our results, we need to introduce some notations. Henceforth  $(\cdot, \cdot)$  denotes the scalar product in  $\mathbb{R}^n$ , and  $\nu = \nu(x)$  the unit outward normal vector to  $\partial\Omega$  at  $x$ . Let the subdomain  $\omega \subset \Omega$  satisfy

$$\partial\omega \supset \{x \in \Omega; ((x - x_0) \cdot \nu(x)) > 0\} \quad (1.5)$$

with some  $x_0 \notin \bar{\Omega}$ . We note that  $\omega \subset \Omega$  cannot be an arbitrary subdomain. For example, in the case of a ball  $\Omega$ , the condition (1.5) requires that  $\omega$  should be a neighborhood of a sub-boundary which is larger than the half of  $\partial\Omega$ . The condition (1.5) is also a sufficient condition for an observability inequality by observations in  $\omega \times (0, T)$  (e.g., Ch VII, section 2.3 in Lions [41]).

We set

$$\Lambda = \left( \sup_{x \in \Omega} |x - x_0|^2 - \inf_{x \in \Omega} |x - x_0|^2 \right)^{\frac{1}{2}}. \quad (1.6)$$

We define admissible sets of unknown coefficients. For arbitrarily fixed functions  $\eta_0 \in C^2(\bar{\Omega})$ ,  $\eta_1 \in (C^2(\bar{\Omega}))^n$  and constants  $M_1 > 0$ ,  $0 < \theta_0 \leq 1$ ,  $\theta_1 > 0$ , we set

$$\mathcal{U}^1 = \mathcal{U}_{M_1, \theta_1, \eta_0, \eta_1}^1 = \left\{ p \in C^2(\bar{\Omega}); p = \eta_0, \nabla p = \eta_1 \quad \text{on } \partial\Omega \right\} \quad (1.7)$$

$$\left\{ \|p\|_{C^2(\bar{\Omega})} \leq M_1, \quad p \geq \theta_1 \quad \text{on } \bar{\Omega} \right\},$$

$$\mathcal{U}^2 = \mathcal{U}_{M_1, \theta_1}^2 = \left\{ \rho \in C^2(\bar{\Omega}); \|\rho\|_{C^2(\bar{\Omega})} \leq M_1, \quad \rho \geq \theta_1 \quad \text{on } \bar{\Omega} \right\},$$

$$\mathcal{U} = \mathcal{U}_{M_1, \theta_0, \theta_1, \eta_0, \eta_1, x_0} = \left\{ (p, \rho) \in \mathcal{U}^1 \times \mathcal{U}^2; \quad \frac{(\nabla(p\rho^{-1}) \cdot (x - x_0))}{2p\rho^{-1}(x)} < 1 - \theta_0, \quad x \in \overline{\Omega \setminus \omega} \right\}.$$

We note that there exists a constant  $M_0 > 0$  such that  $\left\| \nabla \left( \frac{p}{\rho} \right) \right\|_{C(\bar{\Omega})} \leq M_0$  for each  $(p, \rho) \in \mathcal{U}^1 \times \mathcal{U}^2$ .

Then we choose a constant  $\beta > 0$  such that

$$\beta + \frac{M_0 \Lambda}{\sqrt{\theta_1}} \sqrt{\beta} < \theta_0 \theta_1, \quad \theta_1 \inf_{x \in \Omega} |x - x_0|^2 - \beta \Lambda^2 > 0. \quad (1.8)$$

Here we note that such  $\beta > 0$  exists by  $x_0 \notin \bar{\Omega}$ , and in fact  $\beta > 0$  should be sufficiently small.

We are ready to state our first main result.

**Theorem 1.1.** *Let  $q \in \mathcal{U}^1$  be arbitrarily fixed and let  $a_1, a_2 \in C^3(\bar{\Omega})$  satisfy*

$$\begin{cases} |\operatorname{div}(q\nabla a_\ell)| > 0, & \ell = 1 \text{ or } \ell = 2, \\ ((\operatorname{div}(q\nabla a_2)\nabla a_1 - \operatorname{div}(q\nabla a_1)\nabla a_2) \cdot (x - x_0)) > 0 & \text{on } \bar{\Omega}. \end{cases} \quad (1.9)$$

*We further assume that*

$$u(q, \sigma, a_\ell, h_\ell) \in W^{4, \infty}(\Omega \times (0, T)), \quad \ell = 1, 2$$

*and*

$$T > \frac{\Lambda}{\sqrt{\beta}}. \quad (1.10)$$

*Then there exists a constant  $C > 0$  depending on  $\Omega, T, \mathcal{U}, q, \sigma$  and a constant  $M_2 > 0$  such that*

$$\|p - q\|_{H^1(\Omega)} + \|\rho - \sigma\|_{L^2(\Omega)} \leq C \sum_{\ell=1}^2 \|u(p, \rho, a_\ell, h_\ell) - u(q, \sigma, a_\ell, h_\ell)\|_{H^3(0, T; L^2(\omega))} \quad (1.11)$$

for each  $(p, \rho) \in \mathcal{U}$  satisfying

$$\|u(p, \rho, a_\ell, h_\ell)\|_{W^{4,\infty}(\Omega \times (0,T))} \leq M_2. \quad (1.12)$$

The conclusion (1.11) is a Lipschitz stability estimate with twice changed initial displacement satisfying (1.9). In Imanuvilov and Yamamoto [28], by assuming that  $\rho = \sigma \equiv 1$ , a Hölder stability estimate is proved for  $p - q$ , provided that  $p$  and  $q$  vary within a similar admissible set. However, in the case of two unknown coefficients  $p, \rho$ , the condition (1.9) requires us to fix  $q \in \mathcal{U}^1$  and the theorem gives stability only around given  $q$ , in general.

**Remark 1.** In this remark, we will show that with special choice of  $a_1, a_2$ , the condition (1.9) can be satisfied uniformly for  $q \in \mathcal{U}^1$ , which guarantees that the set of  $a_1, a_2$  satisfying (1.9), is not empty. We fix  $a_1, b_2 \in C^2(\bar{\Omega})$  satisfying

$$(\nabla a_1(x) \cdot (x - x_0)) > 0, \quad |\nabla b_2(x)| > 0, \quad x \in \bar{\Omega}. \quad (1.13)$$

We choose  $\gamma > 0$  sufficiently large and we set

$$a_2(x) = e^{\gamma b_2(x)}.$$

Then  $\partial_k a_2 = \gamma(\partial_k b_2)e^{\gamma b_2(x)}$  and

$$\Delta a_2 = (\gamma^2 |\nabla b_2|^2 + \gamma \Delta b_2)e^{\gamma b_2},$$

and so

$$\operatorname{div}(q \nabla a_2) = q \Delta a_2 + \nabla q \cdot \nabla a_2 = (q \gamma^2 |\nabla b_2|^2 + O(\gamma))e^{\gamma b_2}$$

and

$$\begin{aligned} & (\operatorname{div}(q \nabla a_2) \nabla a_1 - \operatorname{div}(q \nabla a_1) \nabla a_2) \cdot (x - x_0) \\ &= e^{\gamma b_2} (\{(q \gamma^2 |\nabla b_2|^2 + O(\gamma)) \nabla a_1 - \operatorname{div}(q \nabla a_1) \gamma \nabla b_2\}) \cdot (x - x_0) \\ &= e^{\gamma b_2} (q \gamma^2 |\nabla b_2|^2 (\nabla a_1 \cdot (x - x_0)) + O(\gamma)) \\ &\geq e^{\gamma \min_{x \in \bar{\Omega}} b_2(x)} (\gamma^2 \theta_1 \min_{x \in \bar{\Omega}} \{|\nabla b_2(x)|^2 (\nabla a_1(x) \cdot (x - x_0))\}) + O(\gamma) \end{aligned}$$

for each  $q \in \mathcal{U}^1$ . Therefore, for large  $\gamma > 0$ , by (1.13) we see that (1.9) is fulfilled. Moreover this choice of  $a_1, a_2$  is independent of choices of  $q \in \mathcal{U}^1$ , and there exists a constant  $C > 0$ , which is dependent on  $\Omega, T, \mathcal{U}, M_2$  but independent of choices  $(p, \rho), (q, \sigma)$ , such that (1.11) holds for each  $(p, \rho), (q, \sigma) \in \mathcal{U}$ .

Without special choice such as (1.13), we consider the stability estimate by not fixing  $q$ . If we can suitably choose initial values  $(n + 1)$ -times, then we can establish the Lipschitz stability for arbitrary  $(p, \rho), (q, \sigma) \in \mathcal{U}$ .

**Theorem 1.2.** Let  $A := \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} \in (C^2(\bar{\Omega}))^{n+1}$  satisfy

$$\det(\partial_1 A(x), \dots, \partial_n A(x), \Delta A(x)) \neq 0, \quad x \in \bar{\Omega}. \quad (1.14)$$

We assume (1.10). Then there exists a constant  $C > 0$  depending on  $\Omega, T, \mathcal{U}, a_\ell, h_\ell, \ell = 1, 2, \dots, n+1$  and a constant  $M_2 > 0$  such that

$$\|p - q\|_{H^1(\Omega)} + \|\rho - \sigma\|_{L^2(\Omega)} \leq C \sum_{\ell=1}^{n+1} \|u(p, \rho, a_\ell, h_\ell) - u(q, \sigma, a_\ell, h_\ell)\|_{H^2(0, T; L^2(\omega))} \quad (1.15)$$

for each  $(p, \rho), (q, \sigma) \in \mathcal{U}$  satisfying

$$\|u(p, \rho, a_\ell, h_\ell)\|_{W^{4, \infty}(\Omega \times (0, T))}, \|u(q, \sigma, a_\ell, h_\ell)\|_{W^{4, \infty}(\Omega \times (0, T))} \leq M_2, \quad \ell = 1, 2, \dots, n+1.$$

**Example 1.** This example illustrates how to choose initial values satisfying (1.14). In Theorem 1.1 it is sufficient to choose only two suitable initial values, while in Theorem 1.2 we need to choose them  $(n+1)$ -times. However, the condition (1.14) on the initial values can be realized more easily than the condition (1.9).

For example, we can choose the following initial displacement  $a_1, \dots, a_{n+1}$ . For an invertible matrix  $D = (d_{ij})_{1 \leq i, j \leq n}$ ,  $d_{ij} \in \mathbb{R}$ , we define  $a_1, \dots, a_n$  by

$$a_\ell(x) = \sum_{k=1}^n d_{\ell k} x_k, \quad \ell = 1, 2, \dots, n$$

and assume  $\Delta a_{n+1}(x) \neq 0$  for  $x \in \bar{\Omega}$ .

Then we can easily verify that this choice  $a_1, \dots, a_{n+1}$  satisfies (1.14).

We note that Theorems 1.1 and 1.2 yield the uniqueness for our inverse problem in the respective case.

## 2 The Carleman estimate for a hyperbolic equation

We show a Carleman estimate for a second-order hyperbolic equation. We recall that  $\mathcal{U}$  is defined by (1.7).

Let us set

$$Q = \Omega \times (-T, T).$$

For  $x_0 \notin \bar{\Omega}$  and  $\beta > 0$  satisfying (1.8), we define the functions  $\psi = \psi(x, t)$  and  $\varphi = \varphi(x, t)$  by

$$\psi(x, t) = |x - x_0|^2 - \beta t^2 \quad (2.1)$$

and

$$\varphi(x, t) = e^{\lambda\psi(x, t)} \quad (2.2)$$

with parameter  $\lambda > 0$ . We add a constant  $C_0 > 0$  if necessary so that we can assume that  $\psi(x, t) \geq 0$  for  $(x, t) \in Q$ , so that

$$\varphi(x, t) \geq 1, \quad (x, t) \in \overline{Q}.$$

We show a Carleman estimate which is derived from Theorem 1.2 in Imanuvilov [24]. See Imanuvilov and Yamamoto [28] for a concrete sufficient condition on the coefficients yielding a Carleman estimate.

**Lemma 2.1.** *We assume  $(\mu, 1) \in \mathcal{U}$ , and that (1.5) holds for some  $x_0 \notin \overline{\Omega}$ . Let  $y \in H^1(Q)$  satisfy*

$$\partial_t^2 y(x, t) - \mu \Delta y = F \quad \text{in } Q \quad (2.3)$$

and

$$y(x, t) = 0, \quad (x, t) \in \partial\Omega \times (-T, T), \quad \partial_t^k y(x, \pm T) = 0, \quad x \in \Omega, \quad k = 0, 1. \quad (2.4)$$

Let

$$T > \frac{\Lambda}{\sqrt{\beta}}. \quad (2.5)$$

We fix  $\lambda > 0$  sufficiently large. Then there exist constants  $s_0 > 0$  and  $C > 0$  such that

$$\int_Q (s|\nabla_{x,t} y|^2 + s^3|y|^2) e^{2s\varphi} dx dt \leq C \int_Q |F|^2 e^{2s\varphi} dx dt + C \int_{-T}^T \int_{\omega} (s|\partial_t y|^2 + s^3|y|^2) e^{2s\varphi} dx dt \quad (2.6)$$

for all  $s > s_0$ .

Henceforth  $C > 0$  denotes generic constants which are independent of choices of  $(p, \rho), (q, \sigma) \in \mathcal{U}$  and the parameter  $s > 0$  in the Carleman estimates, provided that  $s > s_0$ .

Setting  $\Gamma = \{x \in \partial\Omega; (x - x_0) \cdot \nu(x) \geq 0\}$ , one can prove a Carleman estimate whose second term on the right-hand side of (2.6) is replaced by

$$\int_{-T}^T \int_{\Gamma} s |\nabla y \cdot \nu|^2 e^{2s\varphi} dS dt,$$

and as for a direct proof, see Bellassoued and Yamamoto [18], Cheng, Isakov, Yamamoto and Zhou [20]. In Isakov [29], a similar Carleman estimate is established for  $\text{supp } y \subset Q$ , which cannot be applied to the case where we have no Neumann data outside of  $\Gamma$ .

For the Carleman estimate, we have to assume that  $\partial_t^k y(\cdot, \pm T) = 0$  in  $\Omega$  for  $k = 0, 1$ , but  $u(p, \rho, a, h)$ ,  $u(q, \sigma, a, h)$  do not satisfy this condition. Thus we need a cut-off function which is defined as follows.

By (1.10) and the definitions (2.1) and (2.2) of  $\psi, \varphi$ , we can choose  $d_0 \in \mathbb{R}$  such that

$$\varphi(x, 0) > d_0, \quad \varphi(x, \pm T) < d_0, \quad x \in \overline{\Omega}. \quad (2.7)$$

Hence, for small  $\varepsilon_0 > 0$ , we find a sufficiently small  $\varepsilon_1 > 0$  such that

$$\varphi(x, t) \geq d_0 + \varepsilon_0, \quad (x, t) \in \overline{\Omega \times [-\varepsilon_1, \varepsilon_1]} \quad (2.8)$$

and

$$\varphi(x, t) \leq d_0 - \varepsilon_0, \quad (x, t) \in \overline{\Omega} \times ([-T, -T + 2\varepsilon_1] \cup [T - 2\varepsilon_1, T]). \quad (2.9)$$

We define a cut-off function satisfying  $0 \leq \chi \leq 1$ ,  $\chi \in C^\infty(\mathbb{R})$  and

$$\chi(t) = \begin{cases} 0, & -T \leq t \leq -T + \varepsilon_1, \quad T - \varepsilon_1 \leq t \leq T, \\ 1, & -T + 2\varepsilon_1 \leq t \leq T - 2\varepsilon_1. \end{cases} \quad (2.10)$$

Henceforth we write  $\chi'(t) = \frac{d\chi}{dt}(t)$ ,  $\chi''(t) = \frac{d^2\chi}{dt^2}(t)$ .

In view of the cut-off function, we can prove

**Lemma 2.2.** *Let  $(p, \rho) \in \mathcal{U}$  and let (2.5) hold, and we fix  $\lambda > 0$  sufficiently large. Then there exist constants  $s_0 > 0$  and  $C > 0$  such that*

$$\begin{aligned} \int_Q (s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\varphi} dxdt &\leq C \int_Q |\rho\partial_t^2u - \operatorname{div}(p\nabla u)|^2 e^{2s\varphi} dxdt \\ &+ Cs^3e^{2s(d_0-\varepsilon_0)}\|u\|_{H^1(Q)}^2 + C \int_{-T}^T \int_\omega (s|\partial_t u|^2 + s^3|u|^2)e^{2s\varphi} dxdt \end{aligned} \quad (2.11)$$

for all  $s > s_0$  and  $u \in H^1(Q)$  satisfying  $\rho\partial_t^2u - \operatorname{div}(p\nabla u) \in L^2(Q)$  and  $u|_{\partial\Omega} = 0$ .

*Proof.* We notice

$$u = \chi u + (1 - \chi)u.$$

Then

$$\begin{aligned} &\int_Q (s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\varphi} dxdt \\ &\leq 2 \int_Q (s|\nabla_{x,t}(\chi u)|^2 + s^3|\chi u|^2)e^{2s\varphi} dxdt + 2 \int_Q (s|\nabla_{x,t}((1 - \chi)u)|^2 + s^3|(1 - \chi)u|^2)e^{2s\varphi} dxdt. \end{aligned}$$

Since the second term on the right-hand side does not vanish only if  $T - 2\varepsilon_1 \leq |t| \leq T$ , that is, only if  $\varphi(x, t) \leq d_0 - \varepsilon_0$  by (2.9), we obtain

$$\begin{aligned} &\int_Q (s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\varphi} dxdt \\ &\leq 2 \int_Q (s|\nabla_{x,t}(\chi u)|^2 + s^3|\chi u|^2)e^{2s\varphi} dxdt + Cs^3e^{2s(d_0-\varepsilon_0)}\|u\|_{H^1(Q)}^2. \end{aligned} \quad (2.12)$$

On the other hand, we have

$$\begin{cases} \partial_t^2(\chi u)(x, t) = \frac{p}{\rho}\Delta(\chi u) + \frac{\nabla p}{\rho} \cdot \nabla(\chi u) + 2\chi' \partial_t u + \chi'' u & \text{in } Q, \\ \chi u|_{\partial\Omega} = 0, \\ \partial_t^k(\chi u)(\cdot, \pm T) = 0 & \text{in } \Omega, k = 0, 1. \end{cases}$$

Therefore, applying Lemma 1 to  $(\partial_t^2 - \frac{p}{\rho}\Delta)(\chi u)$  by regarding  $\frac{\nabla p}{\rho} \cdot \nabla(\chi u) + 2\chi' \partial_t u + \chi'' u$  as non-homogeneous term, and choosing  $s > 0$  sufficiently large, we obtain

$$\begin{aligned} & \int_Q (s|\nabla_{x,t}(\chi u)|^2 + s^3|\chi u|^2)e^{2s\varphi} dxdt \\ & \leq C \int_Q |\frac{\chi}{\rho}(\rho\partial_t^2 u - \operatorname{div}(p\nabla u))|^2 e^{2s\varphi} dxdt \\ & \quad + C \int_Q |2\chi' \partial_t u + \chi'' u|^2 e^{2s\varphi} dxdt + C \int_{-T}^T \int_{\omega} (s|\partial_t(\chi u)|^2 + s^3|\chi u|^2)e^{2s\varphi} dxdt \\ & \leq C \int_Q |\rho\partial_t^2 u - \operatorname{div}(p\nabla u)|^2 e^{2s\varphi} dxdt \\ & \quad + Ce^{2s(d_0 - \varepsilon_0)} \|u\|_{H^1(Q)}^2 + C \int_{-T}^T \int_{\omega} (s|\partial_t u|^2 + s^3|u|^2)e^{2s\varphi} dxdt. \end{aligned}$$

At the last inequality, we used the same argument as the second term on the right-hand side of (2.12). Substituting this in the first term on the right-hand side of (2.12), we complete the proof of Lemma 2.2. □

We conclude this section with a Carleman estimate for a first-order partial differential equation.

**Lemma 2.3.** *Let  $A \in (C^1(\overline{\Omega}))^n$  and  $B \in C^1(\overline{\Omega})$ , and let*

$$Qf := A(x) \cdot \nabla f(x) + B(x)f, \quad f \in H^1(\Omega).$$

We assume

$$(A(x) \cdot (x - x_0)) \neq 0, \quad x \in \overline{\Omega}. \quad (2.13)$$

Then there exist constants  $s_0 > 0$  and  $C > 0$  such that

$$\int_{\Omega} s^2 |f|^2 e^{2s\varphi(x,0)} dx \leq C \int_{\Omega} |Qf|^2 e^{2s\varphi(x,0)} dx \quad (2.14)$$

for  $s > s_0$  and  $f \in H_0^1(\Omega)$  and

$$\int_{\Omega} s^2 (|f|^2 + |\nabla f|^2) e^{2s\varphi(x,0)} dx \leq C \int_{\Omega} (|Qf|^2 + |\nabla(Qf)|^2) e^{2s\varphi(x,0)} dx \quad (2.15)$$

for  $s > s_0$  and  $f \in H_0^2(\Omega)$ .

The proof can be done directly by integration by parts, and we refer for example to Lemma 2.4 in Bellassoued, Imanuvilov and Yamamoto [14].

### 3 Proofs of Theorems 1.1 and 1.2

#### 3.1 Proof of Theorem 1.1

We divide the proof into three steps. The argument in the second step is a simplification of the corresponding part in [27], while the energy estimate (3.16) in the third step modifies the argument towards the Lipschitz stability in [37].

**the first step: Even extension in  $t$ .**

We set

$$y(a)(x, t) = u(p, \rho, a, h)(x, t) - u(q, \sigma, a, h)(x, t), \quad R(x, t) = u(q, \sigma, a, h)(x, t),$$

and we write  $y$  in place of  $y(a)$ . We define

$$f(x) = p(x) - q(x), \quad g(x) = \rho(x) - \sigma(x), \quad x \in \Omega, \quad 0 < t < T. \quad (3.1)$$

Then we have

$$\rho \partial_t^2 y(x, t) - \operatorname{div}(p(x) \nabla y(x, t)) = \operatorname{div}(f(x) \nabla R) - g \partial_t^2 R(x, t) \quad \text{in } \Omega \times (0, T), \quad (3.2)$$

and

$$y(x, 0) = \partial_t y(x, 0) = 0, \quad x \in \Omega, \quad y|_{\partial\Omega} = 0. \quad (3.3)$$

We take the even extensions of the functions  $R(x, t)$ ,  $y(x, t)$  on  $t \in (-T, 0)$ . For simplicity, we denote the extended functions by the same notations  $R(x, t)$ ,  $y(x, t)$ . Since  $y \in W^{4, \infty}(\Omega \times (0, T))$ ,  $y(\cdot, 0) = \partial_t y(\cdot, 0) = 0$  and  $\partial_t \nabla R(\cdot, 0) = 0$  by  $\partial_t u(q, \sigma, a, h)(\cdot, 0) = 0$  in  $\Omega$ , we see that  $(\partial_t^3 R)(\cdot, 0) = (\partial_t^3 y)(\cdot, 0) = 0$  in  $\Omega$ , and so  $R \in W^{4, \infty}(Q)$ ,

$$y \in W^{4, \infty}(Q)$$

and

$$\left\{ \begin{array}{l} \rho \partial_t^2 y(x, t) - \operatorname{div}(p(x) \nabla y(x, t)) = \operatorname{div}(f(x) \nabla R) - g \partial_t^2 R(x, t) \quad \text{in } Q, \\ y(x, 0) = \partial_t y(x, 0) = 0, \quad x \in \Omega, \\ y = 0 \quad \text{on } \partial\Omega \times (-T, T). \end{array} \right. \quad (3.4)$$

We set

$$y_1 = y_1(a) = \partial_t y(a), \quad y_2 = y_2(a) = \partial_t^2 y(a). \quad (3.5)$$

Henceforth we write  $y_1$  and  $y_2$  in place of  $y_1(a)$  and  $y_2(a)$  when there is no fear of confusion. Then

$$\partial_t^2 R(x, 0) = \partial_t^2 u(q, \sigma, a, h)(x, 0) = \frac{1}{\sigma} \operatorname{div}(q(x) \nabla u(q, \sigma, a, h))|_{t=0} = \frac{\operatorname{div}(q \nabla a)}{\sigma}$$

and  $\partial_t y_2(x, 0) = \partial_t^3 y(x, 0) = 0$  for  $x \in \Omega$ , because we can differentiate the first equation in (3.4) and substitute  $t = 0$  in terms of  $y \in W^{4, \infty}(Q)$ . Hence we have

$$\left\{ \begin{array}{l} \rho \partial_t^2 y_1(x, t) - \operatorname{div}(p(x) \nabla y_1(x, t)) = \operatorname{div}(f(x) \nabla \partial_t R) - g \partial_t^3 R =: G_1 \quad \text{in } Q, \\ y_1(x, 0) = 0, \\ \partial_t y_1(x, 0) = \frac{1}{\rho} \operatorname{div}(f \nabla a) - g \frac{\operatorname{div}(q \nabla a)}{\rho \sigma}, \\ y_1 = 0 \quad \text{on } \partial\Omega \times (-T, T) \end{array} \right. \quad (3.6)$$

and

$$\left\{ \begin{array}{l} \rho \partial_t^2 y_2(x, t) - \operatorname{div}(p(x) \nabla y_2(x, t)) = \operatorname{div}(f(x) \nabla \partial_t^2 R) - g \partial_t^4 R =: G_2 \quad \text{in } Q, \\ y_2(x, 0) = \frac{1}{\rho} \operatorname{div}(f \nabla a) - g \frac{\operatorname{div}(q \nabla a)}{\rho \sigma}, \\ \partial_t y_2(x, 0) = 0, \quad x \in \Omega, \\ y_2 = 0 \quad \text{on } \partial\Omega \times (-T, T). \end{array} \right. \quad (3.7)$$

**the second step: Weighted energy estimate and Carleman estimate.**

Let  $k = 1, 2$ . First, by multiplying the first equations in (3.6) and (3.7) by  $2\partial_t y_k$ , we can readily see

$$\partial_t(\rho|\partial_t y_k|^2 + p|\nabla y_k|^2) - \operatorname{div}(2p(\partial_t y_k) \nabla y_k) = 2(\partial_t y_k) G_k \quad \text{in } Q. \quad (3.8)$$

Multiplying (3.8) by  $\chi(t)e^{2s\varphi}$  and integrating by parts over  $\Omega \times (-T, 0)$ , we have

$$\begin{aligned} & \int_{-T}^0 \int_{\Omega} \{ \chi e^{2s\varphi} \partial_t(\rho|\partial_t y_k|^2) + \chi e^{2s\varphi} \partial_t(p|\nabla y_k|^2) \} dx dt \\ & - \int_{-T}^0 \int_{\omega} \chi e^{2s\varphi} \operatorname{div}(2p(\partial_t y_k) \nabla y_k) dx dt = \int_{-T}^0 \int_{\Omega} \chi e^{2s\varphi} G_k 2(\partial_t y_k) dx dt. \end{aligned} \quad (3.9)$$

For  $k = 2$ , by  $y_2|_{\partial\Omega} = 0$ ,  $\chi(-T) = 0$  and the initial condition of  $y_2$ , we have

$$\begin{aligned} & [\text{the left-hand side of (3.9)}] \\ & = \int_{\Omega} [\chi e^{2s\varphi} \rho |\partial_t y_2|^2]_{t=-T}^{t=0} dx - \int_{-T}^0 \int_{\Omega} (\chi' + 2s\chi \partial_t \varphi) \rho |\partial_t y_2|^2 e^{2s\varphi} dx dt \\ & + \int_{\Omega} [\chi e^{2s\varphi} p |\nabla y_2|^2]_{t=-T}^{t=0} dx - \int_{-T}^0 \int_{\Omega} (\chi' + 2s\chi \partial_t \varphi) p |\nabla y_2|^2 e^{2s\varphi} dx dt \\ & + \int_{-T}^0 \int_{\Omega} 2s\chi (\nabla \varphi \cdot \nabla y_2) 2p(\partial_t y_2) e^{2s\varphi} dx dt \\ & \geq \int_{\Omega} p |\nabla y_2(x, 0)|^2 e^{2s\varphi(x, 0)} dx - C \int_Q s |\nabla_{x,t} y_2|^2 e^{2s\varphi} dx dt. \end{aligned}$$

Here we augmented the integral over  $\Omega \times (-T, 0)$  to  $Q := \Omega \times (-T, T)$ , and used  $|\chi' + 2s\chi \partial_t \varphi| \leq Cs$  in  $Q$  and

$$|2s\chi (\nabla \varphi \cdot \nabla y_2) \partial_t y_2| \leq Cs |\nabla y_2| |\partial_t y_2| \leq Cs |\nabla_{x,t} y_2|^2 \quad \text{in } Q.$$

Moreover

$$[\text{the right-hand side of (3.9)}] \leq C \int_Q |G_2|^2 e^{2s\varphi} dx dt + C \int_Q s |\partial_t y_2|^2 e^{2s\varphi} dx dt. \quad (3.10)$$

Therefore (3.9) and (3.10) yield

$$\int_{\Omega} |\nabla y_2(x, 0)|^2 e^{2s\varphi(x, 0)} dx \leq C \int_Q |G_2|^2 e^{2s\varphi} dx dt + C \int_Q s |\nabla_{x,t} y_2|^2 e^{2s\varphi} dx dt. \quad (3.11)$$

Applying Lemma 2.2 to (3.7) and substituting it into (3.11), we obtain

$$\int_{\Omega} |\nabla y_2(x, 0)|^2 e^{2s\varphi(x, 0)} dx \leq C \int_Q |G_2|^2 e^{2s\varphi} dx dt + C s^3 e^{2s(d_0 - \varepsilon_0)} \|y_2\|_{H^1(Q)}^2 + C D_2^2 \quad (3.12)$$

for  $s \geq s_0$ . Here and henceforth we set

$$D_k^2 := s^3 e^{Cs} \|y_k\|_{H^1(-T, T; L^2(\omega))}^2, \quad k = 1, 2. \quad (3.13)$$

For  $k = 1$ , we can similarly argue to have

$$\begin{aligned} \int_{\Omega} |y_2(x, 0)|^2 e^{2s\varphi(x, 0)} dx &= \int_{\Omega} |\partial_t y_1(x, 0)|^2 e^{2s\varphi(x, 0)} dx \\ &\leq C \int_Q |G_1|^2 e^{2s\varphi} dx dt + C s^3 e^{2s(d_0 - \varepsilon_0)} \|y_1\|_{H^1(Q)}^2 + C D_1^2 \end{aligned} \quad (3.14)$$

Hence (3.12) and (3.14) imply

$$\begin{aligned} &\int_{\Omega} (|\nabla y_2(x, 0)|^2 + |y_2(x, 0)|^2) e^{2s\varphi(x, 0)} dx \\ &\leq C \int_Q (|G_1|^2 + |G_2|^2) e^{2s\varphi} dx dt + C s^3 e^{2s(d_0 - \varepsilon_0)} (\|y_1\|_{H^1(Q)}^2 + \|y_2\|_{H^1(Q)}^2) \\ &\quad + C (D_1^2 + D_2^2) \end{aligned} \quad (3.15)$$

for  $s \geq s_0$ .

**the third step: Energy estimate for  $\|y_1\|_{H^1(Q)}^2$  and  $\|y_2\|_{H^1(Q)}^2$ .**

Applying a usual energy estimate to (3.6) and (3.7), in terms of the Poincaré inequality, we have

$$\begin{aligned} &\int_{\Omega} (|\nabla_{x,t} y_k(x, t)|^2 + |y_k(x, t)|^2) dx \\ &\leq C \int_{\Omega} (|\nabla_{x,t} y_k(x, 0)|^2 + |y_k(x, 0)|^2) dx + C \int_{-T}^T \int_{\Omega} |G_k|^2 dx dt, \quad k = 1, 2, \end{aligned}$$

for  $-T \leq t \leq T$ . Consequently

$$\|y_k\|_{H^1(Q)}^2 \leq C \int_{\Omega} (|\nabla_{x,t} y_k(x, 0)|^2 + |y_k(x, 0)|^2) dx + C \int_Q |G_k|^2 dx dt, \quad k = 1, 2. \quad (3.16)$$

Substituting (3.16) into (3.15) and using  $e^{2s\varphi} \geq 1$ , we obtain

$$\begin{aligned} &\int_{\Omega} (|\nabla y_2(x, 0)|^2 + |y_2(x, 0)|^2) e^{2s\varphi(x, 0)} dx \\ &\leq C \int_Q (|G_1|^2 + |G_2|^2) e^{2s\varphi} dx dt + C s^3 e^{2s(d_0 - \varepsilon_0)} \int_{\Omega} (|\nabla y_2(x, 0)|^2 + |y_2(x, 0)|^2) dx \end{aligned}$$

$$+Cs^3e^{2s(d_0-\varepsilon_0)} \int_Q (|G_1|^2 + |G_2|^2) dxdt + C(D_1^2 + D_2^2),$$

that is,

$$\begin{aligned} & \int_{\Omega} (|\nabla y_2(x,0)|^2 + |y_2(x,0)|^2) e^{2s\varphi(x,0)} (1 - Cs^3e^{2s(d_0-\varepsilon_0-\varphi(x,0))}) dx \\ & \leq Cs^3e^{2s(d_0-\varepsilon_0)} \int_Q (|G_1|^2 + |G_2|^2) dxdt + C \int_Q (|G_1|^2 + |G_2|^2) e^{2s\varphi} dxdt \\ & + C(D_1^2 + D_2^2), \end{aligned}$$

By (2.8), choosing  $s > 0$  sufficiently large, we have

$$1 - Cs^3e^{2s(d_0-\varepsilon_0-\varphi(x,0))} \geq 1 - Cs^3e^{-4\varepsilon_0s} \geq \frac{1}{2}.$$

Hence

$$\begin{aligned} & \int_{\Omega} (|\nabla y_2(x,0)|^2 + |y_2(x,0)|^2) e^{2s\varphi(x,0)} dx \\ & \leq Cs^3e^{2s(d_0-\varepsilon_0)} \int_Q (|G_1|^2 + |G_2|^2) dxdt + C \int_Q (|G_1|^2 + |G_2|^2) e^{2s\varphi} dxdt + C(D_1^2 + D_2^2) \end{aligned}$$

for all large  $s > 0$ . By the definitions of  $G_1$  and  $G_2$  in (3.6) and (3.7), we see that

$$\sum_{k=1}^2 |G_k|^2 \leq C(|\nabla f|^2 + |f|^2 + |g|^2) \quad \text{in } Q.$$

Consequently, recalling (3.5):  $y_1 = y_1(a)$  and  $y_2 = y_2(a)$ , we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla y_2(a)(x,0)|^2 e^{2s\varphi(x,0)} dx \tag{3.17} \\ & \leq Cs^3e^{2s(d_0-\varepsilon_0)} \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) dxdt + C \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dxdt \\ & + C(D_1^2 + D_2^2) \end{aligned}$$

Substituting (3.16) into (3.14), we can similarly argue to have

$$\begin{aligned} & \int_{\Omega} |y_2(a)(x,0)|^2 e^{2s\varphi(x,0)} dx \tag{3.18} \\ & \leq Cs^3e^{2s(d_0-\varepsilon_0)} \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) dxdt + C \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dxdt \\ & + CD_1^2 \end{aligned}$$

for all large  $s > 0$ .

Setting  $a = a_1, a_2$ , by the initial condition in (3.7), we see

$$\rho y_2(a_\ell)(x,0) = \operatorname{div}(f \nabla a_\ell) - \frac{\operatorname{div}(q \nabla a_\ell)}{\sigma} g, \quad \ell = 1, 2. \tag{3.19}$$

Then, eliminating  $g$  in the two equations in (3.19), we obtain

$$\begin{aligned} & (\operatorname{div}(q\nabla a_2)\nabla a_1 - \operatorname{div}(q\nabla a_1)\nabla a_2) \cdot \nabla f + ((\operatorname{div}(q\nabla a_2)\Delta a_1 - (\operatorname{div}(q\nabla a_1)\Delta a_2))f \\ & = \rho \operatorname{div}(q\nabla a_2)y_2(a_1)(x, 0) - \rho \operatorname{div}(q\nabla a_1)y_2(a_2)(x, 0) \quad \text{in } Q. \end{aligned}$$

Applying (2.15) in Lemma 2.3 to this first-order equation in  $f$ , by the second condition in (1.9), we have

$$\begin{aligned} & s^2 \int_{\Omega} (|\nabla f|^2 + |f|^2) e^{2s\varphi(x,0)} dx \tag{3.20} \\ & \leq \int_{\Omega} |\operatorname{div}(q\nabla a_2)y_2(a_1)(x, 0) - \operatorname{div}(q\nabla a_1)y_2(a_2)(x, 0)|^2 e^{2s\varphi(x,0)} dx \\ & + C \int_{\Omega} |\nabla(\operatorname{div}(q\nabla a_2)y_2(a_1)(x, 0) - \operatorname{div}(q\nabla a_1)y_2(a_2)(x, 0))|^2 e^{2s\varphi(x,0)} dx \\ & \leq C \int_{\Omega} \left( \sum_{\ell=1}^2 (|\nabla y_2(a_{\ell})(x, 0)|^2 + |y_2(a_{\ell})(x, 0)|^2) \right) e^{2s\varphi(x,0)} dx. \end{aligned}$$

Moreover, assuming that the first condition in (1.9) holds with  $\ell = 1$  for example, we have

$$g = \frac{\sigma}{\operatorname{div}(q\nabla a_1)} (\operatorname{div}(f\nabla a_1) - \rho y_2(a_1)(x, 0)) \quad \text{on } \overline{\Omega},$$

and so

$$|g(x)| \leq C(|\nabla f(x)| + |f(x)| + |y_2(a_1)(x, 0)|), \quad x \in \overline{\Omega}.$$

Hence, applying (3.20) and (3.17)-(3.18) for  $y_2(a_1)(x, 0)$  and  $y_2(a_2)(x, 0)$ , we obtain

$$\begin{aligned} & \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} dx \tag{3.21} \\ & \leq C s^3 e^{2s(d_0 - \varepsilon_0)} \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) dx \\ & + C \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dx dt + C \tilde{D}^2. \end{aligned}$$

Here we used  $|y_k(x, -t)| = |y_k(x, t)|$ ,  $k = 1, 2$  which is seen by the even extension of  $y(\cdot, t)$  in  $t$ , and recall (3.13), and we set

$$\tilde{D}^2 := \sum_{\ell=1}^2 \|u(p, \rho, a_{\ell}, h_{\ell}) - u(q, \sigma, a_{\ell}, h_{\ell})\|_{H^3(0, T; L^2(\omega))}^2. \tag{3.22}$$

We will estimate the second term on the right-hand side of (3.21) as follows.

$$\begin{aligned} & \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dx dt \\ & = \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} \left( \int_{-T}^T e^{2s(\varphi(x,t) - \varphi(x,0))} dt \right) dx. \end{aligned}$$

Since

$$\begin{aligned}\varphi(x, t) - \varphi(x, 0) &= e^{\lambda|x-x_0|^2} (e^{-\lambda\beta t^2} - 1) \\ &\leq -e^{\lambda \min_{x \in \bar{\Omega}} |x-x_0|^2} (1 - e^{-\lambda\beta t^2}) \leq -C_0(1 - e^{-\lambda\beta t^2}) \quad \text{in } Q,\end{aligned}$$

we have

$$\int_{-T}^T e^{2s(\varphi(x,t) - \varphi(x,0))} dt \leq \int_{-T}^T \exp(-2sC_0(1 - e^{-\lambda\beta t^2})) dt = o(1)$$

as  $s \rightarrow \infty$ , where we used the Lebesgue convergence theorem. Therefore

$$\int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dx dt \leq o(1) \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} dx$$

as  $s \rightarrow \infty$ , and choosing  $s > 0$  sufficiently large, we can absorb the second term on the right-hand side of (3.21) into the left-hand side. By (2.8), we have  $e^{2s\varphi(x,0)} \geq e^{2s(d_0 + \varepsilon_0)}$ , so that from (3.21) we obtain

$$\begin{aligned}&e^{2s(d_0 + \varepsilon_0)} \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) dx \\ &\leq C s^3 e^{2s(d_0 - \varepsilon_0)} \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) dx + C \tilde{D}^2\end{aligned}$$

for all large  $s > 0$ . For large  $s > 0$ , we see that  $e^{2s(d_0 + \varepsilon_0)} - C s^3 e^{2s(d_0 - \varepsilon_0)} > 0$ . Hence fixing such  $s > 0$ , we reach

$$\int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} dx \leq C \tilde{D}^2. \quad (3.23)$$

By the definition (3.22) of  $\tilde{D}^2$ , the proof of Theorem 1.1 is completed.

### 3.2 Proof of Theorem 1.2.

Again we set

$$\begin{aligned}\rho y_2(a_\ell)(x, 0) &= \operatorname{div}(f \nabla a_\ell) - \frac{\operatorname{div}(q \nabla a_\ell)}{\sigma} g \\ &= \sum_{k=1}^n (\partial_k a_\ell) \partial_k f + (\Delta a_\ell) f - \frac{\operatorname{div}(q \nabla a_\ell)}{\sigma} g, \quad \ell = 1, \dots, n+1.\end{aligned}$$

that is,

$$\sum_{k=1}^n (\partial_k a_\ell) \partial_k f - \frac{\operatorname{div}(q \nabla a_\ell)}{\sigma} g = \rho y_2(a_\ell)(x, 0) - (\Delta a_\ell) f, \quad \ell = 1, \dots, n+1. \quad (3.24)$$

We rewrite (3.24) as a linear system with respect to  $(n+1)$  unknowns  $\partial_1 f, \dots, \partial_n f, g$ :

$$\begin{pmatrix} \partial_1 a_1 & \cdots & \partial_n a_1 & -\frac{1}{\sigma} \sum_{k=1}^n (\partial_k q) \partial_k a_1 - \frac{q \Delta a_1}{\sigma} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 a_{n+1} & \cdots & \partial_n a_{n+1} & -\frac{1}{\sigma} \sum_{k=1}^n (\partial_k q) \partial_k a_{n+1} - \frac{q \Delta a_{n+1}}{\sigma} \end{pmatrix} \begin{pmatrix} \partial_1 f \\ \vdots \\ \partial_n f \\ g \end{pmatrix}$$

$$= \begin{pmatrix} \rho y_2(a_1)(x, 0) - (\Delta a_1)f \\ \vdots \\ \rho y_2(a_{n+1})(x, 0) - (\Delta a_{n+1})f \end{pmatrix}.$$

In the coefficient matrix, multiplying the  $j$ -th column by  $\frac{1}{\sigma}\partial_j q$ ,  $j = 1, 2, \dots, n$  and adding them to the  $(n+1)$ -th column, we obtain

$$\begin{aligned} & \text{[the determinant of the coefficient matrix]} \\ & = \det \begin{pmatrix} \partial_1 a_1 & \cdots & \partial_n a_1 & -\frac{q\Delta a_1}{\sigma} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 a_{n+1} & \cdots & \partial_n a_{n+1} & -\frac{q\Delta a_{n+1}}{\sigma} \end{pmatrix} \\ & = -\frac{q}{\sigma} \det \begin{pmatrix} \partial_1 a_1 & \cdots & \partial_n a_1 & \Delta a_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 a_{n+1} & \cdots & \partial_n a_{n+1} & \Delta a_{n+1} \end{pmatrix} \text{ on } \bar{\Omega}. \end{aligned}$$

Therefore by the assumption (1.14), there exists a constant  $C > 0$ , independent of choices of  $(p, \rho)$  and  $(q, \sigma)$ , such that

$$|\nabla f(x)|^2 + |g(x)|^2 \leq C \left( \sum_{\ell=1}^{n+1} |\rho y_2(a_\ell)(x, 0)|^2 + |f(x)|^2 \right), \quad x \in \bar{\Omega},$$

and so

$$\int_{\Omega} (|\nabla f|^2 + |g|^2) e^{2s\varphi(x,0)} dx \leq C \int_{\Omega} \sum_{\ell=1}^{n+1} |y_2(a_\ell)(x, 0)|^2 e^{2s\varphi(x,0)} dx + \int_{\Omega} |f(x)|^2 e^{2s\varphi(x,0)} dx. \quad (3.25)$$

We consider a first-order partial differential operator:

$$(Q_0 f)(x) = (x - x_0) \cdot \nabla f(x), \quad x \in \Omega. \quad (3.26)$$

By  $x_0 \notin \bar{\Omega}$ , the condition (2.13) is satisfied, and (2.14) in Lemma 2.3 yields

$$\begin{aligned} & s^2 \int_{\Omega} |f(x)|^2 e^{2s\varphi(x,0)} dx \leq C \int_{\Omega} |(x - x_0) \cdot \nabla f(x)|^2 e^{2s\varphi(x,0)} dx \\ & \leq C \int_{\Omega} |\nabla f(x)|^2 e^{2s\varphi(x,0)} dx \end{aligned}$$

for all large  $s > 0$ . Therefore

$$\int_{\Omega} |f(x)|^2 e^{2s\varphi(x,0)} dx \leq \frac{C}{s^2} \int_{\Omega} |\nabla f(x)|^2 e^{2s\varphi(x,0)} dx$$

for all large  $s > 0$ . Substituting this inequality into the second term on the right-hand side of (3.25) and absorbing into the left-hand side by choosing  $s > 0$  large, in terms of (3.18) with  $y_2(a_\ell)$ ,  $\ell = 1, 2, \dots, n+1$ ,

$$\int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} dx \leq C \int_{\Omega} \sum_{\ell=1}^{n+1} |y_2(a_\ell)(x, 0)|^2 e^{2s\varphi(x,0)} dx$$

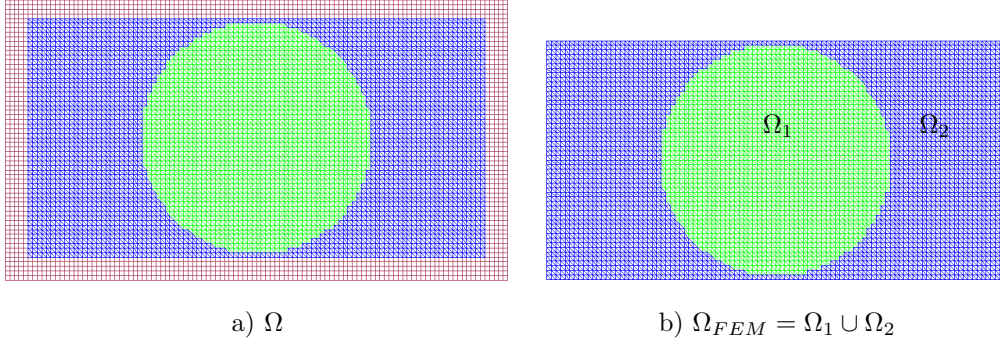


Figure 1: a) Computational mesh used in the domain decomposition of the domain  $\Omega = \Omega_{FEM} \cup \Omega_{FDM}$ . b) The finite element mesh in  $\Omega_{FEM} = \Omega_1 \cup \Omega_2$ .

$$\begin{aligned} &\leq C s^3 e^{2s(d_0 - \varepsilon_0)} \int_{\Omega} (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi(x,0)} dx dt \\ &+ C \int_Q (|\nabla f|^2 + |f|^2 + |g|^2) e^{2s\varphi} dx dt + C \sum_{\ell=1}^{n+1} s^3 e^{C s} \|y_1(a_\ell)\|_{H^1(-T, T; L^2(\omega))}^2 \end{aligned}$$

for all large  $s > 0$ . Similarly to (3.23), we can absorb the first and the second terms on the right-hand side into the left-hand side, so that we can complete the proof of Theorem 1.2.

## 4 Numerical Studies

In this section, we present numerical simulations of the reconstruction of two unknown functions  $\rho(x)$  and  $p(x)$  of the equation (1.1) using the domain decomposition method of [3].

To do that we decompose the computational domain  $\Omega$  into two subregions  $\Omega_{FEM}$  and  $\Omega_{FDM}$  such that  $\Omega = \Omega_{FEM} \cup \Omega_{FDM}$  with two layers of structured overlapping nodes between these domains, see Figure 1 and Figure 2 of [4] for details about communication between  $\Omega_{FEM}$  and  $\Omega_{FDM}$ . We will apply in our computations the finite element method (FEM) in  $\Omega_{FEM}$  and the finite difference method (FDM) in  $\Omega_{FDM}$ . We also decompose the domain  $\Omega_{FEM}$  into two different domains  $\Omega_1, \Omega_2$  such that  $\Omega_{FEM} = \Omega_1 \cup \Omega_2$  which are intersecting only by their boundaries, see Figure 1. We use the domain decomposition approach in our computations since it is efficiently implemented in the high performance software package WavES [47] using C++ and PETSc [45]. For further details about construction of  $\Omega_{FDM}$  and  $\Omega_{FEM}$  domains as well as the domain decomposition method we refer to [3].

The boundary  $\partial\Omega$  of the domain  $\Omega$  is such that  $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega \cup \partial_3\Omega$  where  $\partial_1\Omega$  and  $\partial_2\Omega$  are, respectively, top and bottom parts of  $\Omega$ , and  $\partial_3\Omega$  is the union of left and right sides of this domain. We will collect time-dependent observations  $\Gamma_1 := \partial_1\Omega \times (0, T)$  at the backscattering side  $\partial_1\Omega$  of  $\Omega$ . We also define  $\Gamma_{1,1} := \partial_1\Omega \times (0, t_1]$ ,  $\Gamma_{1,2} := \partial_1\Omega \times (t_1, T)$ ,  $\Gamma_2 := \partial_2\Omega \times (0, T)$  and  $\Gamma_3 := \partial_3\Omega \times (0, T)$ .

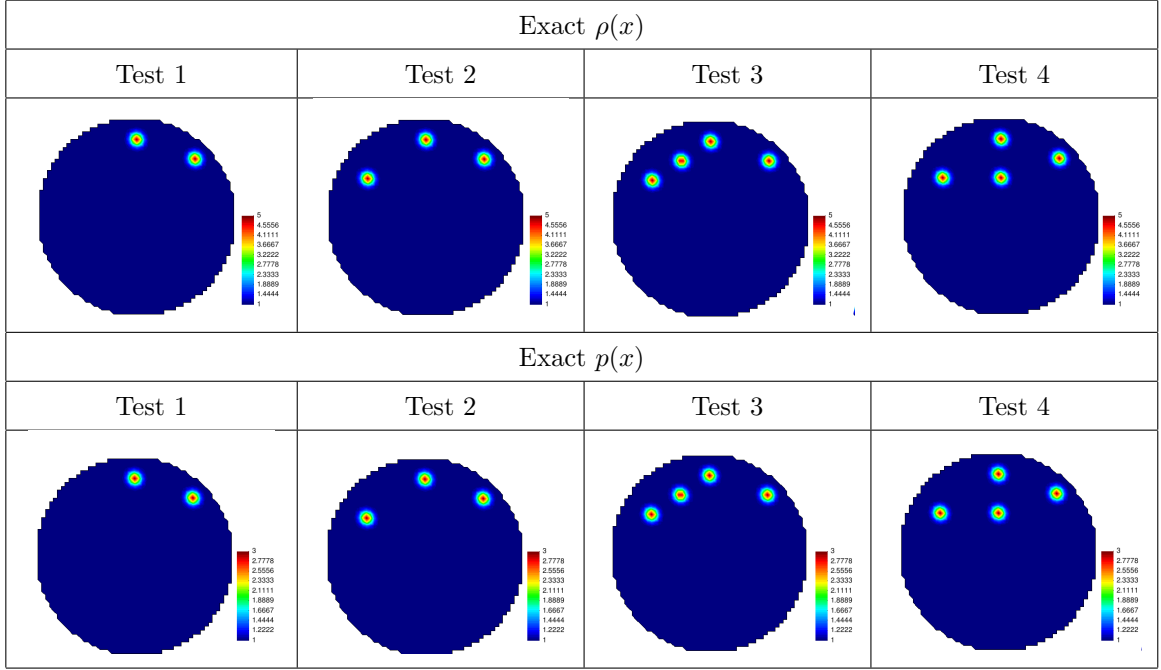


Figure 2: *Exact Gaussian functions  $\rho(x)$  and  $p(x)$  in  $\Omega_1$  in different tests.*

We have used the following model problem in all computations:

$$\begin{aligned}
\rho(x)\partial_t^2 u(x,t) - \operatorname{div}((p(x)\nabla u(x,t))) &= 0 \text{ in } \Omega_T, \\
u(x,0) = a(x), \quad u_t(x,0) &= 0 \text{ in } \Omega, \\
\partial_n u &= f(t) \text{ on } \Gamma_{1,1}, \\
\partial_n u &= -\partial_t u \text{ on } \Gamma_{1,2}, \\
\partial_n u &= -\partial_t u \text{ on } \Gamma_2, \\
\partial_n u &= 0 \text{ on } \Gamma_3.
\end{aligned} \tag{4.1}$$

In (4.1) the function  $f(t)$  represents the single direction of a plane wave which is initialized at  $\partial_1\Omega$  in time  $t = [0, 2.0]$  and is defined as

$$f(t) = \begin{cases} \sin(\omega_f t), & \text{if } t \in \left(0, \frac{2\pi}{\omega_f}\right), \\ 0, & \text{if } t > \frac{2\pi}{\omega_f}. \end{cases} \tag{4.2}$$

We initialize initial condition  $a(x)$  at the boundary  $\partial_1\Omega$  as

$$u(x,0) = f_0(x) = e^{-(x_1^2+x_2^2+x_3^2)} \cdot \cos t|_{t=0} = e^{-(x_1^2+x_2^2+x_3^2)}. \tag{4.3}$$

We assume that both functions  $\rho(x) = p(x) = 1$  are known inside  $\Omega_{FDM} \cup \Omega_2$ . The goal of our numerical tests is to reconstruct simultaneously two smooth functions  $\rho(x), p(x)$  of the domain  $\Omega_1$  of

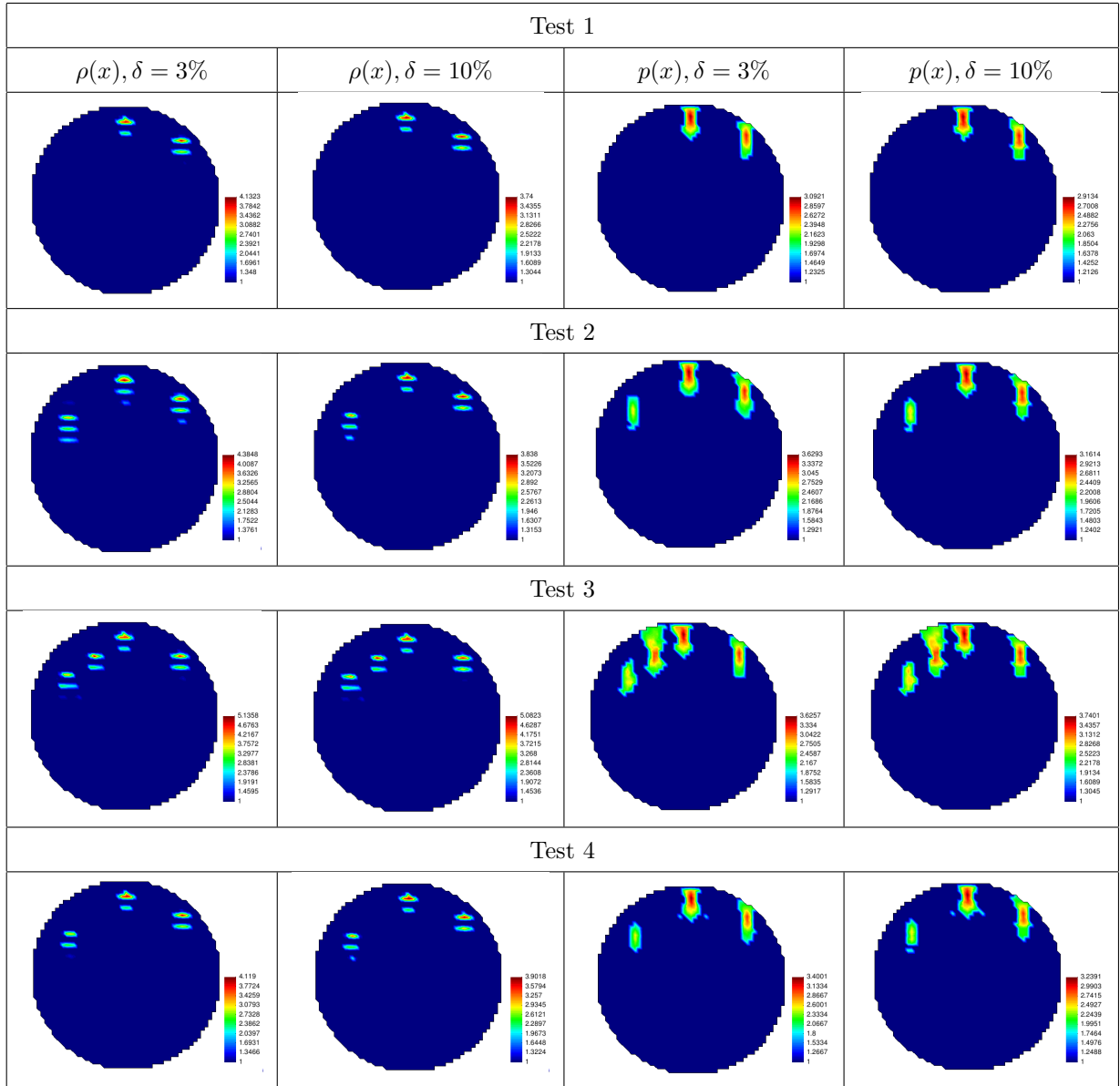


Figure 3: Reconstructions obtained in Tests 1-4 on a coarse mesh for different noise levels  $\delta$  in data.

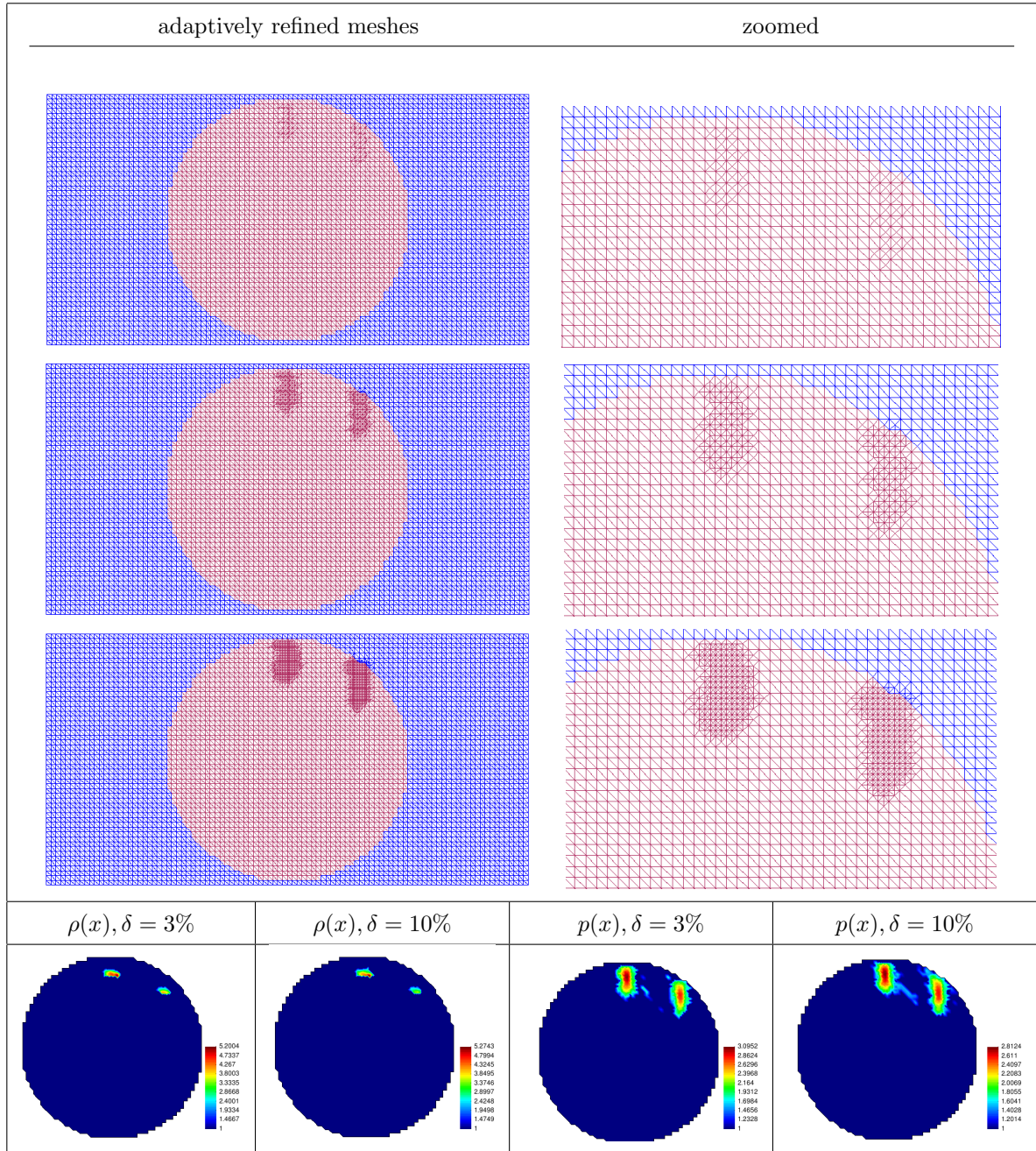


Figure 4: *Test 1: reconstructions obtained on three times adaptively refined mesh for different noise levels  $\delta$  in data.*

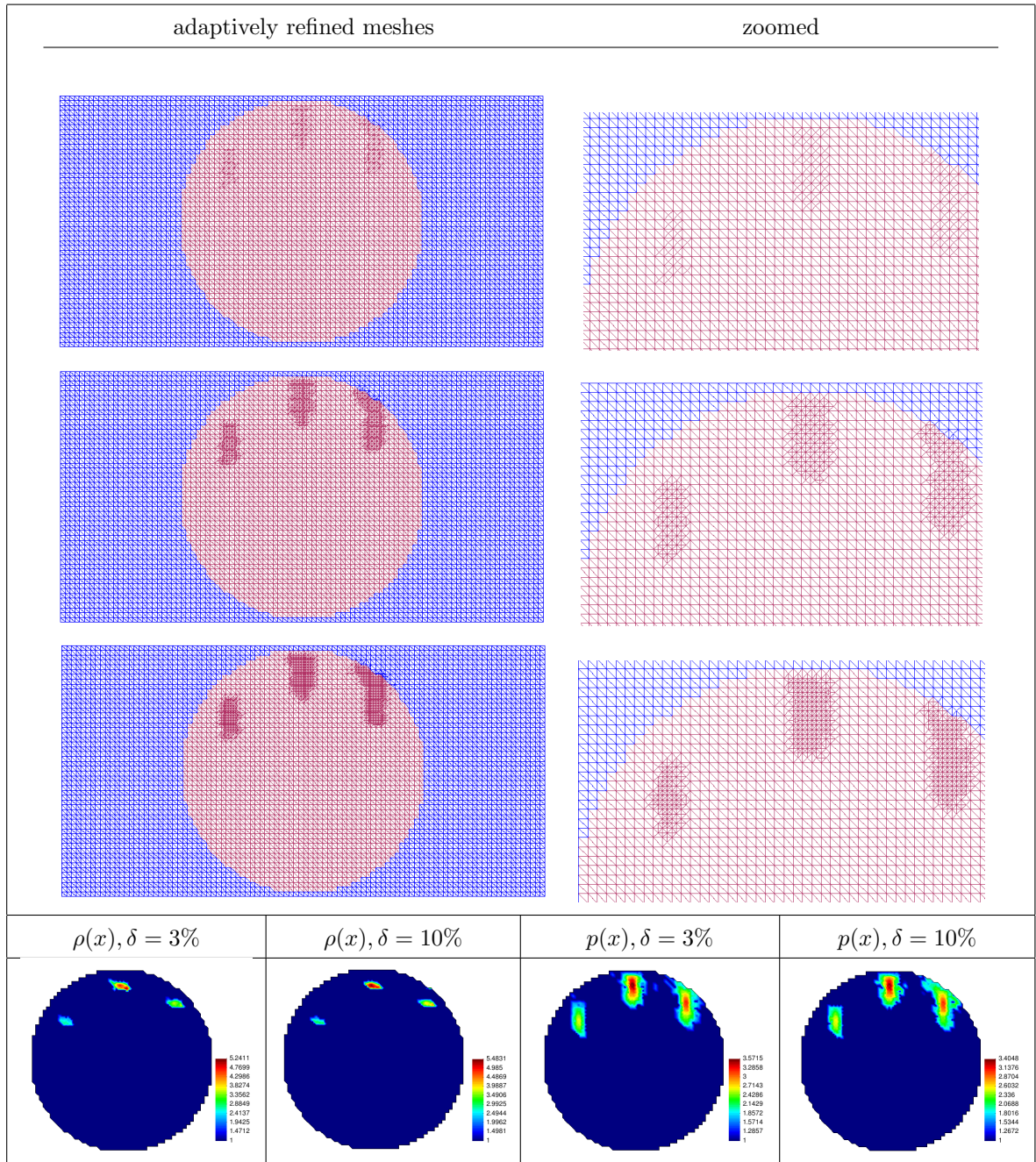


Figure 5: *Test 2: reconstructions obtained on two times adaptively refined mesh for different noise levels  $\delta$  in data.*

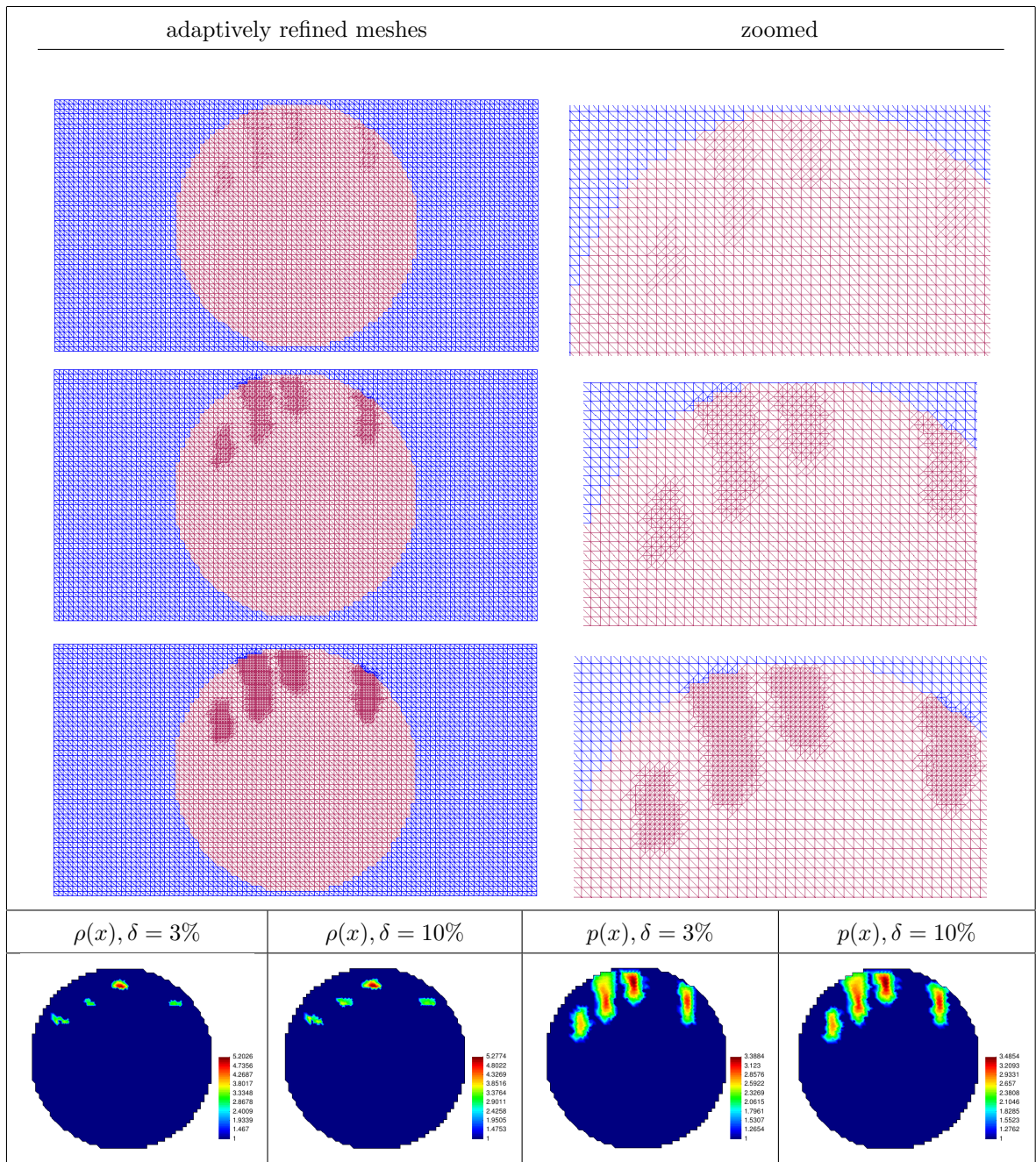


Figure 6: *Test 3: reconstructions obtained on three times adaptively refined mesh for different noise levels  $\delta$  in data.*

Figure 1. The main feature of these functions is that they model inclusions of a very small sizes what can be of practical interest in real-life applications.

We set the dimensionless computational domain  $\Omega$  in the domain decomposition as

$$\Omega = \{x = (x_1, x_2) : x_1 \in (-1.1, 1.1), x_2 \in (-0.62, 0.62)\},$$

and the domain  $\Omega_{FEM}$  as

$$\Omega_{FEM} = \{x = (x_1, x_2) : x_1 \in (-1.0, 1.0), x_2 \in (-0.52, 0.52)\}.$$

We choose the mesh size  $h = 0.02$  in  $\Omega = \Omega_{FEM} \cup \Omega_{FDM}$ , as well as in the overlapping regions between FE/FD domains.

We assume that our two functions  $\rho(x), p(x)$  belongs to the set of admissible parameters

$$\begin{aligned} M_\rho &= \{\rho \in C^2(\bar{\Omega}); 1 \leq \rho(x) \leq 10\}, \\ M_p &= \{p \in C^2(\bar{\Omega}); 1 \leq p(x) \leq 5\}. \end{aligned} \tag{4.4}$$

We define now our coefficient inverse problem which we use in computations.

**Inverse Problem (IP)** *Assume that the functions  $\rho(x), p(x)$  of the model problem (4.1) are unknown. Let these functions satisfy conditions (4.4,) and  $\rho(x) = 1, p(x) = 1$  in the domain  $\Omega \setminus \Omega_{FEM}$ . Determine the functions  $\rho(x), p(x)$  for  $x \in \Omega \setminus \Omega_{FDM}$ , assuming that the following function  $\tilde{u}(x, t)$  is known*

$$u(x, t) = \tilde{u}(x, t), \forall (x, t) \in \Gamma_1. \tag{4.5}$$

To determine both coefficients  $\rho(x), p(x)$  in inverse problem **IP** we minimize the following Tikhonov functional

$$\begin{aligned} J(\rho(x), p(x)) &:= J(u, \rho, p) = \frac{1}{2} \int_{\Gamma_1} (u - \tilde{u})^2 z_\delta(t) ds dt \\ &+ \frac{1}{2} \alpha_1 \int_{\Omega} (\rho - \rho_0)^2 dx + \frac{1}{2} \alpha_2 \int_{\Omega} (p - p_0)^2 dx. \end{aligned} \tag{4.6}$$

Here,  $\tilde{u}$  is the observed function  $u$  in time at the backscattered boundary  $\partial_1 \Omega$ , the function  $u$  satisfy the equations (4.1) and thus depends on  $\rho, p, \rho_0, p_0$  are the initial guesses for  $\rho, p$ , correspondingly, and  $\alpha_i, i = 1, 2$ , are regularization parameters. We take  $\rho_0 = 1, p_0 = 1$  at all points of the computational domain since previous computational works [3, 10, 2, 7] as well as experimental works of [43, 44] have shown that a such choice gives good results of reconstruction. Here,  $z_\delta(t)$  is a cut-off function chosen as in [3, 10, 7]. This function is introduced to ensure the compatibility conditions at  $\bar{\Omega}_T \cap \{t = T\}$  for the adjoint problem, see details in [3, 10, 7].

To solve the minimization problem we take into account conditions (4.4) and introduce the Lagrangian

$$L(v) = J(u, \rho, p) + \int_{\Omega} \int_0^T \lambda \left( \rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div}(p \nabla u) \right) dx dt, \tag{4.7}$$

where  $v = (u, \lambda, \rho, p)$ . Our goal is to find a stationary point of the Lagrangian with respect to  $v$  satisfying  $\forall \bar{v} = (\bar{u}, \bar{\lambda}, \bar{\rho}, \bar{p})$

$$L'(v; \bar{v}) = 0, \quad (4.8)$$

where  $L'(v; \cdot)$  is the Jacobian of  $L$  at  $v$ . To find optimal parameters  $\rho, p$  from (4.8) we use the conjugate gradient method with iterative choice of the regularization parameters  $\alpha_j, j = 1, 2$ , in (4.6). More precisely, in all our computations we choose the regularization parameters iteratively as was proposed in [1], such that  $\alpha_j^n = \alpha_j^0(n+1)^{-q}$ , where  $n$  is the number of iteration in the conjugate gradient method,  $q \in (0, 1)$  and  $\alpha_j^0$  are initial guesses for  $\alpha_j, j = 1, 2$ . Similarly with [35] we take  $\alpha_j = \delta^\zeta$ , where  $\delta$  is the noise level and  $\zeta$  is a small number taken in the interval  $(0, 1)$ . Different techniques for the computation of a regularization parameter are presented in works [23, 30, 31, 46], and checking of performance of these techniques for the solution of our inverse problem can be challenge for our future research.

To generate backscattered data we solve the model problem (4.1) in time  $T = [0, 2.0]$  with the time step  $\tau = 0.002$  which satisfies to the CFL condition [21]. In order to check performance of the reconstruction algorithm we supply simulated backscattered data at  $\partial_1\Omega$  by additive, as in [3, 10, 7], noise  $\delta = 3\%, 10\%$ . Similar results of reconstruction are obtained for random noise and they will be presented in the forthcoming publication.

## 4.1 Test 1

In this test we present numerical results of the simultaneous reconstruction of two functions  $\rho(x)$  and  $p(x)$  given by

$$\begin{aligned} \rho(x) &= 1.0 + 4.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\ &\quad + 4.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001}, \\ p(x) &= 1.0 + 2.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\ &\quad + 2.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001}, \end{aligned} \quad (4.9)$$

which are presented in Figure 2.

Figures 3 show results of the reconstruction on a coarse mesh with additive noise  $\delta = 3\%, 10\%$  in data. We observe that the location of both functions  $\rho, p$  given by (4.9) is imaged correctly. We refer to Table 1 for the reconstruction of the contrast in both functions.

To improve contrast and shape of the reconstructed functions  $\rho(x)$  and  $p(x)$  we run computations again using an adaptive conjugate gradient method similar to the one of [7]. Figure 4 and Table 1 show results of reconstruction on the three times locally refined mesh. We observe that we achieve better contrast for both functions  $\rho(x)$  and  $p(x)$ , as well as better shape for the function  $\rho(x)$ .

Table 1. Computational results of the reconstructions on a coarse and on adaptively refined meshes together with computational errors in the maximal contrast of  $\rho(x), p(x)$  in percents. Here,  $N_\rho^j, N_p^j$  denote the final number of iterations in the conjugate gradient method on  $j$  times refined mesh for reconstructed functions  $\rho$  and  $p$ , respectively.

Coarse mesh							
$\delta = 3\%$				$\delta = 10\%$			
Case	$\max_{\Omega_1} \rho$	error, %	$N_\rho^0$	Case	$\max_{\Omega_1} \rho$	error, %	$N_\rho^0$
Test 1	4.13	17.4	13	Test 1	3.74	25.2	12
Test 2	4.38	12.4	15	Test 2	3.84	23.2	13
Test 3	5.14	2.8	16	Test 3	5.08	1.6	16
Test 4	4.12	17.6	14	Test 4	3.9	22	13
Case	$\max_{\Omega_1} p$	error, %	$N_p^0$	Case	$\max_{\Omega_1} p$	error, %	$N_p^0$
Test 1	3.09	3	13	Test 1	2.9	3.33	12
Test 2	3.63	21	15	Test 2	3.16	5.33	13
Test 3	3.63	21	16	Test 3	3.74	24.67	16
Test 4	3.4	13.3	14	Test 4	3.24	8	13
Adaptively refined mesh							
Case	$\max_{\Omega_1} \rho$	error, %	$N_\rho^j$	Case	$\max_{\Omega_1} \rho$	error, %	$N_\rho^j$
Test 1	5.2	4	$N_\rho^3 = 9$	Test 1	5.3	6	$N_\rho^3 = 7$
Test 2	5.24	4.8	$N_\rho^2 = 6$	Test 2	5.5	10	$N_\rho^2 = 10$
Test 3	5.2	4	$N_\rho^3 = 1$	Test 3	5.28	5.6	$N_\rho^3 = 1$
Test 4	5.5	10	$N_\rho^3 = 8$	Test 4	5.36	7.2	$N_\rho^3 = 8$
Case	$\max_{\Omega_1} p$	error, %	$N_p^j$	Case	$\max_{\Omega_1} p$	error, %	$N_p^j$
Test 1	3.1	3.33	$N_p^3 = 9$	Test 1	2.8	6.67	$N_p^3 = 7$
Test 2	3.57	19	$N_p^2 = 6$	Test 2	3.4	13.3	$N_p^2 = 9$
Test 3	3.39	13	$N_p^3 = 1$	Test 3	3.49	16.3	$N_p^3 = 1$
Test 4	3.4	13.3	$N_p^3 = 14$	Test 4	3.26	8.67	$N_p^3 = 10$

## 4.2 Test 2

In this test we present numerical results of the reconstruction of the functions  $\rho(x)$  and  $p(x)$  given by three Gaussians shown in Figure 2 and given by

$$\begin{aligned}
 \rho(x) &= 1.0 + 4.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
 &\quad + 4.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
 &\quad + 4.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001}, \\
 p(x) &= 1.0 + 2.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
 &\quad + 2.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
 &\quad + 2.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001}.
 \end{aligned} \tag{4.10}$$

Figures 3 show results of the reconstruction on a coarse mesh with additive noise  $\delta = 3\%, 10\%$  in data. We observe that the location of three Gaussians for both functions  $\rho, p$  is imaged correctly, see Table 1 for the reconstruction of contrast in these functions.

To improve contrast and shape of the reconstructed functions  $\rho(x)$  and  $p(x)$  we run computations again using an adaptive conjugate gradient method similar to the one of [7]. Figure 5 and Table 1 show results of reconstruction on the two times locally refined mesh. We observe that we achieve better contrast for both functions  $\rho(x)$  and  $p(x)$ , as well as better shape for the function  $\rho(x)$ . Results on the three times refined mesh were similar to the results obtained on a two times refined mesh, and we are not presenting them here.

## 4.3 Test 3

This test presents numerical results of the reconstruction of the functions  $\rho(x)$  and  $p(x)$  given by four different Gaussians shown in Figure 2 and given by

$$\begin{aligned}
 \rho(x) &= 1.0 + 4.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
 &\quad + 4.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
 &\quad + 4.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001} \\
 &\quad + 4.0 \cdot e^{-((x_1+0.15)^2+(x_2-0.3)^2)/0.001}, \\
 p(x) &= 1.0 + 2.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
 &\quad + 2.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
 &\quad + 2.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001} \\
 &\quad + 2.0 \cdot e^{-((x_1+0.15)^2+(x_2-0.3)^2)/0.001}.
 \end{aligned} \tag{4.11}$$

Figures 3 show results of the reconstruction of four Gaussians on a coarse mesh with additive noise  $\delta = 3\%, 10\%$  in data. We have obtained similar results as in the two previous tests: the location

of four Gaussians for both functions  $\rho, p$  already on a coarse mesh is imaged correctly. However, as follows from the Table 1, the contrast should be improved. Again, to improve the contrast and shape of the Gaussians we run an adaptive conjugate gradient method similar to one of [7]. Figure 6 shows results of reconstruction on the three times locally refined mesh. Using Table 1 we observe that we achieve better contrast for both functions  $\rho(x)$  and  $p(x)$ , as well as better shape for the function  $\rho(x)$ .

#### 4.4 Test 4

In this test we tried to reconstruct four Gaussians shown in Figure 2 and given by

$$\begin{aligned}
\rho(x) &= 1.0 + 4.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
&\quad + 4.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
&\quad + 4.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001} \\
&\quad + 4.0 \cdot e^{-(x_1^2+(x_2-0.2)^2)/0.001}, \\
p(x) &= 1.0 + 2.0 \cdot e^{-((x_1-0.3)^2+(x_2-0.3)^2)/0.001} \\
&\quad + 2.0 \cdot e^{-(x_1^2+(x_2-0.4)^2)/0.001} \\
&\quad + 2.0 \cdot e^{-((x_1+0.3)^2+(x_2-0.2)^2)/0.001} \\
&\quad + 2.0 \cdot e^{-(x_1^2+(x_2-0.2)^2)/0.001}.
\end{aligned} \tag{4.12}$$

We observe that two Gaussians in this example are located one under another one. Thus, backscattered data from these two Gaussians will be superimposed and thus, we expect to reconstruct only three Gaussians from four.

Figure 3 shows results of the reconstruction of these four Gaussians on a coarse mesh with additive noise  $\delta = 3\%, 10\%$  in data. As expected, we could reconstruct only three Gaussians from four, see Table 1 for reconstruction of the contrast in them. Even application of the adaptive algorithm can not give us the fourth Gaussian. However, the contrast in the reconstructed functions is improved, as in Test 3.

## 5 Conclusions

In this work we present theoretical and numerical studies of the reconstruction of two space-dependent functions  $\rho(x)$  and  $p(x)$  in a hyperbolic problem.

In the theoretical part of this work we derive a local Carleman estimate which allows to obtain a conditional Lipschitz stability inequality for the inverse problem formulated in section 1. This stability is very important for our subsequent numerical reconstruction of the two unknown functions  $\rho(x)$  and  $p(x)$  in the hyperbolic model (4.1).

In the numerical part we present a computational study of the simultaneous reconstruction of two

functions  $\rho(x)$  and  $p(x)$  in a hyperbolic problem (4.1) from backscattered data using an adaptive domain decomposition finite element/difference method similar to one developed in [3, 7]. In our numerical tests, we have obtained stable reconstruction of the location and contrasts of both functions  $\rho(x)$  and  $p(x)$  for noise levels  $\delta = 3\%, 10\%$  in backscattered data. Using results of Table 1 and Figures 4–6 we can conclude, that an adaptive domain decomposition finite element/finite difference algorithm significantly improves qualitative and quantitative results of the reconstruction obtained on a coarse mesh.

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