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LOWER CENTRAL AND DERIVED SERIES OF SEMI-DIRECT PRODUCTS, AND APPLICATIONS TO SURFACE BRAID GROUPS

JOHN GUASCHI AND CAROLINA DE MIRANDA E PEREIRO

ABSTRACT. For an arbitrary semi-direct product, we give a general description of its lower central series and an estimation of its derived series. In the second part of the paper, we study these series for the full braid group $B_n(M)$ and pure braid group $P_n(M)$ of a compact surface M , orientable or non-orientable, the aim being to determine the values of n for which $B_n(M)$ and $P_n(M)$ are residually nilpotent or residually soluble. We first solve this problem in the case where M is the 2-torus. We then use the results of the first part of the paper to calculate explicitly the lower central series of $P_n(\mathbb{K})$, where \mathbb{K} is the Klein bottle. Finally, if M is a non-orientable, compact surface without boundary, we determine the values of n for which $B_n(M)$ is residually nilpotent or residually soluble in the cases that were not already known in the literature.

1. INTRODUCTION

Let G be a group. If $g, g' \in G$ then $[g, g'] = gg'g^{-1}g'^{-1}$ denotes their *commutator*, and if H and K are subgroups of G , then the *commutator subgroup of H and K* , denoted by $[H, K]$, is defined by $[H, K] = \langle [h, k] : h \in H \text{ and } k \in K \rangle$, the subgroup of G generated by the commutators of H and K . The *lower central series* $\{\Gamma_i(G)\}_{i \geq 1}$ of G is defined inductively by $\Gamma_1(G) = G$, and for $i \geq 1$, $\Gamma_{i+1}(G) = [\Gamma_i(G), G]$, and the *derived series* $\{G^{(i)}\}_{i \geq 0}$ of G is defined inductively by $G^{(0)} = G$, and for $i \geq 0$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The quotient $G/\Gamma_2(G)$ is the *Abelianisation* of G that we denote by G^{Ab} . Following P. Hall, for any group-theoretic property \mathcal{P} , a group G is said to be residually \mathcal{P} if for any (non-trivial) element $x \in G$, there exists a group H that possesses property \mathcal{P} and a surjective homomorphism $\varphi : G \rightarrow H$ such that $\varphi(x) \neq 1$ (see also [28]). It is well known that a group G is *residually nilpotent* (resp. *residually soluble*) if and only if $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$ (resp. $\bigcap_{i \geq 0} G^{(i)} = \{1\}$). zz

If p is a prime number, the *lower \mathbb{F}_p -linear central filtration* $\{\gamma_i^p(G)\}_{i \geq 1}$ of G is defined inductively by $\gamma_1^p(G) = G$, and for $i \geq 1$, $\gamma_{i+1}^p(G) = \langle [\gamma_i^p(G), G], x^p : x \in \gamma_i^p(G) \rangle$ [30]. If the group G is finitely generated, then G is *residually p -finite* if and only if $\bigcap_{i \geq 1} \gamma_i^p(G) = \{1\}$ [30, Proposition 2.3(2)]. For any group G , $G^{(i)} \subset \Gamma_{i+1}(G) \subset \gamma_{i+1}^p(G)$, so if G is residually p -finite then it is residually nilpotent, which in turn implies that it is residually soluble. The combinatorial study of the lower central and derived series of a group is an interesting and important problem, see [11, 13, 22, 25, 26, 27] for example.

The first part of this paper is devoted to the analysis of the lower central and derived series of arbitrary semi-direct products. Our first main result describes the lower central series of such a group, and gives some information about its derived series.

Theorem 1.1. *Let G and H be groups, and let $\varphi : G \rightarrow \text{Aut}(H)$ be an action of G on H . We define recursively the following subgroups of H : $L_1 = V_1 = H$, and if $n \geq 2$:*

$$\begin{aligned} K_n &= \langle \varphi(g)(h).h^{-1} : g \in \Gamma_{n-1}(G), h \in H \rangle, & H_n &= \langle \varphi(g)(h).h^{-1} : g \in G, h \in L_{n-1} \rangle, \\ \tilde{H}_n &= \langle \varphi(g)(h).h^{-1} : g \in G, h \in V_{n-1} \rangle, & L_n &= \langle K_n, H_n, [H, L_{n-1}] \rangle, \\ V_n &= \langle \tilde{H}_n, [H, V_{n-1}] \rangle. \end{aligned}$$

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Then φ induces an action, which we also denote by φ , of $\Gamma_n(G)$ on L_n (resp. of $G^{(n+1)}$ on V_{n+2}), and for all $n \in \mathbb{N}$, we have:

- (1) $\Gamma_n(H \rtimes_{\varphi} G) = L_n \rtimes_{\varphi} \Gamma_n(G)$.
- (2) $(H \rtimes_{\varphi} G)^{(n-1)} \subset V_n \rtimes_{\varphi} G^{(n-1)}$.

For the case of the commutator subgroup, namely $n = 2$, part (1) was obtained in [15, Proposition 3.3].

In the rest of this paper, we will be interested in computing the lower central and derived series of the full and pure braid groups of compact surfaces without boundary, and we will apply Theorem 1.1 to part of this calculation. We first recall some facts about these braid groups and their lower central and derived series. The braid groups of the disc, also called the Artin braid groups, were introduced by E. Artin [1]. If $n \geq 1$, the n -string Artin braid group, denoted by B_n , is generated by elements $\sigma_1, \dots, \sigma_{n-1}$ that are subject to the Artin relations:

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for all } 1 \leq i \leq n-2 \\ \sigma_j \sigma_i = \sigma_i \sigma_j & \text{if } |i-j| \geq 2 \text{ and } 1 \leq i, j \leq n-1. \end{cases}$$

The notion of braid group was generalised to surfaces by Fox and Neuwirth using configuration spaces as follows [12]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. The n^{th} configuration space of M , denoted by $F_n(M)$, is defined by:

$$F_n(M) = \{(x_1, \dots, x_n) : x_i \in M, \text{ and } x_i \neq x_j \text{ if } i \neq j, i, j = 1, \dots, n\}.$$

The n -string pure braid group $P_n(M)$ of M is defined by $P_n(M) = \pi_1(F_n(M))$. The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates, and the n -string braid group $B_n(M)$ of M is defined by $B_n(M) = \pi_1(F_n(M)/S_n)$. This gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \longrightarrow S_n \longrightarrow 1. \quad (1.1)$$

If $m \geq 1$, the projection $p: F_{n+m}(M) \longrightarrow F_n(M)$ defined by $p(x_1, \dots, x_n, \dots, x_{n+m}) = (x_1, \dots, x_n)$ induces a homomorphism $p_*: P_{n+m}(M) \longrightarrow P_n(M)$. Geometrically, p_* is the homomorphism that ‘forgets’ the last m strings. If M is without boundary, Fadell and Neuwirth showed that p is a locally-trivial fibration [9, Theorem 1], with fibre $F_m(M \setminus \{x_1, \dots, x_n\})$ over the point (x_1, \dots, x_n) , which we consider to be a subspace of the total space via the map $i: F_m(M \setminus \{x_1, \dots, x_n\}) \longrightarrow F_{n+m}(M)$ defined by $i((y_1, \dots, y_m)) = (x_1, \dots, x_n, y_1, \dots, y_m)$. Applying the associated long exact sequence in homotopy to this fibration, we obtain the Fadell-Neuwirth short exact sequence of pure braid groups:

$$1 \longrightarrow P_m(M \setminus \{x_1, \dots, x_n\}) \xrightarrow{i_*} P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1, \quad (1.2)$$

where $n \geq 3$ if M is the sphere \mathbb{S}^2 [8, 10], $n \geq 2$ if M is the projective plane $\mathbb{R}P^2$ [10], and $n \geq 1$ otherwise [9], and i_* is the homomorphism induced by the map i . This sequence has been widely studied. If M is the torus \mathbb{T} or the Klein bottle \mathbb{K} , the existence of a non-vanishing vector field on M allows one to construct a section for p [9, Theorem 5]. This implies that the short exact sequence (1.2) splits for all $n, m \in \mathbb{N}$, and that $P_n(M)$ may be decomposed as an iterated semi-direct product (see Proposition 5.1 for an explicit section for p_* in the case $M = \mathbb{K}$).

We then use the above results to study the derived series of the braid groups of the torus and the lower central series and derived series of non-orientable surfaces. Theorem 1.1 will be used in the computation of the lower central series of $P_n(\mathbb{K})$, but we believe that it is of independent interest, and that it may be applicable to other groups. We first recall some facts about these series for surface braid groups. The lower central series of the Artin braid groups were analysed by Gorin and Lin who gave a presentation of the commutator subgroup $\Gamma_2(B_n)$ of B_n for $n \geq 3$, and who showed that $(B_n)^{(1)} = (B_n)^{(2)}$ for all $n \geq 5$, which implies that $(B_n)^{(1)}$ is perfect [21]. As a consequence, $\Gamma_2(B_n) = \Gamma_3(B_n)$ for all $n \geq 3$, so B_n is not residually nilpotent. The lower central series of the pure braid group P_n was studied by Falk and Randell [11] and by Kohno [25], who proved independently that P_n is residually nilpotent for all $n \geq 1$.

The braid groups of orientable surfaces were studied by Bellingeri, Gervais and Guaschi [5]. If $M_{g,m}$ is a compact, connected, orientable surface of genus $g \geq 1$ with $m \geq 0$ boundary components, then $B_n(M_{g,m})$ is not residually nilpotent if $n \geq 3$, and $B_2(\mathbb{T})$ is residually nilpotent. In the case of the pure braid groups, $P_n(M_{g,m})$ is residually torsion-free nilpotent for all $n \geq 1$ if $m \geq 1$, or if $g = 1$ and $m = 0$ (the torus). If $m = 0$ and $g \geq 1$, Bardakov and Bellingeri proved that $P_n(M_{g,m})$ is residually torsion-free nilpotent for all $n \geq 1$, and the braid group $B_2(M_{g,m})$ is residually 2-finite, in particular, it is residually nilpotent [2]. Gonçalves and Guaschi studied the lower central and derived series of the braid groups of the sphere \mathbb{S}^2 and the projective plane $\mathbb{R}P^2$ [15, 18]. For the sphere, $B_n(\mathbb{S}^2)$ is residually nilpotent if and only if $n \leq 2$, and residually soluble if and only if $n \leq 4$. In the case of the projective plane, $B_n(\mathbb{R}P^2)$ is residually nilpotent if and only if $n \leq 2$, and if $n \neq 4$, $B_n(\mathbb{R}P^2)$ is residually soluble if and only if $n < 4$. More recently, if M is a non-orientable surface different from $\mathbb{R}P^2$, Bellingeri and Gervais showed that $P_n(M)$ is residually 2-finite, and so is residually nilpotent [4].

In the second part of this paper, we study the derived series of the torus and the lower central series and derived series of non-orientable surfaces. Our main results in this direction are as follows.

Theorem 1.2. *The group $B_n(\mathbb{T})$ is residually soluble if and only if $n \leq 4$.*

For non-orientable surfaces, we first analyse the case of the Klein bottle. Using Theorem 1.1, we compute explicitly $\Gamma_n(P_2(\mathbb{K}))$ and $\gamma_n^2(P_2(\mathbb{K}))$ in Theorems 5.4 and 5.13 respectively. From this it will follow that $P_2(\mathbb{K})$ is residually nilpotent and residually 2-finite. In Theorem 5.25, we show that $P_n(\mathbb{K})$ is residually nilpotent for all $n \in \mathbb{N}$. This will allow us to determine the values of n for which $B_n(\mathbb{K})$ is residually nilpotent or residually soluble as follows.

Theorem 1.3. *Let $n \geq 1$. Then:*

- (1) $P_n(\mathbb{K})$ is residually nilpotent for all $n \geq 1$.
- (2) $B_n(\mathbb{K})$ is residually nilpotent if and only if $n \leq 2$, and residually soluble if and only if $n \leq 4$.

For a non-orientable surface M without boundary of higher genus, we may decide whether $B_n(M)$ is residually nilpotent or residually soluble using results of [4, 18].

Theorem 1.4. *Let $n, g \in \mathbb{N}$, and let M be a compact non-orientable surface of genus g without boundary. Then $B_n(M)$ is residually nilpotent if and only if $n \leq 2$, and is residually soluble if and only if $n \leq 4$.*

Although Theorem 1.4 contains Theorem 1.3(2) as a special case, we state the latter separately because the braid groups of the Klein bottle will be the focus of most of the second part of the paper.

The manuscript is organised as follows. In Section 2, we give presentations of the braid groups used in this paper, as well as the statement of Theorem 2.5 due to Gruenberg that will be required in the proofs of some of our results. Theorem 1.1 is proved in Section 3. In Section 4, we study the case of the torus and we prove Theorem 1.2. In Section 5, our focus is on the braid groups of the Klein bottle, and we use Theorem 1.1 in the proof of Theorem 1.3. Theorem 1.4 is proved in Section 6. If M is a compact surface different from \mathbb{K} and the Möbius band, the centre $Z(B_n(M))$ of $B_n(M)$ is known [6, 7, 14, 29, 31, 34]. We determine $Z(B_n(\mathbb{K}))$ in Proposition 5.2, and for the sake of completeness, in Proposition A1 of the Appendix, we compute the centre of the braid groups of the Möbius band.

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2. GENERALITIES

In this section, we give the presentations of the braid and pure braid groups that will be used in this paper. If $M = \mathbb{T}$ or \mathbb{K} , we will make use of the following presentations of $P_n(M)$ and $B_n(M)$.

Theorem 2.1 ([32]). *Let $n \geq 1$, and let M be the torus \mathbb{T} or the Klein bottle \mathbb{K} . The following constitutes a presentation of the pure braid group $P_n(M)$ of M :*

generators: $\{a_i, b_i, i = 1, \dots, n\} \cup \{C_{i,j}, 1 \leq i < j \leq n\}$.

relations:

$$\begin{aligned}
(1) \quad & a_i a_j = a_j a_i, \quad (1 \leq i < j \leq n) \\
(2) \quad & a_i^{-1} b_j a_i = b_j a_j C_{i,j}^{-1} C_{i+1,j} a_j^{-1}, \quad (1 \leq i < j \leq n) \\
(3) \quad & a_i^{-1} C_{j,k} a_i = \begin{cases} C_{j,k}, & (1 \leq i < j < k \leq n) \text{ or } (1 \leq j < k < i \leq n) \\ a_k C_{i+1,k}^{-1} C_{i,k} a_k^{-1} C_{j,k} C_{i,k}^{-1} C_{i+1,k}, & (1 \leq j \leq i < k \leq n) \end{cases} \\
(4) \quad & C_{i,l}^{-1} C_{j,k} C_{i,l} = \begin{cases} C_{j,k}, & (1 \leq i < l < j < k \leq n) \text{ or } (1 \leq j \leq i < l < k \leq n) \\ C_{i,k} C_{l+1,k}^{-1} C_{l,k} C_{i,k}^{-1} C_{j,k} C_{l,k}^{-1} C_{l+1,k}, & (1 \leq i < j \leq l < k \leq n) \end{cases} \\
(5) \quad & \begin{cases} \prod_{j=i+1}^n C_{i,j}^{-1} C_{i+1,j} = a_i b_i C_{1,i} a_i^{-1} b_i^{-1}, & (1 \leq i \leq n), \text{ if } M = \mathbb{T} \\ \prod_{j=i+1}^n C_{i,j} C_{i+1,j}^{-1} = b_i C_{1,i} a_i^{-1} b_i^{-1} a_i^{-1}, & (1 \leq i \leq n), \text{ if } M = \mathbb{K} \end{cases} \\
(6) \quad & \begin{cases} b_j b_i = b_i b_j, & (1 \leq i < j \leq n), \text{ if } M = \mathbb{T} \\ b_j b_i = b_i b_j C_{i,j} C_{i+1,j}^{-1}, & (1 \leq i < j \leq n), \text{ if } M = \mathbb{K} \end{cases} \\
(7) \quad & \begin{cases} b_i^{-1} a_j b_i = a_j b_j C_{i,j} C_{i+1,j}^{-1} b_j^{-1}, & (1 \leq i < j \leq n), \text{ if } M = \mathbb{T} \\ b_i^{-1} a_j b_i = a_j b_j (C_{i,j} C_{i+1,j}^{-1})^{-1} b_j^{-1}, & (1 \leq i < j \leq n), \text{ if } M = \mathbb{K} \end{cases} \\
(8) \quad & \begin{cases} b_i^{-1} C_{j,k} b_i = \begin{cases} C_{j,k}, & (1 \leq i < j < k \leq n) \text{ or } (1 \leq j < k < i \leq n) \\ C_{i+1,k} C_{i,k}^{-1} C_{j,k} b_k C_{i,k} C_{i+1,k}^{-1} b_k^{-1}, & (1 \leq j \leq i < k \leq n) \end{cases} & \text{if } M = \mathbb{T} \\ b_i^{-1} C_{j,k} b_i = \begin{cases} C_{j,k}, & (1 \leq i < j < k \leq n) \text{ or } (1 \leq j < k < i \leq n) \\ C_{i+1,k} C_{i,k}^{-1} C_{j,k} b_k (C_{i,k} C_{i+1,k}^{-1})^{-1} b_k^{-1}, & (1 \leq j \leq i < k \leq n) \end{cases} & \text{if } M = \mathbb{K}. \end{cases}
\end{aligned}$$

Theorem 2.2 ([32]). *Let $n \geq 1$, and let M be the torus \mathbb{T} or the Klein bottle \mathbb{K} . The following constitutes a presentation of the braid group $B_n(M)$ of M :*

generators: $a, b, \sigma_1, \dots, \sigma_{n-1}$.

relations:

$$\begin{aligned}
(1) \quad & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; \\
(2) \quad & \sigma_j \sigma_i = \sigma_i \sigma_j, \text{ if } |i - j| \geq 2; \\
(3) \quad & a \sigma_j = \sigma_j a, \text{ if } j \geq 2; \\
(4) \quad & b \sigma_j = \sigma_j b, \text{ if } j \geq 2; \\
(5) \quad & b^{-1} \sigma_1 a = \sigma_1 a \sigma_1 b^{-1} \sigma_1; \\
(6) \quad & a(\sigma_1 a \sigma_1) = (\sigma_1 a \sigma_1) a; \\
(7) \quad & \begin{cases} b(\sigma_1^{-1} b \sigma_1^{-1}) = (\sigma_1^{-1} b \sigma_1^{-1}) b, & \text{if } M = \mathbb{T}, \\ b(\sigma_1^{-1} b \sigma_1) = (\sigma_1^{-1} b \sigma_1) b, & \text{if } M = \mathbb{K}; \end{cases} \\
(8) \quad & \sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2 \sigma_1 = \begin{cases} b a b^{-1} a^{-1} & \text{if } M = \mathbb{T}, \\ b a^{-1} b^{-1} a^{-1} & \text{if } M = \mathbb{K}. \end{cases}
\end{aligned}$$

We consider the torus and the Klein bottle to be a square whose edges are identified as indicated in Figure 1. Geometric representatives of the generators of $P_n(\mathbb{T})$ and $P_n(\mathbb{K})$ given in Theorem 2.1 are illustrated in Figure 2, and may be interpreted as follows. For $1 \leq i \leq n$, the i^{th} string is the only non-trivial string of the braid a_i (resp. of b_i), and it passes through the edge α (resp. β). If $1 \leq i < j \leq n$, the j^{th} string is the only non-trivial string of the braid $C_{i,j}$, and it encircles all of the basepoints between the i^{th} and j^{th} points. If $i = j$, it will be convenient to define $C_{i,i}$ to be the trivial braid. The figures represent the projection of the braids onto M , so the constant paths in each figure correspond to vertical strings of the braid. The generators of $B_n(\mathbb{T})$ and $B_n(\mathbb{K})$ given in Theorem 2.2 may be taken to be the standard Artin generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n as shown in Figure 3, and $a = a_1$ and $b = b_1$. Various presentations of the braid and pure

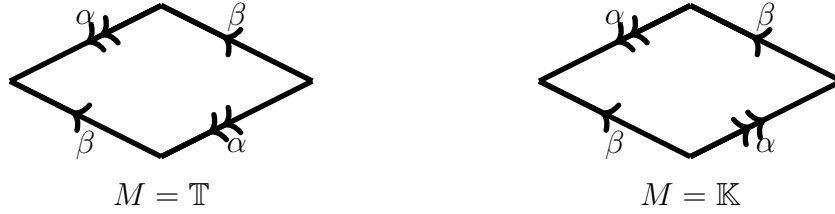


FIGURE 1. Squares representing \mathbb{T} and \mathbb{K}

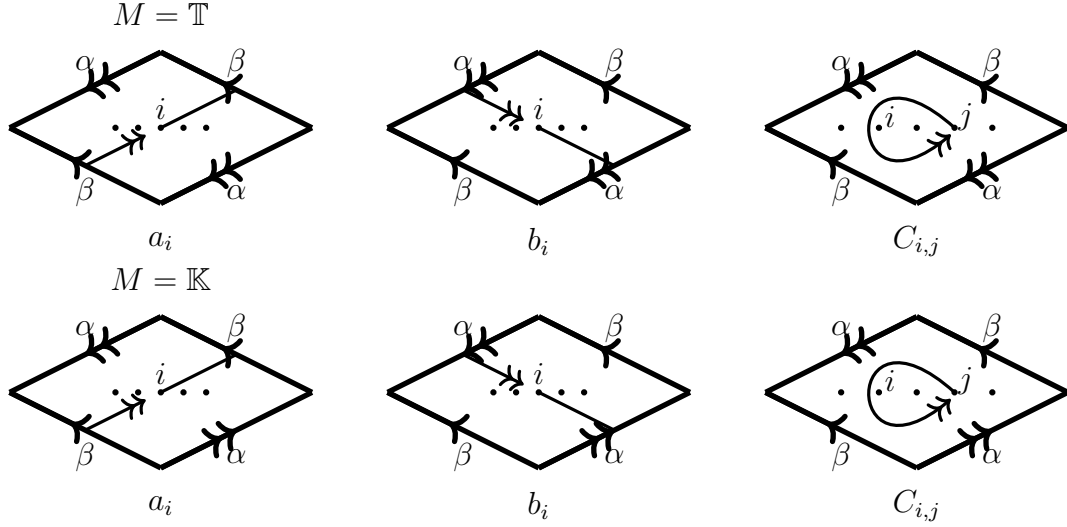


FIGURE 2. The generators of $P_n(\mathbb{T})$ and $P_n(\mathbb{K})$

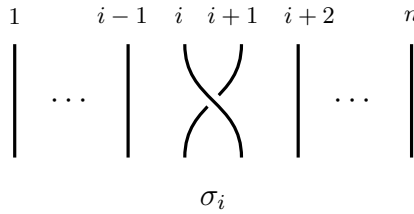


FIGURE 3. The braid σ_i

braid groups of the torus and the Klein bottle may be found in the literature [3, 6, 20, 33], but we choose to work with those of Theorems 2.1 and 2.2 because they highlight the similarities and differences between the braid groups of \mathbb{T} and \mathbb{K} . For example, the word $C_{i,j}C_{i+1,j}^{-1}$ that appears in our presentation of $P_n(\mathbb{T})$ is often replaced by its inverse in $P_n(\mathbb{K})$. To prove Theorem 2.1 (resp. Theorem 2.2), one may use the Fadell-Neuwirth short exact sequence (1.2) (resp. the short exact sequence (1.1)), induction on n , and the following standard method for obtaining a presentation of a group extension [24, Proposition 1, p. 139]. Given a short exact sequence $1 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 1$ and presentations $C = \langle X \mid R \rangle$ and $A = \langle Y \mid S \rangle$, then $B = \langle \tilde{X}, \tilde{Y} \mid \tilde{S}, \tilde{R}, \tilde{T} \rangle$, where $\tilde{Y} = \{\tilde{y} = i(y) : y \in Y\}$, $\tilde{X} = \{\tilde{x} : x \in X\}$ is a transversal for $\text{Im}(i)$ in B such that $p(\tilde{x}) = x$ for all $x \in X$, $\tilde{S} = \{\tilde{s} : s \in S\}$ is the set of words in \tilde{Y} obtained from S by replacing each letter y by \tilde{y} . For each $r \in R$, let \tilde{r} is the word in \tilde{X} obtained from r by replacing each letter x by \tilde{x} . Then $\tilde{r} \in \text{Ker}(p)$, so it may be written as a word, v_r say, in the elements of \tilde{Y} . Since $\text{Im}(i)$ is normal in B , for all $x \in X$ and $y \in Y$, $\tilde{x}^{-1}\tilde{y}\tilde{x} \in \text{Ker}(p)$, so may be written as a word, $w_{x,y}$ say, in the

elements of \tilde{Y} . Then $\tilde{R} = \{\tilde{r}v_r^{-1} : r \in R\}$ and $\tilde{T} = \{\tilde{x}^{-1}\tilde{y}\tilde{x}w_{x,y}^{-1} : x \in X, y \in Y\}$. The details of the proofs of Theorems 2.1 and 2.2 are left to the reader.

Remark 2.3. Using Theorem 2.2, it is straightforward to check that:

$$\begin{aligned} B_n(\mathbb{T})^{\text{Ab}} &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 = \langle \bar{a}, \bar{b}, \sigma : [\bar{a}, \bar{b}] = [\bar{a}, \sigma] = [\bar{b}, \sigma] = \sigma^2 = 1 \rangle \\ B_n(\mathbb{K})^{\text{Ab}} &\cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \bar{a}, \bar{b}, \sigma : [\bar{a}, \bar{b}] = [\bar{a}, \sigma] = [\bar{b}, \sigma] = \sigma^2 = \bar{a}^2 = 1 \rangle, \end{aligned}$$

for all $n \geq 2$, where \bar{a} (resp. \bar{b}, σ) represents the Γ_2 -coset of a (resp. of b, σ_1).

For compact non-orientable surfaces of genus $g \geq 3$ without boundary, we shall make use of the following presentation of their braid groups due to Bellingeri.

Theorem 2.4 ([3]). *Let N_g be a compact, connected non-orientable surface of genus $g \geq 3$ without boundary. The braid group $B_n(N_g)$ admits the following presentation:*

generators: $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g$.

relations:

- (1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
- (2) $\sigma_j \sigma_i = \sigma_i \sigma_j$, if $|i - j| \geq 2$.
- (3) $a_r \sigma_i = \sigma_i a_r$ ($1 \leq r \leq g; i \neq 1$).
- (4) $\sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1$ ($1 \leq r \leq g$).
- (5) $\sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1$ ($1 \leq s < r \leq g$).
- (6) $a_1^2 \cdots a_g^2 = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1$.

To prove some of our results, we will also require the following theorem of Gruenberg.

Theorem 2.5 ([22]). *Let \mathcal{P} denote one of the following classes:*

- (1) *the class of soluble groups.*
- (2) *the class of finite groups.*
- (3) *the class of p -finite groups for a given prime number p .*

Let K and H be groups, and suppose that K is \mathcal{P} and that H is residually \mathcal{P} . Then, for any group extension $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$, the group G is residually \mathcal{P} .

3. THE LOWER CENTRAL AND DERIVED SERIES OF SEMI-DIRECT PRODUCTS

The main aim of this section is to establish the general decomposition of the lower central series and an estimate of the derived series of an arbitrary semi-direct product given in the statement of Theorem 1.1, which will be used in later computations of the lower central and derived series of $P_n(\mathbb{K})$. We first prove two lemmas that will be used in what follows. If x_1, \dots, x_n are elements of a group G , we set:

$$[x_1, x_2, \dots, x_{n-1}, x_n] = \left[x_1, [x_2, \dots, [x_{n-1}, x_n]] \right],$$

and if X is a subset of G then we denote the normal closure of X in G by $\langle\langle X \rangle\rangle_G$.

Lemma 3.1. *Let G be a group, and let $x, y \in G$. For all $n \in \mathbb{N}$, we have:*

$$[x^{2^n}, y] = [x, x, x^2, \dots, x^{2^{n-1}}, y] \cdot [x, x^2, \dots, x^{2^{n-1}}, y]^2 \cdot [x^2, \dots, x^{2^{n-1}}, y]^2 \cdots [x^{2^{n-1}}, y]^2. \quad (3.1)$$

Proof. We prove the lemma by induction on n . Observe that:

$$[x^2, y] = x.x.y.x^{-1}.x^{-1}.y^{-1} = x[x, y]yx^{-1}y^{-1} = x[x, y]x^{-1}[x, y] = [x, x, y] \cdot [x, y]^2, \quad (3.2)$$

which proves (3.1) in the case $n = 1$. Now let $n \geq 2$, and suppose that the result holds for all $1 \leq i \leq n$. Applying (3.2) to the elements x^{2^n} and $[x^{2^n}, y]$, we have:

$$[x^{2^{n+1}}, y] = [(x^{2^n})^2, y] = [x^{2^n}, x^{2^n}, y][x^{2^n}, y]^2,$$

and applying (3.2) to the elements x^{2^n} and y , we obtain:

$$[x^{2^n}, [x^{2^n}, y]] = [x, x, x^2, \dots, x^{2^{n-1}}, [x^{2^n}, y]] [x, x^2, \dots, x^{2^{n-1}}, [x^{2^n}, y]]^2 \cdots [x^{2^{n-1}}, [x^{2^n}, y]]^2.$$

Thus:

$$[x^{2^{n+1}}, y] = [x, x, x^2, \dots, x^{2^{n-1}}, [x^{2^n}, y]] [x, x^2, \dots, x^{2^{n-1}}, [x^{2^n}, y]]^2 \cdots [x^{2^{n-1}}, [x^{2^n}, y]]^2 [x^{2^n}, y]^2,$$

which completes the proof by induction. \square

Remark 3.2. With the notation of Theorem 1.1, In what follows, for the groups K_n , H_n or \tilde{H}_n , we will use the word *generator* to mean a word of the form $\varphi(g)(h).h^{-1}$, where $g \in \Gamma_{n-1}(G)$ and $h \in H$, $g \in G$ and $h \in V_{n-1}$, or $g \in \Gamma_{n-1}(G)$ and $h \in H$ respectively. Similarly, a generator of the group L_n (resp. V_n) will mean either a generator of K_n or H_n , or an element of the form $[h, l]$, where $h \in H$ and $l \in L_{n-1}$ (resp. either a generator of \tilde{H}_n , or an element of the form $[h, v]$, where $h \in H$ and $v \in V_{n-1}$).

Lemma 3.3. *Let $n \geq 2$. With the notation of Theorem 1.1, the subgroups K_n , L_n and V_n are normal in H for all $n \geq 2$, and we have the inclusions $K_{n+1} \subset K_n$, $H_{n+1} \subset H_n$, $\tilde{H}_{n+1} \subset \tilde{H}_n$, $L_{n+1} \subset L_n$ and $V_{n+1} \subset V_n$.*

Proof. The proof is by induction on n . The proof in the case $n = 2$ was given in [15, Proposition 3.3]. So suppose that $n \geq 2$, and assume that L_n (resp. V_n) is a normal subgroup of H , let $x \in L_{n+1}$ (resp. V_{n+1}) and let $h \in H$. Since $L_{n+1} = \langle K_{n+1}, H_{n+1}, [H, L_n] \rangle$ (resp. $V_{n+1} = \langle \tilde{H}_{n+1}, [H, V_n] \rangle$), it suffices to show that $h x h^{-1} \in L_{n+1}$, where x is a generator of K_{n+1}, H_{n+1} or $[H, L_n]$ (resp. of \tilde{H}_{n+1} or $[H, V_n]$), in the sense of Remark 3.2.

- Suppose that $x = \varphi(g)(y)y^{-1} \in K_{n+1}$, where $g \in \Gamma_n(G)$ and $y \in H$. Then $\varphi(g) \in \text{Aut}(H)$, and there exists $h' \in H$ such that $\varphi(g)(h') = h$, so:

$$h x h^{-1} = h(\varphi(g)(y).y^{-1})h^{-1} = (\varphi(g)(h'y)y^{-1}h'^{-1})(\varphi(g)(h')h'^{-1})^{-1} \in K_{n+1} \subset L_{n+1}.$$

This also implies that K_n is a normal subgroup of H for all $n \geq 2$.

- Suppose that $x = \varphi(g)(y)y^{-1}$ is an element of H_{n+1} (resp. of \tilde{H}_{n+1}), where $g \in G$ and $y \in L_n$ (resp. $y \in V_n$), and let $h' \in H$ be such that $\varphi(g)(h') = h$. Then:

$$\begin{aligned} h x h^{-1} &= h(\varphi(g)(y).y^{-1})h^{-1} \\ &= (\varphi(g)(h'yh'^{-1}).(h'yh'^{-1})^{-1})[h', y][y, \varphi(g)(h')] \in L_{n+1} \text{ (resp. } V_{n+1}), \end{aligned}$$

because $h'yh'^{-1} \in L_n$ (resp. V_n) by the normality of L_n (resp. V_n) in H using the induction hypothesis.

- Suppose that $x = [y, l] \in [H, L_n]$ (resp. $[H, V_n]$), where $y \in H$ and $l \in L_n$ (resp. $l \in V_n$). Then:

$$h x h^{-1} = [h y h^{-1}, h l h^{-1}] \in [H, L_n] \subset L_{n+1} \text{ (resp. } [H, V_n] \subset V_{n+1}),$$

because $h l h^{-1} \in L_n$ (resp. V_n) by the normality of L_n (resp. V_n) in H .

This proves that L_n (resp. V_n) is a normal subgroup of H for all $n \geq 2$.

To prove the second part of the statement, notice that the inclusion $\Gamma_n(G) \subset \Gamma_{n-1}(G)$ implies that $K_{n+1} \subset K_n$ for all $n \geq 2$. It is straightforward to see that $H_3 \subset H_2$ (resp. $\tilde{H}_3 \subset \tilde{H}_2$) because $L_2 \subset H$ (resp. $V_2 \subset H$). By induction, suppose that $H_n \subset H_{n-1}$ (resp. $\tilde{H}_n \subset \tilde{H}_{n-1}$) for some $n \geq 3$. Since L_{n-1} (resp. V_{n-1}) is normal in H , we have $[H, L_{n-1}] \subset L_{n-1}$ (resp. $[H, V_{n-1}] \subset V_{n-1}$). Further, using the definitions and the induction hypothesis, we have the inclusions $K_n \subset K_{n-1} \subset L_{n-1}$ and $H_n \subset H_{n-1} \subset L_{n-1}$ (resp. $\tilde{H}_n \subset \tilde{H}_{n-1} \subset V_{n-1}$). It follows that $L_n \subset L_{n-1}$ (resp. $V_n \subset V_{n-1}$), and then that $H_{n+1} \subset H_n$ (resp. $\tilde{H}_{n+1} \subset \tilde{H}_n$). Consequently, $L_{n+1} \subset L_n$ and $V_{n+1} \subset V_n$ for all $n \geq 2$. \square

Proof of Theorem 1.1. The proof is by induction on n . The case $n = 1$ is trivial. If $n = 2$, part (1) was proved in [15, Proposition 3.3], and part (2) follows from part (1) and the fact that $L_2 = V_2$. Now suppose that parts (1) and (2) hold for some $n \geq 2$, and let us prove the result for $n + 1$. Let $\varphi: \Gamma_n(G) \rightarrow \text{Aut}(L_n)$ be the action (also denoted by φ) induced by φ such that

$L_n \rtimes_{\varphi} \Gamma_n(G) = \Gamma_n(H \rtimes_{\varphi} G)$. We claim that φ also induces an action $\varphi: \Gamma_{n+1}(G) \rightarrow \text{Aut}(L_{n+1})$. To see this, let $g \in \Gamma_{n+1}(G)$. To prove that $\varphi(g)(L_{n+1}) \subset L_{n+1}$, it suffices to take $x \in L_{n+1}$ to be of the form $x = \varphi(g')(h).h^{-1}$, where either $g' \in \Gamma_n(G)$ and $h \in H$, or $g' \in G$ and $h \in L_n$, or of the form $x = [h, l] \in [H, L_n]$, where $h \in H$ and $l \in L_n$. The result will then follow for all elements of L_{n+1} because $\varphi(g)$ is a homomorphism.

- If $x = \varphi(g')(h).h^{-1} \in K_{n+1}$, where $g' \in \Gamma_n(G)$ and $h \in H$, or $x = \varphi(g')(h).h^{-1} \in H_{n+1}$, where $g' \in G$ and $h \in L_n$, then:

$$\varphi(g)(x) = \varphi(g) (\varphi(g')(h).h^{-1}) = (\varphi(gg')(h).h^{-1}) (\varphi(g)(h).h^{-1})^{-1}.$$

If $g' \in \Gamma_n(G)$ and $h \in H$ then $\varphi(g)(x) \in K_{n+1} \subset L_{n+1}$ since g and gg' belong to $\Gamma_n(G)$. If $g' \in G$ and $h \in L_n$, then $\varphi(g)(x) \in H_{n+1} \subset L_{n+1}$ because $h \in L_{n+1}$.

- If $x = [h, l] \in [H, L_n]$, where $h \in H$ and $l \in L_n$ then:

$$\varphi(g)(x) = [\varphi(g)(h), \varphi(g)(l)] \in [H, L_n],$$

since $g \in \Gamma_{n+1}(G) \subset \Gamma_n(G)$, so $\varphi(g)(l) \in L_n$.

Since $\varphi(g): L_{n+1} \rightarrow L_{n+1}$ is the restriction of an automorphism, it is injective, so to show that it is an automorphism, it suffices to prove surjectivity. We first consider the following two cases:

- (a) If $x = \varphi(g')(h).h^{-1}$, where either $g' \in \Gamma_n(G)$ and $h \in H$, or $g' \in G$ and $h \in L_n$, let:

$$y = (\varphi(g^{-1}g')(h).h^{-1}) h (\varphi(g^{-1})(h^{-1}).h) h^{-1} \in L_{n+1},$$

because K_{n+1} and L_{n+1} are normal in H , and one may check that $\varphi(g)(y) = x$.

- (b) If $x = [h, l] \in [H, L_n]$, where $h \in H$ and $l \in L_n$, there exist $l' \in L_n$ and $h' \in H$ such that $\varphi(g)(l') = l$ and $\varphi(g)(h') = h$ by the induction hypothesis. Taking $y = [h', l'] \in [H, L_n] \subset L_{n+1}$, we see that $\varphi(g)(y) = x$.

This shows that if x is a generator of K_{n+1}, H_{n+1} or $[H, L_n]$, there exists $y \in L_{n+1}$ such that $\varphi(g)(y) = x$. Given an arbitrary element $x \in L_{n+1}$, there exist x_1, \dots, x_s , each of which satisfies one of the conditions of cases (a) and (b) above, such that $x = x_1 \cdots x_s$. So for $i = 1, \dots, s$, there exists $y_i \in L_{n+1}$ such that $\varphi(g)(y_i) = x_i$, and we have $\varphi(g)(y_1 \cdots y_s) = x$, which proves the surjectivity of $\varphi(g): L_{n+1} \rightarrow L_{n+1}$. Therefore the semi-direct product $L_{n+1} \rtimes_{\varphi} \Gamma_{n+1}(G)$ is well defined. Similar computations show that the same is true for the semi-direct product $V_{n+1} \rtimes_{\varphi} (G)^{(n)}$.

To complete the proof of part (1) of Theorem 1.1, it remains to show that $L_{n+1} \rtimes_{\varphi} \Gamma_{n+1}(G) = \Gamma_{n+1}(H \rtimes_{\varphi} G)$. We first prove that $L_{n+1} \rtimes_{\varphi} \Gamma_{n+1}(G) \subset \Gamma_{n+1}(H \rtimes_{\varphi} G)$. Let $(x, g) \in L_{n+1} \rtimes_{\varphi} \Gamma_{n+1}(G)$, where $x \in L_{n+1}$ and $g \in \Gamma_{n+1}(G)$. Since $(x, g) = (x, 1)(1, g)$, it suffices to show that $(x, 1)$ and $(1, g)$ belong to $\Gamma_{n+1}(H \rtimes_{\varphi} G)$. Clearly, $(1, g) \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$. Further, $(x, 1)$ is a product of elements each of which is of one of the following forms:

- $(\varphi(g)(h).h^{-1}, 1) = [(1, g), (h, 1)]$, where $g \in \Gamma_n(G)$, $h \in H$, and $(1, g) \in \Gamma_n(H \rtimes_{\varphi} G)$. Then $(\varphi(g)(h).h^{-1}, 1) \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$.
- $(\varphi(g)(h).h^{-1}, 1) = [(1, g), (h, 1)]$, where $g \in G$ and $h \in L_n$. Then $(h, 1) \in L_n \rtimes_{\varphi} \Gamma_n(G) = \Gamma_n(H \rtimes_{\varphi} G)$ by the induction hypothesis, and $(\varphi(g)(h).h^{-1}, 1) \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$.
- $([h, l], 1) \in [H, L_n]$, where $h \in H$ and $l \in L_n$. Then $([h, l], 1) = [(h, 1), (l, 1)]$, and $l \in L_n \rtimes_{\varphi} \Gamma_n(G) = \Gamma_n(H \rtimes_{\varphi} G)$ by the induction hypothesis, so $([h, l], 1) \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$.

Since all of these elements belong to $\Gamma_{n+1}(H \rtimes_{\varphi} G)$, it follows that $(x, 1) \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$, whence $L_{n+1} \rtimes_{\varphi} \Gamma_{n+1}(G) \subset \Gamma_{n+1}(H \rtimes_{\varphi} G)$.

For the other inclusion, let $[(h, g), (x, y)] \in \Gamma_{n+1}(H \rtimes_{\varphi} G)$, where $(h, g) \in H \rtimes_{\varphi} G$ and $(x, y) \in \Gamma_n(H \rtimes_{\varphi} G)$. By the induction hypothesis, $\Gamma_n(H \rtimes_{\varphi} G) = L_n \rtimes_{\varphi} \Gamma_n(G)$, so $x \in L_n$ and $y \in \Gamma_n(G)$, and thus:

$$[(h, g), (x, y)] = (h.\varphi(g)(x).\varphi(gyg^{-1})(h^{-1}).\varphi([g, y])(x^{-1}), [g, y]). \quad (3.3)$$

The second factor $[g, y]$ on the right-hand side of (3.3) belongs to $\Gamma_{n+1}(H \rtimes_{\varphi} G)$, and the first factor, denoted by ρ , may be written in the following form:

$$\rho = [h, x].xhx^{-1}(\varphi(g)(x).x^{-1})xh^{-1}x^{-1}.xh(\varphi(gyg^{-1})(h^{-1}).h)h^{-1}x^{-1}.x(\varphi([g, y])(x^{-1}).x)x^{-1}.$$

Note that:

- $[h, x] \in [H, L_n] \subseteq L_{n+1}$, since $h \in H$ and $x \in L_n$.
- $\varphi(g)(x).x^{-1} \in H_{n+1} \subseteq L_{n+1}$, since $x \in L_n$.
- $\varphi(gyg^{-1})(h^{-1}).h \in K_{n+1} \subseteq L_{n+1}$, since $y \in \Gamma_n(G)$, so $gyg^{-1} \in \Gamma_n(G)$ because $\Gamma_n(G)$ is a normal subgroup of G .
- $\varphi([g, y])(x^{-1}).x \in H_{n+1} \cap K_{n+1} \subseteq L_{n+1}$, since $x \in L_n$ and $[g, y] \in \Gamma_n(G)$.

By Lemma 3.3, the conjugates by elements of H of the elements $[h, x]$, $\varphi(g)(x).x^{-1}$, $\varphi(gyg^{-1})(h^{-1}).h$ and $\varphi([g, y])(x^{-1}).x$ also belong to L_{n+1} , therefore $\rho \in L_{n+1}$ as required. This proves part (1) of the statement.

To prove part (2), suppose by induction that $(H \rtimes_{\varphi} G)^{(n-1)} \subset V_n \rtimes_{\varphi} G^{(n-1)}$. Then:

$$(H \rtimes_{\varphi} G)^{(n)} = [(H \rtimes_{\varphi} G)^{(n-1)}, (H \rtimes_{\varphi} G)^{(n-1)}] \subset [V_n \rtimes_{\varphi} G^{(n-1)}, V_n \rtimes_{\varphi} G^{(n-1)}].$$

To show that $[V_n \rtimes_{\varphi} G^{(n-1)}, V_n \rtimes_{\varphi} G^{(n-1)}] \subset V_{n+1} \rtimes_{\varphi} G^{(n)}$, let $(h, g), (x, y) \in V_n \rtimes_{\varphi} G^{(n-1)}$. Then:

- $[h, x] \in [H, V_n] \subseteq V_{n+1}$ because $h, x \in V_n \subset H$.
- the three elements $\varphi(g)(x).x^{-1}$, $\varphi(gyg^{-1})(h^{-1}).h$ and $\varphi([g, y])(x^{-1}).x$ belong to \tilde{H}_{n+1} because $h, x \in V_n$, so they belong to V_{n+1} .

Arguing in a manner similar to that for part (1) from (3.3) onwards, it follows that $[(h, g), (x, y)] \in V_{n+1} \rtimes_{\varphi} G^{(n)}$ as required. \square

The following lemma will help us simplify some of the calculations in the following sections.

Lemma 3.4. *With the notation of Theorem 1.1, let \tilde{G} be a subgroup of G , let \tilde{H} be a subgroup of H , and let X (resp. Y) be a generating set of \tilde{G} (resp. \tilde{H}).*

(1) *The subgroup*

$$\langle \varphi(g)(h).h^{-1} : g \in \tilde{G}, h \in \tilde{H} \rangle$$

is contained in the normal closure $\langle\langle Z \rangle\rangle_H$ of

$$Z = \{ \varphi(g)(h).h^{-1} : g \in X, h \in Y \}$$

in H . In particular, if this subgroup is a normal subgroup of G , it is equal to $\langle\langle Z \rangle\rangle_H$. Consequently, if X (resp. Y) is a generating set of $\Gamma_{n-1}(G)$ (resp. of H) then to calculate the subgroup K_n , it suffices to compute the elements $\varphi(g)(h).h^{-1}$, where $g \in X$ and $h \in Y$.

(2) *Let W be a subset of H such that $L_n = \langle\langle W \rangle\rangle_H$ (resp. $V_n = \langle\langle W \rangle\rangle_H$) is the normal closure of W in H . Let X (resp. Y) be a generating set of G (resp. of H). Then H_{n+1} is contained in $\langle\langle \{ \varphi(g)(w).w^{-1} : g \in X, w \in W \} \rangle\rangle_H \cup [H, L_n]$ (resp. \tilde{H}_{n+1} is contained in $\langle\langle \{ \varphi(g)(w).w^{-1} : g \in X, w \in W \} \rangle\rangle_H \cup [H, V_n]$). Therefore:*

$$L_{n+1} = \langle\langle K_{n+1}, \varphi(g)(w).w^{-1}, [h, w] : g \in X, h \in Y, w \in W \rangle\rangle_H$$

$$V_{n+1} = \langle\langle \varphi(g)(w).w^{-1}, [h, w] : g \in X, h \in Y, w \in W \rangle\rangle_H.$$

Remark 3.5. With the notation of Lemma 3.4(1), we will say that the elements of Z are generators of the subgroup $\langle\langle Z \rangle\rangle_H$. It follows from part (2) that to determine L_{n+1} and V_{n+1} , we need only compute K_{n+1} in the case of L_{n+1} , and calculate the elements of the set $\{ \varphi(g)(w).w^{-1}, [h, w] : g \in X, w \in W \}$.

Proof of Lemma 3.4.

(1) To prove the first part of the statement, note that it suffices to prove the result for elements of the subgroup of the form $\varphi(g)(h).h^{-1}$, where $g \in \tilde{G}$ and $h \in \tilde{H}$. If $g \in G$, there exist $g_1, \dots, g_p \in \tilde{G}$ and $\epsilon_1, \dots, \epsilon_p \in \{1, -1\}$ such that $g_i^{\epsilon_i} \in X$ for all $i = 1, \dots, p$ and $g = g_1^{\epsilon_1} \cdots g_p^{\epsilon_p}$. Now:

$$\varphi(g)(h).h^{-1} = \prod_{i=1}^p \varphi(g_i^{\epsilon_i}) \left(\varphi(g_{i+1}^{\epsilon_{i+1}} \cdots g_p^{\epsilon_p})(h) \right) \cdot \left(\varphi(g_{i+1}^{\epsilon_{i+1}} \cdots g_p^{\epsilon_p})(h) \right)^{-1}$$

$$= \prod_{i=1}^p \varphi(g_i^{\varepsilon_i})(h'_i).h_i'^{-1}, \quad (3.4)$$

where for all $i = 1, \dots, p$, $h'_i = \varphi(g_{i+1}^{\varepsilon_{i+1}} \cdots g_p^{\varepsilon_p})(h)$. Further, for all $h' \in \tilde{H}$, there exist $h_1, \dots, h_q \in \tilde{H}$ and $\delta_1, \dots, \delta_q \in \{1, -1\}$ such that $h_j^{\delta_j} \in Y$ for all $j = 1, \dots, q$ and $h' = h_1^{\delta_1} \cdots h_q^{\delta_q}$. Since

$$\varphi(g')(h').h^{-1} = \prod_{j=1}^q \left(h_1^{\delta_1} \cdots h_{j-1}^{\delta_{j-1}} (\varphi(g')(h_j^{\delta_j}).h_j^{-\delta_j}) h_{j-1}^{-\delta_{j-1}} \cdots h_1^{-\delta_1} \right) \quad (3.5)$$

for all $g' \in G$, the first part of the statement follows by combining (3.4) and (3.5). The second and third parts are consequences of the first part.

- (2) Let $\varphi(g)(h).h^{-1} \in H_{n+1}$ (resp. \tilde{H}_{n+1}), where $g \in G$ and $h \in L_n$ (resp. V_n). As in (1) above, (3.4) holds. For all $h' \in L_n$ (resp. V_n), there exist $x_1, \dots, x_q \in W$, $\delta_1, \dots, \delta_q \in \{1, -1\}$ and $\alpha_1, \dots, \alpha_q \in H$, such that $x_j^{\delta_j} \in W$ and $h' = \alpha_1 x_1^{\delta_1} \alpha_1^{-1} \cdots \alpha_q x_q^{\delta_q} \alpha_q^{-1}$. Then we obtain an equation similar to (3.5), where for all $j = 1, \dots, q$, $h_j^{\delta_j}$ is replaced by $\alpha_j x_j^{\delta_j} \alpha_j^{-1}$. Further, for all $j = 1, \dots, q$, $\varphi(g')(\alpha_j x_j^{\delta_j} \alpha_j^{-1}).(\alpha_j x_j^{\delta_j} \alpha_j^{-1})^{-1}$ is equal to:

$$\varphi(g')(\alpha_j) (\varphi(g')(x_j^{\delta_j}).x_j^{-\delta_j}) \varphi(g')(\alpha_j^{-1}) \cdot \underbrace{\alpha_j [\alpha_j^{-1} \varphi(g')(\alpha_j), x_j^{\delta_j}] \alpha_j^{-1}}_{\in [H, L_n]}. \quad (3.6)$$

Part (1) then follows from (3.4), (3.5) and (3.6). \square

4. THE CASE OF THE TORUS

In this section, we study the derived series of $B_n(\mathbb{T})$, the aim being to prove Theorem 1.2. We shall consider two cases, $n \leq 4$ and $n \geq 5$.

Proposition 4.1. *If $n \leq 4$ then $B_n(\mathbb{T})$ is residually soluble.*

Proof. If $n \leq 4$, the result follows by using the short exact sequence (1.1), Theorem 2.5, the solubility of S_n if $n \leq 4$, and the fact that $P_n(\mathbb{T})$ is residually soluble for all $n \geq 1$ [5, Theorem 4]. \square

To study the case $n \geq 5$, we start by exhibiting a presentation of $(B_n(\mathbb{T}))^{(1)}$.

Proposition 4.2. *A presentation of $(B_n(\mathbb{T}))^{(1)}$ is given by:*

generators: for $k, m \in \mathbb{Z}$ and $i = 1, \dots, n-1$:

- $b_{k,m} = b^k a^m b a^{-m} b^{-k-1}$
- $d_{k,m} = b^k a^m \sigma_1 b \sigma_1^{-1} a^{-m} b^{-1-k}$
- $a_{k,m} = b^k a^m (\sigma_1 a \sigma_1^{-1} a^{-1}) a^{-m} b^{-k}$
- $\theta_{i,k,m} = b^k a^m \sigma_i \sigma_i^{-1} a^{-m} b^{-k}$
- $\rho_{i,k,m} = b^k a^m \sigma_1 \sigma_i a^{-m} b^{-k}$

relations:

- (1) $\begin{cases} \theta_{i,k,m} \rho_{i+1,k,m} \theta_{i,k,m} = \theta_{i+1,k,m} \rho_{i,k,m} \theta_{i+1,k,m} \\ \rho_{i,k,m} \theta_{i+1,k,m} \rho_{i,k,m} = \rho_{i+1,k,m} \theta_{i,k,m} \rho_{i+1,k,m} \end{cases}$
- (2) $\begin{cases} \theta_{i,k,m} \rho_{j,k,m} = \theta_{j,k,m} \rho_{i,k,m} \\ \rho_{i,k,m} \theta_{j,k,m} = \rho_{j,k,m} \theta_{i,k,m} \end{cases} \text{ if } |i-j| \geq 2.$
- (3) $\begin{cases} a_{k,m} = \theta_{j,k,m}^{-1} \theta_{j,k,m+1} \\ a_{k,m} = \rho_{j,k,m} \rho_{j,k,m+1}^{-1} \end{cases} \text{ for } j \geq 2.$
- (4) $\begin{cases} b_{k,m} \theta_{j,k+1,m} = \theta_{j,k,m} d_{k,m} \\ d_{k,m} \rho_{j,k+1,m} = \rho_{j,k,m} b_{k,m} \end{cases} \text{ for } j \geq 2.$
- (5) $\begin{cases} b_{k-1,m}^{-1} a_{k-1,m} b_{k-1,m+1} \rho_{1,k,m+1}^{-1} a_{k,m}^{-1} = 1 \\ d_{k-1,m}^{-1} \rho_{1,k-1,m} \rho_{1,k-1,m+1}^{-1} d_{k-1,m+1} \rho_{1,k,m}^{-1} = 1 \end{cases}$

$$\begin{aligned}
 (6) \quad & \begin{cases} a_{k,m+1}\rho_{1,k,m+2} = a_{k,m}\rho_{1,k,m+1} \\ \rho_{1,k,m}a_{k,m+1} = a_{k,m}\rho_{1,k,m+1} \end{cases} \\
 (7) \quad & \begin{cases} b_{k,m}\rho_{1,k+1,m}^{-1}d_{k+1,m} = \rho_{1,k,m}^{-1}d_{k,m}b_{k+1,m} \\ b_{k,m}\rho_{1,k+1,m}^{-1}d_{k+1,m} = d_{k,m}b_{k+1,m}\rho_{1,k+2,m}^{-1} \end{cases} \\
 (8) \quad & \text{if } n \text{ is odd:} \\
 & \begin{cases} \theta_{1,k,m}\rho_{2,k,m}\theta_{3,k,m} \cdots \rho_{n-1,k,m}\theta_{n-1,k,m} \cdots \rho_{3,k,m}\theta_{2,k,m}\rho_{1,k,m} = b_{k,m}b_{k,m+1}^{-1} \\ \rho_{1,k,m}\theta_{2,k,m}\rho_{3,k,m} \cdots \theta_{n-1,k,m}\rho_{n-1,k,m} \cdots \theta_{3,k,m}\rho_{2,k,m}\theta_{1,k,m} = d_{k,m}a_{k+1,m}d_{k,m+1}^{-1}a_{k,m}^{-1} \end{cases} \\
 (9) \quad & \text{if } n \text{ is even:} \\
 & \begin{cases} \theta_{1,k,m}\rho_{2,k,m}\theta_{3,k,m} \cdots \theta_{n-1,k,m}\rho_{n-1,k,m} \cdots \rho_{3,k,m}\theta_{2,k,m}\rho_{1,k,m} = b_{k,m}b_{k,m+1}^{-1} \\ \rho_{1,k,m}\theta_{2,k,m}\rho_{3,k,m} \cdots \rho_{n-1,k,m}\theta_{n-1,k,m} \cdots \theta_{3,k,m}\rho_{2,k,m}\theta_{1,k,m} = d_{k,m}a_{k+1,m}d_{k,m+1}^{-1}a_{k,m}^{-1} \end{cases}
 \end{aligned}$$

Proof. One applies the Reidemeister-Schreier rewriting process [23, Appendix 1] to the short exact sequence:

$$1 \longrightarrow (B_n(\mathbb{T}))^{(1)} \longrightarrow B_n(\mathbb{T}) \longrightarrow \underbrace{B_n(\mathbb{T})^{\text{Ab}}}_{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2} \longrightarrow 1,$$

using the presentation of the group $B_n(\mathbb{T})$ given in Theorem 2.2, and taking the Schreier transversal to be $\{b^k a^m; b^k a^m \sigma_1 : k, m \in \mathbb{Z}\}$. The details are left to the reader. \square

Proposition 4.3. *If $n \geq 5$, then $B_n(\mathbb{T})$ is not residually soluble. Moreover, $(B_n(\mathbb{T}))^{(2)} = (B_n(\mathbb{T}))^{(3)}$.*

Theorem 1.2 then follows directly from Propositions 4.1 and 4.3.

Proof of Proposition 4.3. The first step is a standard procedure that may be found in [15, Theorem 1.4, p. 3389], and uses just the Artin relations and some properties of the derived series. For future reference, we note that it may also be applied to the braid groups of non-orientable surfaces. If M is a compact surface, consider the following short exact sequence:

$$1 \longrightarrow \frac{(B_n(M))^{(1)}}{(B_n(M))^{(2)}} \xrightarrow{i} \frac{B_n(M)}{(B_n(M))^{(2)}} \xrightarrow{p} B_n(M)^{\text{Ab}} \longrightarrow 1,$$

where p is the canonical projection. By using the above-mentioned procedure, for $i = 1, \dots, n-1$, the $(B_n(M))^{(2)}$ -cosets of the σ_i coincide in $B_n(M)/(B_n(M))^{(2)}$, and are equal to an element that we denote by σ .

Now suppose that $M = \mathbb{T}$. Using relations (3) and (4) of Theorem 2.2, the $(B_n(\mathbb{T}))^{(2)}$ -cosets of a and b commute with σ in $B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(2)}$. Using this fact and relations (5) and (8) of Theorem 2.2, it follows that $\sigma^{-2} = bab^{-1}a^{-1}$ and $\sigma^{2n} = 1$, and so σ has order at most $2n$. To show that the order of σ in $B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(2)}$ is exactly $2n$, using Proposition 4.2, we note that:

$$\frac{(B_n(\mathbb{T}))^{(1)}/(B_n(\mathbb{T}))^{(2)}}{\Theta} = \langle \rho_{1,0,0} \rangle \cong \mathbb{Z}_n, \quad (4.1)$$

where Θ is the normal closure in $(B_n(\mathbb{T}))^{(1)}/(B_n(\mathbb{T}))^{(2)}$ of the $(B_n(\mathbb{T}))^{(2)}$ -cosets of the elements of the set $\{\theta_{i,k,m}, k, m \in \mathbb{Z}, i = 1, \dots, n-1\}$. Let q be the canonical projection of $(B_n(\mathbb{T}))^{(1)}/(B_n(\mathbb{T}))^{(2)}$ onto $\frac{((B_n(\mathbb{T}))^{(1)}/(B_n(\mathbb{T}))^{(2)})}{\Theta}$. The order of σ in $B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(2)}$ is even because $p(\sigma)$ is the generator of \mathbb{Z}_2 . Suppose that the order of σ is $2r$, where $r < n$. Then $i(\rho_{1,0,0}) = \sigma^2$, and $i(\rho_{1,0,0}^r) = \sigma^{2r} = 1$. Since i is injective, $\rho_{1,0,0}^r = 1$, and it follows that $1 = q(\rho_{1,0,0}^r) = \rho_{1,0,0}^r$ in $\frac{((B_n(\mathbb{T}))^{(1)}/(B_n(\mathbb{T}))^{(2)})}{\Theta}$. Thus $\rho_{1,0,0}$ is of order $r < n$, which contradicts (4.1). Hence:

$$B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(2)} = \langle \sigma, a, b : [a, \sigma] = [b, \sigma] = \sigma^{2n} = 1, [b, a] = \sigma^{-2} \rangle.$$

To complete the proof, consider the short exact sequence:

$$1 \longrightarrow \frac{(B_n(\mathbb{T}))^{(2)}}{(B_n(\mathbb{T}))^{(3)}} \longrightarrow \frac{B_n(\mathbb{T})}{(B_n(\mathbb{T}))^{(3)}} \xrightarrow{\tilde{p}} \frac{B_n(\mathbb{T})}{(B_n(\mathbb{T}))^{(2)}} \longrightarrow 1,$$

where \tilde{p} is the canonical projection. Then $\tilde{p}(\sigma_i) = \sigma$ for all $i = 1, \dots, n-1$, and as above, we see that for $i = 1, \dots, n-1$, the $(B_n(\mathbb{T}))^{(3)}$ -cosets of the σ_i coincide in $B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(3)}$, and are equal to an element that we denote by $\tilde{\sigma}$. Using the same relations of $B_n(\mathbb{T})$ as before, it follows that the $(B_n(\mathbb{T}))^{(3)}$ -cosets of a and b commute with $\tilde{\sigma}$ in $B_n(\mathbb{T})/(B_n(\mathbb{T}))^{(3)}$, $[b, a] = \sigma^{-2}$ and $\sigma^{2n} = 1$, so \tilde{p} is an isomorphism, and hence $(B_n(\mathbb{T}))^{(2)} = (B_n(\mathbb{T}))^{(3)}$. We conclude that $B_n(\mathbb{T})$ is not residually soluble if $n \geq 5$, because $(B_n(\mathbb{T}))^{(2)}$ is non trivial. \square

5. THE CASE OF THE KLEIN BOTTLE

In this section, we study the lower central and derived series of the (pure) braid groups of the Klein bottle, and we prove Theorem 1.3. In Section 5.1, we exhibit an algebraic section of the Fadell-Neuwirth short exact sequence (1.2) for the Klein bottle, and we determine the centre of $B_n(\mathbb{K})$. These results will be used in the rest of the section, and we believe that they are interesting in their own right. In Section 5.2, we focus on the case $n = 2$, and in Theorems 5.4 and 5.13, we describe the lower central and the lower \mathbb{F}_2 -linear central filtration respectively of $P_2(\mathbb{K})$ in an explicit manner. In Section 5.4, we study the lower central series of $P_n(\mathbb{K})$, if $n \geq 3$, and in Section 5.5, we complete the proof of Theorem 1.3 by extending our results to $B_n(\mathbb{K})$.

5.1. A section in the case of the Klein bottle. As we mentioned previously, the Fadell-Neuwirth short exact sequence (1.2) admits a section for geometric reasons. We start by giving an explicit section.

Proposition 5.1. *In terms of the presentation of $P_n(\mathbb{K})$ given by Theorem 2.1, the map $s: P_n(\mathbb{K}) \longrightarrow P_{n+1}(\mathbb{K})$ defined on the generators of $P_n(\mathbb{K})$ by:*

$$\left\{ \begin{array}{l} a_i \longmapsto a_i, \quad i = 1, \dots, n-1 \\ b_i \longmapsto b_i, \quad i = 1, \dots, n-1 \\ C_{i,j} \longmapsto C_{i,j}, \quad 1 \leq i < j \leq n-1 \\ C_{i,n} \longmapsto C_{i,n} C_{i,n+1} C_{n,n+1}^{-1}, \quad 1 \leq i \leq n-1 \\ a_n \longmapsto a_n a_{n+1} \\ b_n \longmapsto b_{n+1} b_n \end{array} \right.$$

extends to an algebraic section for the Fadell-Neuwirth short exact sequence arising from equation (1.2):

$$1 \longrightarrow \pi_1(\mathbb{K} \setminus \{x_1, \dots, x_n\}) \longrightarrow P_{n+1}(\mathbb{K}) \xrightarrow{p_*} P_n(\mathbb{K}) \longrightarrow 1. \quad (5.1)$$

The geometric idea behind this section is to make use of the non-vanishing vector field on \mathbb{K} to duplicate the last string of the elements of $P_n(\mathbb{K})$.

Proof of Proposition 5.1. In what follows, the numbers (1)–(8) refer to the relations of $P_n(\mathbb{K})$ given by Theorem 2.1. To prove the proposition, it suffices to check that the images under s of these relations remain valid in $P_{n+1}(\mathbb{K})$. We do this for relations (2) and (6). For relation (2), which is $a_i^{-1} b_n a_i = b_n a_n C_{i,n}^{-1} C_{i+1,n} a_n^{-1}$, we consider two cases.

- If $i+1 = n$ then:

$$\begin{aligned} s(a_i^{-1} b_n a_i) &= a_i^{-1} \underbrace{b_{n+1} b_n}_{(6)} a_i = a_i^{-1} (b_n b_{n+1} \underbrace{C_{n,n+1}}_{(3)}) a_i = \underbrace{a_i^{-1} b_n a_i}_{(2)} \underbrace{a_i^{-1} b_{n+1} (a_i C_{n,n+1})}_{(2)} \\ &= (b_n a_n C_{i,n}^{-1} \underbrace{a_n^{-1}}_{(2)}) (b_{n+1} a_{n+1} C_{i,n+1}^{-1} C_{n,n+1} a_{n+1}^{-1}) C_{n,n+1} \\ &= b_n a_n \underbrace{C_{i,n}^{-1} (b_{n+1} a_{n+1} C_{n,n+1}^{-1})}_{(3) \text{ and } (8)} \underbrace{a_{n+1}^{-1} a_n^{-1}}_{(1)} a_{n+1} C_{i,n+1}^{-1} C_{n,n+1} a_{n+1}^{-1} C_{n,n+1} \end{aligned}$$

$$\begin{aligned}
 &= b_n \underbrace{a_n(b_{n+1} a_{n+1} C_{i,n}^{-1})}_{(**)} C_{n,n+1}^{-1} \underbrace{(a_n^{-1}) C_{i,n+1}^{-1} C_{n,n+1}}_{(*)} a_{n+1}^{-1} C_{n,n+1} \\
 &= b_n (b_{n+1} C_{n,n+1} a_n) a_{n+1} C_{i,n}^{-1} \underbrace{C_{n,n+1}^{-1} (C_{n,n+1} C_{i,n+1}^{-1} a_n^{-1})}_{(3)} a_{n+1}^{-1} C_{n,n+1} \\
 &= b_n b_{n+1} C_{n,n+1} \cdot a_n a_{n+1} \cdot \underbrace{C_{i,n}^{-1} C_{i,n+1}^{-1} (C_{n,n+1} a_n^{-1} a_{n+1}^{-1})}_{(4)} \\
 &= b_n b_{n+1} C_{n,n+1} \cdot a_n a_{n+1} \cdot (C_{n,n+1} C_{i,n}^{-1} C_{i,n+1}^{-1}) \cdot a_n^{-1} a_{n+1}^{-1} = s(b_n a_n C_{i,n}^{-1} a_n^{-1}),
 \end{aligned}$$

where $(*)$ (resp. $(**)$) is a consequence of relation (3) (resp. relations (2) and (3)).

- If $i + 1 < n$ then:

$$\begin{aligned}
 s(a_i^{-1} b_n a_i) &= a_i^{-1} \underbrace{b_{n+1} b_n}_{(6)} a_i = a_i^{-1} (b_n b_{n+1} \underbrace{C_{n,n+1}}_{(3)}) a_i = \underbrace{a_i^{-1} b_n a_i}_{(2)} \underbrace{a_i^{-1} b_{n+1} a_i}_{(2)} C_{n,n+1} \\
 &= (b_n a_n C_{i,n}^{-1} C_{i+1,n}^{-1} \underbrace{a_n^{-1}}_{(2)}) (b_{n+1} a_{n+1} C_{i,n+1}^{-1} C_{i+1,n+1}^{-1} a_{n+1}^{-1}) C_{n,n+1} \\
 &= b_n a_n \underbrace{C_{i,n}^{-1} C_{i+1,n}^{-1} (b_{n+1} a_{n+1} C_{n,n+1}^{-1} a_{n+1}^{-1} a_n^{-1})}_{(3) \text{ and } (8)} a_{n+1} \underbrace{C_{i,n+1}^{-1} C_{i+1,n+1}^{-1} a_{n+1}^{-1}}_{(1)} C_{n,n+1} \\
 &= b_n \underbrace{a_n (b_{n+1} a_{n+1} C_{i,n}^{-1} C_{i+1,n}^{-1})}_{(**)} C_{n,n+1}^{-1} \underbrace{(a_n^{-1}) C_{i,n+1}^{-1} C_{i+1,n+1}^{-1} a_{n+1}^{-1}}_{(*)} C_{n,n+1} \\
 &= b_n (b_{n+1} C_{n,n+1} a_n) a_{n+1} C_{i,n}^{-1} C_{i+1,n}^{-1} \underbrace{C_{n,n+1}^{-1} (C_{n,n+1} C_{i,n+1}^{-1} C_{i+1,n+1}^{-1} C_{n,n+1}^{-1} a_n^{-1})}_{(3)} a_{n+1}^{-1} C_{n,n+1} \\
 &= b_n b_{n+1} C_{n,n+1} a_n a_{n+1} \underbrace{C_{i,n}^{-1} C_{i+1,n}^{-1} C_{i,n+1}^{-1}}_{(4)} C_{i+1,n+1}^{-1} \underbrace{C_{n,n+1}^{-1} (C_{n,n+1} a_n^{-1} a_{n+1}^{-1})}_{(3)} \\
 &= b_n b_{n+1} C_{n,n+1} \cdot a_n a_{n+1} \cdot C_{i,n}^{-1} (C_{i,n+1}^{-1} C_{i+1,n}^{-1}) C_{i+1,n+1}^{-1} \cdot a_n^{-1} a_{n+1}^{-1} = s(b_n a_n C_{i,n}^{-1} C_{i+1,n}^{-1} a_n^{-1}),
 \end{aligned}$$

where $(*)$ is a consequence of relation (3), similar to that in the case $i + 1 = n$, and $(**)$ is the same relation as in the case $i + 1 = n$.

Thus s respects relation (2). For relation (6), which is $b_n b_i = b_i b_n C_{i,n} C_{i+1,n}^{-1}$, we also consider two cases.

- if $i + 1 = n$ then:

$$\begin{aligned}
 s(b_n b_i) &= \underbrace{b_{n+1} b_n}_{(6)} b_i = (b_n b_{n+1} \underbrace{C_{n,n+1}}_{(8)}) b_i = b_n \underbrace{b_{n+1} (b_i C_{n,n+1})}_{(6)} = \underbrace{b_n (b_i b_{n+1} C_{i,n+1} C_{n,n+1}^{-1})}_{(6)} C_{n,n+1} \\
 &= (b_i b_n \underbrace{C_{i,n}}_{(8)}) b_{n+1} C_{i,n+1} = b_i b_n (b_{n+1} C_{i,n}) C_{i,n+1} = b_i b_n b_{n+1} (C_{n,n+1} \underbrace{C_{n,n+1}^{-1} C_{i,n} C_{i,n+1}}_{(4)}) \\
 &= b_i b_n b_{n+1} C_{n,n+1} (C_{i,n} C_{i,n+1} C_{n,n+1}^{-1}) = s(b_i b_n C_{i,n}).
 \end{aligned}$$

- if $i + 1 < n$:

$$\begin{aligned}
 s(b_n b_i) &= \underbrace{b_{n+1} b_n}_{(6)} b_i = (b_n b_{n+1} \underbrace{C_{n,n+1}}_{(8)}) b_i = b_n \underbrace{b_{n+1} (b_i C_{n,n+1})}_{(6)} = \underbrace{b_n (b_i b_{n+1} C_{i,n+1} C_{i+1,n+1}^{-1})}_{(6)} C_{n,n+1} \\
 &= (b_i b_n \underbrace{C_{i,n} C_{i+1,n}^{-1}}_{(8)}) b_{n+1} C_{i,n+1} C_{i+1,n+1}^{-1} C_{n,n+1} = b_i b_n (b_{n+1} C_{i,n} \underbrace{C_{i+1,n}^{-1} C_{i,n+1} C_{i+1,n+1}^{-1}}_{(4)}) C_{n,n+1}
 \end{aligned}$$

$$\begin{aligned}
&= b_i b_n b_{n+1} \underbrace{(C_{n,n+1} C_{n,n+1}^{-1})}_{(4)} C_{i,n} (C_{i,n+1} \underbrace{C_{i+1,n}^{-1} C_{i+1,n+1}^{-1}}_{(4)} C_{n,n+1}) \\
&= b_i b_n b_{n+1} C_{n,n+1} (C_{i,n} C_{i,n+1} C_{n,n+1}^{-1}) (C_{n,n+1} C_{i+1,n+1}^{-1} C_{i+1,n}^{-1}) = s(b_i b_n C_{i,n} C_{i+1,n}^{-1}).
\end{aligned}$$

Thus s respects relation (6). The computations for the other relations are similar, and are left to the reader. \square

As we mentioned at the end of the introduction, for any compact surface M and for all $n \in \mathbb{N}$, the centre $Z(B_n(M))$ of $B_n(M)$ is known, with the exception of the Klein bottle and the Möbius band. The section given by Proposition 5.1 allows us to determine $Z(B_n(M))$ if $M = \mathbb{K}$. For the sake of completeness, in Proposition A1 of the Appendix of this paper, we also compute the centre of the braid groups of the Möbius band.

Proposition 5.2. *For all $n \in \mathbb{N}$, the centre of $B_n(\mathbb{K})$ is equal to $\langle (b_n \cdots b_1)^2 \rangle$, and is isomorphic to \mathbb{Z} .*

Proof. The idea of the proof is similar to that of [31, Proposition 4.2]. Let $\beta_n = b_n \cdots b_1$, and let $Z_n = \langle \beta_n^2 \rangle$. We will show by induction on n that $Z_n = Z(B_n(\mathbb{K}))$. Arguing as in [31, Proposition 4.2, step 4], we see that $Z(B_n(\mathbb{K})) \subset P_n(\mathbb{K})$, so $Z(B_n(\mathbb{K})) \subset Z(P_n(\mathbb{K}))$. Thus it suffices to show that $Z(P_n(\mathbb{K})) = Z_n$ and $Z_n \subset Z(B_n(\mathbb{K}))$. We prove that $Z(P_n(\mathbb{K})) = Z_n$ by induction on n . If $n = 1$, by Theorem 2.1, we have:

$$\pi_1(\mathbb{K}) = \langle a_1, b_1 : a_1 b_1 = b_1 a_1^{-1} \rangle, \quad (5.2)$$

and it is well known that $Z(\pi_1(\mathbb{K})) = Z_1$. Now suppose by induction that $Z(P_{n-1}(\mathbb{K})) = Z_{n-1}$ for some $n \geq 2$.

We first prove that $Z(P_n(\mathbb{K})) \subset Z_n$ in a manner similar to that of [31, Proposition 4.2, step 3]. Let $g \in Z(P_n(\mathbb{K}))$, and consider the Fadell-Neuwirth short exact sequence arising from (1.2):

$$1 \longrightarrow \pi_1(\mathbb{K} \setminus \{x_1, \dots, x_{n-1}\}) \longrightarrow P_n(\mathbb{K}) \xrightarrow{p_*} P_{n-1}(\mathbb{K}) \longrightarrow 1.$$

Since p_* is surjective, $p_*(g) \in Z(P_{n-1}(\mathbb{K})) = Z_{n-1}$, and since $p_*(Z_n) = Z_{n-1}$, there exists $h \in Z_n$ such that $p_*(h) = p_*(g)$. If $g' = gh^{-1}$ then g' belongs to $Z(P_n(\mathbb{K}))$ and to the free group $\pi_1(\mathbb{K} \setminus \{x_1, \dots, x_{n-1}\})$ by exactness. Hence $g' \in Z(\pi_1(\mathbb{K} \setminus \{x_1, \dots, x_{n-1}\})) = \{1\}$, so $g' = 1$, and thus $g = h \in Z_n$, which shows that $Z(P_n(\mathbb{K})) \subset Z_n$.

Still under the above induction hypothesis, we now prove that $Z_n \subset Z(P_n(\mathbb{K}))$. Using the section $s: P_{n-1}(\mathbb{K}) \longrightarrow P_n(\mathbb{K})$ given by Proposition 5.1, we have $s(\beta_{n-1}^2) = \beta_n^2$. By the induction hypothesis, $\beta_{n-1}^2 a_i = a_i \beta_{n-1}^2$ and $\beta_{n-1}^2 b_i = b_i \beta_{n-1}^2$ for all $1 \leq i \leq n-1$, and $\beta_{n-1}^2 C_{i,j} = C_{i,j} \beta_{n-1}^2$ for all $1 \leq i < j \leq n-1$ in $P_{n-1}(\mathbb{K})$. Taking the image of both sides of these equations by s , we obtain the following relations:

$$\begin{cases} \beta_n^2 a_i = a_i \beta_n^2 \text{ and } \beta_n^2 b_i = b_i \beta_n^2 & \text{if } 1 \leq i < n-1 \\ \beta_n^2 \cdot a_{n-1} a_n = a_{n-1} a_n \cdot \beta_n^2 \text{ and } \beta_n^2 \cdot b_n b_{n-1} = b_n b_{n-1} \cdot \beta_n^2 & \text{if } i = n-1, \end{cases} \quad (5.3)$$

and

$$\begin{cases} \beta_n^2 C_{i,j} = C_{i,j} \beta_n^2 & \text{if } 1 \leq i < j < n-1 \\ \beta_n^2 \cdot C_{i,n-1} C_{i,n} C_{n-1,n}^{-1} = C_{i,n-1} C_{i,n} C_{n-1,n}^{-1} \cdot \beta_n^2 & \text{if } 1 \leq i < j = n-1. \end{cases} \quad (5.4)$$

Let us prove that a_n and b_n commute with β_n^2 , from which it will follow from (5.3) that β_n^2 commutes with a_i and b_i for all $1 \leq i \leq n$. Using relations (6)–(8) of Theorem 2.1, we have:

$$C_{1,n} a_n^{-1} b_i = C_{i+1,n} b_i C_{i,n}^{-1} C_{1,n} a_n^{-1} \quad (5.5)$$

$$b_n C_{i+1} b_i = b_i b_n C_{i,n} \quad (5.6)$$

for all $i = 1, \dots, n-1$. To prove relation (5.5), notice that by relations (7) and (8) we have:

$$b_i^{-1} a_n b_i = a_n \underbrace{b_n (C_{i,n} C_{i+1,n}^{-1})^{-1} b_n^{-1}}_{(8)} = a_n (C_{i+1,n}^{-1} b_i^{-1} C_{i,n} b_i),$$

and taking the inverse of both sides, it follows that $a_n^{-1}b_i = C_{i,n}^{-1}b_iC_{i+1,n}a_n^{-1}$. We thus obtain:

$$\begin{aligned} C_{1,n}\underbrace{a_n^{-1}b_i}_{(8)} &= C_{1,n}(C_{i,n}^{-1}b_iC_{i+1,n}a_n^{-1}) = \underbrace{(b_i b_i^{-1})}_{(8)}C_{1,n}\underbrace{(b_i b_i^{-1})}_{(8)}C_{i,n}^{-1}b_i C_{i+1,n}a_n^{-1} \\ &= \underbrace{b_i(C_{i+1,n}C_{i,n}^{-1}C_{1,n}b_nC_{i+1,n}C_{i,n}^{-1}b_n^{-1})}_{(8)}(b_nC_{i,n}C_{i+1,n}b_n^{-1}C_{i+1,n}^{-1})C_{i+1,n}a_n^{-1} \\ &= (C_{i+1,n}b_i)C_{i,n}^{-1}C_{1,n}a_n^{-1}. \end{aligned}$$

To prove relation (5.6), one may use relation (6) and the fact that b_i commutes with $C_{i+1,n}$ by relation (8). We now claim that:

$$a_n\beta_n = b_n \cdots b_{i+1}C_{i+1,n}^{-1}C_{1,n}a_n^{-1}b_i b_{i-1} \cdots b_1 \text{ and } b_n\beta_n = b_n \cdots b_{i+1}b_nC_{i+1,n}b_i \cdots b_1$$

for all $i = 0, \dots, n-1$. We shall prove the claim by reverse induction on i . First, we have $a_n\beta_n = a_n(b_n b_{n-1} \cdots b_1) \stackrel{(5)}{=} (b_n C_{1,n} a_n^{-1}) b_{n-1} \cdots b_1$ and $b_n\beta_n = b_n b_n \cdots b_1$, so the claim is valid if $i = n-1$. Suppose that it holds for some $1 \leq i \leq n-1$. Then:

$$\begin{aligned} a_n\beta_n &= b_n \cdots b_{i+1}C_{i+1,n}^{-1}C_{1,n}a_n^{-1}b_i b_{i-1} \cdots b_1 \stackrel{(5.5)}{=} b_n \cdots b_{i+1}C_{i+1,n}^{-1}C_{i+1,n}b_i C_{i,n}^{-1}C_{1,n}a_n^{-1}b_{i-1} \cdots b_1 \\ &= b_n \cdots b_{i+1}b_i C_{i,n}^{-1}C_{1,n}a_n^{-1}b_{i-1} \cdots b_1, \text{ and} \end{aligned}$$

$$b_n\beta_n = b_n \cdots b_{i+1}b_nC_{i+1,n}b_i \cdots b_1 \stackrel{(5.6)}{=} b_n \cdots b_{i+1}b_i b_nC_{i,n}b_{i-1} \cdots b_1$$

so the claim holds. Taking $i = 0$, we obtain $a_n\beta_n = \beta_n a_n^{-1}$ and $b_n\beta_n = \beta_n b_n C_{1,n}$. Hence $a_n\beta_n^2 = \beta_n a_n^{-1}\beta_n = \beta_n^2 a_n$, and applying (5) with $i = n$ twice, we obtain:

$$b_n\beta_n^2 = \beta_n b_n C_{1,n}\beta_n = \beta_n a_n b_n a_n \beta_n = \beta_n^2 a_n^{-1} b_n C_{1,n} a_n^{-1} = \beta_n^2 b_n.$$

Thus β_n^2 commutes with a_n and b_n , and so with a_i and b_i for all $i \in \{1, \dots, n\}$. Finally, by relation (5), $C_{1,n} = b_n^{-1}a_n b_n a_n$ and $C_{i+1,n} = C_{i,n}a_n^{-1}b_n^{-1}a_i^{-1}b_n a_i a_n$ by relation (2). Since β_n^2 commutes with a_i and b_i for all $1 \leq i \leq n$, it follows that β_n^2 commutes with $C_{1,n}$ by relation (5), and by induction that β_n^2 commutes with $C_{i,n}$ for all $1 \leq i < n$. So by (5.4), β_n^2 commutes with $C_{i,j}$ for all $1 \leq i < j \leq n$, and therefore commutes with all of the elements of a generating set of $P_n(\mathbb{K})$, hence $Z_n = Z(P_n(\mathbb{K}))$.

To complete the proof of the proposition, it remains to show that $Z_n \subset Z(B_n(\mathbb{K}))$. To do so, from above, it suffices to prove that σ_i commutes with β_n^2 for all $i = 1, \dots, n-1$. One may see by Figure 2 that $\sigma_i^2 = C_{i,i+1}$. Further:

$$\sigma_i^{-1}b_j\sigma_i = \begin{cases} b_{i+1}\sigma_i^2 & \text{if } j = i \\ \sigma_i^{-2}b_i & \text{if } j = i+1 \\ b_j & \text{otherwise.} \end{cases} \quad (5.7)$$

The case $j = i$ is illustrated in Figure 4. If $j = i+1$ then $\sigma_i^{-1}b_{i+1}\sigma_i = \sigma_i^{-1} \cdot \sigma_i^{-1}b_i\sigma_i \cdot \sigma_i^{-2} \cdot \sigma_i = \sigma_i^{-2}b_i$ using the case $j = i$. For $i = 1, \dots, n-1$, using (5.7) and relation (6) of Theorem 2.1, we have:

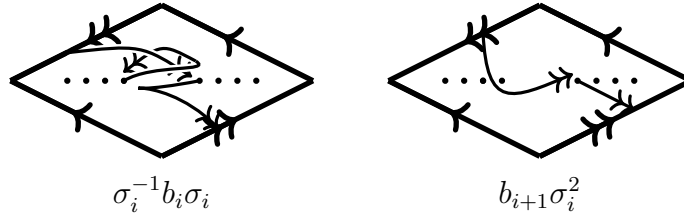


FIGURE 4. The relation $\sigma_i^{-1}b_i\sigma_i = b_{i+1}\sigma_i^2$.

$$\sigma_i^{-1}(b_n \cdots b_1)\sigma_i = b_n \cdots b_{i+2} \cdot \underbrace{C_{i,i+1}^{-1} b_i \cdot b_{i+1} C_{i,i+1}} \cdot b_{i-1} \cdots b_1 = C_{i,i+1}^{-1} b_n \cdots b_1 = \sigma_i^{-2} b_n \cdots b_1,$$

from which it follows that $(b_n \cdots b_1)^2 \sigma_i = \sigma_i (b_n \cdots b_1)^2$ as required. \square

Remark 5.3. For $n = 2$, we modify slightly the presentation of $P_2(\mathbb{K})$ given by Theorem 2.1 by removing the generator $C_{1,2}$ using relation (5), so $C_{1,2} = b_2^{-1} a_2 b_2 a_2$. Hence $P_2(\mathbb{K})$ is generated by a_1, a_2, b_1 and b_2 , subject to the relations:

- (1) $a_1^{-1} a_2 a_1 = a_2$.
- (2) $a_1^{-1} b_2 a_1 = a_2^{-1} b_2 a_2^{-1}$.
- (3) $b_1^{-1} a_2 b_1 = a_2 b_2 a_2^{-1} b_2^{-1} a_2^{-1}$.
- (4) $b_1^{-1} b_2 b_1 = a_2 b_2 a_2$.
- (5) $b_2^{-1} a_2 b_2 a_2 = b_1 a_1^{-1} b_1^{-1} a_1^{-1}$.

Using Proposition 5.1 and the Fadell-Neuwirth short exact sequence (5.1), $P_{n+1}(\mathbb{K})$ may be written as a semi-direct product of the free group $\pi_1(\mathbb{K} \setminus \{x_1, \dots, x_n\})$ by $s(P_n(\mathbb{K}))$ for all $n \in \mathbb{N}$. In particular, if $n = 1$ then:

$$P_2(\mathbb{K}) \cong \pi_1(\mathbb{K} \setminus \{x_1\}) \rtimes_{\varphi} s(P_1(\mathbb{K})), \quad (5.8)$$

where $\pi_1(\mathbb{K} \setminus \{x_1\}) = \langle a_2, b_2 \rangle$ is a free group of rank 2,

$$s(P_1(\mathbb{K})) = \langle a_1 a_2, b_2 b_1 : (b_2 b_1)(a_1 a_2) = (a_1 a_2)^{-1} (b_2 b_1) \rangle$$

by (5.2), and where the action $\varphi: s(P_1(\mathbb{K})) \rightarrow \text{Aut}(\pi_1(\mathbb{K} \setminus \{x_1\}))$ is given by:

$$\begin{cases} \varphi(a_1 a_2) : \begin{cases} a_2 \mapsto a_2 \\ b_2 \mapsto a_2^{-2} b_2 \end{cases} \\ \varphi(b_2 b_1) : \begin{cases} a_2 \mapsto a_2^{-1} \\ b_2 \mapsto a_2 b_2 a_2, \end{cases} \end{cases} \quad (5.9)$$

using Remark 5.3.

5.2. The lower central series of $P_2(\mathbb{K})$. In this section, we use Theorem 1.1 to calculate explicitly the lower central series of $P_2(\mathbb{K})$. This will enable us to prove that $P_2(\mathbb{K})$ is residually nilpotent.

Theorem 5.4. *Let $n \geq 2$. With the notation of Theorem 2.1, we have:*

$$\Gamma_n(P_2(\mathbb{K})) = \langle\langle a_2^{2^{n-1}}, x^{2^{n-i}} : x \in \Gamma_i(\pi_1(\mathbb{K} \setminus \{x_1\})), 2 \leq i \leq n \rangle\rangle_{\pi_1(\mathbb{K} \setminus \{x_1\})} \rtimes_{\varphi} \langle (a_1 a_2)^{2^{n-1}} \rangle,$$

where φ is as defined in equation (5.9). Consequently, $P_2(\mathbb{K})$ is residually nilpotent.

The idea of the proof is to apply Theorem 1.1(1) to the semi-direct product structure of $P_2(\mathbb{K})$ given by (5.8). From now on, we shall make use of the notation of that theorem, taking $H = \pi_1(\mathbb{K} \setminus \{x_1\})$, $G = s(P_1(\mathbb{K}))$ and $P_2(\mathbb{K}) = H \rtimes_{\varphi} G$. In the lemmas that follow, we first compute the subgroups K_n , H_n and L_n for these choices of G and H , and in Proposition 5.12, we calculate the subgroup L_n , which will enable us to prove Theorem 5.4. We first need to know the lower central series of $P_1(\mathbb{K})$. Applying Theorem 1.1 to the semi-direct product structure $\langle a_1 \rangle \rtimes \langle b_1 \rangle$ of $\pi_1(\mathbb{K})$ given by (5.2), it is straightforward to see that $K_2 = H_2 = L_2 = \langle a_1^2 \rangle$, and if $n \geq 3$, $K_n = \{1\}$ and $H_n = L_n = \langle a_1^{2^{n-1}} \rangle$, therefore the lower central series of $s(P_1(\mathbb{K}))$ is given by:

$$\Gamma_n(s(P_1(\mathbb{K}))) = \langle (a_1 a_2)^{2^{n-1}} \rangle \quad (5.10)$$

for all $n > 1$. We now turn to the case of $P_2(\mathbb{K})$. We first determine K_n .

Lemma 5.5. *With the notation of Theorem 1.1, and taking $H = \pi_1(\mathbb{K} \setminus \{x_1\})$, $G = s(P_1(\mathbb{K}))$ and $P_2(\mathbb{K}) = H \rtimes_{\varphi} G$, the subgroup K_n is equal to $\langle\langle a_2^2, \Gamma_2(H) \rangle\rangle_H$ if $n = 2$, and to $\langle\langle a_2^{2^{n-1}} \rangle\rangle_H$ if $n \geq 3$.*

Proof. First suppose that $n = 2$. By (5.9), we have:

$$\begin{cases} \varphi(a_1 a_2)(a_2).a_2^{-1} = 1 \\ \varphi(a_1 a_2)(b_2).b_2^{-1} = \varphi(b_2 b_1)(a_2).a_2^{-1} = a_2^{-2} \\ \varphi(b_2 b_1)(b_2).b_2^{-1} = a_2 b_2 a_2 b_2^{-1} = a_2^2 [a_2^{-1}, b_2], \end{cases} \quad (5.11)$$

and so $\{a_2^2, [a_2^{-1}, b_2]\} \subset K_2$. Since K_2 is normal in H by Lemma 3.3, $\langle\langle a_2^2, [a_2^{-1}, b_2] \rangle\rangle_H$ is a subgroup of K_2 , and therefore $\langle\langle a_2^2, \Gamma_2(H) \rangle\rangle_H \subset K_2$ because $\Gamma_2(H) = \langle\langle [a_2^{-1}, b_2] \rangle\rangle_H$. For the other inclusion, (5.11) implies that $\varphi(g)(h).h^{-1} \in \langle\langle a_2^2, \Gamma_2(H) \rangle\rangle_H$ for all h (resp. all g) belonging to the generating set $\{a_2, b_2\}$ (resp. $\{a_1 a_2, b_2 b_1\}$) of H (resp. of G). The inclusion $K_2 \subset \langle\langle a_2^2, \Gamma_2(H) \rangle\rangle_H$ then follows from Lemma 3.4(1). This proves the result for $n = 2$.

Now assume that $n \geq 3$. Then $\Gamma_{n-1}(G) = \langle\langle (a_1 a_2)^{2^{n-2}} \rangle\rangle$ by (5.10). Using (5.9), we have $\varphi((a_1 a_2)^{2^{n-2}})(a_2).a_2^{-1} = 1$ and:

$$\varphi((a_1 a_2)^2)(b_2) = \varphi(a_1 a_2)(a_2^{-2} b_2) = a_2^{-2} (a_2^{-2} b_2) = a_2^{-4} b_2.$$

Suppose by induction that $\varphi((a_1 a_2)^j)(b_2) = a_2^{-2j} b_2$ for some $j \geq 2$. Then:

$$\varphi((a_1 a_2)^{j+1})(b_2) = \varphi(a_1 a_2) \varphi((a_1 a_2)^j)(b_2) = \varphi(a_1 a_2)(a_2^{-2j} b_2) = a_2^{-2j} (a_2^{-2} b_2) = a_2^{-2(j+1)} b_2.$$

In particular, if $j = 2^{n-2}$, we have $\varphi((a_1 a_2)^{2^{n-2}})(b_2).b_2^{-1} = (a_2^{-2(2^{n-2})} b_2).b_2^{-1} = a_2^{-2^{n-1}}$, and hence $\langle\langle a_2^{2^{n-1}} \rangle\rangle_H \subset K_n$. Conversely, taking $\tilde{G} = \{(a_1 a_2)^{2^{n-2}}\}$ and $\tilde{H} = \{a_2, b_2\}$ in Lemma 3.4(1) and using (5.10), we obtain $K_n \subset \langle\langle Z \rangle\rangle_H$, where:

$$Z = \left\{ \varphi(g)(h).h^{-1} : g \in \{(a_1 a_2)^{2^{n-2}}\}, h \in \{a_2, b_2\} \right\} = \{a_2^{-2^{n-1}}\},$$

whence the inclusion $K_n \subset \langle\langle a_2^{2^{n-1}} \rangle\rangle_H$, and this proves the lemma. \square

If $\beta \in H$ then $\beta a_2^2 \beta^{-1} = [\beta, a_2^2] a_2^2 \in \langle a_2^2, \Gamma_2(H) \rangle$, and since $\Gamma_2(H)$ is normal in H , it follows that the subgroup $\langle a_2^2, \Gamma_2(H) \rangle$ is also normal in H , so $\langle a_2^2, \Gamma_2(H) \rangle = \langle\langle a_2^2, \Gamma_2(H) \rangle\rangle_H$. Using the relations $K_2 = H_2$ and $[H, L_1] = \Gamma_2(H)$, it follows from Lemma 5.5 that:

$$L_2 = \langle a_2^2, \Gamma_2(H) \rangle = \langle\langle a_2^2, [a_2, b_2] \rangle\rangle_H. \quad (5.12)$$

Let $W_2 = \widetilde{W}_2 = L_2$, and for $n \geq 3$, define:

$$W_n = \langle \Gamma_n(H), x^2 : x \in W_{n-1} \rangle \quad (5.13)$$

$$\widetilde{W}_n = \langle\langle a_2^{2^{n-1}}, x^{2^{n-i}} : x \in \Gamma_i(H), 2 \leq i \leq n \rangle\rangle_H. \quad (5.14)$$

Note that W_n is normal in H for all $n \geq 2$. This follows from the fact that $\Gamma_n(H)$ is normal in H for all $n \geq 2$ and arguing by induction on n as follows. If $n = 2$ then $W_2 = L_2$ by Lemma 3.3, so suppose that $n \geq 3$, and that W_{n-1} is normal in H . Then $h x h^{-1} \in W_{n-1}$ for all $x \in W_{n-1}$ and $h \in H$, thus $h x^2 h^{-1} = (h x h^{-1})^2 \in W_n$, and W_n is normal in H as claimed. If $x \in \Gamma_i(H)$ and $2 \leq i \leq n$, we will refer to the elements $a_2^{2^{n-1}}$ and $x^{2^{n-i}}$ as generators of \widetilde{W}_n . In order to prove Theorem 5.4, we will show in Proposition 5.12 that $L_n = W_n = \widetilde{W}_n$ for all $n \geq 2$. Supposing this to be the case, the following lemma implies that to determine the lower central series of $P_2(\mathbb{K})$ using Theorem 1.1, it is not necessary to calculate the subgroups H_n .

Lemma 5.6. *Suppose that $L_i = W_i = \widetilde{W}_i$ for all $2 \leq i \leq n$. Then $H_{n+1} \subset \langle K_{n+1}, [H, L_n] \rangle$. In particular, $L_{n+1} = \langle K_{n+1}, [H, L_n] \rangle$.*

Proof. We prove the given inclusion by induction on n . If $n = 2$, using (5.9) notice that:

- $\varphi(a_1 a_2)(a_2^2) a_2^{-2} = 1$.
- $\varphi(b_2 b_1)(a_2^2) a_2^{-2} = a_2^{-4} \in K_3$ by Lemma 5.5.
- $\varphi(a_1 a_2)([a_2, b_2])[b_2, a_2] = [a_2^{-2}, a_2 b_2 a_2^{-1} b_2^{-1} a_2^{-1}] \in [L_2, H]$ by (5.12).

- $\varphi(b_2 b_1)([a_2, b_2])[b_2, a_2] = \underbrace{[b_2, a_2^2]}_{\in [H, L_2]} \underbrace{[a_2, [a_2, b_2]]}_{\in \Gamma_3(H)} \in [L_2, H]$ by (5.12).

Using these calculations and the description of L_2 given in (5.12), it follows from the first part of Lemma 3.4(2) and Lemma 5.5 that:

$$H_3 = \langle\langle [H, L_2], \varphi(g)(w).w^{-1} : g \in \{a_1 a_2, b_2 b_1\}, w \in \{a_2^2, [a_2, b_2]\} \rangle\rangle_H \subset \langle K_3, [H, L_2] \rangle,$$

which proves the result if $n = 2$.

Now suppose that the given inclusion holds for $n - 1$ for some $n \geq 3$, and assume that $L_i = W_i = \widetilde{W}_i$ for all $2 \leq i \leq n$. Let $\varphi(g)(x).x^{-1}$ be an element of H_{n+1} , where $g \in G$ and $x \in L_n = \langle K_n, [H, L_{n-1}] \rangle$ by the induction hypothesis. We wish to show that $\varphi(g)(x).x^{-1} \in \langle K_{n+1}, [H, L_n] \rangle$. By Lemma 3.4(1), we only need to check the following two possibilities:

- $x = a_2^{2^{n-1}}$. Then $x \in K_n$ by Lemma 5.5, $\varphi(a_1 a_2)(x).x^{-1} = 1$ and $\varphi(b_2 b_1)(x).x^{-1} = a_2^{-2^n} \in K_{n+1}$ by Lemma 5.5 and (5.11).
- $x = [h, l] \in [H, L_{n-1}]$, where $h \in H$ and $l \in L_{n-1}$. Then $\varphi(g)([h, l])[h, l]^{-1}$ may be written in the following form:

$$\left(\varphi(g)(h).l \underbrace{[(\varphi(g)(l^{-1}).l)^{-1}, \varphi(g)(h^{-1})] l^{-1} \varphi(g)(h^{-1})}_{\in [L_n, H]} \right) \underbrace{h [(\varphi(g)(h^{-1}).h)^{-1}, l] h^{-1}}_{\in [L_2, L_{n-1}]} \quad (5.15)$$

To complete the proof, it suffices to show that the subgroup $[L_2, L_{n-1}]$ is contained in $[H, L_n]$. To do so, first note that $[L_2, L_{n-1}]$ is normal in H because L_j is normal in H for all $j \geq 2$ by Lemma 3.3. Using the fact that $L_2 = \langle a_2^2, \Gamma_2(H) \rangle$ by (5.12), it suffices to show that the following elements belong to $[H, L_n]$:

- $[[y, z], l] \in [\Gamma_2(H), L_{n-1}]$, where $y, z \in H$ and $l \in L_{n-1}$. Then:

$$[[y, z], l] = y \underbrace{[z, [y^{-1}, l]]}_{\in [H, L_n]} y^{-1} . l y z \underbrace{[[z^{-1}, l^{-1}], y^{-1}]}_{\in [H, L_n]} z^{-1} y^{-1} l^{-1},$$

because $[H, L_{n-1}] \subset L_n$. Thus $[[y, z], l] \in [H, L_n]$ because $[H, L_n]$ is a normal subgroup of H , and therefore $[\Gamma_2(H), L_{n-1}] \subset [H, L_n]$.

- $[a_2^2, l] \in [a_2^2, L_{n-1}]$, where $l \in L_{n-1}$. Then using (3.2), we have:

$$[a_2^2, l] = \underbrace{[a_2, [a_2, l]]}_{\in [H, L_n]} [a_2, l]^2, \quad (5.16)$$

since $[H, L_{n-1}] \subset L_n$. Further $[a_2, l]^2 \in [H, L_n]$ because $l \in L_{n-1} = W_{n-1}$ by hypothesis, so $l^2 \in W_n = L_n$, and

$$\underbrace{[l^2, a_2]}_{\in [L_n, H]} = \underbrace{[l, [l, a_2]]}_{\in [H, L_n]} [l, a_2]^2$$

by (3.2). So $[l, a_2]^2 \in [H, L_n]$, and thus $[a_2^2, l] \in [H, L_n]$ by (5.16).

This shows that $[L_2, L_{n-1}] \subset [H, L_n]$, hence $H_{n+1} \subset \langle K_{n+1}, [H, L_n] \rangle$ as desired, which concludes the proof of the first part of the statement. The second part follows from the first part and the definition of L_{n+1} . \square

In order to prove Proposition 5.12, we shall require a couple of intermediate results. Let A be either the empty set or a normal subgroup of H , and for $m \geq 1$, let:

$$B_m = \left\langle [x_1, \dots, x_i]^{2^{m-i-k}} \in \Gamma_i(H) : \begin{array}{l} \exists 1 \leq j_1 < \dots < j_k \leq i : x_{j_1}, \dots, x_{j_k} \in A, \\ 0 \leq k \leq m - i \text{ and } i = 1, \dots, m \end{array} \right\rangle.$$

For $1 \leq l \leq m$, let:

$$\mathcal{E}_{l,m} = \left\langle [x_1, \dots, x_i]^{2^{m-i-k}} \in \Gamma_i(H) : \begin{array}{l} \exists 1 \leq j_1 < \dots < j_k \leq i : x_{j_1}, \dots, x_{j_k} \in A, \\ 0 \leq k \leq m - i \text{ and } l \leq i \leq m \end{array} \right\rangle. \quad (5.17)$$

In the case that $A = \emptyset$, we will denote the subgroup $\mathcal{E}_{l,m}$ by $\tilde{\mathcal{E}}_{l,m}$. This corresponds to taking $k = 0$ in (5.17), so:

$$\tilde{\mathcal{E}}_{l,m} = \langle [x_1, \dots, x_i]^{2^{m-i}} \in \Gamma_i(H) : l \leq i \leq m \rangle. \quad (5.18)$$

If $x = [x_1, \dots, x_i] \in \Gamma_i(H)$, the elements $x^{2^{m-i-k}}$ (resp. $x^{2^{m-i}}$) of $\mathcal{E}_{l,m}$ (resp. of $\tilde{\mathcal{E}}_{l,m}$) given in (5.17) (resp. in (5.18)) will be termed *generators* of $\mathcal{E}_{l,m}$ (resp. of $\tilde{\mathcal{E}}_{l,m}$). Note that $\Gamma_m(H) = \mathcal{E}_{m,m} \subset \mathcal{E}_{l+1,m} \subset \mathcal{E}_{l,m} \subset \mathcal{E}_{1,m} = B_m$ by [27, Problem 3, Section 5.3, p. 297] for all $l = 1, \dots, m-1$, and that for all $1 \leq l \leq m$, $\mathcal{E}_{l,m}$ is normal in H , since if $h \in H$ and $x^{2^{m-i-k}}$ is a generator of $\mathcal{E}_{l,m}$, where $x = [x_1, \dots, x_i] \in \Gamma_i(H)$, then $h[x_1, \dots, x_i]^{2^{m-i-k}}h^{-1} = [hx_1h^{-1}, \dots, hx_ih^{-1}]^{2^{m-i-k}} \in \mathcal{E}_{l,m}$ because A is normal in H or is empty. In particular, taking $A = \emptyset$, we have:

$$\Gamma_m(H) = \tilde{\mathcal{E}}_{m,m} \subset \tilde{\mathcal{E}}_{l+1,m} \subset \tilde{\mathcal{E}}_{l,m} \subset \tilde{\mathcal{E}}_{1,m} = B_m, \quad (5.19)$$

and that $\tilde{\mathcal{E}}_{l,m}$ is normal in H for all $1 \leq l \leq m$.

Lemma 5.7. *Let $y \in H$, let $m \geq 1$, and let $1 \leq i \leq m$. If $x = [x_1, \dots, x_i]$ is an element of $\Gamma_i(H)$ for which $x^{2^{m-i-k}}$ is a generator of $\mathcal{E}_{i,m}$, where $0 \leq k \leq m-i$ (resp. $x^{2^{m-i}}$ is a generator of $\tilde{\mathcal{E}}_{i,m}$), then:*

$$[x^{2^{m-i-k}}, y] \equiv [x, y]^{2^{m-i-k}} \pmod{\mathcal{E}_{i+1,m+1}} \text{ and} \quad (5.20)$$

$$[x^{2^{m-i}}, y] \equiv [x, y]^{2^{m-i}} \pmod{\tilde{\mathcal{E}}_{i+1,m+1}} \quad (5.21)$$

respectively.

Remark 5.8. Let $m \geq 1$, and let $1 \leq i \leq m$. Since $\mathcal{E}_{i+1,m+1} \subset B_{m+1}$ (resp. $\tilde{\mathcal{E}}_{i+1,m+1} \subset \tilde{\mathcal{E}}_{2,m+1} \subset \tilde{W}_{m+1}$ by (5.14)), the congruence (5.20) (resp. (5.21)) is also valid modulo B_{m+1} (resp. modulo \tilde{W}_{m+1}).

Proof. It suffices to prove (5.20), since then the congruence (5.21) follows by taking $A = \emptyset$. We will do so by induction on m . If $m = 1$ then $i = 1$ and $k = 0$, and the congruence is in fact an equality. If $m = 2$, we consider two cases:

- if $i = 2$, or if $i = k = 1$, then $x \in \Gamma_2(H)$, and the two sides of (5.20) are equal.
- if $i = 1$ and $k = 0$ then $x \in \Gamma_1(H) = H$. Thus $[x^2, y] = [x, x, y][x, y]^2$ by (3.2), and $[x, x, y] \in \Gamma_3(H)$, so $[x, x, y] \in \mathcal{E}_{2,3}$, and we obtain (5.20).

This proves the result if $m = 2$. We now consider the general case.

Induction hypothesis 1: suppose that the congruence (5.20) holds for some $m \geq 1$. Let us show by induction that the result holds for $m+1$ i.e. if $1 \leq i \leq m+1$, $x \in \Gamma_i(H)$, $x^{2^{m+1-i-k}}$ is a generator of $\mathcal{E}_{i,m+1}$, $0 \leq k \leq m+1-i$ and $y \in H$, then:

$$[x^{2^{m+1-i-k}}, y] \equiv [x, y]^{2^{m+1-i-k}} \pmod{\mathcal{E}_{i+1,m+2}}. \quad (5.22)$$

This will be achieved by making a second induction hypothesis as follows.

Induction hypothesis 2: let $1 \leq i \leq m+1$ be such that:

$$[x^{2^{m+1-j-k}}, y] \equiv [x, y]^{2^{m+1-j-k}} \pmod{\mathcal{E}_{j+1,m+2}} \quad (5.23)$$

for all $i \leq j \leq m+1$, where $x \in \Gamma_j(H)$, $x^{2^{m+1-j-k}}$ is a generator of $\mathcal{E}_{j,m+1}$, $0 \leq k \leq m+1-j$ and $y \in H$. If $i = m+1$ then (5.23) holds trivially. So suppose that (5.23) is valid for some $2 \leq i \leq m+1$, and let us prove by reverse induction on i that it also holds for $i-1$. By induction hypothesis 2, it suffices to prove (5.23) for $j = i-1$. Let $y \in H$, and let $x = [x_1, \dots, x_{i-1}] \in \Gamma_{i-1}(H)$, where $x^{2^{m+1-(i-1)-k}}$ is a generator of $\mathcal{E}_{i-1,m+1}$ and $0 \leq k \leq m+1-(i-1)$. Then $[x^{2^{m+1-(i-1)-k}}, y] =$

$[x^{2^{m+2-i-k}}, y]$. If $k = m + 2 - i$ then (5.23) is an equality. So assume that $0 \leq k \leq m + 1 - i$. By Proposition 3.1, $[x^{2^{(m+1)-(i-1)-k}}, y]$ is equal to:

$$\underbrace{[x, x, x^2, \dots, x^{2^{m+1-i-k}}, y]}_{(*)} [x, x^2, \dots, x^{2^{m+1-i-k}}, y]^2 \dots [x^{2^{m-i-k}}, x^{2^{m+1-i-k}}, y]^2 \underbrace{[x^{2^{m+1-i-k}}, y]^2}_{(**)}. \quad (5.24)$$

Using induction hypotheses 1 and 2, we will first show that the expression (**) is congruent to $[x, y]^{2^{m+2-i-k}}$ modulo $\mathcal{E}_{i, m+2}$, and then that the expression (*) belongs to $\mathcal{E}_{i, m+2}$, from which we will conclude by induction that $[x^{2^{m+2-i-k}}, y] \equiv [x, y]^{2^{m+2-i-k}}$ modulo $\mathcal{E}_{i, m+2}$ for all $i = 1, \dots, m + 1$.

To show that $[x^{2^{m+1-i-k}}, y]^2 \equiv [x, y]^{2^{m+2-i-k}}$ modulo $\mathcal{E}_{i, m+2}$, since $x = [x_1, \dots, x_{i-1}] \in \Gamma_{i-1}(H)$ and $x^{2^{m+1-(i-1)-k}}$ is a generator of $\mathcal{E}_{i-1, m+1}$, there exist $1 \leq j_1 < \dots < j_k \leq i - 1$ such that $x_{j_1}, \dots, x_{j_k} \in A$, and therefore $[x_1, \dots, x_{i-1}]^{2^{m-(i-1)-k}}$ is a generator of $\mathcal{E}_{i-1, m}$. Applying induction hypothesis 1, we have:

$$[x^{2^{m+1-i-k}}, y] \equiv [x, y]^{2^{m+1-i-k}} \pmod{\mathcal{E}_{i, m+1}}.$$

It follows that there exist generators $\alpha_1, \dots, \alpha_t$ of $\mathcal{E}_{i, m+1}$ and $\delta_1, \dots, \delta_t \in \{1, -1\}$ for which $[x^{2^{m+1-i-k}}, y] = [x, y]^{2^{m+1-i-k}} \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}$, and so:

$$\begin{aligned} [x^{2^{m+1-i-k}}, y]^2 &= ([x, y]^{2^{m+1-i-k}} \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t})^2 = [x, y]^{2^{m+2-i-k}} [[x, y]^{-2^{m+1-i-k}}, \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}]. \\ &\alpha_1^{2\delta_1} [\alpha_1^{-\delta_1}, \alpha_2^{\delta_2} \dots \alpha_t^{\delta_t}] \alpha_2^{2\delta_2} \dots [\alpha_{t-1}^{-\delta_{t-1}}, \alpha_t^{\delta_t}] \alpha_t^{2\delta_t}. \end{aligned} \quad (5.25)$$

We claim that:

- (1) $\alpha_1^{2\delta_1}, \dots, \alpha_t^{2\delta_t} \in \mathcal{E}_{i, m+2}$.
- (2) $[\alpha_s, \theta], [\alpha_s^{-1}, \theta] \in \mathcal{E}_{i, m+2}$ for all $s = 1, \dots, t$ and all $\theta \in H$.
- (3) $[x, y]^{-2^{m+1-i-k}}, \theta] \in \mathcal{E}_{i, m+2}$ for all $\theta \in H$.

Claim (1) follows from (5.17). To prove (2), let $s \in \{1, \dots, t\}$. Since α_s is a generator of $\mathcal{E}_{i, m+1}$, $\alpha_s = \beta^{2^{m+1-l-\tilde{k}}}$, where $\beta = [\beta_1, \dots, \beta_l] \in \Gamma_l(H)$, $\beta_{j_1}, \dots, \beta_{j_{\tilde{k}}} \in A$, $i \leq l \leq m + 1$ and $0 \leq \tilde{k} \leq m + 1 - l$. By induction hypothesis 2,

$$[\alpha_s, \theta] = [\beta^{2^{m+1-l-\tilde{k}}}, \theta] \equiv [\beta, \theta]^{2^{m+1-l-\tilde{k}}} \pmod{\mathcal{E}_{l+1, m+2}}. \quad (5.26)$$

Moreover, $[\theta, \beta] = [\theta, \beta_1, \dots, \beta_l] \in \Gamma_{l+1}(H)$, where $\beta_{j_1}, \dots, \beta_{j_{\tilde{k}}} \in A$, so $[\theta, \beta]^{2^{(m+2)-(l+1)-\tilde{k}}} \in \mathcal{E}_{l+1, m+2}$ since $i \leq l + 1 \leq m + 2$ and $0 \leq \tilde{k} \leq (m + 2) - (l + 1)$. Hence:

$$[\beta, \theta]^{2^{m+1-l-\tilde{k}}} = ([\theta, \beta]^{2^{m+1-l-\tilde{k}}})^{-1} \in \mathcal{E}_{l+1, m+2}.$$

The fact that $\mathcal{E}_{l+1, m+2} \subset \mathcal{E}_{i, m+2}$ implies that the congruence (5.26) is also valid modulo $\mathcal{E}_{i, m+2}$, from which it follows using (5.26) that $[\alpha_s, \theta] \in \mathcal{E}_{l+1, m+2} \subset \mathcal{E}_{i, m+2}$. Further, $[\alpha_s^{-1}, \theta] = \alpha_s^{-2} [\alpha_s, \theta] (\theta \alpha_s^2 \theta^{-1}) \in \mathcal{E}_{i, m+2}$, using also (1) and the fact that $\mathcal{E}_{i, m+2}$ is normal in H , which proves (2). To prove (3), since $x^{2^{m+1-(i-1)-k}} \in \mathcal{E}_{i-1, m+1}$, where $x = [x_1, \dots, x_{i-1}] \in \Gamma_{i-1}(H)$ and $0 \leq k \leq m + 1 - i$, and there exist $1 \leq j_1 < \dots < j_k \leq i - 1$ such that $x_{j_1}, \dots, x_{j_k} \in A$, it follows that $[y, x] = [y, x_1, \dots, x_{i-1}] \in \Gamma_i(H)$ and $[y, x]^{2^{(m+1)-i-k}} \in \mathcal{E}_{i, m+1}$. So applying induction hypothesis 2, we have:

$$[[x, y]^{-2^{m+1-i-k}}, \theta] = [[y, x]^{2^{m+1-i-k}}, \theta] \equiv [[y, x], \theta]^{2^{m+1-i-k}} \pmod{\mathcal{E}_{i+1, m+2}}. \quad (5.27)$$

Now

$$[[y, x], \theta] = [\theta, y, x]^{-1} = [[y, x], \theta] = [\theta, y, x_1, \dots, x_{i-1}]^{-1},$$

and since $[\theta, y, x_1, \dots, x_{i-1}]^{2^{(m+2)-(i+1)-k}} \in \mathcal{E}_{i+1, m+2}$, we conclude that $[[x, y]^{-2^{m+1-i-k}}, \theta] \in \mathcal{E}_{i+1, m+2} \subset \mathcal{E}_{i, m+2}$, and this proves claim (3). Thus it follows from (5.25) and claims (1), (2) and (3) that:

$$[x^{2^{m+1-i-k}}, y]^2 \equiv [x, y]^{2^{m+2-i-k}} \pmod{\mathcal{E}_{i, m+2}},$$

which proves that the expression (**) is congruent to $[x, y]^{2^{m+2-i-k}}$ modulo $\mathcal{E}_{i, m+2}$.

To see that the expression $(*)$ belongs to $\mathcal{E}_{i,m+2}$, notice that each of its terms is a commutator, so can be written as a product of conjugates of the element $[x^{2^{m-i-k}}, [x^{2^{m+1-i-k}}, y]]$ or its inverse. Since $\mathcal{E}_{i,m+2}$ is normal in H , it thus suffices to show that $[\theta, [x^{2^{m+1-i-k}}, y]]^{-1} = [[x^{2^{m+1-i-k}}, y], \theta] \in \mathcal{E}_{i,m+2}$ for all $y, \theta \in H$ and for all $x = [x_1, \dots, x_{i-1}] \in \Gamma_{i-1}(H)$ for which $x^{2^{m+1-(i-1)-k}} \in \mathcal{E}_{i-1,m+1}$, where $i \geq 3$, $0 \leq k \leq m+1-i$, $x_{j_1}, \dots, x_{j_k} \in A$ and $1 \leq j_1 < \dots < j_k \leq i-1$. To do so, note that $x^{2^{m+1-i-k}} = x^{2^{m-(i-1)-k}} \in \mathcal{E}_{i-1,m}$, so $[x^{2^{m+1-i-k}}, y] \equiv [x, y]^{2^{m+1-i-k}} \pmod{\mathcal{E}_{i,m+1}}$ by induction hypothesis 1. Thus there exist generators $\alpha_1, \dots, \alpha_t$ of $\mathcal{E}_{i,m+1}$, and $\delta_1, \dots, \delta_t \in \{1, -1\}$ such that $[x^{2^{m+1-i-k}}, y] = [x, y]^{2^{m+1-i-k}} \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}$, and hence:

$$\begin{aligned} [[x^{2^{m+1-i-k}}, y], \theta] &= [[x, y]^{2^{m+1-i-k}} \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}, \theta] \\ &= ([x, y]^{2^{m+1-i-k}} [\alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}, \theta] [x, y]^{-2^{m+1-i-k}}) [[x, y]^{2^{m+1-i-k}}, \theta]. \end{aligned} \quad (5.28)$$

Further, $[\alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}, \theta]$ belongs to the normal closure of $\{[\alpha_1^{\delta_1}, \theta], \dots, [\alpha_t^{\delta_t}, \theta]\}$ in H . This may be seen by applying reverse induction on $1 \leq s \leq t$, and by noting that for $s \geq 2$,

$$[\alpha_{s-1}^{\delta_{s-1}} \alpha_s^{\delta_s} \dots \alpha_t^{\delta_t}, \theta] = (\alpha_{s-1}^{\delta_{s-1}} [\alpha_s^{\delta_s} \dots \alpha_t^{\delta_t}, \theta] \alpha_{s-1}^{-\delta_{s-1}}) [\alpha_{s-1}^{\delta_{s-1}}, \theta]. \quad (5.29)$$

Then $[[x^{2^{m+1-i-k}}, y], \theta] \in \langle\langle [[x, y]^{2^{m+1-i-k}}, \theta], [\alpha_1^{\delta_1}, \theta], \dots, [\alpha_t^{\delta_t}, \theta] \rangle\rangle_H$ by (5.28) and (5.29). Now by claims (2) and (3), the elements $[\alpha_1^{\delta_1}, \theta], \dots, [\alpha_t^{\delta_t}, \theta]$ and $[[x, y]^{2^{m+1-i-k}}, \theta]$ belong to $\mathcal{E}_{i,m+2}$, and since $\mathcal{E}_{i,m+2}$ is normal in H , we conclude that the expression $(*)$ belongs to $\mathcal{E}_{i,m+2}$. This completes the proof of (5.23) for $i-1$, and so by induction, (5.23) holds for all $1 \leq i \leq m+1$, which is exactly (5.22). By induction, we conclude that (5.20) holds, and this completes the proof of the lemma. \square

Lemma 5.7 has the following consequences.

Corollary 5.9. *For all $1 \leq l \leq m$, $[\mathcal{E}_{l,m}, H] \subset \mathcal{E}_{l+1,m+1}$ and $[\tilde{\mathcal{E}}_{l,m}, H] \subset \tilde{\mathcal{E}}_{l+1,m+1}$.*

Proof. Let $l \leq i \leq m$. First assume that $x = [x_1, \dots, x_i] \in \Gamma_i(H)$ is such that $x^{2^{m-i-k}}$ is a generator of $\mathcal{E}_{l,m}$, and let $y \in H$. Then by (5.20), $[x^{2^{m-i-k}}, y] \equiv [x, y]^{2^{m-i-k}} \pmod{\mathcal{E}_{l+1,m+1}}$. Since $[y, x] = [y, x_1, \dots, x_i] \in \Gamma_{i+1}(H)$, it follows that $[y, x]^{2^{(m+1)-(i+1)-k}} \in \mathcal{E}_{l+1,m+1}$, so $[x^{2^{m-i-k}}, y] \in \mathcal{E}_{l+1,m+1}$.

Now suppose that $x = \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}$, where for all $i = 1, \dots, t$, α_i is a generator of $\mathcal{E}_{l,m}$, and $\delta_i \in \{1, -1\}$. Then $[x, y]$ belongs to the normal closure of $\{[\alpha_i^{\delta_i}, y], i = 1, \dots, t\}$ in H by (5.29), and so $[x, y] \in \mathcal{E}_{l+1,m+1}$ for all $x \in \mathcal{E}_{l,m}$, $y \in H$ by the first paragraph of the proof and the fact that $\mathcal{E}_{l+1,m+1}$ is normal in H . Once more, the result for $\tilde{\mathcal{E}}_{l,m}$ is obtained from that for $\mathcal{E}_{l,m}$ by taking $A = \emptyset$. \square

Corollary 5.10. *If $m \geq 2$, then $\tilde{W}_m = \langle\langle a_2^{2^{m-1}}, \tilde{\mathcal{E}}_{2,m} \rangle\rangle_H$.*

Proof. It suffices to prove that

$$\tilde{\mathcal{E}}_{2,m} = \langle x^{2^{m-i}} : x \in \Gamma_i(H), 2 \leq i \leq m \rangle \quad (5.30)$$

for all $m \geq 2$. If $m = 2$, (5.30) follows from (5.12) and (5.18). Suppose by induction on m that (5.30) holds for some $m \geq 2$. It is clear from (5.18) that $\tilde{\mathcal{E}}_{2,m+1} \subset \langle x^{2^{m-i}} : x \in \Gamma_i(H), 2 \leq i \leq m \rangle$. To prove the converse, let $y = x^{2^{m+1-i}}$, where $x \in \Gamma_i(H)$ and $2 \leq i \leq m+1$. If $i = m+1$ then $y \in \Gamma_{m+1}(H) \subset \tilde{\mathcal{E}}_{2,m+1}$ by (5.19). So suppose that $2 \leq i \leq m$. Then $x^{2^{m-i}} \in \tilde{\mathcal{E}}_{2,m}$ by the induction hypothesis, so $x^{2^{m-i}} = \alpha_1^{\delta_1} \dots \alpha_t^{\delta_t}$, where for all $i = 1, \dots, t$, α_i is a generator of $\tilde{\mathcal{E}}_{2,m}$ and $\delta_i \in \{1, -1\}$. Hence:

$$y = (x^{2^{m-i}})^2 = (\alpha_1^{\delta_1} \dots \alpha_t^{\delta_t})^2 = \alpha_1^{2\delta_1} [\alpha_1^{-\delta_1}, \alpha_2^{\delta_2} \dots \alpha_t^{\delta_t}] \alpha_2^{2\delta_2} [\alpha_2^{-\delta_2}, \alpha_3^{\delta_3} \dots \alpha_t^{\delta_t}] \dots [\alpha_{t-1}^{-\delta_{t-1}}, \alpha_t^{\delta_t}] \alpha_t^{2\delta_t}.$$

So $y \in \tilde{\mathcal{E}}_{2,m+1}$ because for all $i = 1, \dots, t$, $\alpha_i^{2\delta_i} \in \tilde{\mathcal{E}}_{2,m+1}$ by (5.18), and $[\alpha_i^{-\delta_i}, \theta] \in [\tilde{\mathcal{E}}_{2,m}, H] \subset \tilde{\mathcal{E}}_{3,m+1} \subset \tilde{\mathcal{E}}_{2,m+1}$ by Corollary 5.9 and (5.19). The inclusion $\langle x^{2^{m-i}} : x \in \Gamma_i(H), 2 \leq i \leq m \rangle \subset \tilde{\mathcal{E}}_{2,m+1}$ then follows. \square

Corollary 5.11. *For all $m \geq 2$, $[H, \widetilde{W}_m] \subset \widetilde{W}_{m+1}$.*

Proof. Since $[\widetilde{\mathcal{E}}_{2,m}, H] \subset \widetilde{\mathcal{E}}_{2,m+1}$ using Corollary 5.9, by Corollary 5.10, it suffices to prove that $[h, a_2^{2^{m-1}}] \in \widetilde{W}_{m+1}$ for all $h \in H$. To see this, observe that $[h, a_2^{2^{m-1}}] \equiv [h, a_2]^{2^{m-1}} \pmod{\widetilde{W}_{m+1}}$ by taking $i = 1$ and $x = a_2$ in (5.21) and using Remark 5.8. So there exists $\tilde{w} \in \widetilde{W}_{m+1}$ such that $[h, a_2^{2^{m-1}}] = [h, a_2]^{2^{m-1}} \cdot \tilde{w}$. Now $[h, a_2] \in \Gamma_2(H)$, hence $[h, a_2]^{2^{m-1}} \in \widetilde{W}_{m+1}$ by (5.14), and we conclude that $[h, a_2^{2^{m-1}}] \in \widetilde{W}_{m+1}$ as required. \square

The following result will enable us to obtain the explicit characterisation of $\Gamma_n(P_2(\mathbb{K}))$ given in Theorem 5.4.

Proposition 5.12. *For all $n \geq 2$, $L_n = W_n = \widetilde{W}_n$.*

Proof. If $n = 2$, the statement is true by definition. So suppose by induction that $L_n = W_n = \widetilde{W}_n$ for some $n \geq 2$. Then we have the following inclusions:

- $\widetilde{W}_{n+1} \subset W_{n+1}$. To see this, let $y \in \widetilde{W}_{n+1}$. If y is a generator of \widetilde{W}_{n+1} , then by (5.14), y either belongs to $\Gamma_{n+1}(H)$, and so belongs to W_{n+1} by (5.13), or is of the form $x^{2^{n+1-i}}$, where $1 \leq i \leq n$, $x \in \Gamma_i(H)$, and $x = a_2$ if $i = 1$. Hence $x^{2^{n-i}} \in \widetilde{W}_n = W_n$ by induction, and thus $y = x^{2^{n+1-i}} = (x^{2^{n-i}})^2 \in W_{n+1}$ by (5.13). If y is an arbitrary element of \widetilde{W}_{n+1} , it may be written as a product of conjugates of generators and their inverses, so it belongs to W_{n+1} because W_{n+1} is normal in H .
- $L_{n+1} \subset \widetilde{W}_{n+1}$, since $K_{n+1} \subset \widetilde{W}_{n+1}$ by Lemma 5.5 and (5.14), and $[H, L_n] = [H, \widetilde{W}_n] \subset \widetilde{W}_{n+1}$ by induction and Corollary 5.11, so $L_{n+1} = \langle K_{n+1}, [H, L_n] \rangle \subset \widetilde{W}_{n+1}$ using Lemma 5.6.
- $W_{n+1} \subset L_{n+1}$. First, $\Gamma_{n+1}(H) = [H, \Gamma_n(H)] \subset [H, W_n] = [H, L_n] \subset L_{n+1}$ by (5.13) and induction. Secondly, let $x^2 \in W_{n+1}$, where $x \in W_n = L_n$ is a generator. If $n = 2$, by (5.12), $x = a_2^2$ or $x = [b_2, a_2]$, then $x^2 = a_2^4 \in K_3 \subset L_3$ by Lemma 5.5, or $x^2 = [b_2, \underbrace{a_2^2}_{\in L_2}][a_2, \underbrace{[a_2, b_2]}_{\in L_2}] \in$

$[H, L_2] \subset L_3$. So assume that $n \geq 3$. By Lemma 5.6, there are two possibilities for x :

- if $x \in K_n$, then $x = a_2^{2^{n-1}}$ by Lemma 5.5, and $x^2 = a_2^{2^n} \in K_{n+1} \subset L_{n+1}$.
- if $x = [h, l] \in [H, L_{n-1}]$, where $h \in H$ and $l \in L_{n-1} = W_{n-1}$, then $x^2 = [h, l]^2 = [h, l^2][l, [l, h]]$. Now $l^2 \in W_n = L_n$ by induction and $[l, h] \in L_n$, so $[h, l^2]$ and $[l, [l, h]]$ belong to $[H, L_n]$, which is contained in L_{n+1} .

Finally, let $x^2 \in W_{n+1}$, where x is an arbitrary element of W_n . Then there exists $q \in \mathbb{N}$ such that $x = x_1 \cdots x_q$, where for all $i = 1, \dots, q$, x_i is a generator of $W_n = L_n$. Then as in (5.25), we have:

$$x^2 = x_1^2[x_1^{-1}, x_2 \cdots x_q]x_2^2[x_2^{-1}, x_3 \cdots x_q] \cdots [x_{q-1}^{-1}, x_q]x_q^2. \quad (5.31)$$

From the second case above, for all $i = 1, \dots, q$, $x_i^2 \in L_{n+1}$. Further, for all $i = 1, \dots, q-1$, $[x_i^{-1}, x_{i+1} \cdots x_q] \in [L_n, H] \subset L_{n+1}$. It then follows from (5.31) that $x^2 \in L_{n+1}$.

It follows from these three inclusions that $L_{n+1} = W_{n+1} = \widetilde{W}_{n+1}$. \square

Proof of Theorem 5.4. The result is a consequence of Theorem 1.1(1), Proposition 5.12, and equations (5.8), (5.10) and (5.14). To see that $P_2(\mathbb{K})$ is residually nilpotent, note first that $W_n \subset \gamma_n^2(H)$ for all $n \geq 2$ by (5.13). Since H is a free group of finite rank, it is residually 2-finite, and it follows using Proposition 5.12 that $\bigcap_{n \geq 1} L_n = \bigcap_{n \geq 2} W_n = \{1\}$. The residual nilpotence of $P_2(\mathbb{K})$ is then a consequence of the first part of the statement, equation (5.14) and Proposition 5.12. \square

5.3. The lower \mathbb{F}_2 -linear central filtration of $P_2(\mathbb{K})$. Now that we have a good description of $\Gamma_n(P_2(\mathbb{K}))$, we may obtain the following decomposition of $\gamma_n^2(P_2(\mathbb{K}))$.

Theorem 5.13. *Let $n \geq 2$. With the notation of Theorem 2.1, $\gamma_n^2(P_2(\mathbb{K}))$ is equal to:*

$$\langle\langle a_2^{2^{n-1}}, b_2^{2^{n-1}}, x^{2^{n-i}} : x \in \Gamma_i(\pi_1(\mathbb{K} \setminus \{x_1\})) \rangle\rangle_{\pi_1(\mathbb{K} \setminus \{x_1\})} \rtimes \langle (a_1 a_2)^{2^{n-1}}, (b_2 b_1)^{2^{n-1}} \rangle.$$

Consequently, $P_2(\mathbb{K})$ is residually 2-finite.

Remark 5.14. Using the action given by (5.9) and the description of $\Gamma_n(P_2(\mathbb{K}))$ given by Theorem 5.4, it is straightforward to see that:

$$\langle\langle a_2^{2^{n-1}}, b_2^{2^{n-1}}, x^{2^{n-i}} : x \in \Gamma_i(\pi_1(\mathbb{K} \setminus \{x_1\})), 2 \leq i \leq n \rangle\rangle_{\pi_1(\mathbb{K} \setminus \{x_1\})} \rtimes \langle\langle (a_1 a_2)^{2^{n-1}}, (b_2 b_1)^{2^{n-1}} \rangle\rangle$$

is equal to U_n , where:

$$U_n = \langle\langle \Gamma_n(P_2(\mathbb{K})), b_2^{2^{n-1}}, (b_2 b_1)^{2^{n-1}} \rangle\rangle_{P_2(\mathbb{K})}. \quad (5.32)$$

Proof of Theorem 5.13. Let $n \geq 2$. By Remark 5.14, it suffices to prove by induction on n that $U_n = \gamma_n^2(P_2(\mathbb{K}))$, where U_n is defined by (5.32). If $n = 2$, $U_2 \subset \gamma_2^2(P_2(\mathbb{K}))$ because b_2^2 and $(b_2 b_1)^2$ belong to $\{x^2 : x \in P_2(\mathbb{K})\}$, which is contained in $\gamma_2^2(P_2(\mathbb{K}))$, $\Gamma_2(P_2(\mathbb{K}))$ is contained in $\gamma_2^2(P_2(\mathbb{K}))$, and $\gamma_2^2(P_2(\mathbb{K}))$ is normal in $P_2(\mathbb{K})$. For the converse inclusion, we know that $\Gamma_2(P_2(\mathbb{K})) \subset U_2$, that $P_2(\mathbb{K})$ is generated by $\{a_2, a_2 a_1, b_2, b_2 b_1\}$ by Remark 5.3, and that the square of each element of this set belongs to U_2 , since by Theorem 5.4, $a_2^2 \in \Gamma_2(P_2(\mathbb{K}))$ and $(a_2 a_1)^2 \in \Gamma_2(s(\pi_1(\mathbb{K}))) \subset \Gamma_2(P_2(\mathbb{K}))$. Further, if $x = x_1 \cdots x_q$, where for $i = 1, \dots, q$, $x_i \in \{a_2^{\pm 1}, (a_2 a_1)^{\pm 1}, b_2^{\pm 1}, (b_2 b_1)^{\pm 1}\}$, then using the decomposition given in (5.31), we have $x^2 \in U_2$ because $x_i^2 \in U_2$ for all $1 \leq i \leq q$ and $[x_i^{-1}, x_{i+1} \cdots x_q] \in \Gamma_2(P_2(\mathbb{K})) \subset U_2$ for all $1 \leq i \leq q-1$. Thus $\gamma_2^2(P_2(\mathbb{K})) \subset U_2$, and therefore $\gamma_2^2(P_2(\mathbb{K})) = U_2$.

Now assume that $n \geq 2$, and suppose by induction that $U_i = \gamma_i^2(P_2(\mathbb{K}))$ for all $2 \leq i \leq n$. Then $U_{n+1} \subset \gamma_{n+1}^2(P_2(\mathbb{K}))$ since $\Gamma_{n+1}(P_2(\mathbb{K})) \subset \gamma_{n+1}^2(P_2(\mathbb{K}))$, $\Gamma_{n+1}(P_2(\mathbb{K}))$ and $\gamma_{n+1}^2(P_2(\mathbb{K}))$ are normal in $P_2(\mathbb{K})$, and if $x \in \{b_2, b_2 b_1\}$ then $x^{2^n} = (x^{2^{n-1}})^2$, where $x^{2^{n-1}} \in U_n = \gamma_n^2(P_2(\mathbb{K}))$ by the induction hypothesis, so $x^{2^n} \in \gamma_{n+1}^2(P_2(\mathbb{K}))$.

To prove that $\gamma_{n+1}^2(P_2(\mathbb{K})) \subset U_{n+1}$, using the induction hypothesis and the fact that $\gamma_{n+1}^2(P_2(\mathbb{K}))$ is generated by $[P_2(\mathbb{K}), \gamma_n^2(P_2(\mathbb{K}))] \cup \{x^2 : x \in \gamma_n^2(P_2(\mathbb{K}))\}$, it suffices to show that $[P_2(\mathbb{K}), U_n] \subset U_{n+1}$, and that $x^2 \in U_{n+1}$ for all $x \in U_n$. We first show that $[P_2(\mathbb{K}), U_n] \subset U_{n+1}$. Let $x \in P_2(\mathbb{K})$, and let $u \in U_n$.

- (1) If $u \in \Gamma_n(P_2(\mathbb{K}))$ then $[x, u] \in U_{n+1}$.
- (2) If $u = (b_2 b_1)^{2^{n-1}}$ then $[x, u] = 1$ because $(b_2 b_1)^{2^{n-1}} \in Z(P_2(\mathbb{K}))$ by Proposition 5.2.
- (3) If $u = b_2^{2^{n-1}}$, we claim that $[b_2^{2^{n-1}}, x] \in L_{n+1}$ for all $x \in \{a_2, (a_2 a_1)^{-1}, b_2, (b_2 b_1)^{-1}\}$. The result is clear if $x = b_2$, so we consider the three other cases. We proceed by induction on n . Suppose first that $n = 2$. If $x = a_2$ then by (3.2), (5.13) and Proposition 5.12, we have:

$$[u, x] = [b_2^2, a_2] = \underbrace{[b_2, b_2, a_2]}_{\in W_3=L_3} \underbrace{[b_2, a_2]^2}_{\in W_3=L_3} \in L_3.$$

In the remaining two cases, by (5.9), (5.12), (5.14) and Proposition 5.12, we have:

$$[(a_2 a_1)^{-1}, u] = [(a_2 a_1)^{-1}, b_2^2] = (a_2^{-2} b_2)(a_2^{-2} b_2) b_2^{-2} = \underbrace{a_2^{-4}}_{\in \widetilde{W}_3=L_3} \underbrace{[a_2^2, b_2]}_{\in [L_2, H] \subset L_3} \in L_3$$

$$\begin{aligned} [(b_2 b_1)^{-1}, u] &= [(b_2 b_1)^{-1}, b_2^2] = (a_2 b_2 a_2)(a_2 b_2 a_2) b_2^{-2} \\ &= \underbrace{[a_2 b_2, a_2^2]}_{\in [H, L_2] \subset L_3} \underbrace{a_2^4}_{\in \widetilde{W}_3=L_3} \underbrace{[a_2^{-1}, b_2]^2}_{\in \widetilde{W}_3=L_3} \underbrace{[[b_2, a_2^{-1}], b_2]}_{\in \Gamma_3(H) \subset L_3} \in L_3, \end{aligned}$$

which proves the claim in the case $n = 2$. Now suppose that $[b_2^{2^{j-1}}, x] \in L_{j+1}$ for all $2 \leq j \leq n$ and $x \in \{a_2, (a_2 a_1)^{-1}, b_2, (b_2 b_1)^{-1}\}$. Then by Proposition 5.12 and (5.13), $[b_2^{2^{n-1}}, x] \in L_{n+1} = W_{n+1}$, and hence $[b_2^{2^{n-1}}, x]^2 \in W_{n+2} = L_{n+2}$. So by (3.2), we have:

$$[b_2^{2^n}, x] = \underbrace{[b_2^{2^{n-1}}, [b_2^{2^{n-1}}, x]]}_{\in L_{n+1}} \underbrace{[b_2^{2^{n-1}}, x]^2}_{\in L_{n+2}} \in L_{n+2},$$

$\underbrace{\hspace{10em}}_{\in [H, L_{n+1}] \subset L_{n+2}}$

which proves the claim for all $n \geq 2$. Now let x be an arbitrary element of $P_2(\mathbb{K})$. Since the set $\{a_2, (a_2a_1)^{-1}, b_2, (b_2b_1)^{-1}\}$ generates $P_2(\mathbb{K})$, for some $t \geq 0$, there exist $x_1, \dots, x_t \in \{a_2, (a_2a_1)^{-1}, b_2, (b_2b_1)^{-1}\}$ and $\varepsilon_1, \dots, \varepsilon_t \in \{1, -1\}$ for which $x = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_t^{\varepsilon_t}$. As in (5.29), we have the following relation:

$$[u, x] = [u, x_1^{\varepsilon_1}] (x_1^{\varepsilon_1} [u, x_2^{\varepsilon_2}] x_1^{-\varepsilon_1}) \cdots (x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{t-1}^{\varepsilon_{t-1}} [u, x_t^{\varepsilon_t}] x_{t-1}^{-\varepsilon_{t-1}} \cdots x_2^{-\varepsilon_2} x_1^{-\varepsilon_1}). \quad (5.33)$$

Using the fact that L_{n+1} is normal in H by Lemma 3.3, it follows from (5.33) that $[b_2^{2^{n-1}}, x] \in L_{n+1}$, and since $L_{n+1} \subset \Gamma_{n+1}(P_2(\mathbb{K}))$ by Theorem 1.1, we deduce that $[b_2^{2^{n-1}}, x] \in U_{n+1}$.

This concludes the proof of the inclusion $[P_2(\mathbb{K}), U_n] \subset U_{n+1}$. It remains to prove that $\{x^2 : x \in U_n\} \subset U_{n+1}$. If $x = b_2^{2^{n-1}}$ or $x = (b_2b_1)^{2^{n-1}}$ then clearly $x^2 \in U_{n+1}$. Using Theorems 1.1(1) and 5.4, if $x \in \Gamma_n(P_2(\mathbb{K})) = L_n \rtimes \langle (a_1a_2)^{2^{n-1}} \rangle$, then in terms of this semi-direct product, $x = (x_1, x_2)$, where $x_1 \in L_n$ and $x_2 \in \langle (a_1a_2)^{2^{n-1}} \rangle$, and $x^2 = (x_1, x_2)(x_1, x_2) = (x_1 \cdot \varphi(x_2)(x_1), x_2^2)$. Now $x_2^2 \in \langle (a_1a_2)^{2^n} \rangle$, and since $x_1 \in L_n = W_n$, we have $x_1^2 \in W_{n+1} = L_{n+1}$ by Proposition 5.12, and therefore $x_1 \cdot \varphi(x_2)(x_1) = x_1^2 (\varphi(x_2)(x_1^{-1}) \cdot x_1)^{-1} \in L_{n+1}$ and $x^2 \in L_{n+1} \rtimes \langle (a_1a_2)^{2^n} \rangle = \Gamma_{n+1}(P_2(\mathbb{K})) \subset U_{n+1}$. If x is a product of conjugates of generators of U_n then $x^2 \in U_{n+1}$ using (5.31). This shows that $\{x^2 : x \in U_n\} \subset U_{n+1}$. It follows that $\gamma_{n+1}^2(P_2(\mathbb{K})) \subset U_{n+1}$, so $\gamma_{n+1}^2(P_2(\mathbb{K})) = U_{n+1}$. Using the explicit characterisation of $\Gamma_n(P_2(\mathbb{K}))$ given by Theorem 5.4 and Remark 5.14, the first part of the statement follows. The second part is a consequence of the fact that $\bigcap_{n \geq 2} U_n = \{1\}$ using the first part. \square

5.4. The lower central series of $P_n(\mathbb{K})$. For $n \geq 3$, the situation is more complicated due to the complexity of the lower central series of $P_2(\mathbb{K})$. The idea is to use the Fadell-Neuwirth short exact sequence (5.1) and to calculate recursively $\Gamma_m(P_n(\mathbb{K}))$ for $m \geq 1$ and $n \geq 3$. Thus will allows us to prove that $P_n(\mathbb{K})$ is residually nilpotent for all $n \geq 1$ in Theorem 5.25.

With the notation of Theorem 1.1 and equation (5.1), we may write $P_{n+1}(\mathbb{K}) = H \rtimes_{\varphi} G$, where $H = \pi_1(\mathbb{K} \setminus \{x_1, \dots, x_n\})$, $G = s(P_n(\mathbb{K}))$, and s is the section for p_* given by Proposition 5.1.

Remark 5.15. Notice that $H_m = [G, L_{m-1}]$ and $K_m = [\Gamma_{m-1}(G), H]$, for all $m \geq 2$, since the action φ of G on H is defined by conjugation.

In what follows, we will take G to be equipped with the generating set:

$$\begin{aligned} X &= \{s(z) : z \in \{a_i, b_i, C_{j,k} : 1 \leq i \leq n \text{ and } 1 \leq j < k \leq n\}\} \\ &= \{a_i, b_i, C_{i,k}, a_n a_{n+1}, b_n b_{n+1} C_{n,n+1}, C_{i,n} C_{i,n+1} C_{n,n+1}^{-1} : 1 \leq i \leq k \leq n-1\}. \end{aligned} \quad (5.34)$$

In $P_{n+1}(\mathbb{K})$, for $1 \leq j \leq n$, let $D_j = C_{j,n+1}^{-1} C_{j+1,n+1}$. Since $C_{j,n+1} = D_n^{-1} \cdots D_{j+1}^{-1} D_j^{-1}$ for all $j = 1, \dots, n$, the set:

$$Y = \{a_{n+1}, b_{n+1}, D_j : j = 1, \dots, n\} \quad (5.35)$$

generates H , and using Theorem 2.1, the action $\varphi: s(P_n(\mathbb{K})) \rightarrow \text{Aut}(\pi_1(\mathbb{K} \setminus \{x_1, \dots, x_n\}))$ is given by:

$$\left\{ \begin{array}{l} \varphi(a_i)(z) = \begin{cases} a_{n+1} & \text{if } z = a_{n+1} \\ b_{n+1}a_{n+1}D_i a_{n+1}^{-1} & \text{if } z = b_{n+1} \\ \alpha_{i,j}D_j\alpha_{i,j}^{-1} & \text{if } z = D_j \end{cases} \\ \varphi(b_i)(z) = \begin{cases} a_{n+1}b_{n+1}C_{i,n+1}D_i C_{i,n+1}^{-1}b_{n+1}^{-1} & \text{if } z = a_{n+1} \\ b_{n+1}C_{i,n+1}D_i^{-1}C_{i,n+1}^{-1} & \text{if } z = b_{n+1} \\ \beta_{i,i}D_i^{-1}\beta_{i,i}^{-1} & \text{if } z = D_i \\ \beta_{i,j}D_j\beta_{i,j}^{-1} & \text{if } z = D_j, j \neq i \end{cases} \\ \varphi(C_{i,k})(z) = \begin{cases} a_{n+1} & \text{if } z = a_{n+1} \\ b_{n+1} & \text{if } z = b_{n+1} \\ \delta_{i,j,k}D_j\delta_{i,j,k}^{-1} & \text{if } z = D_j \end{cases} \\ \varphi(a_n a_{n+1})(z) = \begin{cases} a_{n+1} & \text{if } z = a_{n+1} \\ a_{n+1}^{-1}b_{n+1}a_{n+1}D_n & \text{if } z = b_{n+1} \\ \tilde{\alpha}_j D_j \tilde{\alpha}_j^{-1} & \text{if } z = D_j \end{cases} \\ \varphi(b_n b_{n+1} C_{n,n+1})(z) = \begin{cases} D_n b_{n+1}^{-1} a_{n+1} b_{n+1} & \text{if } z = a_{n+1} \\ b_{n+1} D_n^{-1} & \text{if } z = b_{n+1} \\ D_n^{-1} & \text{if } z = D_n \\ b_{n+1}^{-1} D_j b_{n+1} & \text{if } z = D_j, j \neq n \end{cases} \\ \varphi(C_{i,n} C_{i,n+1} C_{n,n+1}^{-1})(z) = \begin{cases} C_{n,n+1} C_{i,n+1}^{-1} a_{n+1} C_{i,n+1} C_{n,n+1}^{-1} & \text{if } z = a_{n+1} \\ C_{n,n+1} C_{i,n+1}^{-1} b_{n+1} C_{i,n+1} C_{n,n+1}^{-1} & \text{if } z = b_{n+1} \\ \tilde{\delta}_{i,j} D_j \tilde{\delta}_{i,j}^{-1} & \text{if } z = D_j, \end{cases} \end{array} \right. \quad (5.36)$$

where:

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } i < j \\ C_{j+1,n+1}^{-1} a_{n+1} & \text{if } i = j \\ C_{i+1,n+1}^{-1} C_{i,n+1} & \text{if } i > j \end{cases} \quad \beta_{i,j} = \begin{cases} 1 & \text{if } i < j \\ b_{n+1} C_{i,n+1} & \text{if } i = j \\ b_{n+1} C_{i,n+1} C_{i+1,n+1}^{-1} b_{n+1}^{-1} & \text{if } i > j \end{cases}$$

$$\delta_{i,j,k} = \begin{cases} 1 & \text{if } k < j \text{ or } i > j \\ C_{j+1,n+1}^{-1} C_{i,n+1} & \text{if } k = j \\ C_{k+1,n+1}^{-1} C_{k,n+1} & \text{if } k > j \geq i \end{cases}$$

for all $1 \leq i \leq k \leq n-1$ and $1 \leq j \leq n$, and where $\tilde{\alpha}_j = a_{n+1}^{-1} \alpha_{n,j}$ and $\tilde{\delta}_{i,j} = C_{n,n+1} C_{i,n+1}^{-1} \delta_{i,j,n}$.

Our aim is to determine the subgroups L_m that were defined in the statement of Theorem 1.1 for all $m \geq 2$. For $i \geq 0$, let:

$$A_n^{2^i} = \langle\langle D_j^{2^i} : j = 1, \dots, n \rangle\rangle_H. \quad (5.37)$$

If $i = 0$, we write $A_n = A_n^1$. To compute K_2 , recall that $K_2 = H_2$, and that this subgroup is normal in H by Lemma 3.3. Using (5.36), observe that $\varphi(a_i)(b_{n+1})b_{n+1}^{-1} = b_{n+1}a_{n+1}D_i a_{n+1}^{-1}b_{n+1}^{-1}$ for all $1 \leq i \leq n-1$, and $\varphi(b_n b_{n+1} C_{n,n+1})(b_{n+1})b_{n+1}^{-1} = b_{n+1}D_n^{-1}b_{n+1}^{-1}$. So by normality of K_2 , D_j belongs to K_2 for all $1 \leq j \leq n$, and therefore $A_n \subset K_2$ by (5.37). Moreover, $\varphi(b_n b_{n+1} C_{n,n+1})(a_{n+1})a_{n+1}^{-1} = D_n b_{n+1}^{-1} a_{n+1} b_{n+1} a_{n+1}^{-1}$, so $[b_{n+1}^{-1}, a_{n+1}]$ belongs to K_2 also. Applying Lemma 3.4(1) with X and Y as defined in (5.34) and (5.35), and using (5.36), we see that $K_2 = \langle\langle A_n, [a_{n+1}, b_{n+1}] \rangle\rangle_H$, and therefore:

$$L_2 = \langle \Gamma_2(H), A_n \rangle. \quad (5.38)$$

Let $Y_1 = L_1 = H$, and for $m \geq 2$, let:

$$Y_m = \langle A_n^{2^{m-2}}, [Y_i, Y_k] : 1 \leq i \leq k < m, i + k = m \rangle. \quad (5.39)$$

In what follows, we will refer to the elements of the set

$$\{D_j^{2^{m-2}}, [y_i, y_k] : j = 1, \dots, n, y_i \in Y_i, y_k \in Y_k \text{ and } i + k = m\}$$

as *generators* of Y_m . Since $\Gamma_m(P_{n+1}(\mathbb{K})) = L_m \rtimes \Gamma_m(P_n(\mathbb{K}))$ by Theorem 1.1(1), to prove that $P_{n+1}(\mathbb{K})$ is residually nilpotent by induction on n , it will suffice to show that $L_m \subset Y_m$ for all $m \geq 2$ (we will show in fact that $L_m = Y_m$), and then that $\bigcap_{m \geq 1} Y_m = \{1\}$.

Remark 5.16. For all $m \geq 1$, Y_m is a normal subgroup of H , and $Y_{m+1} \subset Y_m$ by induction on m . Further, we claim that $\varphi(g)(Y_i) \subset Y_i$ for all $i \geq 1$ and $g \in G$. To see this, observe that $\varphi(g)(D_j)$ is a conjugate of D_j or D_j^{-1} for all $j = 1, \dots, n$ by (5.36), so $\varphi(g)(A_n^{2^i}) \subset A_n^{2^i}$ for all $i \geq 1$. Also, if $y_k \in Y_k$ and $y_l \in Y_l$ then $\varphi(g)([y_k, y_l]) = [\varphi(g)(y_k), \varphi(g)(y_l)]$, and the claim follows by induction on i .

Lemma 5.17. $[G, Y_m] \subset Y_{m+1}$ for all $m \geq 1$.

Proof. If $m = 1$, then $[G, Y_1] = [G, H] = H_2 \subset L_2 = Y_2$. Now, by induction on m , suppose that $[G, Y_i] \subset Y_{i+1}$, for all $1 \leq i \leq m$. Let us prove that $[G, Y_{m+1}] \subset Y_{m+2}$. To do so, let $[g, h] = \varphi(g)(h).h^{-1} \in [G, Y_{m+1}]$, where $g \in G$ and $h \in Y_{m+1}$. By Lemma 3.4(2), we need only analyse the following cases where $g \in X$ and h is a generator of Y_{m+1} :

- (1) $h = D_j^{2^{m-1}}$, where $j \in \{1, \dots, n\}$. By (5.36), if $j \in \{1, \dots, n-1\}$ then:

$$\varphi(b_j)(D_j^{2^{m-1}}).D_j^{-2^{m-1}} = [\beta_{j,j}, D_j^{-2^{m-1}}]D_j^{-2^m} \in Y_{m+2},$$

because $D_j^{2^{m-1}} \in Y_{m+1}$, and if $j = n$ then:

$$\varphi(b_n b_{n+1} C_{n,n+1})(D_n^{2^{m-1}}).D_n^{-2^{m-1}} = D_n^{-2^m} \in Y_{m+2}.$$

Similarly, by (5.36), one may check that if g is any other element of X then $\varphi(g)(D_j^{2^{m-1}}).D_j^{-2^{m-1}}$ is a commutator of $D_j^{2^{m-1}}$ with an element of H , and so belongs to $[H, Y_{m+1}]$, which is contained in Y_{m+2} by (5.39).

- (2) $h = [y_i, y_j]$, where $1 \leq i, j \leq m, i + j = m + 1, y_i \in Y_i$ and $y_j \in Y_j$. By Remark 5.16, we have that $\varphi(g)(Y_l) \subset Y_l$ for all $l \geq 1$. By the induction hypothesis $(\varphi(g)(y_j^{-1}).y_j)^{-1} \in [G, Y_j] \subset Y_{j+1}$ and $(\varphi(g)(y_i^{-1}).y_i)^{-1} \in [G, Y_i] \subset Y_{i+1}$. Therefore $[(\varphi(g)(y_j^{-1}).y_j)^{-1}, \varphi(g)(y_i^{-1})] \in [Y_{j+1}, Y_i] \subset Y_{m+2}$ and $[(\varphi(g)(y_i^{-1}).y_i)^{-1}, y_j] \in [Y_{i+1}, Y_j] \subset Y_{m+2}$ by (5.39), and writing $[g, [y_i, y_j]] = \varphi(g)([y_i, y_j]).[y_j, y_i]$ in the form of (5.15) and Remark 5.16, where we replace l by y_j and h by y_i , and using the fact that Y_{m+2} is normal in H , it follows that $\varphi(g)([y_i, y_j]).[y_j, y_i] \in Y_{m+2}$. \square

Remark 5.18. For all $m \geq 1$, Y_m is a normal subgroup of $H \rtimes_\varphi G$. To see this, recall that Y_m is normal in H by Remark 5.16, and if $y \in Y_m$ and $g \in G$, $[g, y] \in Y_{m+1} \subset Y_m$ by Lemma 5.17, and therefore $gyg^{-1} = [g, y]y \in Y_m$.

To prove the results that follow, we will make use of two commutator identities [27, Theorem 5.1]:

$$[a, bc] = [a, b][b, [a, c]][a, c] \quad (5.40)$$

$$[[a, b], c^b][[b, c], a^c][[c, a], b^a] = 1 \quad (5.41)$$

where $c^b = bcb^{-1}$. The following two lemmas are inspired by those of [11, Section 3].

Lemma 5.19. *Let $x \in Y_p$, and $y, w \in H \rtimes_\varphi G$. Suppose that $[w, y] \in Y_q$. Then $[x, y] \in Y_{p+q}$ if and only if $[x, y^w] \in Y_{p+q}$.*

Proof. Set $z = [w, y]$. Then $y^w = zy$, so by (5.40), $[x, y^w] = [x, zy] = [x, z][z, [x, y]][x, y]$. Since $z \in Y_q$ by hypothesis, we have that $[x, z] \in [Y_p, Y_q] \subset Y_{p+q}$ by Remark 5.16. Also, $[x, y] \in Y_p$ because Y_p is normal in $H \rtimes_{\varphi} G$ by Remark 5.18, and so $[z, [x, y]] \in [Y_q, Y_p] \subset Y_{p+q}$ by Remark 5.16. The result then follows. \square

Lemma 5.20. *For all $k, m \geq 1$, $[\Gamma_k(G), Y_m] \subset Y_{k+m}$.*

Proof. If $k = 1$, the result is a consequence of Lemma 5.17. Now suppose by induction that $[\Gamma_k(G), Y_m] \subset Y_{m+k}$ for some $k \geq 1$ and all $m \geq 1$, and let us prove that $[\Gamma_{k+1}(G), Y_m] \subset Y_{k+m+1}$ for all $m \geq 1$. Applying (5.41) to elements $g \in G$, $g_k \in \Gamma_k(G) \subset G$ and $h \in Y_m$ of the commutator $[\Gamma_{k+1}(G), Y_m] = [[\Gamma_k(G), G], Y_m]$, we obtain:

$$[[g_k, g], h^g] [[g, h], g_k^h] [[h, g_k], g^{g_k}] = 1. \quad (5.42)$$

By Lemma 5.17, $[g, h] \in Y_{m+1}$, and $[[g, h], g_k] \in [Y_{m+1}, \Gamma_k(G)] \subset Y_{m+k+1}$ using also the induction hypothesis. Further, $[h, g_k] \in [Y_m, \Gamma_k(G)] \subset Y_{m+k} \subset Y_k$ by the induction hypothesis and Remark 5.16, and consequently $[[g, h], g_k^h] \in Y_{m+k+1}$ by Lemma 5.19. Also, $[[h, g_k], g^{g_k}] \in [[Y_m, \Gamma_k(G)], G] \subset [Y_{m+k}, G] \subset Y_{m+k+1}$ by the induction hypothesis and Lemma 5.17. Thus $[[g_k, g], h^g] \in Y_{m+k+1}$ by (5.42). By (5.40), we see that:

$$[[g_k, g], h^g] = [[g_k, g], [g, h]h] = [[g_k, g], [g, h]] [[g, h], [[g_k, g], h]] [[g_k, g], h]. \quad (5.43)$$

Now $[[g_k, g], [g, h]] \in [\Gamma_k(G), Y_{m+1}] \subset Y_{m+k+1}$ by the induction hypothesis, and by Remark 5.16, we see that

$$[[g, h], [[g_k, g], h]] \in [Y_{m+1}, [\Gamma_k(G), Y_m]] \subset [Y_{m+1}, Y_{k+m}] \subset Y_{2m+k+1} \subset Y_{m+k+1}.$$

We conclude from (5.43) that $[[g_k, g], h] \in Y_{m+k+1}$, and the result follows. \square

Lemma 5.21. *For all $m \geq 1$, $L_m = Y_m$.*

Proof. We prove the lemma by induction on $m \geq 1$. For $m = 1, 2$, the result follows from (5.38) and (5.39). Suppose that $m \geq 2$ is such that $L_i = Y_i$ for all $1 \leq i \leq m$.

We first show that $Y_{m+1} \subset L_{m+1}$. For all $i, j \geq 0$, it follows from [27, Theorem 5.3] and Theorem 1.1(1) that:

$$\begin{aligned} [L_i \rtimes_{\varphi} \Gamma_i(G), L_j \rtimes_{\varphi} \Gamma_j(G)] &= [\Gamma_i(H \rtimes_{\varphi} G), \Gamma_j(H \rtimes_{\varphi} G)] \\ &\subset \Gamma_{i+j}(H \rtimes_{\varphi} G) = L_{i+j} \rtimes_{\varphi} \Gamma_{i+j}(G). \end{aligned} \quad (5.44)$$

Let $x_i \in L_i$ and $x_j \in L_j$. Then $(x_i, 1) \in L_i \rtimes_{\varphi} \Gamma_i(G)$ and $(x_j, 1) \in L_j \rtimes_{\varphi} \Gamma_j(G)$, and using (3.3) and (5.44), we obtain:

$$([x_i, x_j], 1) = [(x_i, 1), (x_j, 1)] \in L_{i+j} \rtimes_{\varphi} \Gamma_{i+j}(G).$$

Hence $[x_i, x_j] \in L_{i+j}$, then $[L_i, L_j] \subset L_{i+j}$. So if $1 \leq i, j \leq m$, where $i + j = m + 1$, we see that $[Y_i, Y_j] = [L_i, L_j] \subset L_{m+1}$ using the induction hypothesis. To prove that $A_n^{2^{m-1}} \subset L_{m+1}$, by induction, we have $D_j^{2^{m-2}} \in Y_m = L_m$ for all $j = 1, \dots, n$. Therefore $\varphi(b_j)(D_j^{2^{m-2}}).D_j^{-2^{m-2}}$ and $\varphi(b_n b_{n+1} C_{n, n+1})(D_n^{2^{m-2}}).D_n^{-2^{m-2}}$ belong to $H_{m+1} \subset L_{m+1}$ for all $j = 1, \dots, n - 1$. Consequently $[\beta_{j,j}, D_j^{-2^{m-2}}]D_j^{-2^{m-1}}$ and $D_n^{-2^{m-1}}$ belong to L_{m+1} for all $j = 1, \dots, n - 1$. Now $[\beta_{j,j}, D_j^{-2^{m-2}}] \in [H, L_m] \subset L_{m+1}$ for all $j = 1, \dots, n - 1$, and thus $D_j^{2^{m-1}} \in L_{m+1}$. Using the fact that L_{m+1} is normal in H by Remark 5.16, Lemma 3.3 and (5.39), we conclude that $Y_{m+1} \subset L_{m+1}$.

To prove that $L_{m+1} \subset Y_{m+1}$, the induction hypothesis implies that $[H, L_m] = [H, Y_m] \subset Y_{m+1}$. By Remark 5.15 and the induction hypothesis, we have $H_{m+1} = [G, L_m] = [G, Y_m]$ and $K_{m+1} = [\Gamma_m(G), H]$. So by Lemma 5.20 H_{m+1} and K_{m+1} are contained in Y_{m+1} . Since $L_{m+1} = \langle [H, L_m], H_{m+1}, K_{m+1} \rangle$, it follows that $L_{m+1} \subset Y_{m+1}$, and hence $L_{m+1} = Y_{m+1}$. \square

To prove that $P_{n+1}(\mathbb{K})$ is residually nilpotent, it remains to show that $\bigcap_{m \geq 1} Y_m = \{1\}$. To do so, we define two families $(Z_m)_{m \geq 1}$ and $(\tilde{Z}_m)_{m \geq 1}$ of subgroups of H as follows. Let $Z_1 = \tilde{Z}_1 = H$, $Z_2 = V_2 = Y_2$, and if $m \geq 3$, let:

$$Z_m = \langle\langle \{x^2 : x \in Z_{m-1}\} \cup X_m \rangle\rangle_H \text{ and } \tilde{Z}_{m-1} = \langle\langle A_n^{2^{m-3}} \cup \tilde{X}_{m-1} \rangle\rangle_H, \text{ where:} \quad (5.45)$$

$$X_m = \left\{ [x_1, \dots, x_i] \in \Gamma_i(H) : \exists 1 \leq j_1 < \dots < j_{m-i} \leq i, x_{j_1}, \dots, x_{j_{m-i}} \in A_n \right. \\ \left. \text{for all } i = 2, \dots, m \right\} \text{ for all } m \geq 3$$

$$\tilde{X}_m = \left\{ [x_1, \dots, x_i]^{2^{m-i-k}} \in \Gamma_i(H) : \exists 1 \leq j_1 < \dots < j_k \leq i, x_{j_1}, \dots, x_{j_k} \in A_n \right. \\ \left. \text{for all } 0 \leq k \leq m-i, \text{ and } i = 2, \dots, m \right\} \text{ for all } m \geq 2.$$

Note that if $i = m$ (resp. $k = 0$), the elements x_1, \dots, x_i of H that appear in the definition of X_m (resp. of \tilde{X}_m) are arbitrary. If $m \geq 3$ (resp. $m \geq 2$), we will refer to the elements of $X_m \cup \{x^2 : x \in Z_{m-1}\}$ (resp. of $\tilde{X}_m \cup \{D_j^{2^{m-2}} : j = 1, \dots, n\}$) as *generators* of Z_m (resp. of \tilde{Z}_m).

Proposition 5.22. *Let $m \geq 1$. Then $\tilde{Z}_m \subset \gamma_{\lceil m/2 \rceil}^2(H)$, where $[x]$ denotes the least integer greater than or equal to x . In particular, $\bigcap_{m \geq 1} \tilde{Z}_m = \{1\}$.*

Proof. If $m \in \{1, 2\}$ then $\lceil m/2 \rceil = 1$, $\gamma_1^2(H) = H$ and thus $\tilde{Z}_m \subset \gamma_1^2(H)$. So suppose by induction on m that $\tilde{Z}_i \subset \gamma_{\lceil i/2 \rceil}^2(H)$ for some $m \geq 2$ and all $1 \leq i \leq m$. Since $\gamma_{\lceil (m+1)/2 \rceil}^2(H)$ is normal in H , by (5.37) and (5.45), it suffices to show that $\{D_j^{2^{m-1}} : j = 1, \dots, n\} \cup \tilde{X}_{m+1} \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$. If $j = 1, \dots, n$ then $D_j^{2^{m-1}} = (D_j^{2^{m-2}})^2$, and since $D_j^{2^{m-2}} \in A_n^{2^{m-2}}$ and $A_n^{2^{m-2}} \subset \tilde{Z}_m$, it follows by the induction hypothesis that $D_j^{2^{m-2}} \in \gamma_{\lceil m/2 \rceil}^2(H)$, and hence $D_j^{2^{m-1}} \in \gamma_{\lceil m/2 \rceil + 1}^2(H)$. The fact that $\lceil m/2 \rceil + 1 \geq \lceil (m+1)/2 \rceil$ implies that $\gamma_{\lceil m/2 \rceil + 1}^2(H) \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$, whence $D_j^{2^{m-1}} \in \gamma_{\lceil (m+1)/2 \rceil}^2(H)$. Now let $x \in \tilde{X}_{m+1}$, and let $x = [x_1, \dots, x_i]^{2^{m+1-i-k}} \in \Gamma_i(H)$, where $2 \leq i \leq m+1$ and there exist $1 \leq j_1 < \dots < j_k \leq i$, such that $x_{j_1}, \dots, x_{j_k} \in A_n$, for all $0 \leq k \leq m+1-i$. If $i = m+1$ then $x \in \gamma_{m+1}^2(H) \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$ because $m+1 \geq \lceil (m+1)/2 \rceil$. So suppose that $2 \leq i \leq m$. If $0 \leq k \leq m-i$ then $x = ([x_1, \dots, x_i]^{2^{m-i-k}})^2$, where $[x_1, \dots, x_i]^{2^{m-i-k}} \in \tilde{X}_m \subset \gamma_{\lceil m/2 \rceil}^2(H)$ using the induction hypothesis, and thus $x \in \gamma_{\lceil m/2 \rceil + 1}^2(H) \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$. Finally, if $k = m+1-i$ then $x = [x_1, \dots, x_i] \in \Gamma_i(H)$. Since $k \leq i$, we have $i \geq (m+1)/2 \geq \lceil (m+1)/2 \rceil$, and hence $x \in \Gamma_i(H) \subset \Gamma_{\lceil (m+1)/2 \rceil}(H) \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$ as required, and this completes the proof of the inclusion $\{D_j^{2^{m-1}} : j = 1, \dots, n\} \cup \tilde{X}_{m+1} \subset \gamma_{\lceil (m+1)/2 \rceil}^2(H)$. Finally, since H is a free group of finite rank, it is residually 2-finite and $\bigcap_{m \geq 1} \gamma_m^2(H) = \{1\}$, so $\bigcap_{m \geq 1} \tilde{Z}_m = \{1\}$. \square

The aim now is to prove that $Y_m = Z_m = \tilde{Z}_m$ for all $m \geq 2$, from which we will conclude that $\bigcap_{m \geq 1} Y_m = \{1\}$ and that $P_{n+1}(\mathbb{K})$ is residually nilpotent.

Remark 5.23. Let $m \geq 2$. Taking $A = A_n$ in (5.17), we have $\langle \tilde{X}_m \rangle = \mathcal{E}_{2,m}$. Further, since $D_j \in A_n$ for all $j = 1, \dots, n$, we see that $D_j^{2^{m-2}} \in \mathcal{E}_{1,m}$, where we take $i = k = 1$, so $A_n^{2^{m-2}} \subset \mathcal{E}_{1,m}$. Therefore $\tilde{Z}_m \subset \mathcal{E}_{1,m}$, and Corollary 5.9 then implies that

$$[H, \tilde{Z}_m] \subset [H, \mathcal{E}_{1,m}] \subset \mathcal{E}_{2,m+1} = \langle \tilde{X}_{m+1} \rangle \subset \tilde{Z}_{m+1}.$$

Note also that if $m = 1$ then $[H, \tilde{Z}_1] = \Gamma_2(H) = \langle \tilde{X}_2 \rangle \subset \tilde{Z}_2$.

Lemma 5.24. *For all $m \geq 1$, $Y_m = Z_m = \tilde{Z}_m$.*

Proof. If $m = 1$, the given equality holds by definition. If $m = 2$ then $\langle \tilde{X}_2 \rangle = \Gamma_2(H)$, and $\tilde{Z}_2 = \langle\langle A_n \cup \Gamma_2(H) \rangle\rangle_H = V_2$ using (5.38) and the fact that V_2 is normal in H by Lemma 3.3. So suppose by induction that:

$$Y_i = Z_i = \tilde{Z}_i \text{ for some } m \geq 2 \text{ and all } 1 \leq i \leq m. \quad (5.46)$$

To prove that $\tilde{Z}_{m+1} \subset Z_{m+1}$, let us show that $A_n^{2^{m-1}}$ and \tilde{X}_{m+1} are contained in Z_{m+1} . For all $j = 1, \dots, n$, we have $D_j^{2^{m+1-2}} = (D_j^{2^{m-2}})^2$ and $D_j^{2^{m-2}} \in \tilde{Z}_m = Z_m$ by induction, so $A_n^{2^{m-1}} \subset Z_{m+1}$ by (5.45). Now suppose that $[x_1, \dots, x_i]^{2^{m+1-i-k}} \in \tilde{X}_{m+1}$, so k elements of $\{x_1, \dots, x_i\}$ belong to A_n , where $0 \leq k \leq m+1-i$. If $m+1-i > k$ then $[x_1, \dots, x_i]^{2^{m-i-k}}$ belongs to $\tilde{Z}_m = Z_m$ by induction, so $[x_1, \dots, x_i]^{2^{m+1-i-k}} = ([x_1, \dots, x_i]^{2^{m-i-k}})^2$ belongs to Z_{m+1} . If $m+1-i = k$ then $[x_1, \dots, x_i]^{2^{m+1-i-k}} = [x_1, \dots, x_i] \in \Gamma_i(H)$, where $k = (m+1) - i$ elements of this commutator belong to A_n , so $[x_1, \dots, x_i] \in Z_{m+1}$. Hence $\tilde{X}_{m+1} \subset Z_{m+1}$, and thus $\tilde{Z}_{m+1} \subset Z_{m+1}$.

We now show that $Z_{m+1} \subset \tilde{Z}_{m+1}$. First note that $X_{m+1} \subset \tilde{X}_{m+1}$, for if the commutator $[x_1, \dots, x_i]$ belongs to X_{m+1} then $m+1-i$ of its elements belong to A_n , and so $[x_1, \dots, x_i] = [x_1, \dots, x_i]^{2^{m+1-i-k}} \in \tilde{X}_{m+1}$, where $k = m+1-i$. Now let $y \in Z_{m+1}$ be of the form $y = x^2$, where $x \in Z_m = \tilde{Z}_m$ by induction. If x is a generator of \tilde{Z}_m then $x^2 \in \tilde{Z}_{m+1}$ from the definition of \tilde{Z}_{m+1} . If $x = x_1 \cdots x_l$, where x_i is a generator of \tilde{Z}_m for all $1 \leq i \leq l$, then making use of a decomposition of x analogous to that of (5.31), the previous sentence and the fact that $[H, \tilde{Z}_m] \subset \tilde{Z}_{m+1}$ by Remark 5.23, it follows that $y \in \tilde{Z}_{m+1}$. We conclude that $Z_{m+1} \subset \tilde{Z}_{m+1}$, and hence $Z_{m+1} = \tilde{Z}_{m+1}$.

To show that $Z_{m+1} \subset Y_{m+1}$, let $y \in Z_{m+1}$. We first consider the following two possibilities:

- (i) $y = x^2$, where $x \in Z_m = Y_m$ by induction. If $x = D_j^{2^{m-2}}$ for some $j = 1, \dots, n$, it follows from the definition of Y_{m+1} that $y = x^2 = D_j^{2^{m-1}} \in Y_{m+1}$. If $x = [x_i, x_j]$, where $1 \leq i \leq j < m$, $i+j = m$, $x_i \in Y_i$ and $x_j \in Y_j$, then:

$$y = [x_i, x_j]^2 = \underbrace{\left[x_i, \underbrace{[x_j, x_i^{-1}]}_{\in [Y_i, Y_j] \subset Y_m} \right]}_{\in [H, Y_m] \subset Y_{m+1}} x_j x_i^{-1} \underbrace{\left[x_j^{-1}, \underbrace{x_i^2}_{\in Z_{i+1} = Y_{i+1}} \right]}_{\in [Y_j, Y_{i+1}] \subset Y_{m+1}} x_i x_j^{-1}.$$

Note that to obtain $x_i^2 \in Y_{i+1}$, we have $i < m$, so $Y_i = Z_i$ and $Y_{i+1} = Z_{i+1}$ by the induction hypothesis. Therefore $y \in Y_{m+1}$ since Y_{m+1} is normal in H by Remark 5.16.

- (ii) $y = [x_1, \dots, x_i] \in \Gamma_i(H)$, where $2 \leq i \leq m+1$, $x_1, \dots, x_i \in H$, and there exist $1 \leq j_1 < \dots < j_{m+1-i} \leq i$ such that $x_{j_1}, \dots, x_{j_{m+1-i}} \in A_n$. If $j_1 = 1$, i.e. $x_1 \in A_n$, then $x_1 \in Y_2$, $[x_2, \dots, x_i] \in \Gamma_{i-1}(H)$, and $m-i = (m-1) - (i-1)$ elements of $\{x_2, \dots, x_i\}$ belong to A_n , so $[x_2, \dots, x_i] \in Z_{m-1} = Y_{m-1}$ by induction. Hence:

$$y = [x_1, [x_2, \dots, x_i]] \in [Y_2, Y_{m-1}] \subset Y_{m+1}.$$

If $j_1 > 1$ then $[x_2, \dots, x_i] \in \Gamma_{i-1}(H)$ and $m+1-i = m - (i-1)$ elements of $\{x_2, \dots, x_i\}$ belong to A_n . Therefore $[x_2, \dots, x_i] \in Z_m = Y_m$ by induction, and thus:

$$y = [x_1, [x_2, \dots, x_i]] \in [H, Y_m] \subset Y_{m+1}.$$

For the general case, if y is a product of conjugates of the two types of elements described in (i) and (ii) above, then $y \in Y_{m+1}$ because Y_{m+1} is normal in H by Remark 5.16, and we conclude that $Z_{m+1} \subset Y_{m+1}$.

To complete the proof, it remains to see that $Y_{m+1} \subset \tilde{Z}_{m+1}$. From the definition of \tilde{Z}_{m+1} given in (5.45), it is clear that $A_n^{2^{m-1}} \subset \tilde{Z}_{m+1}$. Applying the induction hypothesis, we have $[Y_i, Y_j] = [\tilde{Z}_i, \tilde{Z}_j]$ for all $1 \leq i, j \leq m$ for which $i+j = m+1$, so it suffices to show that $[\tilde{Z}_i, \tilde{Z}_j] \subset \tilde{Z}_{i+j}$. We shall prove by induction on i that $[\tilde{Z}_i, \tilde{Z}_j] \subset \tilde{Z}_{i+j}$ for all $1 \leq i \leq m$ and $j \geq 1$. The inclusion that we require then follows as a special case. So let $j \geq 1$. If $i = 1$ then $[\tilde{Z}_1, \tilde{Z}_j] = [H, \tilde{Z}_j] \subset \tilde{Z}_{j+1}$ by Remark 5.23. So assume that $1 < i \leq m$, and suppose by induction that:

$$[\tilde{Z}_k, \tilde{Z}_j] \subset \tilde{Z}_{k+j} \text{ for all } j \geq 1 \text{ and } 1 \leq k < i \leq m. \quad (5.47)$$

The aim is to prove that $[\tilde{Z}_i, \tilde{Z}_j] \subset \tilde{Z}_{i+j}$. Let $x \in \tilde{Z}_i$ and $y \in \tilde{Z}_j$. Then $x \in Y_i$ by the induction hypothesis (5.46) and the fact that $1 < i \leq m$. Assume first that x is a generator of Y_i , so that one of the following conditions holds:

- (1) $x \in [Y_l, Y_k]$, where $1 \leq l \leq k < i \leq m$ and $l + k = i$. Then $x \in [\tilde{Z}_l, \tilde{Z}_k]$ by the induction hypothesis (5.46).
(2) $x \in A_n^{2^{i-2}}$,

and let us prove that $[x, y] \in \tilde{Z}_{i+j}$. In case (1), [27, Theorem 5.2] implies that:

$$[[\tilde{Z}_l, \tilde{Z}_k], \tilde{Z}_j] = [\tilde{Z}_j, [\tilde{Z}_l, \tilde{Z}_k]] \subset [\tilde{Z}_l, [\tilde{Z}_k, \tilde{Z}_j]] \cdot [\tilde{Z}_k, [\tilde{Z}_j, \tilde{Z}_l]]. \quad (5.48)$$

By the induction hypothesis (5.47) and the fact that $k, l < m$, we have:

$$\begin{cases} [\tilde{Z}_l, [\tilde{Z}_k, \tilde{Z}_j]] \subset [\tilde{Z}_l, \tilde{Z}_{k+j}] \subset \tilde{Z}_{l+k+j} = \tilde{Z}_{i+j} \\ [\tilde{Z}_k, [\tilde{Z}_j, \tilde{Z}_l]] = [\tilde{Z}_k, [\tilde{Z}_l, \tilde{Z}_j]] \subset [\tilde{Z}_k, \tilde{Z}_{l+j}] \subset \tilde{Z}_{k+l+j} = \tilde{Z}_{i+j}, \end{cases} \quad (5.49)$$

and thus $[x, y] \in \tilde{Z}_{i+j}$ for all $y \in \tilde{Z}_j$ using (5.48) and (5.49). In case (2), let $x = D_k^{2^{i-2}}$, where $k \in \{1, \dots, n\}$. We consider the following cases:

- $i = 2$. Then $x \in A_n$. Let us show by induction on j that $[A_n, Z_j] \subset Z_{j+2}$ for all $j \geq 1$. Suppose first that $j = 1$, let $x \in A_n$, and let $y \in Z_1 = H$. Then $[x, y] \in \Gamma_2(H)$, and the set $\{x, y\}$ has one element, x , in A_n , therefore $[x, y] \in X_3 \subset Z_3$, and thus $[A_n, Z_1] \subset Z_3$. Now suppose that $j > 1$ and that $[A_n, Z_{j-1}] \subset Z_{j+1}$ by induction. Let $x \in A_n$, and let $y \in Z_j$. First assume that y is a generator of Z_j . If $y = [y_1, \dots, y_l] \in \Gamma_l(H)$ belongs to X_j then $j - l$ elements of $\{y_1, \dots, y_l\}$ belong to A_n , therefore $[x, y] \in \Gamma_{l+1}(H)$, where $j - l + 1 = (j + 2) - (l + 1)$ elements of $\{x, y_1, \dots, y_l\}$ belong to A_n . So $[x, y] \in X_{j+2} \subset Z_{j+2}$. If $y = z^2$, where $z \in Z_{j-1}$, then:

$$[x, y] = [x, z^2] = [x, z]^2 [[z, x], z],$$

by (3.2). Since $[x, z] \in [A_n, Z_{j-1}] \subset Z_{j+1}$ by induction, it follows that $[x, z]^2 \in Z_{j+2}$ from the definition of Z_{j+2} in (5.45), and $[[z, x], z] \in [Z_{j+1}, H] \subset Z_{j+2}$ by Remark 5.23 and the fact that $\tilde{Z}_l = Z_l$ for all $l \geq 1$. Therefore $[x, y] \in Z_{j+2}$. Now if $y = (\alpha_1 y_1^{\delta_1} \alpha_1^{-1}) \cdots (\alpha_s y_s^{\delta_s} \alpha_s^{-1})$, where for all $k = 1, \dots, s$, y_k is a generator of Z_j , $\alpha_k \in H$ and $\delta_k \in \{1, -1\}$, then applying (5.29) and induction on s , we see that $[x, y]$ may be written as a product of conjugates of commutators of the form $[x, y_k^{\delta_k}]$. Then $[x, y] \in \tilde{Z}_{j+2}$ using the above computations, the normality of Z_{j+2} in H , and the fact $\tilde{Z}_l = Z_l$ for all $l \geq 1$ from the first part of the proof.

- $i \geq 3$. Then:

$$[x, y] = [D_k^{2^{i-2}}, y] = [D_k^{2^{i-3}}, [D_k^{2^{i-3}}, y]] [D_k^{2^{i-3}}, y]^2$$

by (3.2). Now $[D_k^{2^{i-3}}, y] \in [\tilde{Z}_{i-1}, \tilde{Z}_j] \subset \tilde{Z}_{i-1+j}$ using (5.47), and we deduce from Remark 5.23 that $[D_k^{2^{i-3}}, [D_k^{2^{i-3}}, y]] \in [H, \tilde{Z}_{i-1+j}] \subset \tilde{Z}_{i+j}$. Since $[D_k^{2^{i-3}}, y] \in \tilde{Z}_{i-1+j} = Z_{i-1+j}$, we have $[D_k^{2^{i-3}}, y]^2 \in Z_{i+j} = \tilde{Z}_{i+j}$, so $[x, y] \in \tilde{Z}_{i+j}$.

If now $x \in Y_i$ is a product of conjugates of generators of Y_i , then $[x, y] \in \tilde{Z}_{i+j}$ using the above computations, (5.29), and the normality of \tilde{Z}_{i+j} . This shows that $[\tilde{Z}_i, \tilde{Z}_j] = [Y_i, \tilde{Z}_j] \subset \tilde{Z}_{i+j}$ as claimed, and so $Y_{m+1} \subset \tilde{Z}_{m+1}$ as required. \square

Theorem 5.25. *For all $n, m \geq 1$, we have $\Gamma_m(P_{n+1}(\mathbb{K})) = Z_m \rtimes \Gamma_m(P_n(\mathbb{K}))$.*

Proof. If $n = 1, 2$ then $P_n(\mathbb{K})$ is residually nilpotent by Theorem 5.4 and (5.10). Suppose by induction that $P_n(\mathbb{K})$ is residually nilpotent for some $n \geq 2$. Using the Fadell-Neuwirth split short exact sequence (5.1), the result follows by induction, and by applying Theorem 1.1, Lemmas 5.21 and 5.24. \square

Proof of Theorem 1.3(1). If $n = 1$ (resp. $n = 2$), the result is a consequence of (5.10) (resp. Theorem 5.4). If $n \geq 3$, the result follows by induction on n , Proposition 5.22 and Theorem 5.25. \square

5.5. **The case of the braid group $B_n(\mathbb{K})$.** In this section, we prove Theorem 1.3(2). We start by giving two propositions in the cases where $B_n(\mathbb{K})$ is not residually nilpotent or is not residually soluble.

Proposition 5.26. *If $n \geq 3$ then $B_n(\mathbb{K})$ is not residually nilpotent. Further, for all $i \geq 3$, $\Gamma_2(B_n(\mathbb{K})) = \Gamma_i(B_n(\mathbb{K})) = \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$.*

Proof. In this proof, the relation numbers are those of Theorem 2.2. Let $n \geq 3$, and consider the following short exact sequence:

$$1 \longrightarrow \frac{\Gamma_2(B_n(\mathbb{K}))}{\Gamma_3(B_n(\mathbb{K}))} \longrightarrow \frac{B_n(\mathbb{K})}{\Gamma_3(B_n(\mathbb{K}))} \xrightarrow{p_*} \frac{B_n(\mathbb{K})}{\Gamma_2(B_n(\mathbb{K}))} \longrightarrow 1, \quad (5.50)$$

where p_* is the canonical projection. Using relations (1) and (2), and arguing as in [16, p. 680] or [5, Proposition 3], it follows that the $\Gamma_3(B_n(\mathbb{K}))$ -cosets of $\sigma_1, \dots, \sigma_{n-1}$ in $B_n(\mathbb{K})/\Gamma_3(B_n(\mathbb{K}))$ are all identified to a single element, which we denote by σ . Since $n \geq 3$, it follows from relations (3) and (4) that the $\Gamma_3(B_n(\mathbb{K}))$ -cosets of a and b commute with σ . By relation (7) and (5.50), σ is of order 2, by relation (5), the $\Gamma_3(B_n(\mathbb{K}))$ -cosets of a and b commute, and from relation (8), the $\Gamma_3(B_n(\mathbb{K}))$ -coset of a is of order 2 in $B_n(\mathbb{K})/\Gamma_3(B_n(\mathbb{K}))$. By Remark 2.3, $\bar{\sigma}$ and the $\Gamma_2(B_n(\mathbb{K}))$ -cosets of a and b are non trivial in $B_n(\mathbb{K})/\Gamma_2(B_n(\mathbb{K}))$, therefore σ and the $\Gamma_3(B_n(\mathbb{K}))$ -cosets of a and b are also non trivial in $B_n(\mathbb{K})/\Gamma_2(B_n(\mathbb{K}))$ and satisfy the same relations in $B_n(\mathbb{K})/\Gamma_3(B_n(\mathbb{K}))$ as their images in $B_n(\mathbb{K})/\Gamma_2(B_n(\mathbb{K}))$ under p_* . Hence p_* is an isomorphism, and $\Gamma_2(B_n(\mathbb{K})) = \Gamma_3(B_n(\mathbb{K}))$, so $\Gamma_2(B_n(\mathbb{K})) = \Gamma_i(B_n(\mathbb{K}))$ for all $i \geq 3$. Since $\Gamma_2(B_n(\mathbb{K}))$ is non trivial, we see that $B_n(\mathbb{K})$ is not residually nilpotent.

It remains to show that $\Gamma_2(B_n(\mathbb{K})) = \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$. From relations (1) and (2), for all $i = 1, \dots, n-2$, we have $[\sigma_i, \sigma_{i+1}] = \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} = \sigma_{i+1}^{-1} \sigma_i$, in particular, $\langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})} \subset \Gamma_2(B_n(\mathbb{K}))$. To prove the other inclusion, for all $i = 2, \dots, n-2$, note that:

$$\begin{aligned} [\sigma_i, \sigma_{i+1}] &= \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1}^{-1} \underbrace{\sigma_i (\sigma_{i-1} \sigma_i \sigma_i^{-1} \sigma_{i-1}^{-1})}_{(1)} = \underbrace{\sigma_{i+1}^{-1} (\sigma_{i-1} \sigma_i \sigma_{i-1})}_{(2)} \sigma_i^{-1} \sigma_{i-1}^{-1} \\ &= (\sigma_{i-1} \underbrace{\sigma_{i+1}^{-1}}_{(1)}) \sigma_i \sigma_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} = \sigma_{i-1} (\sigma_i \sigma_{i+1} \sigma_i^{-1} \underbrace{\sigma_{i+1}^{-1}}_{(2)}) \sigma_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} \\ &= \sigma_{i-1} \sigma_i \sigma_{i+1} (\sigma_i^{-1} \sigma_{i-1}) \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i-1}^{-1}. \end{aligned}$$

It follows by induction on i that $[\sigma_i, \sigma_{i+1}] \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$ for all $i = 1, \dots, n-2$. Further,

$$[a, \sigma_1] = a \sigma_1 (\underbrace{\sigma_2^{-1} \sigma_2}_{(3)}) a^{-1} \sigma_1^{-1} = a (\sigma_2 \sigma_1^{-1})^{-1} a^{-1}. \quad (\sigma_2 \sigma_1^{-1}) \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}, \quad (5.51)$$

and similarly, $[b, \sigma_1] \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$ using relation (4). To see that $[b, a] \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$, first note that:

$$\sigma_1^2 = \sigma_1 b^{-1} \sigma_1 b^{-1} \sigma_1^{-1} b \sigma_1^{-1} b = (\sigma_1 b^{-1} [\sigma_1, b^{-1}] b \sigma_1^{-1}) [\sigma_1, b^{-1}] \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})} \quad (5.52)$$

using relation (7), and:

$$\begin{aligned} bab^{-1}a^{-1} &= ba (\underbrace{\sigma_1^{-1} \sigma_1}_{(5)}) b^{-1} (\sigma_1 \sigma_1^{-1}) a^{-1} = ba \sigma_1^{-1} (a^{-1} \sigma_1^{-1} b^{-1} \sigma_1 a) \sigma_1^{-1} a^{-1} \\ &= b [a, \sigma_1^{-1}] \sigma_1^{-2} b^{-1} [\sigma_1, a] \in \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}, \end{aligned}$$

by (5.51) and (5.52). Since the result is valid for the generators of $B_n(\mathbb{K})$, the result follows for an arbitrary element of $\Gamma_2(B_n(\mathbb{K}))$ using the formula given by (5.29) and by the normality of $\langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$. We conclude that $\Gamma_2(B_n(\mathbb{K})) \subset \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$, and hence that $\Gamma_2(B_n(\mathbb{K})) = \langle\langle \sigma_2^{-1}\sigma_1 \rangle\rangle_{B_n(\mathbb{K})}$. \square

Proposition 5.27. *If $n \geq 5$ then $B_n(\mathbb{K})$ is not residually soluble. Further, $(B_n(\mathbb{K}))^{(1)} = (B_n(\mathbb{K}))^{(i)}$ for all $i \geq 2$.*

Proof. Let $n \geq 5$. Once more, the relation numbers will refer to those of Theorem 2.2. As in the case of the torus (Proposition 4.3), first consider the following short exact sequence:

$$1 \longrightarrow \frac{(B_n(\mathbb{K}))^{(1)}}{(B_n(\mathbb{K}))^{(2)}} \xrightarrow{i} \frac{B_n(\mathbb{K})}{(B_n(\mathbb{K}))^{(2)}} \xrightarrow{p_*} B_n(\mathbb{K})^{\text{Ab}} \longrightarrow 1,$$

where p_* is the canonical projection. Using relations (1) and (2), for all $i = 1, \dots, n-1$, the σ_i belong to the same $(B_n(\mathbb{K}))^{(2)}$ -coset, denoted by σ , in $B_n(\mathbb{K})/(B_n(\mathbb{K}))^{(2)}$ (the hypothesis that $n \geq 5$ is used here, see [15, Theorem 1.4, p. 3389]). By Remark 2.3, σ and the $(B_n(\mathbb{K}))^{(2)}$ -cosets of a and b are non trivial. From relations (3) and (4), the $(B_n(\mathbb{K}))^{(2)}$ -cosets of a and b commute with σ . Relation (7) implies that σ is of order 2, and so from relation (5), the $(B_n(\mathbb{K}))^{(2)}$ -coset of a commutes with that of b . By relation (8), the $(B_n(\mathbb{K}))^{(2)}$ -coset of a is of order 2. So p_* is a isomorphism, and $(B_n(\mathbb{K}))^{(1)} = (B_n(\mathbb{K}))^{(2)}$. The second part then follows. \square

Proof of Theorem 1.3(2). By Proposition 5.26 (resp. Proposition 5.27), if $n \geq 3$ (resp. $n \geq 5$), $B_n(\mathbb{K})$ is not residually nilpotent (resp. not residually soluble). Conversely, using Theorem 2.5, the short exact sequence (1.1), and the fact that $P_2(\mathbb{K})$ is residually 2-finite by Theorem 5.13, $B_2(\mathbb{K})$ is residually 2-finite. In particular, $B_2(\mathbb{K})$ is residually nilpotent. By (5.10), $B_1(\mathbb{K}) = \pi_1(\mathbb{K})$ is residually nilpotent. The fact that $P_n(\mathbb{K})$ is residually soluble for all $n \geq 1$ by Theorem 5.25 implies that $B_n(\mathbb{K})$ is residually soluble for all $n \leq 4$ using Theorem 2.5. \square

6. THE CASE OF NON-ORIENTABLE SURFACES OF HIGHER GENUS

In this short section, we prove Theorem 1.4, by generalising Propositions 5.26 and 5.27 to non-orientable surfaces of higher genus.

Theorem 6.1. *Let M be a compact, connected non-orientable surface without boundary and of genus $g \geq 3$. Then $B_n(M)$ is not residually nilpotent if $n \geq 3$, and is not residually soluble if $n \geq 5$.*

Proof. The relation numbers will refer to those of Theorem 2.4. Using Theorem 2.4 it is easy to see that

$$B_n(M)^{\text{Ab}} = \langle \sigma, a_i, i = 1, \dots, g : a_1^2 \cdots a_g^2 = \sigma^{2(n-1)}, [\sigma, a_i] = [a_i, a_j] = 1, i, j = 1, \dots, n \rangle,$$

where σ denotes the $\Gamma_2(B_n(M))$ -coset of σ_i for all $i = 1, \dots, n-1$. As in the proof of Proposition 5.26 (resp. Proposition 5.27), using relations (1) and (2), one may show that for all $i = 1, \dots, n-1$, the σ_i belong to the same $\Gamma_3(B_n(M))$ -coset (resp. $(B_n(M))^{(2)}$ -coset), which we also denote by σ , in $B_n(M)/\Gamma_3(B_n(M))$ (resp. in $B_n(M)/(B_n(M))^{(2)}$). If $n \geq 3$ (resp. $n \geq 5$), by relation (3), the $\Gamma_3(B_n(M))$ -coset (resp. $(B_n(M))^{(2)}$ -coset) of a_r commutes with σ for all $1 \leq r \leq g$. By relation (5), the $\Gamma_3(B_n(M))$ -coset (resp. $(B_n(M))^{(2)}$ -coset) of a_r commutes with that of a_s for all $1 \leq r, s \leq g$. By (6), $a_1^2 \cdots a_g^2 = \sigma^{2(n-1)}$, and relation (4) does not give any new information. Thus $B_n(M)/\Gamma_3(B_n(M))$ (resp. $B_n(M)/(B_n(M))^{(2)}$) is isomorphic to $B_n(M)^{\text{Ab}}$. Hence $B_n(M)$ is not residually nilpotent if $n \geq 3$ (resp. is not residually soluble if $n \geq 5$). \square

Proof of Theorem 1.4. If $M = \mathbb{K}$, the result follows from Theorem 1.3(2), and if M is a compact surface without boundary of genus $g \geq 3$, the conclusion follows from [4] and from Theorem 6.1. If $M = \mathbb{R}P^2$, by [18], $B_n(\mathbb{R}P^2)$ is residually nilpotent if $n \leq 2$, and if $n \neq 4$, $B_n(\mathbb{R}P^2)$ is residually soluble if $n < 4$. The result in the case $n = 4$ may be obtained by using Theorem 2.5 and the following Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\}) \longrightarrow P_{m+n}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2) \longrightarrow 1,$$

where $n \geq 2$. Note that if $n = 2$, $P_2(\mathbb{R}P^2)$ is the quaternion group of order 8 [34], which is 2-finite, and $P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_n\})$ is residually 2-finite by [4]. Therefore $P_{m+2}(\mathbb{R}P^2)$ is residually 2-finite

for all $m \geq 1$, in particular $P_4(\mathbb{R}P^2)$ is residually soluble. Applying Theorem 2.5 to the short exact sequence (1.1), we see that $B_4(\mathbb{R}P^2)$ is residually soluble. \square

APPENDIX

Let M be the Möbius band, and let $n \geq 1$. The braid groups of M are those of $\mathbb{R}P^2$ with a single point removed [17, proof of Theorem 2(a)], and so $P_n(M)$ is the group $\Gamma_{n,1}(\mathbb{R}P^2)$ of [19, Proposition 11]. We use the notation and results of that proposition in what follows. In particular, $P_n(M)$ is generated by the set $\{A_{i,j}, \rho_j \mid 1 \leq i < j \text{ and } 2 \leq j \leq n+1\}$.

Proposition A1. *Let $n \geq 1$. Then $Z(P_n(M)) = Z(B_n(M))$ is infinite cyclic, generated by ρ_2 if $n = 1$, and by the full twist Δ_{n+1}^2 if $n \geq 2$.*

Proof. If $n = 1$ then $P_1(M) = B_1(M)$ is infinite cyclic, generated by ρ_2 , and the result follows in this case. So suppose that $n \geq 2$. Using the short exact sequence (1.1), we see that $B_n(M)$ is generated by $\{\sigma_2, \dots, \sigma_n, \rho_2, \dots, \rho_{n+1}\}$. The braid Δ_{n+1}^2 generates the centre of B_{n+1} , thus σ_i commutes with Δ_{n+1}^2 for all $i = 2, \dots, n$. Further, using [19, Proposition 11, relation (V)], Δ_{n+1}^2 may be written in the following form:

$$\begin{aligned} \Delta_{n+1}^2 &= (A_{1,2})(A_{1,3}A_{2,3}) \cdots (A_{1,n}A_{2,n} \cdots A_{n-1,n})(A_{1,n+1}A_{2,n+1} \cdots A_{n,n+1}) \\ &= (A_{1,2})(A_{1,3}A_{2,3}) \cdots (A_{1,n}A_{2,n} \cdots A_{n-1,n})\rho_{n+1}^{-2}. \end{aligned} \quad (\text{A1})$$

Since ρ_{n+1} commutes with $A_{i,j}$ for all $1 \leq i < j < n+1$ [19, Proposition 11, relation (II)], it follows that ρ_{n+1} commutes with Δ_{n+1}^2 . Now the relation $\rho_{i+1} = \sigma_i^{-1}\rho_i\sigma_i^{-1}$ of [34, p. 83] for $B_{n+1}(\mathbb{R}P^2)$ also holds in $B_n(M)$ for all $i = 2, \dots, n$, so $\rho_i = \sigma_i \cdots \sigma_n \rho_{n+1} \sigma_n \cdots \sigma_i$, from which we conclude that ρ_i commutes with Δ_{n+1}^2 . Thus Δ_{n+1}^2 commutes with all of the elements of a generating set of $B_n(M)$, hence $\langle \Delta_{n+1}^2 \rangle \subset Z(B_n(M))$, and $\langle \Delta_{n+1}^2 \rangle \subset Z(P_n(M))$ since $\Delta_{n+1}^2 \in P_n(M)$. To prove the converse, we consider the following two cases:

(a) $n = 2$. Let $p: P_2(M) \rightarrow P_1(M)$ denote the surjective homomorphism given geometrically by forgetting the second string. The kernel of p is a free group of rank 2 for which $(A_{2,3}, \rho_3)$ is a basis, and $P_1(M) = \langle \rho_2 \rangle$ is infinite cyclic. Let $Q = \langle \rho_2^2 \rangle$ be the index 2 subgroup of $P_1(M)$, and let $G = p^{-1}(Q)$ be the index 2 subgroup of $P_2(M)$. Then we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & \text{Ker}(p) & \longrightarrow & G & \xrightarrow{p|_G} & Q & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{Ker}(p) & \longrightarrow & P_2(M) & \xrightarrow{p} & P_1(M) & \longrightarrow & 1, \\ & & & & \downarrow q' & & \downarrow q & & \\ & & & & \mathbb{Z}_2 & \xlongequal{\quad} & \mathbb{Z}_2 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \quad (\text{A2})$$

where $q: P_1(M) \rightarrow \mathbb{Z}_2$ is defined by $q(\rho_2) = \bar{1}$, and $q': P_2(M) \rightarrow \mathbb{Z}_2$ is given by $q' = q \circ p$. From (A1), we have $\Delta_3^2 = A_{1,2}\rho_3^{-2}$, and since $A_{1,2} = \rho_2^{-1}A_{2,3}\rho_2^{-1}$ by [19, Proposition 11, relation (V)], we see that:

$$\Delta_3^2 = \rho_2^{-1}A_{2,3}\rho_2^{-1}\rho_3^{-2} = \rho_2^{-2} \cdot \underbrace{\rho_2 A_{2,3} \rho_2^{-1}}_{\in \text{Ker}(p)} \rho_3^{-2} \quad (\text{A3})$$

using exactness of (A2). So the restriction $p|_G: G \rightarrow Q$ admits a section given by sending ρ_2^2 to Δ_3^{-2} , and from this and the fact that $\Delta_3^2 \in Z(P_2(M))$, the upper row of (A2) splits as a direct product. In particular, G is the internal direct product of $\text{Ker}(p)$ and $\langle \Delta_3^2 \rangle$, and $Z(G) = \langle \Delta_3^2 \rangle$. Now $P_2(M)$ (resp. G) is an index 2 subgroup of $B_2(M)$ (resp. of $P_2(M)$), and a transversal is given by $\{1, \sigma_2\}$ (resp. by $\{1, \rho_2\}$). So every element z of $B_2(M)$ may be written as $z = a\rho_2^i\sigma_2^j$, where $a \in G$ and $i, j \in \{0, 1\}$ are unique. We shall refer to this expression as the *normal form* of z . Let $z \in Z(B_2(M))$, and assume first that $z \notin P_2(M)$, so $j = 1$. Consider the basis (u, v) of $\text{Ker}(p)$, where $u = A_{2,3}\rho_3$ and $v = \rho_3$. By [19, Proposition 11, relations (III) and (IV)], we have:

$$\rho_2 u \rho_2^{-1} = \rho_3^{-1} A_{2,3}^{-1} \rho_3. A_{2,3} \rho_3 = u^{-1} v u \text{ and } \rho_2 v \rho_2^{-1} = u. \quad (\text{A4})$$

Hence the action by conjugation of ρ_2 on $\text{Ker}(p)$ is given by composing the involution that exchanges u and v with conjugation by u^{-1} . The relation $\rho_3 = \sigma_2^{-1} \rho_2 \sigma_2^{-1}$ implies that:

$$\sigma_2 v \sigma_2^{-1} = \sigma_2 \rho_3 \sigma_2^{-1} = \rho_2 \sigma_2^{-2} = \rho_2 A_{2,3}^{-1} = \rho_2 v u^{-1} = \rho_2 v u^{-1} \rho_2^{-1}. \rho_2 = v^{-1} u \rho_2$$

by (A4). Since $z \in Z(B_2(M))$, z and v commute, so:

$$a \rho_2^i \sigma_2 = z = v z v^{-1} = v a \rho_2^i \sigma_2 v^{-1} \sigma_2^{-1}. \sigma_2 = v a \rho_2^i \rho_2^{-1} u^{-1} v \sigma_2. \quad (\text{A5})$$

If $i = 1$ then the left- and right-hand sides of (A5) are in normal form, and they clearly differ. If $i = 0$ then using the fact that $\rho_2^2 = u^{-1} v^{-1} \Delta_3^{-2} \in G$ by (A3) and (A4), equation (A5) may be written as:

$$a \sigma_2 = v a \rho_2^{-1} u^{-1} v \sigma_2 = v a \rho_2^{-2}. \rho_2 u^{-1} v \rho_2^{-1}. \rho_2 \sigma_2 = v a \Delta_3^2 u^2 \rho_2 \sigma_2.$$

Again the left- and right-hand sides are in normal form, and they differ also. In both cases, this contradicts the fact that $z \in Z(B_2(M))$, and so we conclude that $j = 0$. Hence $z \in P_2(M)$, and $\langle \Delta_3^2 \rangle \subset Z(B_2(M)) \subset Z(P_2(M))$. It remains to show that $z \in \langle \Delta_3^2 \rangle$. Suppose that $i = 1$. Since $z \in Z(P_2(M))$, z commutes with u , so:

$$a \rho_2 = z = u z u^{-1} = u a \rho_2 u^{-1} = u a \rho_2 u^{-1} \rho_2^{-1}. \rho_2 = u a u^{-1} v^{-1} u \rho_2$$

by (A4). Both sides are in normal form, and thus $a = u a u^{-1} v^{-1} u$ in $\text{Ker}(p)$, which gives rise to a contradiction under Abelianisation in this free group. Hence $i = 0$, and thus $z \in G$. So $z \in Z(G)$, and therefore $z \in \langle \Delta_3^2 \rangle$. We conclude that $\langle \Delta_3^2 \rangle = Z(B_2(M)) = Z(P_2(M))$ as required.

- (b) $n \geq 3$. Then $Z(S_n) = \{\text{Id}\}$, and since the homomorphism $B_n(M) \rightarrow S_n$ of (1.1) is surjective, it follows that $\langle \Delta_{n+1}^2 \rangle \subset Z(B_n(M)) \subset Z(P_n(M))$. The proof of the fact that $Z(P_n(M)) \subset \langle \Delta_{n+1}^2 \rangle$ is analogous to that of the inclusion $Z(P_n(\mathbb{K})) \subset Z_n$ given in the second paragraph of Proposition 5.2, where the kernel of the Fadell-Neuwirth short exact sequence involving the pure braid groups of M is a free group with trivial centre. \square

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