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► **To cite this version:**

Edoardo Bocchi. Floating structures in shallow water: local well-posedness in the axisymmetric case. 2018. hal-01714437v1

HAL Id: hal-01714437

<https://hal.science/hal-01714437v1>

Preprint submitted on 21 Feb 2018 (v1), last revised 15 Oct 2019 (v3)

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Floating structures in shallow water: local well-posedness in the axisymmetric case

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Abstract

The floating structure problem describes the interaction between surface water waves and a floating body, generally a boat or a wave energy converter. As shown by Lannes in [18] the equations for the fluid motion can be reduced to a set of two evolution equations on the surface elevation and the horizontal discharge. The presence of the object is accounted for by a constraint on the discharge under the object; the pressure exerted by the fluid on this object is then the Lagrange multiplier associated to this constraint. Our goal in this paper is to prove the well-posedness of this fluid-structure interaction problem in the shallow water approximation under the assumption that the flow is axisymmetric without swirl. We write the fluid equations as a quasilinear hyperbolic mixed initial boundary value problem and the solid equation as a second order ODE coupled to the fluid equations. Finally we prove the local in time well-posedness for this coupled problem, provided some compatibility conditions on the initial data are satisfied.

1 Introduction

The floating structure problem was first formulated by John in his two famous papers [14, 15], in which he considered a linear flow. It is a particular example of fluid-structure interaction, when a partially immersed body is floating on the fluid free surface. The fluid is supposed to be incompressible and the flow irrotational, and we can consider both the case of a prescribed solid motion or of a free motion governed by Newton's law for the solid object. In this problem we have to treat two free boundary problems.

The first free boundary problem is the typical water waves problem, which consists in describing the evolution of the fluid surface in contact with the air (or another fluid whose density can be neglected). In the absence of floating bodies, this is the standard water waves problem, which has been studied by many authors in the last years and whose local well-posedness theory is well-known. For instance we refer to Wu [27, 28], Lannes [16, 17], Alazard, Burq and Zuily [2, 3] and Iguchi [12]. A notable formulation is the one introduced by Zakharov, Craig and Sulem [29, 7]: they consider the potential velocity and they remove the dependence on the vertical variable z working with new unknowns, the free surface elevation and the trace of the potential on the free surface.

The second free boundary problem is given by the fact that the portion of the body in contact with the fluid depends on time, so that the contact line between is a free boundary problem. For this difficulty John studied a more simplified problem. He considered a linear model in order to describe the evolution of the free surface waves and he used the potential velocity formulation. Then he assumed that the motion of the solid is of small amplitude and he neglected the variations of the contact line in time. These assumptions permitted him to avoid the free boundary problem associated to the contact line. Moreover he studied a one-dimensional problem (where d is the horizontal dimension). Another way to avoid this free boundary problem is to consider a structure with vertical side-walls and to assume its motion to be only vertical.

Even if John's approach is simplified because it does not take into account nonlinear effects, the linear approach has been used extensively in hydrodynamic engineering. In particular we refer to Cummins who, dealing with ship motion, proposed in [8] his celebrated delay differential equation on the six modes of response: surge, sway, heave, roll, pitch and yaw.

In order to take into account nonlinear effects to better describe the real motion of floating bodies, Lannes proposed a different approach in his recent paper [18]. He modelled the problem not with the velocity potential theory but using a new formulation. In order to remove the dependence on the vertical variable z , he considered the horizontal discharge, i.e. the horizontal component of the velocity field integrated vertically between the free surface and the fluid domain bottom. He showed that the equations of the problem have a "compressible-incompressible" structure, in which the interior pressure exerted by the fluid on the body is a Lagrange multiplier that one can determine via the resolution of a d -dimensional elliptic equation. He also implemented the same approach on asymptotic models, such as the nonlinear shallow water equations and Boussinesq equations.

In this paper we address the two-dimensional floating body problem, where

a cylindrically symmetric structure with vertical side-walls is floating only vertically on an incompressible fluid with irrotational motion. These assumptions on the shape of the solid and its motion permit us to avoid the free boundary problem associated to the contact line and to simplify the problem. Indeed in this case the projection of the portion of the body in contact with the fluid does not depend on time. We suppose that the flow is axisymmetric and without swirl, i.e. we consider a rotation-invariant velocity field with no azimuthal component. Moreover we consider the shallow water regime, which means that the wavelength of the waves are larger than the depth. Consequently, we work with the nonlinear shallow water equations for the flow model, instead of the much more involved free surface Euler equations.

The aim of this paper is to prove the local in time well-posedness of this coupled fluid-structure problem in Sobolev spaces. We need enough regular initial data, satisfying some compatibility conditions, in order to get the solution. We consider here a two-dimensional problem, but the axisymmetry keeps the boundary condition maximally dissipative. We use this condition to get better trace estimates which in general we do not have in a two-dimensional case. With all these assumptions we can reduce the problem to a one-dimensional radial problem, and we adapt the classical theory with a reformulation for weighted spaces. The important point is that, instead of John's model, in this paper we take into account nonlinear terms. On the other hand our model still has some limitations. For instance, one should allow the solid to move also in the horizontal direction and to rotate. In these cases, one cannot bypass the study of the evolution of the contact line; moreover the flow would cease to be axisymmetric. The axisymmetric situation considered here is however relevant and can be used to validate the shallow water approach to the floating body problem: indeed, several experimental data with an axisymmetric geometry are available.

1.1 Outline of the paper

In Section 2 we write the free surface Euler equations with the constraint that the solid must be in contact with the fluid during all the motion, avoiding air holes between them. Then we write the nonlinear shallow water approximation for this floating structure problem using the same formulation as Lannes [18], with the introduction of the horizontal discharge. Once we have assumed the axisymmetry and the absence of swirl, introducing cylindrical coordinates we reformulate the problem to get a one-dimensional set of equations.

In Section 3 we focus on the "fluid part" of the problem. We write the floating structure problem in the exterior domain $r > R$ as a quasilinear hyperbolic

initial boundary value problem, namely

$$\begin{cases} \partial_t h_e + \partial_r q_e + \frac{1}{r} q_e = 0 \\ \partial_t q_e + \partial_r \left(\frac{q_e^2}{h_e} \right) + \frac{q_e^2}{r h_e} + g h_e \partial_r h_e = 0 \\ q_e|_{r=R} = q_i|_{r=R}. \end{cases} \quad (1)$$

where $h_e = \zeta_e + h_0$ and ζ_e are the fluid height ($-h_0$ is the parametrization of the flat bottom) and the free surface in the exterior domain respectively. The exterior horizontal discharge q_e is defined as

$$q_e(t, r) = \int_{-h_0}^{\zeta_e} u_r(t, r, z) dz \quad \text{for } r > R,$$

where u_r is the radial component of the fluid velocity field \mathbf{U} . In the boundary condition in (1) q_i is the interior horizontal discharge, defined as before but for in the interior domain $r < R$.

As usual we first show local in time existence and uniqueness of the solution to the associated linear problem in $L^2(rdr)$. In order to have more regular solutions, the data of the problem must satisfy some compatibility conditions. Then we apply a standard iterative scheme argument on the nonlinear problem in order to find the local in time existence and uniqueness of the solution.

We treat the solid motion in Section 4. We write the vertical component of Newton's law for the conservation of linear momentum, which describes the motion of the structure, as a nonlinear second order ODE on the displacement δ_G between the vertical position of the centre of mass at time t and at its equilibrium. We show that this ODE can be written under the form

$$(m + m_a(\delta_G)) \ddot{\delta}_G(t) = -c \delta_G(t) + c \zeta_e(t, R) + \left(\frac{\mathbf{b}}{h_e^2(t, R)} + \beta(\delta_G) \right) (\dot{\delta}_G(t))^2. \quad (2)$$

In this equation the terms $\zeta_e(t, R)$ and $h_e(t, R)$ are responsible for the coupling with the fluid equations (1). An important point is the presence of the added mass term $m_a(\delta_G)$ which can be explained by the fact that, in order to move, the solid has to accelerate itself and also the portion of fluid around it. We can find the added mass effect also in other fluid-structure interaction problems. In the case of a totally submerged solid we refer to Glass, Sueur and Takahashi [11] and Glass, Munnier and Sueur [10]. It has an important role for the stability of numerical simulations [6].

The value of the elevation of the fluid surface at the boundary $\zeta_e(t, R)$ in (2) is the coupling term with the fluid motion. On the other hand we show that the

boundary condition in (1) can be written as

$$q_e|_{r=R} = q_i|_{r=R} = -\frac{R}{2}\dot{\delta}_G,$$

showing the retro-action of the solid on the fluid motion.

Finally in Section 5 we write the coupled system modelling the problem and we show the local in time existence and uniqueness introducing an iterative scheme on the coupled fluid-structure system, looking for the solution via a fixed point argument, and we get the following result (see Theorem 5.3 for the rigorous statement):

Theorem 1.1. *The coupled system (1) - (2) is local in time well-posed, provided the initial data are regular enough and verify some compatibility conditions.*

In Appendix A we show the details in the case of a non-flat solid bottom, considering that the contact between the solid and the fluid is still on the vertical side-walls, and we derive the corresponding solid motion ODE.

Acknowledgements

I address my sincere gratitude to D. Lannes for the supervision of this paper and his precious advices. I want to acknowledge C. Prange for his useful remarks and T. Iguchi for some interesting conversations. I also want to thank M. Tucsnak, who noticed that, in the configuration we consider in this paper, a correction pressure term must be introduced to have exact energy conservation.

2 Floating structure equations

2.1 Constrained free surface Euler equations

Let us consider a floating body, typically a wave energy converter, with vertical side-walls and a cylindrical symmetry, forced to move only in the vertical direction. We call $C(t)$ the region occupied by the solid at time t , $\partial C(t)$ the boundary and $\partial_w C(t)$ the portion of the boundary in contact with the fluid, called the *wetted surface*. The presence of the solid naturally allows to divide the horizontal plane \mathbb{R}^2 into two regions, the projection \mathcal{I} , of the wetted surface on it, and $\mathcal{E} := \mathbb{R}^2 \setminus \mathcal{I}$. We call them *interior* and *exterior domain* respectively. The boundary $\Gamma := \partial\mathcal{I} = \partial\mathcal{E}$ is called the projection of the *contact line*, where the solid, the fluid and the exterior air interact. For simplicity we call Γ itself the contact line. These domains do not depend on time since the solid is moving only vertically and is assumed to have vertical side-walls. We consider a wetted surface that

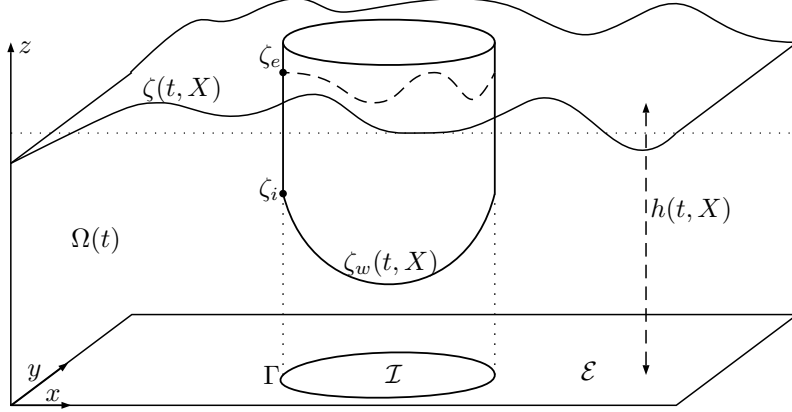


Figure 1: A cylindrically symmetric floating structure with vertical side-walls

can be parametrized by a graph of some function $\zeta_w(t, X)$ for $X \in \mathcal{I}$ and, like in the water waves theory, we assume that the surface of the fluid is the graph of a function $\zeta(t, X)$ for $X \in \mathbb{R}^2$, as shown in Figure 1.

We assume that the fluid is incompressible, irrotational, with constant density ρ and inviscid. For simplicity we consider a flat bottom which can be parametrized by $-h_0$ with $h_0 > 0$ and the fluid domain is

$$\Omega(t) = \{(X, z) \in \mathbb{R}^{2+1} \mid -h_0 < z < \zeta(t, X)\}.$$

Then the motion of the fluid is given by the incompressible Euler equation

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \quad \text{in } \Omega(t) \quad (3)$$

$$\operatorname{div} \mathbf{U} = 0 \quad (4)$$

$$\operatorname{curl} \mathbf{U} = 0. \quad (5)$$

The boundary conditions for the Euler equation on the velocity field \mathbf{U} in the fluid domain are the traditional kinematic equation at the surface and the impermeability condition at the bottom, respectively

$$z = \zeta, \quad \partial_t \zeta - \mathbf{U} \cdot \mathbf{N} = 0 \quad \text{with } \mathbf{N} = \begin{pmatrix} -\nabla \zeta \\ 1 \end{pmatrix} \quad (6)$$

$$z = -h_0, \quad \mathbf{U} \cdot \mathbf{e}_z = 0 \quad (7)$$

We consider a configuration when the fluid is completely attached to the solid. Hence we have the following contact constraint:

$$\zeta(t, X) = \zeta_w(t, X) \quad \text{in } \mathcal{I}.$$

Let us denote the restrictions to the interior domain and the exterior domain of a function f defined on \mathbb{R}^2 as

$$f_i := f|_{\mathcal{I}} \quad f_e := f|_{\mathcal{E}}.$$

According to this notation the contact constraint becomes the following

$$\zeta_i = \zeta_w \quad \text{in } \mathcal{I} \quad (8)$$

Remark 2.1. *Since the solid has vertical side-walls, the free surface is not continuous across Γ , i.e.*

$$\zeta_e \neq \zeta_i \quad \text{on } \Gamma.$$

In the presence of a floating structure we have to change the standard condition on the value of the pressure on the free surface. In the exterior domain it is given by the constant atmospheric pressure P_{atm} , i.e.

$$\underline{P}_e = P_{atm}. \quad (9)$$

with $\underline{P} = P|_{z=\zeta}$. In the interior domain the pressure on the free surface is an unknown of the problem, depending on the dynamics of the solid but we know its value on Γ . Indeed, by integrating the vertical component of Euler's equation (3) between $z = \zeta_i$ and $z = \zeta_e$, we have

$$\underline{P}_i(t, \cdot) = P_{atm} + \rho g(\zeta_e - \zeta_i) + \rho \int_{\zeta_i}^{\zeta_e} (\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w) \quad \text{on } \Gamma, \quad (10)$$

where w is the vertical component of the velocity field \mathbf{U} . The second and the third term do not vanish due to the discontinuity of the free surface on Γ (see Remark 2.1).

Moreover one has the continuity of the normal velocity at the vertical side-walls, i.e.

$$V \cdot \nu = V_C \cdot \nu \quad (11)$$

where ν is the unit normal vector to Γ pointing towards \mathcal{E} , V and V_C are the horizontal velocities of the fluid and the solid respectively.

As in the standard water waves theory we suppose that the height of the fluid $h_e(t)$ does not vanish during all the motion. Hence we have the following assumption:

$$\exists h_m > 0 : h_e(t, X) \geq h_m \quad \forall t \in [0, +\infty), \forall X \in \mathcal{E}. \quad (12)$$

From the physics of the floating structure problem we suppose also that the solid does not touch the bottom of the domain during its motion. This is equivalent to assuming that the height of the fluid $h_i(t, r)$ under the solid does not vanish, i.e.

$$\exists h_{min} > 0 : h_w(t, X) \geq h_{min} \quad \forall t \in [0, +\infty), \forall X \in \mathcal{I}. \quad (13)$$

with $h_w(t, X) = h_i(t, X)$ in \mathcal{I} due to (8). This assumption is completely relevant for the situation investigated here; we refer to [9, 22] (Euler equation) and [19] (shallow water and Green-Naghdi equations) for the analysis of the vanishing depth problem.

2.2 Averaged free surface Euler equations with a floating structure

Because of the difficulty to deal with a moving domain, we want to obtain a set of evolution equations on \mathbb{R}^2 , without the dependence on the vertical variable z . This idea has been used also for the water waves problem without a floating structure by Zakharov-Craig-Sulem (see [29], [7] and [17]), who introduce the trace of the velocity potential on the free surface. In the presence of a floating structure we use another formulation, the one that Lannes implemented in his paper [18]. We will see that this formulation permits us to write directly the boundary condition (11). We define the horizontal discharge

$$Q(t, X) := \int_{-h_0}^{\zeta(t, X)} V(t, X, z) dz = h(t, X) \bar{V}(t, X)$$

where $h(t, X) = h_0 + \zeta(t, X)$ is the fluid height and \bar{V} is the vertical average of V . Then we can reformulate the problem:

Proposition 2.2. *Using the (ζ, Q) -formulation, the water waves problem with a floating structure is modelled by the following hyperbolic system*

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \left(\frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta + \nabla \cdot \mathbf{R} + h \mathbf{a}_{NH}(h, Q) = -\frac{h}{\rho} \nabla \underline{P}, \end{cases} \quad (14)$$

with the “Reynolds” tensor \mathbf{R} and the non-hydrostatic acceleration \mathbf{a}_{NH} as

$$\mathbf{R}(h, Q) = \int_{-h_0}^{\zeta} (V - \bar{V}) \otimes (V - \bar{V}),$$

$$\mathbf{a}_{NH}(h, Q) = \frac{1}{h} \int_{-h_0}^{\zeta} \nabla \left[\int_z^{\zeta} (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U}) \cdot \mathbf{e}_z \right].$$

The surface pressure \underline{P} is given by

$$\underline{P}_e = P_{atm} \quad \text{and} \quad \begin{cases} -\nabla \cdot \left(\frac{h}{\rho} \nabla \underline{P}_i \right) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS}(h, Q) & \text{on } \mathcal{I} \\ \underline{P}_i|_{\Gamma(t)} = P_{atm} + \rho g(\zeta_e - \zeta_i) + P_{NH} & \text{on } \Gamma, \end{cases} \quad (15)$$

where

$$\mathbf{a}_{FS}(h, Q) = \nabla \cdot \left(\nabla \cdot \left(\frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta + \nabla \cdot \mathbf{R} + h \mathbf{a}_{NH}(h, Q) \right),$$

$$P_{NH} = \rho \int_{\zeta_i}^{\zeta_e} (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U}) \cdot \mathbf{e}_z$$

and the transition condition at the contact line is

$$Q_e \cdot \nu = Q_i \cdot \nu \quad \text{on } \Gamma. \quad (16)$$

Remark 2.3. Note that the expressions for \mathbf{R} and \mathbf{a}_{NH} in the statement above involve the velocity field $\mathbf{U} = \begin{pmatrix} V \\ w \end{pmatrix}$. It is shown in [18] that the velocity field is fully determined by the knowledge of ζ and Q , hence the notations $\mathbf{R}(h, Q)$ and $\mathbf{a}_{NH}(h, Q)$.

Sketch of the proof. In order to get (14) we just need to integrate over the vertical coordinate the horizontal component of the Euler equation (3). The condition on the surface in the exterior domain is (9). One can note that, in the interior domain, \underline{P}_i is the Lagrange multiplier associated to the constraint (8). Then, using the standard argument for incompressible Euler, the elliptic system in (15) is obtained by applying the horizontal divergence to (14) in the interior domain. The boundary condition for the elliptic system comes from (10). Finally for (16) we use the definition of Q_e and Q_i and (11) (see Proposition 10 in [18]). \square

Remark 2.4. The fluid energy is

$$E_{fluid} = \frac{\rho}{2} g \int_{\mathbb{R}^2} \zeta^2 + \frac{\rho}{2} \int_{\Omega(t)} |\mathbf{U}|^2.$$

2.3 The shallow water regime

We are interested here in a simplified model of these equations in the shallow water regime (i.e. when the horizontal scale of the problem is much larger than the depth). As in [18], the same approximations as in the case without floating object are made,

$$\mathbf{R} \approx 0 \quad \text{and} \quad \mathbf{a}_{NH} \approx 0.$$

Then we get the following equations:

Proposition 2.5. *The nonlinear shallow water equations with a floating structure in the (ζ, Q) -formulation are*

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot \left(\frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta = -\frac{h}{\rho} \nabla \underline{P}. \end{cases} \quad (17)$$

with the surface pressure \underline{P} given by

$$\underline{P}_e = P_{atm} \quad \text{and} \quad \begin{cases} -\nabla \cdot \left(\frac{h}{\rho} \nabla \underline{P}_i \right) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS}(h, Q) \quad \text{on } \mathcal{I} \\ \underline{P}_i|_{\Gamma(t)} = P_{atm} + \rho g (\zeta_e - \zeta_i) + P_{cor} \quad \text{on } \Gamma, \end{cases} \quad (18)$$

where

$$\mathbf{a}_{FS}(h, Q) = \nabla \cdot \left(\nabla \cdot \left(\frac{1}{h} Q \otimes Q \right) + gh \nabla \zeta \right),$$

coupled with the transition condition at the contact line

$$Q_e \cdot \nu = Q_i \cdot \nu \quad \text{on } \Gamma. \quad (19)$$

Unlike the case that Lannes considered in [18], where the jump of pressure $\underline{P}_i|_{\Gamma(t)} - P_{atm}$ at the boundary of the object is assumed to be only due to the hydrostatic pressure, *i.e.*

$$\underline{P}_i|_{\Gamma(t)} - P_{atm} = \rho g (\zeta_e - \zeta_i)|_{\Gamma(t)},$$

we add here a non-hydrostatic correction term P_{cor} . This corrector is determined later in Proposition 2.7 below to ensure exact energy conservation (the mathematical analysis if we remove this term can be performed in the same way). As for the kinematic condition (6), we have that

$$\partial_t \zeta_w - \underline{U}_w \cdot N_w \quad \text{in } \mathcal{I} \quad \text{with} \quad N_w = \begin{pmatrix} -\nabla \zeta_w \\ 1 \end{pmatrix} \quad (20)$$

where \underline{U}_w is the velocity of the solid on the wetted surface. Let us denote the centre of mass of the solid $G(t) = (X_G(t), z_G(t))$ and $\mathbf{U}_G(t) = (V_G(t), w_G(t))$ its velocity and ω the angular velocity of the solid. From the solid mechanics we have

$$\underline{U}_w = \mathbf{U}_G + \omega \times \mathbf{r}_G \quad \text{with} \quad \mathbf{r}_G(t, X) = \begin{pmatrix} X - X_G(t) \\ \zeta_w(t, X) - z_G(t) \end{pmatrix}.$$

Then (20) gives

$$\partial_t \zeta_w = (\mathbf{U}_G + \omega \times \mathbf{r}_G) \cdot \mathbf{N}_w \quad \text{in } \mathcal{I}. \quad (21)$$

Since we are considering a solid moving only vertically $V_G = 0$ and $\omega = 0$. Taking the time derivative we finally get

$$\partial_t^2 \zeta_w = \dot{w}_G.$$

Because of the linearity of the elliptic problem we can decompose the interior pressure as $\underline{P}_i = \underline{P}_i^I + \underline{P}_i^{II} + \underline{P}_i^{III}$ where:

- \underline{P}_i^I is the pressure we would have in the case of a fixed solid, solution to

$$\begin{cases} -\nabla \cdot \left(\frac{h}{\rho} \nabla \underline{P}_i^I \right) = \mathbf{a}_{FS}(h, Q) & \text{on } \mathcal{I}, \\ \underline{P}_i^I|_{\Gamma(t)} = P_{atm}, \end{cases} \quad (22)$$

where $\mathbf{a}_{FS}(h, Q)$ is the free surface acceleration in the absence of a floating structure;

- \underline{P}_i^{II} is the part of the pressure due to the acceleration of the solid

$$\begin{cases} -\nabla \cdot \left(\frac{h}{\rho} \nabla \underline{P}_i^{II} \right) = -\dot{w}_G, & \text{on } \mathcal{I}, \\ \underline{P}_i^{II}|_{\Gamma} = 0, \end{cases} \quad (23)$$

where w_G is the vertical component of the velocity of the centre of mass $G(t)$ of the solid;

- \underline{P}_i^{III} is the part of the pressure due to the pressure discontinuity at the contact line

$$\begin{cases} -\nabla \cdot \left(\frac{h}{\rho} \nabla \underline{P}_i^{III} \right) = 0 & \text{on } \mathcal{I}, \\ \underline{P}_i^{III}|_{\Gamma} = \rho g (\zeta_e - \zeta_i)|_{r=R} + P_{cor}. \end{cases} \quad (24)$$

2.4 Axisymmetric without swirl setting

Without loss of generality we suppose the centre of mass to have coordinates $G(t) = (0, 0, z_G(t))$ and let R be the radius of the interior domain \mathcal{I} . Introducing a cylindrical coordinates system with the z -axis coincident with the axis of symmetry of the solid (see Figure 2) we write the velocity field \mathbf{U} as

$$\mathbf{U}(t, r, \theta, z) = (u_r(t, r, \theta, z), u_\theta(t, r, \theta, z), u_z(t, r, \theta, z)).$$

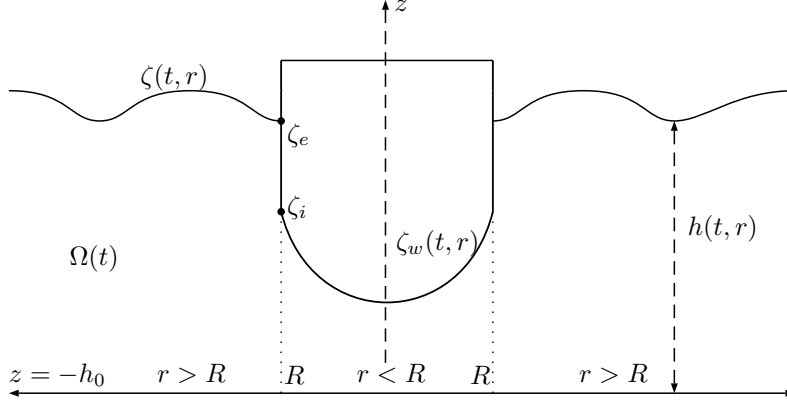


Figure 2: A vertical cross section of the floating structure configuration in the axisymmetric case

From now on and throughout the paper we consider an axisymmetric flow without swirl, which means that the flow has no dependence on the angular variable θ , i.e. $\mathbf{U} = \mathbf{U}(t, r, z)$, and $u_\theta = 0$ respectively. Hence the horizontal discharge can be written as

$$\mathbf{Q}(t, r) = (q_r(t, r), 0)$$

with

$$q_r(t, r) = \int_{-h_0}^{\zeta} u_r(t, r, z) dz$$

and the tangential component vanishes since

$$q_\theta(t, r) = \int_{-h_0}^{\zeta} u_\theta(t, r, z) dz = 0.$$

For simplicity we write q instead of q_r for the radial component of the horizontal discharge. In the new system of reference the shallow water model (14) - (16) becomes

$$\begin{cases} \partial_t h + \partial_r q + \frac{q}{h} = 0 \\ \partial_t q + \partial_r \left(\frac{q^2}{h} \right) + \frac{q^2}{rh} + gh \partial_r h = -\frac{h}{\rho} \partial_r \underline{P} \end{cases} \quad (25)$$

coupled with the transition condition

$$q_e|_{r=R} = q_i|_{r=R}. \quad (26)$$

We have $\underline{P}_e = P_{atm}$ and (22) - (24) become

$$\begin{cases} -(\partial_r + \frac{1}{r})(\frac{h_w}{\rho}\partial_r \underline{P}_i^I) = \\ (\partial_r + \frac{1}{r})(\partial_r(\frac{q_i^2}{h_w}) + \frac{q_i^2}{rh_w} + gh_w\partial_r h_w) \text{ on } r < R \\ \underline{P}_i^I|_{r=R} = P_{atm}, \end{cases} \quad (27)$$

$$\begin{cases} -(\partial_r + \frac{1}{r})(\frac{h_w}{\rho}\partial_r \underline{P}_i^{II}) = -w_G \text{ on } r < R \\ \underline{P}_i^{II}|_{r=R} = 0, \end{cases} \quad (28)$$

$$\begin{cases} -(\partial_r + \frac{1}{r})(\frac{h_w}{\rho}\partial_r \underline{P}_i^{III}) = 0 \text{ on } r < R \\ \underline{P}_i^{III}|_{r=R} = \rho g(\zeta_e - \zeta_i)|_{r=R} + P_{cor}, \end{cases} \quad (29)$$

where we replace $h_i = \zeta_i + h_0$ with $h_w = \zeta_w + h_0$ due to the contact constraint (8). Using axisymmetry and absence of swirl together with this change of coordinates we pass from a two-dimensional to a one-dimensional problem, where explicit calculations can be done (see Section 4). With these assumptions, the horizontal discharge is no more a vectorial quantity but a scalar quantity, making the problem easier to handle.

Remark 2.6. *Under the shallow water approximation and in the axisymmetric without swirl setting the fluid energy in Remark 2.4 becomes*

$$E_{SW} = 2\pi\frac{\rho}{2}g \int_0^{+\infty} \zeta^2 r dr + 2\pi\frac{\rho}{2} \int_0^{+\infty} \frac{q^2}{h} r dr. \quad (30)$$

In the presence of a floating structure the fluid energy E_{SW} is no more conserved by the equations in (17). Let us define the energy for a solid moving only vertically as

$$E_{sol} = \frac{1}{2}mw_G^2 + mgz_G. \quad (31)$$

and the total fluid-structure energy

$$E_{tot} := E_{SW} + E_{sol}. \quad (32)$$

Notation 2.1.

$$\llbracket f \rrbracket := f_{e|_{r=R}} - f_{i|_{r=R}}$$

is the jump between the exterior and the interior domain at the contact line $r = R$.

We can now state the following proposition:

Proposition 2.7. *Choosing $P_{cor} = \frac{\rho}{2} q_{i|_{r=R}}^2 \left[\frac{1}{h^2} \right]$, the total fluid-structure energy is conserved, i.e.*

$$\frac{d}{dt} E_{tot} = 0. \quad (33)$$

Proof. Multiplying the first equation of (25) by $\frac{q^2}{h^2} r$ and the second by $\frac{q}{h} r$ and, subtracting the first from the second, we can write the system under a conservative form

$$\partial_t e + \partial_r F = -r q \partial_r \underline{P},$$

where e is the local fluid energy

$$e = \frac{\rho}{2} g \zeta^2 r + \frac{\rho}{2} \frac{q^2}{h} r$$

and F is the flux

$$F = \rho \left(\frac{q^3}{2h^2} r + g \zeta q r \right).$$

Integrating on $r > 0$ and multiplying by 2π we obtain

$$\frac{d}{dt} E_{SW} - 2\pi \rho R \left[\frac{q^3}{2h^2} + g \zeta q \right] = -2\pi \int_0^R r q_i \partial_r (\underline{P}_i - P_{atm}) dr, \quad (34)$$

where we used the fact that $\underline{P}_e = P_{atm}$. By integration by parts we get

$$\begin{aligned} \frac{d}{dt} E_{SW} &= 2\pi \rho R \left[\frac{q^3}{2h^2} + g \zeta q \right] - 2\pi R (\underline{P}_i - P_{atm})|_{r=R} q_{i|_{r=R}} \\ &\quad + 2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_r (r q_i) dr. \end{aligned} \quad (35)$$

On the other hand, from the definition of E_{sol} , we have

$$\begin{aligned} \frac{d}{dt} E_{sol} &= m w_G \dot{w}_G + m g w_G = w_G (m \dot{w}_G + m g) \\ &= w_G 2\pi \int_0^R (\underline{P}_i - P_{atm}) r dr \\ &= 2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_t \zeta_w r dr \end{aligned}$$

where we used Newton's law for the conservation of the linear momentum and (21). From the contact constraint (8) and the mass conservation equation in (25) the following yields:

$$\frac{d}{dt}E_{sol} = -2\pi \int_0^R (\underline{P}_i - P_{atm}) \partial_r(rq_i)dr. \quad (36)$$

Therefore

$$\frac{d}{dt}E_{sol} = -\frac{d}{dt}E_{SW} + 2\pi\rho R \left[\frac{q^3}{2h^2} + g\zeta q \right] - 2\pi R (\underline{P}_i - P_{atm})|_{r=R} q_{i|_{r=R}}.$$

Using the expression of the interior pressure \underline{P}_i on the boundary $r = R$ in (27) - (29) and the transition condition (26) we obtain

$$\frac{d}{dt}(E_{SW} + E_{sol}) = 2\pi R \left(\frac{\rho}{2} q_{i|_{r=R}}^3 \left[\frac{1}{h^2} \right] - q_{i|_{r=R}} P_{cor} \right).$$

If $q_{i|_{r=R}} = 0$ the result follows directly. Otherwise we choose the pressure corrector term P_{cor} in (29) as

$$P_{cor} = \frac{\rho}{2} q_{i|_{r=R}}^2 \left[\frac{1}{h^2} \right], \quad (37)$$

and we get (33). □

3 The fluid equations

In this section we focus on the “fluid part” of the coupled problem. We show that the exterior part of (25) can be written as a one-dimensional quasilinear hyperbolic initial boundary value problem in an exterior domain and we shall prove the local in time well-posedness. Like frequently in the literature, throughout this paper we also use the term *mixed* problem: this comes from the fact that we have, as data of the problem, both the initial (in time) and the boundary (in space) values.

Hyperbolic problems in exterior domains have been treated in many works. Métivier [21], Benzoni and Serre [5] have studied hyperbolic initial boundary value problems in exterior domains with constant coefficients and maximally dissipative boundary condition. Isozaki [13] and Alazard [1] have studied the singular incompressible limit for the compressible Euler equation in an exterior domain. Concerning the quasilinear hyperbolic mixed problems, Schochet [24] has proved the local in time existence in the case of bounded domains and Shibata and Kikuchi [25] have showed the local in time existence for some second

order problem in bounded and unbounded domains. Differentiability of solutions to hyperbolic mixed problems has also been studied by Rauch and Massey [23].

The case we are considering here has not been treated in the literature yet. We consider a two-dimensional problem, but the axisymmetry keeps the boundary condition maximally dissipative which in general for a two-dimensional problem is not true. This property is essential for the coupling with the solid motion as it provides us better trace estimates than the ones for the general two-dimensional shallow water equations, but in other cases it is not necessary (see [20] for elementary examples). Hence we reduce the problem to a one-dimensional radial problem, then we must adapt the classical theory.

Let us recall that in the exterior domain $r > R$ one has

$$\begin{cases} \partial_t h_e + \partial_r q_e + \frac{q_e}{r} = 0 \\ \partial_t q_e + \partial_r \left(\frac{q_e^2}{h_e} \right) + \frac{q_e^2}{r h_e} + g h_e \partial_r h_e = 0 \\ q_e|_{r=R} = q_i|_{r=R}. \end{cases} \quad (38)$$

Defining $u = (h_e, q_e)^T$ and adding the Cauchy data we can write (38) as the following quasilinear hyperbolic mixed problem

$$\begin{cases} \partial_t u + A(u) \partial_r u + B(u, r) u = 0 \\ \mathbf{e}_2 \cdot u|_{r=R} = q_i|_{r=R} \\ u(t=0) = u_0 \end{cases} \quad (39)$$

with

$$A(u) = \begin{pmatrix} 0 & 1 \\ g h_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}, \quad B(u, r) = \begin{pmatrix} 0 & \frac{1}{r} \\ 0 & \frac{q_e}{r h_e} \end{pmatrix}$$

and

$$u_0 = (h_e(0), q_e(0))^T.$$

3.1 The hyperbolic mixed problem

Let us now treat the quasilinear hyperbolic mixed exterior problem (39) we have obtained from the fluid equations.

We consider $u : \mathbb{R} \times (R, +\infty) \rightarrow \mathbb{R}^2$ and we look for the solution in $L_r^2((R, +\infty))$, the weighted Lebesgue space $L^2((R, +\infty), r dr)$.

First let us define the following classical assumptions for a general hyperbolic mixed problem as (39) (see [21, 5]):

Definition 3.1.

- (P1) The system is *Friedrichs symmetrizable*, i.e. there exists a symmetric matrix $S = S(u)$, called the symmetrizer, such that $S(u)$ is uniformly positive definite on $L_r^2((R, +\infty))$ and $S(u)A(u)$ is symmetric.
- (P2) The boundary condition is *maximally dissipative*: $S(u)A(u)$ is negative definite on \mathbf{e}_2^\perp , where \mathbf{e}_2^\perp is the orthogonal complement of \mathbf{e}_2 .

Let us assume that we are in the subsonic regime, which means that

$$\frac{q^2}{h^2} < gh. \quad (40)$$

In the case of water waves in oceans, where the water depth is much bigger than the water velocity, this assumption is satisfied. Then for the floating structure initial boundary problem (39) we have the following result:

Proposition 3.2. *The exterior hyperbolic initial boundary value problem (39) coming from the fluid equations of the floating structure problem satisfies the following properties:*

- (P3) $A(u)$ has one strictly positive eigenvalue and one strictly negative eigenvalue,
- (P4) $P_- \mathbf{e}_2^\perp \neq 0$, where P_- is the projector on the eigenspace associated to the negative eigenvalue of $A(u)$.

Proof. The assumption (40) gives directly property (P3). We prove now property (P4) by contradiction. Let us suppose that $P_- \mathbf{e}_2^\perp = 0$. Since \mathbf{e}_2^\perp is of the form $a\mathbf{e}_1$, with $a \in \mathbb{R}$, this implies that the unit eigenvector $e_+(u)$ associated to the positive eigenvalue of $A(u)$ has to be \mathbf{e}_1 , but this is not possible because one can show that $e_+(u) = (1, \frac{q}{h} + \sqrt{gh})^T$ and the second component is not zero in the subsonic regime. Hence property (P4) is satisfied. \square

We can actually prove that properties (P1) and (P2) are direct consequences of properties (P3) and (P4) in the situation we are considering here.

Lemma 3.3. *The quasilinear hyperbolic exterior mixed problem (39) satisfies property (P1) and property (P2).*

Proof. From property (P3) of $A(u)$ there exist $\lambda_+(u)$ and $\lambda_-(u)$, the positive and the negative eigenvalue. We denote $e_+(u)$ and $e_-(u)$ the unit eigenvectors associated, $P_+(u)$ and $P_-(u)$ the projectors on the associated eigenspaces and

we define $S(u) := MP_-(u) + P_+(u)$ for some constant $M > 0$. Decomposing $v = c_-e_-(u) + c_+e_+(u)$ we have, for $S = S(u), P_\pm = P_\pm(u), e_\pm = e_\pm(u)$,

$$\begin{aligned} (Sv, v) &= ((MP_- + P_+)v, v) = (Mc_-e_- + c_+e_+, c_-e_- + c_+e_+) \\ &= Mc_-^2 + c_+^2 \geq m|v|^2, \end{aligned}$$

with $m = \frac{\min(M, 1)}{2}$. The symmetry of $S(u)$ is trivial and the symmetry of $S(u)A(u)$ comes from the fact that $A(u)$ and $P_\pm(u)$ commute and we get property (P1). We refer to Taylor (see Prop. 2.2 of [26]) for a different proof with a general notion of symmetrizer involving pseudo-differential operators.

Let us consider \mathbf{e}_2^\perp , the one dimensional orthogonal complement of $\mathbf{e}_2 \in \mathbb{R}^2$. A vector in this subspace of \mathbb{R}^2 can always be written as a linear combination of e_\pm , $\mathbf{e}_2^\perp = c_-e_- + c_+e_+$, with $\frac{c_+}{c_-}$ equal to a fixed constant since the subspace is a line (one dimensional subspace). Therefore we have

$$\begin{aligned} (SA\mathbf{e}_2^\perp, \mathbf{e}_2^\perp) &= (M\lambda_-c_-e_- + \lambda_+c_+e_+, c_-e_- + c_+e_+) \\ &= M\lambda_-c_-^2 + \lambda_+c_+^2. \end{aligned}$$

Because of the property (P4), $c_- \neq 0$ and we obtain property (P2) by choosing $M > -\frac{\lambda_+c_+^2}{\lambda_-c_-^2}$. \square

3.1.1 The linearized problem

In order to construct the solution to the floating structure problem, which is a quasilinear mixed problem of the form (39) coupled with Newton's equation for the solid motion, we shall use an iterative scheme based on the following linearization of (39),

$$\begin{cases} L(\bar{u})u = \partial_t u + A(\bar{u})\partial_r u + B(\bar{u}, r)u = f, \\ \mathbf{e}_2 \cdot u|_{r=R} = g, \\ u(t=0) = u_0, \end{cases} \quad (41)$$

with some \bar{u} and $g = q_{i|_{r=R}}$. Let us introduce the following space:

$$X^k(T) := \bigcap_{j=0}^k C^j([0, T], H_r^{k-j}((R, +\infty)))$$

endowed with the norm

$$\|u\|_{X^k(T)} := \sup_{[0,T]} \|u(t)\|_{X^k}, \quad \|u(t)\|_{X^k} = \sum_{j=0}^k \|\partial_t^j u(t)\|_{H_r^{k-j}((R,+\infty))} \quad (42)$$

with $\|\cdot\|_{H_r^k((R,+\infty))}$ the norm of the weighted Sobolev space $H^k((R,+\infty), r dr)$. We first show an important a priori estimate useful to find strong solutions of the problem (41).

Proposition 3.4. *Provided $\bar{u} \in X^2(T)$, then for all $T > 0$ there is a constant $C_T(\bar{u})$ such that for $u \in H_r^1([0, T] \times (R, +\infty))$ solution to (41) the following holds:*

$$\|u(t)\|_{L_r^2((R,+\infty))}^2 + \|u|_{r=R}\|_{L^2([0,t])}^2 \leq C_T(\bar{u}) \left(\|u_0\|_{L_r^2((R,+\infty))}^2 + \|g\|_{L^2([0,t])}^2 \right) \quad (43)$$

$$+ \int_0^t \|f(\tau)\|_{L_r^2((R,+\infty))}^2 d\tau \quad (44)$$

for all $t \in [0, T]$, with $C_T(\bar{u}) = C(\beta, R)e^{T(1+\|\bar{u}\|_{X^2(T)})}$.

Proof. Following Proposition 2.2 of [5], we have from property (P1) and by integrations by parts, for $S = S(\bar{u})$, $A = A(\bar{u})$, $B = B(\bar{u}, r)$,

$$\begin{aligned} \frac{d}{dt}(Su, u)_{L_r^2((R,+\infty))} &= -2(SA\partial_r u, u)_{L_r^2((R,+\infty))} - 2(SBu, u)_{L_r^2((R,+\infty))} \\ &+ ((\partial_t S)u, u)_{L_r^2((R,+\infty))} + 2(Sf, u)_{L_r^2((R,+\infty))} \leq \\ &\leq SAu|_{r=R} \cdot u|_{r=R} R + (1 + \beta^{-1}\|W\|_\infty) (S(t)u, u)_{L^2((R,+\infty))} + (S(t)f, f)_{L^2((R,+\infty))} \end{aligned}$$

with

$$\|W\|_\infty = \|\partial_t S + \partial_r(SA) + \frac{1}{r}SA - 2SB\|_{L^\infty([0,T] \times (R,+\infty))} \quad (45)$$

and using the fact that we can always find a β such that, in the sense of quadratic form,

$$\beta \leq S(t) \leq \beta^{-1} \quad (46)$$

since the symmetrizer is uniformly positive definite. Note that we have

$$\|W\|_\infty \lesssim \|\bar{u}\|_{X^2(T)}.$$

Property (P2) permits us to control the first term on the right-hand side of the inequality, using the following Lemma from Métivier [21]:

Lemma 3.5. *The symmetric matrix SA is negative definite on \mathbf{e}_2^\perp , the set of all vectors orthogonal to \mathbf{e}_2 , if and only if there are constants $c_1, c_2 > 0$ such that for each vector $h \in \mathbb{C}^2$:*

$$-(SAh, h) \geq c_1|h|^2 - c_2|\mathbf{e}_2 \cdot h|^2.$$

Choosing $h = u|_{r=R}$, integrating in time and using property (H1) we have

$$\begin{aligned} (S(t)u(t), u(t))_{L^2((R, +\infty))} &\leq (S(0)u_0, u_0)_{L^2((R, +\infty))} + \\ &+ \int_0^t (S(\tau)f(\tau), f(\tau))_{L^2((R, +\infty))} + c_2R|g(\tau)|^2 - c_1R|u|_{r=R(\tau)}|^2 d\tau + \\ &+ \int_0^t \left(1 + \beta^{-1}\|\bar{u}\|_{X^2(T)}\right) (S(t)u, u)_{L^2((R, +\infty))} d\tau. \end{aligned} \quad (47)$$

By Gronwall's Lemma and (46) we have

$$\begin{aligned} \beta\|u(t)\|_{L_r^2((R, +\infty))}^2 + c_1R\|u|_{r=R}\|_{L^2([0, t])}^2 &\leq e^{T(1 + \beta^{-1}\|\bar{u}\|_{X^2(T)})} \left(\beta^{-1}\|u_0\|_{L_r^2((R, +\infty))}^2 + \right. \\ &\left. + c_2R\|g\|_{L^2([0, t])}^2 + \beta^{-1} \int_0^t \|f(\tau)\|_{L_r^2((R, +\infty))}^2 d\tau \right), \end{aligned}$$

and we get (44) for $C_T(\bar{u}) = \frac{\max(\beta^{-1}, c_2R)}{\min(\beta, c_1R)} e^{T(1 + \beta^{-1}\|\bar{u}\|_{X^2(T)})}$. \square

Then, following Theorem 2.3.6 of [21], one can show that there is a unique solution $u \in C^0([0, T], L_r^2((R, +\infty)))$ for the initial datum u_0 in $L_r^2((R, +\infty))$ and boundary value g in $L^2([0, T])$. This solution satisfies the energy estimate (44).

Regular solutions. To solve the mixed problem in Sobolev spaces we need some compatibility conditions. For instance, the initial and the boundary conditions imply that necessarily

$$\mathbf{e}_2 \cdot u_0|_{r=R} = g|_{t=0} = \mathbf{e}_2 \cdot u|_{t=0, r=R}, \quad (48)$$

if the traces are defined. Let us consider the generic equation

$$\partial_t u = f - A(\bar{u})\partial_r u - B(\bar{u}, r)u,$$

then we can formally (this is the meaning of the brackets """) define

$$u_1 := "\partial_t u|_{t=0}" = "f|_{t=0} - A(\bar{u})|_{t=0}\partial_r u_0 - B(\bar{u}, r)|_{t=0}u_0".$$

Hence, provided traces are defined,

$$\begin{aligned}\mathbf{e}_2 \cdot u_1|_{r=R} &= \mathbf{e}_2 \cdot \left("f|_{t=0} - A(\bar{u})|_{t=0} \partial_r u_0 - B(\bar{u}, r)|_{t=0} u_0 " \right)|_{r=R} \\ &= g_1 := \partial_t g|_{t=0} = \mathbf{e}_2 \cdot \left(" \partial_t u|_{t=0} " \right)|_{r=R}.\end{aligned}$$

These conditions are necessary for the existence of a smooth solution. We can continue the expansion to higher orders looking for more compatibility conditions. We denote $u_j := " \partial_t^j u|_{t=0} "$ and $g_j := \partial_t^j g|_{t=0}$, with u and g smooth enough. $" \partial_t^j u|_{t=0} "$ is defined by formally taking $j - 1$ time derivatives of the system, solving for $\partial_t^j u$ and evaluating at time $t = 0$, *i.e.*

$$u_j = " \partial_t^{j-1} (f - A(\bar{u}) \partial_r u - B(\bar{u}, r) u) |_{t=0} ". \quad (49)$$

Note that u_j is not the derivative of a known function but rather the value that the derivative of u will have if u exists.

Therefore necessarily smooth enough solutions to (41) must satisfy

$$\mathbf{e}_2 \cdot \left(" \partial_t^{j-1} (f - A(\bar{u}) \partial_r u - B(\bar{u}, r) u) |_{t=0} " \right)|_{r=R} = g_j \quad (50)$$

Definition 3.6. *The data $u_0 \in H_r^k(\mathbb{R}_+)$, $g \in H^k([0, T])$ and $f \in H_r^k([0, T] \times \mathbb{R}_+)$ satisfy the compatibility conditions up to order $s \leq k - 1$ if (50) holds for each $j = 0, \dots, s$.*

For the floating structure mixed problem (39) the compatibility conditions (50) on the initial data $u_0 = (h_e(0), q_e(0))$ and the boundary value $g = q_i|_{r=R}$ can be written as:

$$\begin{aligned}\bullet q_e(0)|_{r=R} &= q_i|_{r=R, t=0} \\ \bullet " \partial_t^{j-1} \left(-\partial_r \frac{q_e^2}{h_e} - \frac{q_e^2}{rh_e} - gh_e \partial_r h_e \right) |_{t=0} " |_{r=R} &= \partial_t^j q_i|_{r=R, t=0} \quad j \geq 1.\end{aligned} \quad (51)$$

As in the L^2 -case, our goal is to find an a priori estimate for the linear problem (41) in order to get existence and uniqueness of the solution in some more regular space.

Proposition 3.7. *Let $k \geq 1$. For all $T > 0$, provided $\bar{u} \in X^s(T)$ with integer $s = \max(k, 2)$, there is a constant $C_{s,T}(\bar{u})$ such that for $u \in H_r^{k+1}([0, T] \times (R, +\infty))$ solution to (41) the following holds:*

$$\begin{aligned}\|u(t)\|_{X^k}^2 + \|u|_{r=R}\|_{H^k([0,t])}^2 &\leq C_{s,T}(\bar{u}) \left(\|u(0)\|_{X^k}^2 + \|g\|_{H^k([0,t])}^2 \right. \\ &\quad \left. + \int_0^t \|f(\tau)\|_{X^k}^2 d\tau \right)\end{aligned} \quad (52)$$

for all $t \in [0, T]$ with $C_{s,T}(\bar{u}) = C(\beta, R, \bar{u})e^{T(2+\beta^{-1}\|\bar{u}\|_{X^s(T)}+k\beta^{-2}\|\bar{u}\|_{X^s(T)})}$.

Proof. We adapt here the argument presented in [21]. We denote $u^i := \partial_t^i u$ for $i \leq k$ the tangential derivative, which in the one dimensional case it is simply the time derivative, and we introduce the tangential norm

$$\|u(t)\|'_{X^k} := \sum_{i=0}^k \|\partial_t^i u(t)\|_{L_r^2((R,+\infty))}$$

We apply ∂_t^i to the equation of (41) and we get

$$\begin{aligned} \partial_t u^i + A(\bar{u})\partial_r u^i + B(\bar{u})u^i &= [A(\bar{u}), \partial_t^i]\partial_r u + [B(\bar{u}), \partial_t^i]u + f^i, \\ \mathbf{e}_2 \cdot u^i|_{r=R} &= g^i. \end{aligned} \quad (53)$$

As we have done in the previous L^2 case we consider $\frac{d}{dt}(u^i, Su^i)_{L_r^2((R,+\infty))}$, with $S = S(\bar{u})$. The only difference from the previous case is the presence of the two commutator terms in (53). We need to control their L_r^2 norms in a different way. The first term can be written under the form $\sum_{\alpha=1, \dots, i} \|\partial_t^\alpha(A(\bar{u}))\partial_t^{i-\alpha}\partial_r u\|_{L_r^2}$. For $\alpha \leq k-1$ every term of the sum is controlled by

$$\begin{aligned} \|\partial_t^\alpha(A(\bar{u}))\|_{L^\infty} \|\partial_t^{i-\alpha}\partial_r u\|_{L_r^2} &\leq C_R \|\partial_t^\alpha(A(\bar{u}))\|_{H_r^{k-\alpha}} \|u\|_{X^{k-\alpha+1}} \\ &\leq C_R \|A(\bar{u})\|_{X^k} \|u\|_{X^k} \end{aligned} \quad (54)$$

using the fact that $1 \leq \alpha \leq k-1$ for the Sobolev embedding and that $\|\cdot\|_{X^\lambda} \leq \|\cdot\|_{X^\delta}$ if $\lambda < \delta$. Recall that Sobolev embeddings $H_r^k \hookrightarrow C^s$ still hold for the weighted spaces H_r^k since we are considering the exterior domain $r > R$. For $\alpha = k$ we directly have

$$\|\partial_t^k(A(\bar{u}))\|_{L_r^2} \|\partial_r u\|_{L^\infty} \leq C_R \|A(\bar{u})\|_{X^k} \|u\|_{X^k}$$

since $u \in H_r^{k+1}$ with $k \geq 1$. We can find the same estimate for the commutator term with $B(\bar{u})$.

We have

$$\|A(\bar{u}), B(\bar{u})\|_{X^k(T)} \lesssim \|\bar{u}\|_{X^k(T)}$$

and, as in Proposition 3.4,

$$\|W\|_\infty \lesssim \|\bar{u}\|_{X^2(T)},$$

with $\|W\|_\infty$ as in (45). We get the following inequality:

$$\begin{aligned}
& \left(S(t)u^i(t), u^i(t) \right)_{L^2((R,+\infty))} \leq \left(S(0)u_0^i, u_0^i \right)_{L^2((R,+\infty))} \\
& + \int_0^t \left(S(\tau)f^i(\tau), f^i(\tau) \right)_{L^2((R,+\infty))} + c_2R \left| g^i(\tau) \right|^2 - c_1R \left| u_{|r=R}^i(\tau) \right|^2 d\tau \\
& + \int_0^t \left(2 + \beta^{-1} \|\bar{u}\|_{X^2(T)} \right) \left(S(\tau)u^i(\tau), u^i(\tau) \right)_{L^2((R,+\infty))} + \beta^{-1} \|\bar{u}\|_{X^k(T)}^2 \|u(\tau)\|_{X^k}^2 d\tau.
\end{aligned} \tag{55}$$

For u satisfying the equation (41) we can show that the X^k norm is controlled by the tangential one. The following holds:

$$\begin{aligned}
\|u(\tau)\|_{X^k}^2 & \leq C(\bar{u}, f) \|u(\tau)\|_{X^k}'^2 \\
& = C(\bar{u}, f) \sum_{i=0}^k \|u^i(t)\|_{L_r^2((R,+\infty))}^2 \\
& \leq C(\bar{u}, f) \beta^{-1} \sum_{i=0}^k \left(S(t)u^i, u^i \right)_{L^2((R,+\infty))}.
\end{aligned}$$

By taking the sum for i from 0 to k and by applying Gronwall's Lemma we obtain

$$\begin{aligned}
\frac{\beta}{C(\bar{u}, f)} \|u(t)\|_{X^k}'^2 + c_1R \left\| u_{|r=R} \right\|_{H^k([0,t])}^2 & \leq e^{T(2+\beta^{-1}\|\bar{u}\|_{X^s(T)} + kC(\bar{u})\beta^{-2}\|\bar{u}\|_{X^s(T)}^2)} \\
& \left(\beta^{-1} \|u_0\|_{X^k}'^2 + c_2R \|g\|_{H^k([0,t])}^2 + \beta^{-1} \int_0^t \|f(\tau)\|_{X^k}'^2 \right),
\end{aligned} \tag{56}$$

with $s = \max(k, 2)$. By definition the tangential norm is dominated by the X^k norm. Then we can write both the sides of (56) without the prime and (52) follows for

$$C_{s,T}(\bar{u}) = \frac{\max(\beta^{-1}, c_2R)}{\min\left(\frac{\beta}{C(\bar{u}, f)}, c_1R\right)} e^{T(2+\beta^{-1}\|\bar{u}\|_{X^s(T)} + kC(\bar{u}, f)\beta^{-2}\|\bar{u}\|_{X^s(T)}^2)}.$$

□

Remark 3.8. We note that for solutions to (41), we have

$$\|u(0)\|_{X^k} = \sum_{j=0}^k \|u_j\|_{H^{k-j}}$$

where the u_j are defined in (49).

Equivalently to the L^2 -case, we can state the following theorem:

Theorem 3.9. *Let k be an integer with $k \geq 1$. Suppose $u_0 \in H_r^k((R, +\infty))$, $g \in H^k([0, T])$ and $f \in H_r^k([0, T] \times (R, +\infty))$ satisfy the compatibility conditions (50) up to the order $k - 1$. Moreover assume $\bar{u} \in X^s(T)$ with $s = \max(k, 2)$. Then there is a unique solution $u \in X^k(T)$ to (41). Its trace on $r = R$ belongs to $H^k([0, T])$ and u satisfies the energy estimate (52).*

Proof. We show only the idea of the proof of the existence. For more details and the proof of uniqueness see Chapter 2 of [21]. First we solve the equation with a loss of smoothness. We consider the data u_0, f and g in H_r^{k+2} satisfying the compatibility conditions up to order k . One can prove that there is a solution in $H_r^{k+1}([0, T] \times (R, +\infty)) \subset X^k(T)$, by extending the data by 0 for $t < 0$ and then by applying the existence result for the mixed problem in $(-\infty, T] \times (R, +\infty)$ of [21].

The second step is to consider H^k -data: we use the compatibility conditions up to order $k - 1$ to approximate u_0, f and g in H_r^k and H^k with sequences $u_0^n \in H_r^{k+2}((R, +\infty))$, $f^n \in H_r^{k+2}([0, T] \times (R, +\infty))$ and $g^n \in H^{k+2}([0, T])$ satisfying the compatibility conditions up to order $k + 1$. From the previous argument and the energy estimate (52) we have that u^n is a Cauchy sequence in $X^k(T)$ and therefore converges to the limit $u \in X^k(T)$, which is a solution to (41) since $k \geq 1$. \square

3.1.2 The quasilinear problem and application to the case of a solid with forced motion

In the particular case of the floating structure problem, the boundary condition in (39) is $g = q_{i_{|r=R}}$, the value of the horizontal discharge in the interior domain at the boundary $r = R$. We will see in the next section that this quantity is strictly linked to the solid motion, in particular to the vertical component of the velocity of the centre of mass $w_G(t)$.

In the case of a solid with forced motion, the boundary condition g is a datum of the problem. Hence, after having studied the linear problem (41), one can use a standard iterative scheme argument in order to get existence and uniqueness of the solution to (39).

Theorem 3.10. *Consider a solid with a forced motion. For $k \geq 2$, let $u_0 \in H_r^k((R, +\infty))$ and $w_G \in H^k([0, T])$ satisfy the compatibility conditions in Definition 5.2 up to order $k - 1$. Then the fluid problem (64) admits a unique solution $u \in X^k(T)$ with $X^k(T)$ as in (42).*

Sketch of the proof. We introduce the iterative scheme by defining the sequence $(u^n)_n$ with u^n solution to the linear problem $L(u^{n-1})u^n = 0$. The existence of

such a sequence is given by Theorem 3.9. Once we have showed the control of the sequence in some “big norm” and the convergence in some “small norm”, the limit u of $(u^n)_n$ is the solution to (39). For more details we refer to [21]. We will show a detailed proof in the case of a free motion in Theorem 5.3 below. \square

From this point on we consider a solid with free motion. Therefore the boundary condition is still an unknown of the problem and we must adapt the classical argument used in Theorem 3.10 to our problem introducing an iterative scheme for the fluid-structure coupled system. The details of this coupled iterative scheme argument are given in Section 5. Before, we deal with the solid problem and we deduce an ordinary differential equation describing the motion of its centre of mass.

4 The solid equation

In this section we address the motion of the solid. We recall that we are considering a floating structure moving only vertically.

Denoting m the mass of the body, g the gravity acceleration and z_G the vertical position of the centre of mass, we consider only the vertical component of Newton’s law for the conservation of linear momentum:

$$m\ddot{z}_G = -mg + F_{fluid} \quad (57)$$

where $F_{fluid} = \int_{r < R} (\underline{P}_i - P_{atm})$ is the resulting vertical force exerted by the fluid on the solid.

Let us introduce the displacement $\delta_G(t) := z_G(t) - z_{G,eq}$ between the vertical position of the centre of mass at time t and at its equilibrium. In the case of vertical motion $h_w(r, t) = h_{w,eq}(r) + \delta_G(t)$, where $h_{w,eq}$ is the fluid height at the equilibrium. For simplicity we consider a cylindrically symmetric solid with flat bottom, which means that the wetted surface ζ_w (therefore h_w) does not depend on the spatial coordinate in the interior domain $r < R$. See Appendix B for the general case with a cylindrically symmetric solid with a non-flat bottom.

Proposition 4.1. *Newton’s law (57) can be written under the following form:*

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -c\delta_G(t) + c\zeta_e(t, R) + \left(\frac{\mathfrak{b}}{h_e^2(t, R)} + \beta(\delta_G) \right) (\dot{\delta}_G(t))^2 \quad (58)$$

with

$$c = \rho g \pi R^2, \quad \mathfrak{b} = \frac{\pi \rho R^4}{8},$$

$$m_a(\delta_G) = \frac{\mathfrak{b}}{h_w(\delta_G)} = \frac{\mathfrak{b}}{(h_{w,eq} + \delta_G(t))},$$

$$\beta(\delta_G) = \frac{\mathfrak{b}}{2h_w^2(\delta_G)} = \frac{\mathfrak{b}}{2(h_{w,eq} + \delta_G(t))^2}.$$

Remark 4.2. From (12) and (13) we have

$$0 < m_a(\delta_G) \leq \frac{\mathfrak{b}}{h_{min}}, \quad 0 < \frac{\mathfrak{b}}{h_e^2} \leq \frac{\mathfrak{b}}{h_m^2},$$

$$0 < \beta(\delta_G) \leq \frac{\mathfrak{b}}{2h_{min}^2}.$$

Remark 4.3. In (58) $m_a(\delta_G)$ is called the added mass term and it represents the fact that, in order to move in the fluid, the solid has to accelerate itself but also the portion of fluid next to it. This effect appears in other hydrodynamical configurations, in particular for totally submerged solids studied by Glass, Sueur and Takahashi [11] and Glass, Munnier and Sueur [10]. It has an important role for the stability of numerical simulations of fluid-structure interactions [6].

The coupling with the fluid motion is given by the term $\zeta_e(\mathbf{R}, t)$, which means that the solid motion depends on the value of the elevation of the exterior free surface at the boundary $r = R$.

Proof. We decompose F_{fluid} according to the decomposition (27) - (29) of the pressure

$$F_{fluid} = F_{fluid}^I + F_{fluid}^{II} + F_{fluid}^{III}$$

with

$$F_{fluid}^I = \int_{r < R} (\underline{P}_i^I - P_{atm}), \quad F_{fluid}^j = \int_{r < R} \underline{P}_i^j, \quad j = II, III.$$

Using the elementary potential $\Phi_{\mathcal{I}}^r$ defined in [18] we can write

$$F_{fluid}^{II} = - \int_{r < R} \underline{P}_i^{II} \left(\partial_r + \frac{1}{r} \right) (h_w \partial_r \Phi_{\mathcal{I}}^r),$$

and, after integration by parts,

$$\begin{aligned} F_{fluid}^{II} &= - \int_{r < R} \left(\partial_r + \frac{1}{r} \right) (h_w \partial_r \underline{P}_i^{II}) \Phi_{\mathcal{I}}^r \\ &= -\rho \int_{r < R} \dot{w}_G \Phi_{\mathcal{I}}^r, \end{aligned}$$

where the second equality comes from the definition (28) of $\underline{P}_i^{\text{II}}$. Using again the definition of elementary potential we obtain

$$\begin{aligned} F_{fluid}^{\text{II}} &= \left[\int_{r < R} \Phi_{\mathcal{I}}^r \left(\partial_r + \frac{1}{r} \right) (h_w \Phi_{\mathcal{I}}^r) \right] \dot{w}_G \\ &= - \left[\int_{r < R} \frac{1}{h_w} (h_w \partial_r \Phi_{\mathcal{I}}^r)^2 \right] \dot{w}_G. \end{aligned}$$

From the definition of the elementary potential we explicitly have that

$$h_w \partial_r \Phi_{\mathcal{I}}^r = -\frac{r}{2}.$$

It follows that

$$F_{fluid}^{\text{II}} = -m_a(h_w) \dot{w}_G,$$

with $m_a(h_w)$ as in (58). Proceeding similarly we can write also

$$F_{fluid}^{\text{I}} = -\rho \int_{r < R} \frac{r}{2h_w} \frac{h_w}{\rho} \partial_r \underline{P}_i^{\text{I}}.$$

Then (57) becomes

$$(m + m_a(h_w)) \dot{w}_G = -mg + F_{fluid}^{\text{I}} + \int_{r < R} \underline{P}_i^{\text{III}}. \quad (59)$$

Solving the elliptic problem (29) we have

$$\underline{P}_i^{\text{III}}(r, t) = \rho g (\zeta_e(t, R) - \zeta_i(t, R)) + \frac{\rho}{2} q_i^2(t, R) \left(\frac{1}{h_e(t, R)^2} - \frac{1}{h_w(t, R)^2} \right),$$

where we used the expression of P_{cor} in (37) and the contact constraint (8). Moreover (27) can be written as

$$\partial_r y(r) + \frac{y(r)}{r} = -\frac{b(r)}{r}$$

with

$$y(r) = \frac{h_w}{\rho} \partial_r \underline{P}_i^{\text{I}} + \partial_r \left(\frac{q_i^2}{h_w} \right) \quad \text{and} \quad b(r) = \partial_r \left(\frac{q_i^2}{h_w} \right).$$

Hence we have

$$y(r) = -\frac{1}{r} \left(\frac{q_i(r)^2}{h_w(r)} - \frac{q_i(0)^2}{h_w(0)} \right). \quad (60)$$

Because of the constraint (8), the mass conservation equation of (25) in the interior domain $r < R$ becomes

$$\partial_r q_i + \frac{1}{r} q_i = -\dot{\delta}_G(t),$$

then we have

$$q_i(r, t) = -\frac{r}{2} \dot{\delta}_G(t). \quad (61)$$

Hence $q_i(0, t) = 0$ and (60) becomes

$$\frac{h_w}{\rho} \partial_r P_i^I = -\partial_r \left(\frac{q_i^2}{h_w} \right) - \frac{1}{r} \frac{q_i^2}{h_w} = -\frac{3}{4h_w} r \dot{\delta}_G^2.$$

Replacing the expression of $\frac{h_w}{\rho} \partial_r P_i^I$ in F_{fluid}^I we get

$$F_{fluid}^I = \frac{3\pi\rho R^4}{16h_w^2} \dot{\delta}_G^2$$

and, by definition of the equilibrium state, we have

$$-mg - \rho g \int_{r<R} \zeta_{w,eq} = 0. \quad (62)$$

Since the solid has vertical side-walls the following equality holds

$$\rho g \int_{r<R} \zeta_w(t) - \rho g \int_{r<R} \zeta_{w,eq} = c\delta_G(t). \quad (63)$$

These two equalities, together with the constraint $\zeta_i = \zeta_w$, give

$$-mg = c\zeta_i(R, t) - c\delta_G(t),$$

hence we obtain the nonlinear second order ordinary differential equation (58). \square

Recall that in (61) we have

$$q_{i|_{r=R}}(t) = -\frac{R}{2} \dot{\delta}_G(t).$$

This term is the boundary value in the fluid mixed problem (38). It follows that this is the coupling term between the fluid and the solid motion in the fluid system, as $\zeta_e(t, R)$ has the same property in the solid equation (see Remark 4.3).

5 Fluid-structure coupling

From the previous two sections, it follows that the fluid-structure interaction problem considered in this paper is described by the following mathematical model:

Proposition 5.1. *The nonlinear shallow water equations with a floating structure for an axisymmetric flow without swirl take the form*

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u, r)u = 0, & r > R \\ \mathbf{e}_2 \cdot u|_{r=R} = -\frac{R}{2}\dot{\delta}_G(t) \\ u(t=0) = u_0 := (h(0), q(0))^T, \end{cases} \quad (64)$$

with $A(u), B(u, r)$ as in (39). Moreover the solid motion is given by the Cauchy problem

$$\begin{cases} (m + m_a(\delta_G))\ddot{\delta}_G(t) = \\ = -\mathbf{c}\delta_G(t) + \mathbf{c}(\mathbf{e}_1 \cdot u|_{r=R} - h_0) + (\mathbf{b}(u) + \beta(\delta_G))(\dot{\delta}_G(t))^2, \\ \delta_G(0) = \delta_0 := z_G(0) - z_{G,eq}, \\ \dot{\delta}_G(0) = \delta_1 := \dot{z}_G(0), \end{cases} \quad (65)$$

with $m_a(\delta_G), \beta(\delta_G)$ as in (58),

$$\mathbf{b}(u) = \frac{\mathbf{b}}{(\mathbf{e}_1 \cdot u|_{r=R})^2} = \frac{\mathbf{b}}{h_e(t, R)^2},$$

using the fact that

$$h_e(t, R) = \mathbf{e}_1 \cdot u|_{r=R} \quad \text{and} \quad \zeta_e(t, R) = h_e(t, R) - h_0.$$

Let us give the notion of compatibility conditions in the case of this particular fluid-structure coupled problem:

Definition 5.2. *The data $u_0 \in H_r^k((R, +\infty))$, $\delta_0 \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}$ of the floating structure coupled system (64) - (65) satisfy the compatibility conditions up to order $k - 1$ if, for $1 \leq j \leq k - 1$, the following holds:*

- $\mathbf{e}_2 \cdot u_0|_{r=R} = -\frac{R}{2}\delta_1,$
- $\mathbf{e}_2 \cdot \left(\partial_t^{j-1} (-A(u)\partial_r u - B(u, r)u) \Big|_{t=0} \right) \Big|_{r=R}$
 $= \partial_t^{j-1} \left[-\frac{R}{2(m + m_a(\delta_G))} \left(-\mathbf{c}\delta_G + \mathbf{c}(\mathbf{e}_1 \cdot u|_{r=R} - h_0) + (\mathbf{b}(u) + \beta(\delta_G))\delta_G^2 \right) \right] \Big|_{t=0}.$

With the following statement we prove that the coupled model (64) - (65) is locally in time well-posed:

Theorem 5.3. For $k \geq 2$, let $u_0 \in H_r^k((R, +\infty))$, δ_0 and δ_1 satisfy the compatibility conditions in Definition 5.2 up to order $k - 1$. Then the coupled problem (64) - (65) admits a unique solution $(u, \delta_G) \in X^k(T) \times H^{k+1}([0, T])$ with $X^k(T)$ as in (42).

Remark 5.4. Considering an initial datum $u_0 \in H_r^2((R, +\infty))$, we need the following compatibility conditions satisfied:

$$q_e(0, R) = -\frac{R}{2}\delta_1$$

and

$$-\partial_r \left(\frac{q_e^2}{h_e} \right) (0, R) - \frac{1}{R} \frac{q_e}{h_e} (0, R) - gh_e(0, R) \partial_r \zeta_e(0, R) \quad (66)$$

$$= -\frac{R}{2(m + m_a(\delta_0))} \left(-c\delta_0 + c\zeta_e(0, R) + \left(\frac{\mathbf{b}}{h_e^2(0, R)} + \beta(\delta_0) \right) \delta_1^2 \right). \quad (67)$$

For instance let us take the initial configuration of the fluid-structure interaction as the following: the solid displaced from its equilibrium position with no initial velocity, which means

$$\delta_0 \neq 0 \text{ and } \delta_1 = 0,$$

and the fluid such that

$$h_e(0, R) = h_0, \quad q_e(0, R) = 0, \quad \partial_r \zeta_e(0, R) = -\frac{c\delta_0 R}{2(m + m_a(\delta_0))gh_0}.$$

Then the initial conditions are compatible and we can apply Theorem 5.3.

Proof. We adapt here the argument that Métivier used in [21] for the existence and uniqueness of the solution to the fluid mixed problem. We couple it with the solid equation.

First we introduce the iterative scheme for the coupled system (64) - (65). Let $u_0 \in H_r^k((R, +\infty))$, $\delta_G \in H^k([0, T])$. Let us write the nonlinear fluid equation in the following way:

$$\partial_t u = -F(u, r)$$

with $F(u, r) = A(u)\partial_r u + B(u, r)u$. We define $u_j \in H_r^{k-j}((R, +\infty))$ by

$$u_j = -\mathcal{F}_{j-1}(u_0, \dots, u_{j-1}) \quad (68)$$

with

$$\mathcal{F}_{j-1}(u_0, \dots, u_{j-1}) := \partial_t^j F(u)|_{t=0} = \sum_{k=1}^j \sum_{j^1 + \dots + j^k = j} c(k, j^1, \dots, j^k) F^k(u_0)(u_{j^1}, \dots, u_{j^k}).$$

We choose the first element of the sequence $u^0 \in H_r^k(\mathbb{R} \times (R, +\infty))$ such that

$$\partial_t^j u^0|_{t=0} = u_j, \quad 0 \leq j \leq k.$$

We can assume that u^0 vanishes for $|t| \geq 1$, hence $u^0 \in X^k(T)$ for all T . Assume that u^{n-1} and δ_G^{n-1} are constructed. For $n \geq 1$ we solve by induction the linear mixed problems

$$\begin{cases} \partial_t u^n + A(u^{n-1}) \partial_r u^n + B(u^{n-1}, r) u^n = 0, & r > R \\ \mathbf{e}_2 \cdot u^n|_{r=R} = -\frac{R}{2} \delta_G^{n-1}(t) \\ u^n(t=0) = u_0. \end{cases} \quad (69)$$

We suppose that u^{n-1} belongs to $X^k(T)$ with

$$\partial_t^j u^{n-1}|_{t=0} = u_j, \quad j \leq k. \quad (70)$$

For $n = 1$ this is true. By definition of \mathcal{F}_j and by (70),

$$\partial_t F(u^{n-1})|_{t=0} = \mathcal{F}_j(u_0, \dots, u_j).$$

Now we consider the linear problem (69). We construct the u_j^n using (50). We can see that $u_j^n = u_j$ with the u_j defined before. Then the compatibility conditions imply that the data $-F(u^{n-1}), \delta_G^{n-1}$ and u_0 are compatible for the linear problem. From Theorem 3.9 the system (69) has a unique solution $u^n \in X^k(T)$ and $\partial_t^j u^n|_{t=0} = u_j^n = u_j$. Therefore we can continue the construction and this permits to define a sequence $u^n \in X^k(T)$ solving the linear problem (69).

Define the sequence $V^n = (u^n, \delta_G^n)$ with u^n as before and δ_G^n solution to the linear ODE

$$\begin{cases} (m + m_a(\delta_G^{n-1})) \ddot{\delta}_G^n(t) \\ = -\mathbf{c} \delta_G^n(t) + \mathbf{c} \left(\mathbf{e}_1 \cdot u^n|_{r=R} - h_0 \right) + \left(\mathbf{b}(u^n) + \beta(\delta_G^{n-1}) \right) \delta_G^n(t)^2, \\ \delta_G^n(0) = \delta_0, \\ \dot{\delta}_G^n(0) = \delta_1, \end{cases} \quad (71)$$

Our goal is to apply the fixed point method. Hence we need to show the existence and the convergence of the sequence V^n . First we need the control of u^n in some "big" norm and then the convergence in some "small" norm, as presented by Alinhac and Gérard in [4].

We consider the product space $X^k(T) \times H^{k+1}([0, T])$ endowed with the norm defined by the sum of the X^k norm defined before and the Sobolev H^{k+1} norm

$$\|V^n\|_{coup} := \sup_{0 \leq t \leq T} \|u^n(t)\|_{X^k} + \|\delta_G^n\|_{H^{k+1}([0, T])}.$$

Let us suppose that we have

$$\|V^{j-1} - V^0\|_{coup} \leq \tilde{R}, \quad 1 \leq j \leq n \quad (72)$$

with $V^0 = (u^0, \delta_0)^T$ for some \tilde{R} to determine later. For $j = 1$ the latter is trivially true. We want to show that it is true also for $j = n + 1$. Then $u^n - u^0$ satisfies

$$\begin{cases} \partial_t u^n + A(u^{n-1}) \partial_r (u^n - u^0) + B(u^{n-1}, r) (u^n - u^0) \\ = -A(u^{n-1}) \partial_r u^0 - B(u^{n-1}, r) u^0, \\ \mathbf{e}_2 \cdot (u^n - u^0)|_{r=R} = -\frac{R}{2} (\delta^{n-1}_G(t) - \delta_1) \\ (u^n - u^0)(t=0) = 0. \end{cases}$$

The source term $f := -A(u^{n-1}) \partial_r u^0 - B(u^{n-1}, r) u^0$ belongs to $X^k(T)$. Indeed we choose $u^0 \in X^{2k}(T) \subset X^k(T)$ and we have

$$\|A(u^{n-1}) \partial_r u^0\|_{X^k} \leq \sum_{j=0}^k \left\| \partial_t^j (A(u^{n-1}) \partial_r u^0) \right\|_{H_r^{k-j}}.$$

We can control all the terms of the sum as in (54) except the terms where time derivatives are only applied to $\partial_r u^0$. For them we have for $j \leq k$

$$\begin{aligned} & \left\| A(u^{n-1}) \partial_r \partial_t^j u^0 \right\|_{H_r^{k-j}} \leq \left\| A(u^{n-1}) \right\|_{H_r^{k-j}} \left\| \partial_r \partial_t^j u^0 \right\|_{L^\infty} \\ & \lesssim \left\| A(u^{n-1}) \right\|_{X^k} \left\| \partial_t^j u^0 \right\|_{H_r^k} \lesssim \left\| A(u^{n-1}) \right\|_{X^k} \left\| u^0 \right\|_{X^{2k}} \end{aligned}$$

for $u^0 \in X^{2k}$. We can do the same estimates for $B(u^{n-1}, r) u^0$. Then the source term is in X^k and by (52) we have

$$\begin{aligned} & \|(u^n - u^0)(t)\|_{X^k}^2 + \|(u^n - u^0)|_{r=R}\|_{H^k([0, t])}^2 \leq \\ & C_{k, T}(u^{n-1}) \left(\|\delta_G^{n-1} - \delta_1\|_{H^k([0, t])}^2 + \int_0^t \|f(\tau)\|_{X^k}^2 d\tau \right) \end{aligned}$$

with the source term $f = -A(u^{n-1})\partial_r u^0 - B(u^{n-1}, r)u^0$ and $C_{k,T}(u^{n-1})$ as in (52) (here $s = \max(k, 2)$ is equal to k since $k \geq 2$). On the other hand, we have

$$\|\delta_G^n - \delta_0\|_{H^{k+1}([0,T])}^2 \leq C_1(T) + C_2(T)\|\delta_G^n\|_{H^{k-1}([0,T])}^2 \quad (73)$$

with $C_1(T) = \delta_1^2 \frac{2T^3}{3} + 2\delta_1^2 T$ and $C_2(T) = \frac{2T^4}{9} + T^2 + 1$.

Using the equation on δ_G^n and the following estimate

$$\begin{aligned} \|fg\|_{H^k} &\leq 2\sqrt{T}\|f\|_{H^k}\|g\|_{H^k} + \left(|f(0)| + |\partial_t f(0)| + \dots + |\partial_t^{k-1} f(0)|\right)\|g\|_{H^k} \\ &\quad + \left(|g(0)| + |\partial_t g(0)| + \dots + |\partial_t^{k-1} g(0)|\right)\|f\|_{H^k} \end{aligned}$$

for $f, g \in H^k([0, T])$ with $k \geq 2$, we have

$$\begin{aligned} \|\delta_G^n\|_{H^{k-1}} &\leq 2\sqrt{T} \left\| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} \left(\|\delta_G^n\|_{H^{k-1}} + \|\mathbf{e}_1 \cdot u_{|r=R}^n - h_0\|_{H^{k-1}} \right) \\ &\quad + 2\sqrt{T} \left\| \frac{\beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} \|(\delta_G^n)^2\|_{H^{k-1}} \\ &\quad + C_0 \left\| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} + D_0 \left(\|\delta_G^n\|_{H^{k-1}} + \|\mathbf{e}_1 \cdot u_{|r=R}^n - h_0\|_{H^{k-1}} \right) \\ &\quad + C_1 \left\| \frac{\mathbf{b}(u^n) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})} \right\|_{H^{k-1}} + D_1 \|(\delta_G^n)^2\|_{H^{k-1}} \\ &\lesssim \sqrt{T} \left\| \delta_G^{n-1} \right\|_{H^{k-1}} \left(\|\delta_G^n\|_{H^{k-1}} + \|(u^n - u^0)_{|r=R}\|_{H^k} \right) \\ &\quad + \|u_{|r=R}^0\|_{H^k} + h_0\sqrt{T} + \|\delta_G^n\|_{H^k}^2 + (C_0 + C_1) \left\| \delta_G^{n-1} \right\|_{H^{k-1}} \\ &\quad + D_0 \left(\|\delta_G^n\|_{H^{k-1}} + \|(u^n - u^0)_{|r=R}\|_{H^k} + \|u_{|r=R}^0\|_{H^k} + h_0\sqrt{T} \right) \\ &\quad + D_1 \|\delta_G^n\|_{H^k}^2, \end{aligned}$$

with

$$C_0 = C_0 \left(|\delta_G^n(0)|, \dots, |\partial_t^{k-1} \delta_G^n(0)|, |\mathbf{e}_1 \cdot u_{|r=R}^n(0) - h_0|, \dots, |\partial_t^{k-2} \mathbf{e}_1 \cdot u_{|r=R}^n(0)| \right),$$

$$D_0 = D_0 \left(\left| \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})}(0) \right|, \dots, \left| \partial_t^{k-2} \frac{\mathbf{c}}{m + m_a(\delta_G^{n-1})}(0) \right| \right),$$

$$C_1 = C_1 \left(|\delta_G^n(0)|^2, \dots, |\partial_t^{k-2} \delta_G^n(0)|^2 \right),$$

$$D_1 = D_1 \left(\left| \frac{\mathbf{b}(u^n) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})}(0) \right|, \dots, \left| \partial_t^{k-2} \frac{\mathbf{b}(u^n) + \beta(\delta_G^{n-1})}{m + m_a(\delta_G^{n-1})}(0) \right| \right),$$

where we used Remark 4.2 to control the three H^{k-1} norms of the terms at step $n-1$. We have $\|\delta_G^n\|_{H^{k-1}} \leq |\delta_0| + C(T) \|\delta_G^n\|_{H^{k+1}}$ with $C(T) \rightarrow 0$ for $T \rightarrow 0$. Then, writing $\delta_G^n = \delta_G^n - \delta_0 + \delta_0$, we can move to the left of the inequality (73) the terms with the H^{k+1} norm of δ_G^n and for T small enough we get

$$\begin{aligned} & \|\delta_G^n - \delta_0\|_{H^{k+1}([0,T])} \\ & \leq \alpha(T) + \beta(T) \|\delta_G^{n-1} - \delta_0\|_{H^{k+1}([0,T])} + \gamma(T) \|(u^n - u^0)|_{r=R}\|_{H^k([0,T])} \end{aligned}$$

for some $\alpha(T), \beta(T)$ depending on T, δ_0, δ_1 and $\|u^0|_{r=R}\|_{H^k}$. Then we have the following estimate for the coupled norm:

$$\begin{aligned} \|V^n - V^0\|_{coup} &= \sup_{0 \leq t \leq T} \|u^n(t) - u^0(t)\|_{X^k} + \|\delta_G^n - \delta_0\|_{H^{k+1}([0,T])} \\ &\leq \sup_{0 \leq t \leq T} \|u^n(t) - u^0(t)\|_{X^k} + \gamma(T) \|(u^n - u^0)|_{r=R}\|_{H^k([0,T])} \\ &\quad + \beta(T) \|\delta_G^{n-1} - \delta_0\|_{H^{k+1}([0,T])} + \alpha(T) \\ &\leq C(T) \left(\|\delta_G^{n-1} - \delta_1\|_{H^k([0,T])} + \int_0^T \|f(s)\|_{X^k} ds \right) \\ &\quad + \beta(T) \|\delta_G^{n-1} - \delta_0\|_{H^{k+1}([0,T])} + \alpha(T), \end{aligned}$$

with $f = -A(u^{n-1})\partial_r u^0 - B(u^{n-1})u^0$. We have

$$\|A(u^{n-1}), B(u^{n-1})\|_{X^k(T)} \lesssim \|u^{n-1}\|_{X^k(T)},$$

then applying the inductive hypothesis (72) for $j = n$ we can find some \tilde{R} such that for T small enough

$$\|V^n - V^0\|_{coup} \leq \tilde{R}.$$

Now we look for the convergence of the sequence V^n in a "smaller" norm. We consider the space

$$X^0(T) \times H^1([0, T]) = C^0([0, T], L_r^2((R, +\infty))) \times H^1([0, T]).$$

We have that $u^n - u^{n-1}$ satisfies

$$\begin{cases} \partial_t (u^n - u^{n-1}) + A(u^{n-1})\partial_r (u^n - u^{n-1}) + B(u^{n-1}, r) (u^n - u^{n-1}) \\ = - (A(u^{n-1}) - A(u^{n-2})) \partial_r u^{n-1} - (B(u^{n-1}, r) - B(u^{n-2}, r)) u^{n-1}, \\ \mathbf{e}_2 \cdot (u^n - u^{n-1})|_{r=R} = -\frac{R}{2} (\delta^{n-1}_G(t) - \delta^{n-2}_G(t)) \\ (u^n - u^{n-1})(t=0) = 0. \end{cases}$$

Since A and B are continuous from L_r^2 to L_r^2 and the fact that, from the previous argument, we have

$$\|\partial_r u^{n-1}\|_{L^\infty} \lesssim \|u^{n-1}\|_{X^k} \lesssim \tilde{R},$$

the L_r^2 norm of the source term is controlled by

$$\begin{aligned} & \left\| - \left(A(u^{n-1}) - A(u^{n-2}) \right) \partial_r u^{n-1} + \right. \\ & \left. - \left(B(u^{n-1}, r) - B(u^{n-2}, r) \right) u^{n-1} \right\|_{L_r^2} \lesssim \tilde{R} \|u^{n-1} - u^{n-2}\|_{L_r^2}. \end{aligned}$$

Then by (44) we have

$$\begin{aligned} & \| (u^n - u^{n-1})(t) \|_{L_r^2((R, +\infty))}^2 + \| (u^n - u^{n-1})|_{r=R} \|_{L^2([0, t])}^2 \\ & \leq C_{k, T}(u^{n-1}) \left(\| \delta^{n-1}_G(t) - \delta^{n-2}_G(t) \|_{L^2([0, t])}^2 \right. \\ & \left. + \int_0^t \| u^{n-1} - u^{n-2} \|_{L_r^2((R, +\infty))}^2 d\tau \right), \end{aligned}$$

with $C_{k, T}(u^{n-1})$ as before. On the other hand we have

$$\| \delta_G^n - \delta_G^{n-1} \|_{H^1([0, T])}^2 \leq C_1(T) \| \delta_G^n - \delta_G^{n-1} \|_{L^2([0, T])}^2, \quad (74)$$

with $C_1(T) = \frac{T^4}{9} + \frac{T^2}{2}$. Since in the ODE (65) the terms

$$\frac{\mathbf{c}}{m + m_a(\delta_G)}, \quad \frac{\mathbf{b}(u)}{m + m_a(\delta_G)}, \quad \frac{\beta(\delta_G)}{m + m_a(\delta_G)}$$

are Lipschitz continuous on δ_G and u from L^2 to L^2 and considering the equation for δ_G^n and δ_G^{n-1} , we obtain the following estimate for T small enough:

$$\begin{aligned} \| \delta_G^n - \delta_G^{n-1} \|_{H^1([0, T])} & \leq \tilde{\alpha}(T, \tilde{R}) \| (u^n - u^{n-1})|_{r=R} \|_{L^2([0, T])} \\ & \quad + \tilde{\beta}(T, \tilde{R}) \| \delta_G^{n-1} - \delta_G^{n-2} \|_{L^2([0, T])}. \end{aligned}$$

for some constants $\tilde{\alpha}(T, \tilde{R}), \tilde{\beta}(T, \tilde{R})$. Therefore

$$\begin{aligned}
& \|V^n - V^{n-1}\|_{X^0(T) \times H^{k+1}([0, T])} \\
&= \sup_{0 \leq t \leq T} \|u^n(t) - u^{n-1}(t)\|_{L_r^2(\mathbb{R}_+^R)} + \|\delta_G^n - \delta_G^{n-1}\|_{H^1([0, T])} \\
&\leq \sup_{0 \leq t \leq T} \|u^n(t) - u^{n-1}(t)\|_{L_r^2(\mathbb{R}_+^R)} + \tilde{\alpha}(T, \tilde{R}) \|(u^n - u^{n-1})|_{r=R}\|_{L^2([0, T])} \\
&+ \tilde{\beta}(T, \tilde{R}) \|\delta_G^{n-1} - \delta_G^{n-2}\|_{H^{k+1}([0, T])} \\
&\leq C(T, \tilde{R}) \left(\|\delta_G^{n-1} - \delta_G^{n-2}\|_{H^1([0, T])} + \int_0^T \|u^{n-1}(t) - u^{n-2}(t)\|_{L_r^2(\mathbb{R}_+^R)} dt \right) \\
&\leq K(T, \tilde{R}) \|V^{n-1} - V^{n-2}\|_{X^0(T) \times H^{k+1}([0, T])}.
\end{aligned}$$

Then we can choose T small enough such that $K(T, \tilde{R}) < 1$ and we obtain that V^n is a convergent sequence in $C^0([0, T], L_r^2(\mathbb{R}_+^R)) \times H^1([0, T])$ with limit $V = (u, \delta_G)$. Since the V^n are bounded in $X^k(T) \times H^{k+1}([0, T])$, there exists a subsequence which converges weakly in $X^k(T) \times H^{k+1}([0, T])$ and via interpolation inequalities one has the strong convergence in intermediate spaces. Choosing the appropriate subsequence, by uniqueness of the limit we get $V \in X^k(T) \times H^{k+1}([0, T])$. \square

Remark 5.5. Since $k \geq 2$, from the Sobolev embedding $H^{k+1}([0, T]) \hookrightarrow C^2([0, T])$ we have that the displacement δ_G belongs to $C^2([0, T])$.

A Solid with non-flat bottom

We derive here the equation for the solid motion, as in Section 4, in the more general case of a solid with non-flat bottom. Due to the fact that the interior and exterior domains do not change during the motion, we suppose that the contact between the free surface and the floating structure takes place on the vertical side-walls during all the motion. Then we can state the following proposition:

Proposition A.1. *In the case of a solid with non-flat bottom, Newton's law (57) can be written under the following form:*

$$(m + m_a^{NF}(\delta_G)) \ddot{\delta}_G(t) = -c \delta_G(t) + c \zeta_e(t, R) + \left(\frac{\mathfrak{b}}{h_e^2(t, R)} + \beta^{NF}(\delta_G) \right) (\dot{\delta}_G(t))^2 \tag{75}$$

with

$$c = \rho g \pi R^2, \quad \mathfrak{b} = \frac{\pi \rho R^4}{8},$$

$$m_a^{NF}(\delta_G) = \frac{\rho\pi}{8} \int_0^R \frac{r^4}{h_w^2(\delta_G, r)} \partial_r h_w(\delta_G, r) dr,$$

$$\beta^{NF}(\delta_G) = \frac{\mathfrak{b}}{2h_w^2(\delta_G, R)} + \frac{\pi\rho}{8} \int_0^R \frac{r^4}{h_w^3(\delta_G, r)} \partial_r h_w(\delta_G, r) dr,$$

and the dependence on δ_G given by

$$h_w(\delta_G, r) = \delta_G(t) + h_{w,eq}(r).$$

Remark A.2. One can note that in the case of a solid with flat bottom ($\partial_r h_w(\delta_G, r) = 0$)

$$m_a^{NF}(\delta_G) = m_a(\delta_G), \quad \beta^{NF}(\delta_G) = \beta(\delta_G),$$

and (75) coincides with (58).

Proof. As in Proposition 4.1 we derive only the new expression of F_{fluid}^I and F_{fluid}^{III} . For the added mass term we have

$$F_{fluid}^I = -m_a^{NF}(h_w) \dot{w}_G$$

with

$$m_a^{NF}(h_w) = \frac{\rho\pi}{2} \int_0^R \frac{r^3}{h_w} dr.$$

By definition,

$$F_{fluid}^I = \rho \int_{r < R} \frac{r}{2h_w} \left(-\frac{h_w}{\rho} \partial_r \underline{P}_i^I \right)$$

with \underline{P}_i^I defined as the solution to (27). Since we want

$$\underline{P}_i^I - P_{atm} \in H_{0,r}^1((0, R))$$

we get

$$-\frac{h_w}{\rho} \partial_r \underline{P}_i^I = \partial_r \left(\frac{q_i^2}{h_w} \right) + \frac{q_i^2}{rh_w} + gh_w \partial_r h_w.$$

Using the formula for the horizontal discharge in the interior domain $q_i(r, t) = -\frac{r}{2} \dot{\delta}_G(t)$, we obtain that

$$F_{fluid}^I = \pi\rho \int_0^R \left(\frac{3r^3}{4h_w^2} \delta_G^2 + \left(gr^2 - \frac{r^4}{4h_w^3} \delta_G^2 \right) \partial_r h_w \right) dr.$$

Also in the case of a solid with non-flat bottom, (29) admits the unique constant solution

$$\underline{P}_i^{III}(t, r) = \rho g (\zeta_e(r, R) - \zeta_i(t, R)) + \frac{\rho}{2} q_i^2(t, R) \left(\frac{1}{h_e(t, R)^2} - \frac{1}{h_w(t, R)^2} \right),$$

with $q_i(t, R) = -\frac{R}{2}\dot{\delta}_G(t)$. By definition of F_{fluid}^{III} we have

$$F_{fluid}^{III} = \mathfrak{c}(\zeta_e(t, R) - \zeta_i(t, R)) + \mathfrak{b} \left(\frac{1}{h_e(t, R)^2} - \frac{1}{h_w(t, R)^2} \right) \dot{\delta}_G(t)^2.$$

The relations (62) and (63) still hold but in this case we obtain

$$-mg = \rho g 2\pi \int_0^R r \zeta_i(t, r) dr - \mathfrak{c} \delta_G(t).$$

Now we observe that the term $\pi \rho \int_0^R g r^2 \partial_r h_w dr$ can be written by integration by parts as

$$\pi \rho \int_0^R g r^2 \partial_r h_w dr = \mathfrak{c} \zeta_i(R, t) - \rho g 2\pi \int_0^R r \zeta_i(r, t) dr$$

where $\partial_r h_w = \partial_r \zeta_w = \partial_r \zeta_i$ using the constraint (8). Putting all these expressions in Newton's law (57) and integrating by parts, we get (75). \square

References

- [1] T. ALAZARD, *Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions*, *Adv. Differential Equations*, 10 (2005), pp. 19–44.
- [2] T. ALAZARD, N. BURQ, AND C. ZUILY, *On the water-wave equations with surface tension*, *Duke Math. J.*, 158 (2011), pp. 413–499.
- [3] —, *On the Cauchy problem for gravity water waves*, *Invent. Math.*, 198 (2014), pp. 71–163.
- [4] S. ALINHAC AND P. GÉRARD, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, *Savoirs Actuels. [Current Scholarship]*, InterEditions, Paris; Éditions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991.
- [5] S. BENZONI-GAVAGE AND D. SERRE, *Multi-dimensional hyperbolic partial differential equations: First-order Systems and Applications*, Oxford University Press on Demand, 2007.
- [6] P. CAUSIN, J. F. GERBEAU, AND F. NOBILE, *Added-mass effect in the design of partitioned algorithms for fluid-structure problems*, *Comput. Methods Appl. Mech. Engrg.*, 194 (2005), pp. 4506–4527.
- [7] W. CRAIG AND C. SULEM, *Numerical simulation of gravity waves*, *J. Comput. Phys.*, 108 (1993), pp. 73–83.
- [8] W. CUMMINS, *The Impulse Response Function and Ship Motions*, Report (David W. Taylor Model Basin), Navy Department, David Taylor Model Basin, 1962.
- [9] T. DE POYFERRÉ, *A priori estimates for water waves with emerging bottom*, arXiv preprint arXiv:1612.04103, (2016).
- [10] O. GLASS, A. MUNNIER, AND F. SUEUR, *Dynamics of a point vortex as limits of a shrinking solid in an irrotational fluid*, ArXiv e-prints, (2014).
- [11] O. GLASS, F. SUEUR, AND T. TAKAHASHI, *Smoothness of the motion of a rigid body immersed in an incompressible perfect fluid*, *Ann. Sci. Éc. Norm. Supér.* (4), 45 (2012), pp. 1–51.
- [12] T. IGUCHI, *A shallow water approximation for water waves*, *J. Math. Kyoto Univ.*, 49 (2009), pp. 13–55.

- [13] H. ISOZAKI, *Singular limits for the compressible Euler equation in an exterior domain. II. Bodies in a uniform flow*, Osaka J. Math., 26 (1989), pp. 399–410.
- [14] F. JOHN, *On the motion of floating bodies. I*, Comm. Pure Appl. Math., 2 (1949), pp. 13–57.
- [15] —, *On the motion of floating bodies. II. Simple harmonic motions*, Comm. Pure Appl. Math., 3 (1950), pp. 45–101.
- [16] D. LANNES, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc., 18 (2005), pp. 605–654.
- [17] —, *The water waves problem*, vol. 188 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics.
- [18] —, *On the dynamics of floating structures*, Ann. PDE, 3 (2017), pp. Art. 11, 81.
- [19] D. LANNES AND G. METIVIER, *The shoreline problem for the one-dimensional shallow water and green-naghdi equations*, arXiv preprint arXiv:1710.03651, (2017).
- [20] A. MAJDA AND S. OSHER, *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math., 28 (1975), pp. 607–675.
- [21] G. MÉTIVIER, *Small viscosity and boundary layer methods*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser Boston, Inc., Boston, MA, 2004. Theory, stability analysis, and applications.
- [22] M. MING AND C. WANG, *Elliptic estimates for the Dirichlet-Neumann operator on a corner domain*, Asymptot. Anal., 104 (2017), pp. 103–166.
- [23] J. B. RAUCH AND F. J. MASSEY, III, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc., 189 (1974), pp. 303–318.
- [24] S. SCHOCHET, *The compressible euler equations in a bounded domain: existence of solutions and the incompressible limit*, Communications in Mathematical Physics, 104 (1986), pp. 49–75.
- [25] Y. SHIBATA AND M. KIKUCHI, *On the mixed problem for some quasilinear hyperbolic system with fully nonlinear boundary condition*, J. Differential Equations, 80 (1989), pp. 154–197.

- [26] M. E. TAYLOR, *Partial differential equations III. Nonlinear equations*, vol. 117 of Applied Mathematical Sciences, Springer, New York, second ed., 2011.
- [27] S. WU, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, *Invent. Math.*, 130 (1997), pp. 39–72.
- [28] ———, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, *J. Amer. Math. Soc.*, 12 (1999), pp. 445–495.
- [29] V. E. ZAKHAROV, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, *Journal of Applied Mechanics and Technical Physics*, 9 (1968), pp. 190–194.