



HAL
open science

Radial extensions in fractional Sobolev spaces

Haïm Brezis, Petru Mironescu, Itai Shafrir

► **To cite this version:**

Haïm Brezis, Petru Mironescu, Itai Shafrir. Radial extensions in fractional Sobolev spaces. RACSAM. Real Academia de Ciencias. Serie A, Matemáticas - Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas, 2019, 113 (2), pp.707-714. 10.1007/s13398-018-0510-3 . hal-01713818v2

HAL Id: hal-01713818

<https://hal.science/hal-01713818v2>

Submitted on 2 Mar 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Radial extensions in fractional Sobolev spaces

Haim Brezis^{(1),(2)}, Petru Mironescu⁽³⁾, Itai Shafrir⁽⁴⁾

February 20, 2018

Abstract

Given $f : \partial(-1, 1)^n \rightarrow \mathbb{R}$, consider its radial extension $Tf(X) := f(X/\|X\|_\infty)$, $\forall X \in [-1, 1]^n \setminus \{0\}$. In “On some questions of topology for S^1 -valued fractional Sobolev spaces” (RACSAM 2001), the first two authors (HB and PM) stated the following auxiliary result (Lemma D.1). If $0 < s < 1$, $1 < p < \infty$ and $n \geq 2$ are such that $1 < sp < n$, then $f \mapsto Tf$ is a bounded linear operator from $W^{s,p}(\partial(-1, 1)^n)$ into $W^{s,p}((-1, 1)^n)$. The proof of this result contained a flaw detected by the third author (IS). We present a correct proof. We also establish a variant of this result involving higher order derivatives and more general radial extension operators. More specifically, let B be the unit ball for the standard Euclidean norm $|\cdot|$ in \mathbb{R}^n , and set $U_a f(X) := |X|^a f(X/|X|)$, $\forall X \in \overline{B} \setminus \{0\}$, $\forall f : \partial B \rightarrow \mathbb{R}$. Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$ be such that $(s - a)p < n$. Then $f \mapsto U_a f$ is a bounded linear operator from $W^{s,p}(\partial B)$ into $W^{s,p}(B)$.

In [1], the first two authors stated the following

Lemma 1. ([1, Lemma D.1]) Let $0 < s < 1$, $1 < p < \infty$ and $n \geq 2$ be such that $1 < sp < n$. Let

$$Q := (-1, 1)^n. \tag{1}$$

Set

$$Tf(X) := f(X/\|X\|_\infty), \forall X \in \overline{Q} \setminus \{0\}, \forall f : \partial Q \rightarrow \mathbb{R}; \tag{2}$$

here, $\|\cdot\|_\infty$ is the sup norm in \mathbb{R}^n . Then $f \mapsto Tf$ is a bounded linear operator from $W^{s,p}(\partial Q)$ into $W^{s,p}(Q)$.

The argument presented in [1] does not imply the conclusion of Lemma 1. Indeed, it is established in [1] (see estimate (D.3) there) that

$$|Tf|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n+sp}} d\sigma(x) d\sigma(y).$$

However, this does not imply the desired conclusion in Lemma 1, for which we need the stronger estimate

$$|Tf|_{W^{s,p}(Q)}^p \leq C \int_{\partial Q} \int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n-1+sp}} d\sigma(x) d\sigma(y).$$

In what follows, we establish the following slight generalization of Lemma 1.

Keywords: Sobolev spaces; fractional Sobolev spaces; radial extensions
MSC 2000 classification: 46E35

Lemma 2. Let $0 < s \leq 1$, $1 \leq p < \infty$ and $n \geq 2$ be such that $sp < n$. Let Q, T be as in (1)–(2). Then $f \mapsto Tf$ is a bounded linear operator from $W^{s,p}(\partial Q)$ into $W^{s,p}(Q)$.

Lemma 2 can be generalized beyond one derivative, but for this purpose it is necessary to work on unit spheres arising from norms smoother than $\|\cdot\|_\infty$. We consider for example maps $f : \partial B \rightarrow \mathbb{R}$, with

$$B := \text{the Euclidean unit ball in } \mathbb{R}^n. \quad (3)$$

For $a \in \mathbb{R}$, set

$$U_a f(X) := |X|^a f(X/|X|), \quad \forall X \in \overline{B} \setminus \{0\}, \quad \forall f : \partial B \rightarrow \mathbb{R}; \quad (4)$$

here, $|\cdot|$ is the standard Euclidean norm in \mathbb{R}^n .

We will prove the following

Lemma 3. Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$ be such that $(s-a)p < n$. Then $f \mapsto U_a f$ is a bounded linear operator from $W^{s,p}(\partial B)$ into $W^{s,p}(B)$.

It is possible to establish directly Lemma 2 by adapting some arguments presented in Step 3 in the proof of Lemma 4.1 in [2]. However, we will derive it from Lemma 3.

Proof of Lemma 2 using Lemma 3 Let

$$\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \Phi(X) := \begin{cases} \frac{|X|}{\|X\|_\infty} X, & \text{if } X \neq 0 \\ 0, & \text{if } X = 0 \end{cases}, \quad \Lambda := \Phi|_{\overline{B}} \text{ and } \Psi := \Phi|_{\partial B}.$$

Clearly,

$$\Lambda : \overline{B} \rightarrow \overline{Q}, \quad \Psi : \partial B \rightarrow \partial Q \text{ are bi-Lipschitz homeomorphisms} \quad (5)$$

and

$$Tf = [U_0(f \circ \Psi)] \circ \Lambda^{-1}. \quad (6)$$

Using (5) and the fact that $0 < s \leq 1$, we find that

$$f \mapsto f \circ \Psi \text{ is a bounded linear operator from } W^{s,p}(\partial Q) \text{ into } W^{s,p}(\partial B) \quad (7)$$

and

$$g \mapsto g \circ \Lambda^{-1} \text{ is a bounded linear operator from } W^{s,p}(B) \text{ into } W^{s,p}(Q). \quad (8)$$

We obtain Lemma 2 from (6)–(8) and Lemma 3 (with $a = 0$). The same argument shows that the conclusion of Lemma 2 holds for the unit sphere and ball of any norm in \mathbb{R}^n . \square

Proof of Lemma 3 Consider a, s, p and n such that

$$a \in \mathbb{R}, \quad s > 0, \quad 1 \leq p < \infty, \quad n \geq 2 \text{ and } (s-a)p < n. \quad (9)$$

Considering spherical coordinates on B , we obtain that

$$\begin{aligned} \|U_a f\|_{L^p(B)}^p &= \int_0^1 \int_{\partial B} r^{n-1} |U_a f(rx)|^p d\sigma(x) dr \\ &= \int_0^1 \int_{\partial B} r^{n-1+a p} |f(x)|^p d\sigma(x) dr = \frac{1}{n+ap} \|f\|_{L^p(\partial B)}^p. \end{aligned} \quad (10)$$

Here, we have used the fact that, by (9), we have $n + ap > n - (s - a)p > 0$.

In view of (10), it suffices to establish the estimate

$$|U_a f|_{W^{s,p}(B)}^p \leq C \|f\|_{W^{s,p}(\partial B)}^p, \quad \forall f \in W^{s,p}(\partial B), \quad (11)$$

for some appropriate $C = C_{a,s,p,n}$ and semi-norm $|\cdot|_{W^{s,p}}$ on $W^{s,p}(B)$.

Step 1. Proof of (11) when $0 < s < 1$. We consider the standard Gagliardo semi-norm on $W^{s,p}(B)$. We have

$$\begin{aligned} |U_a f|_{W^{s,p}(B)}^p &= \int_B \int_B \frac{|U_a f(X) - U_a f(Y)|^p}{|X - Y|^{n+sp}} dX dY \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|U_a f(rx) - U_a f(\rho y)|^p}{|rx - \rho y|^{n+sp}} d\sigma(x) d\sigma(y) dr d\rho \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|r^\alpha f(x) - \rho^\alpha f(y)|^p}{|rx - \rho y|^{n+sp}} d\sigma(x) d\sigma(y) dr d\rho \\ &= 2 \int_{\partial B} \int_{\partial B} \int_0^1 \int_0^r r^{n-1} \rho^{n-1} \frac{|r^\alpha f(x) - \rho^\alpha f(y)|^p}{|rx - \rho y|^{n+sp}} d\rho dr d\sigma(x) d\sigma(y). \end{aligned}$$

With the change of variable $\rho = tr$, $t \in [0, 1]$, we find that

$$\begin{aligned} |U_a f|_{W^{s,p}(B)}^p &= 2 \int_0^1 r^{n-(s-a)p-1} dr \int_{\partial B} \int_{\partial B} \int_0^1 t^{n-1} \frac{|f(x) - t^\alpha f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \\ &= \frac{2}{n - (s-a)p} \int_{\partial B} \int_{\partial B} \int_0^1 k(x, y, t) dt d\sigma(x) d\sigma(y), \end{aligned}$$

with

$$k(x, y, t) := t^{n-1} \frac{|f(x) - t^\alpha f(y)|^p}{|x - ty|^{n+sp}}, \quad \forall x, y \in \partial B, \quad \forall t \in [0, 1].$$

In order to complete this step, it thus suffices to establish the estimates

$$I_1 := \int_{\partial B} \int_{\partial B} \int_0^{1/2} k(x, y, t) dt d\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p, \quad (12)$$

$$I_2 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|f(x) - f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \leq C \|f\|_{W^{s,p}(\partial B)}^p, \quad (13)$$

$$I_3 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|(1 - t^\alpha) f(y)|^p}{|x - ty|^{n+sp}} dt d\sigma(x) d\sigma(y) \leq C \|f\|_{L^p(\partial B)}^p; \quad (14)$$

here, $|\cdot|_{W^{s,p}(\partial B)}$ is the standard Gagliardo semi-norm on ∂B .

In the above and in what follows, C denotes a generic finite positive constant independent of f , whose value may change with different occurrences.

Using the obvious inequalities

$$\begin{aligned} |x - ty| &\geq 1 - t \geq 1/2, \quad \forall x, y \in \partial B, \quad \forall t \in [0, 1/2], \\ |f(x) - t^\alpha f(y)| &\leq (1 + t^\alpha)(|f(x)| + |f(y)|), \end{aligned}$$

and the fact that, by (9), we have $n + ap > 0$, we find that

$$I_1 \leq C \int_0^{1/2} (t^{n-1} + t^{n-1+ap}) dt \|f\|_{L^p(\partial B)}^p \leq C \|f\|_{L^p(\partial B)}^p,$$

so that (12) holds.

In order to obtain (13), it suffices to establish the estimate

$$\int_{1/2}^1 \frac{1}{|x-ty|^{n+sp}} dt \leq \frac{C}{|x-y|^{n-1+sp}}, \quad \forall x, y \in \partial B. \quad (15)$$

Set $A := \langle x, y \rangle \in [-1, 1]$. If $A \leq 0$, then $|x-ty| \geq 1$, $\forall t \in [1/2, 1]$, and then (15) is clear. Assuming $A \geq 0$ we find, using the change of variable $t = A + (1-A^2)^{1/2} \tau$,

$$\begin{aligned} \int_{1/2}^1 \frac{1}{|x-ty|^{n+sp}} dt &\leq \int_{\mathbb{R}} \frac{1}{|x-ty|^{n+sp}} dt = \int_{\mathbb{R}} \frac{1}{(t^2+1-2At)^{(n+sp)/2}} dt \\ &= \frac{1}{(1-A^2)^{(n-1+sp)/2}} \int_{\mathbb{R}} \frac{1}{(\tau^2+1)^{(n+sp)/2}} d\tau \\ &= \frac{C}{(1-A^2)^{(n-1+sp)/2}} \leq \frac{C}{(2-2A)^{(n-1+sp)/2}} = \frac{C}{|x-y|^{n-1+sp}}, \end{aligned}$$

and thus (15) holds again. This completes the proof of (13).

In order to prove (14), we note that

$$|1-t^a|^p \leq C(1-t)^p, \quad \forall t \in [1/2, 1],$$

and that the integral

$$J := \int_{1/2}^1 \int_{\partial B} \frac{(1-t)^p}{|x-ty|^{n+sp}} d\sigma(x) dt$$

does not depend on $y \in \partial B$.

By the above, we have

$$I_3 \leq C \int_{1/2}^1 \int_{\partial B} \int_{\partial B} \frac{(1-t)^p |f(y)|^p}{|x-ty|^{n+sp}} d\sigma(x) d\sigma(y) dt = C J \|f\|_{L^p(\partial B)}^p,$$

and thus (14) amounts to proving that $J < \infty$. Since J does not depend on y , we may assume that $y = (0, \dots, 0, 1)$. Expressing J in spherical coordinates and using the change of variable $t = 1 - \tau$, $\tau \in [0, 1/2]$, we find that

$$J = C \int_{1/2}^1 \int_0^\pi \frac{\tau^p \sin^{n-1} \theta}{(\tau^2 + 4(1-\tau) \sin^2 \theta/2)^{(n+sp)/2}} d\theta d\tau.$$

When $\tau \in [0, 1/2]$ and $\theta \in [0, \pi]$, we have

$$\begin{aligned} \frac{\tau^p \sin^{n-1} \theta}{(\tau^2 + 4(1-\tau) \sin^2 \theta/2)^{(n+sp)/2}} &\leq C \frac{\tau^p \sin^{n-1} \theta}{(\tau + \sin \theta/2)^{n+sp}} \leq C \frac{\tau^p \sin^{n-1} \theta/2 \cos \theta/2}{(\tau + \sin \theta/2)^{n+sp}} \\ &\leq C (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2. \end{aligned}$$

Inserting the last inequality into the formula of J , we find that

$$\begin{aligned} J &\leq C \int_0^{1/2} \int_0^\pi (\tau + \sin \theta/2)^{p-sp-1} \cos \theta/2 d\theta d\tau \\ &= C \int_0^{1/2} \int_0^1 (\tau + \xi)^{p-sp-1} d\xi d\tau < \infty, \end{aligned}$$

the latter inequality following from $p - sp > 0$. This completes the proof of (14) and Step 1.

Step 2. Proof of (11) when $s \geq 1$. We will reduce the case $s \geq 1$ to the case $0 \leq s < 1$. Using the linearity of $f \mapsto U_a f$ and a partition of unity, we may assume with no loss of generality that $\text{supp} f$ is contained in a spherical cap of the form $\{x \in \partial B; |x - \mathbf{e}| < \varepsilon\}$ for some $\mathbf{e} \in \partial B$ and sufficiently small ε . We may further assume that $\mathbf{e} = (0, 0, \dots, 0, 1)$, and thus

$$f \in W^{s,p}(\partial B; \mathbb{R}), \text{supp} f \subset \mathcal{E} := \{x \in \partial B; |x - (0, 0, \dots, 0, 1)| < \varepsilon\}. \quad (16)$$

Let

$$\mathcal{S} := \{x \in \partial B; |x - (0, 0, \dots, 0, 1)| \leq 2\varepsilon\} \text{ and } \mathcal{H} := \mathbb{R}^{n-1} \times \{1\}.$$

Consider the projection Θ with vertex 0 of

$$\mathbb{R}_+^n := \{X = (X', X_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; X_n > 0\}$$

onto \mathcal{H} , given by the formula $\Theta(X', X_n) = (X'/X_n, 1)$. The restriction Π of Θ to \mathcal{S} maps \mathcal{S} onto $\mathcal{N} := \mathcal{B} \times \{1\}$, with

$$\mathcal{B} := \{X' \in \mathbb{R}^{n-1}; |X'| \leq r := 2\varepsilon \sqrt{1 - \varepsilon^2}/(1 - 2\varepsilon^2)\},$$

and is a smooth diffeomorphism between these two sets. We choose ε such that $r = 1/2$, and thus $\mathcal{B} \subset \{X' \in \mathbb{R}^{n-1}; \|X'\|_\infty \leq 1/2\}$.

Set

$$g(X') := \begin{cases} |X'|^\alpha f(\Pi^{-1}(X', 1)), & \text{if } X' \in \mathcal{B} \\ 0, & \text{otherwise} \end{cases}. \quad (17)$$

By the above, there exist $C, C' > 0$ such that for every $f \in W^{s,p}(\partial B)$ satisfying (16), the function g defined in (17) satisfies

$$C \|g\|_{W^{s,p}(\mathbb{R}^{n-1})} \leq \|f\|_{W^{s,p}(\partial B)} \leq C' \|g\|_{W^{s,p}(\mathbb{R}^{n-1})}. \quad (18)$$

On the other hand, set $\mathcal{C} := \{(tY', t); Y' \in \mathcal{B}, t > 0\}$ and

$$V_a g(X', X_n) := \begin{cases} (X_n)^\alpha g(X'/X_n), & \text{if } (X', X_n) \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

Then we have $U_a f(X', X_n) = V_a g(X', X_n), \forall (X', X_n) \in \overline{\mathcal{B}} \setminus \{0\}$.

Write now $s = m + \sigma$, with $m \in \mathbb{N}$ and $0 \leq \sigma < 1$. When $s = m$, we consider, on $W^{s,p}(B)$, the semi-norm

$$|F|_{W^{s,p}(B)}^p = \sum_{\substack{\alpha \in \mathbb{N}^n \setminus \{0\} \\ |\alpha| \leq m}} \|\partial^\alpha F\|_{L^p(B)}^p. \quad (19)$$

When s is not an integer, we consider the semi-norm

$$|F|_{W^{s,p}(B)}^p = \sum_{\substack{\alpha \in \mathbb{N}^n \setminus \{0\} \\ |\alpha| \leq m}} \|\partial^\alpha F\|_{L^p(B)}^p + \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} |\partial^\alpha F|_{W^{\sigma,p}(B)}^p \quad (20)$$

(the semi-norm on $W^{\sigma,p}(B)$ is the standard Gagliardo one.)

By the above discussion, in order to obtain (11) it suffices to establish the estimate

$$\|V_a g\|_{W^{s,p}(B)}^p \leq C \|g\|_{W^{s,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{s,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (21)$$

Let $\alpha \in \mathbb{N}^n \setminus \{0\}$ be such that $|\alpha| \leq m$. By a straightforward induction on $|\alpha|$, the distributional derivative $\partial^\alpha[V_a g]$ satisfies

$$\partial^\alpha[V_a g](X', X_n) = \sum_{|\beta'| \leq |\alpha|} V_{a-|\alpha|} [P_{\alpha,\beta'} \partial^{\beta'} g](X', X_n) \text{ in } \mathcal{D}'(B \setminus \{0\}), \quad (22)$$

for some appropriate polynomials $P_{\alpha,\beta'}(Y')$, $Y' \in \mathbb{R}^{n-1}$, depending only on $a \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$ and $\beta' \in \mathbb{N}^{n-1}$.

Thanks to the fact that $g(X'/X_n) = 0$ when $(X', X_n) \notin \mathcal{C}$, we find that for any such α we have

$$\begin{aligned} \int_B |\partial^\alpha[V_a g]|^p dx &\leq C \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{C} \cap \mathcal{Q}} (X_n)^{(a-|\alpha|)p} |\partial^{\beta'} g(X'/X_n)|^p dX' dX_n \\ &= \frac{C}{n + (a - |\alpha|)p} \sum_{|\beta'| \leq |\alpha|} \int_{\mathcal{B}} |\partial^{\beta'} g(Y')|^p dY'. \end{aligned} \quad (23)$$

Here, we rely on

$$\int_0^1 (X_n)^{n-1+(a-|\alpha|)p} dX_n = \frac{1}{n + (a - |\alpha|)p} < \infty,$$

thanks to the assumption (9), which implies that $(|\alpha| - a)p < n$.

Using (23), the fact that $V_a g \in W_{loc}^{m,p}(B \setminus \{0\})$ and the assumption that $n \geq 2$, we find that the equality (22) holds also in $\mathcal{D}'(B)$, that $V_a g \in W^{m,p}(B)$ and that

$$\|V_a g\|_{W^{m,p}(B)}^p \leq C \|g\|_{W^{m,p}(\mathbb{R}^{n-1})}^p, \quad \forall g \in W^{m,p}(\mathbb{R}^{n-1}) \text{ with } \text{supp } g \subset \mathcal{B}. \quad (24)$$

In particular, (21) holds when s is an integer.

Assume next that s is not an integer. In view of (18), (22) and (24), estimate (21) will be a consequence of

$$\begin{aligned} \|V_b[Ph]\|_{W^{\sigma,p}(B)}^p &\leq C \|h\|_{W^{\sigma,p}(\mathbb{R}^{n-1})}^p, \quad \forall h \in W^{\sigma,p}(\mathbb{R}^{n-1}) \\ &\text{with } \text{supp } h \subset \mathcal{B}, \end{aligned} \quad (25)$$

under the assumptions

$$0 < \sigma < 1, \quad 1 \leq p < \infty, \quad n \geq 2, \quad (\sigma - b)p < n \quad (26)$$

and

$$P \in C^\infty(\mathbb{R}^{n-1}). \quad (27)$$

(Estimate (25) is applied with $b := a - m$, $P := P_{\alpha,\beta'}$ and $h := \partial^{\beta'} g$.)

In turn, estimate (25) follows from Step 1. Indeed, consider $k : \partial B \rightarrow \mathbb{R}$ such that $\text{supp} k \subset \mathcal{B}$ and $U_b k = V_b[Ph]$. (The explicit formula of k can be obtained by “inverting” the formula (17).) By Step 1 and (18), we have

$$\begin{aligned} |V_b[Ph]|_{W^{s,p}(B)}^p &= |U_b k|_{W^{s,p}(B)}^p \leq C \|k\|_{W^{s,p}(\partial B)}^p \leq C \|Ph\|_{W^{s,p}(\mathbb{R}^{n-1})}^p \\ &\leq C \|h\|_{W^{s,p}(\mathbb{R}^{n-1})}^p. \end{aligned}$$

This completes Step 2 and the proof of Lemma 3. \square

Finally, we note that the assumptions of Lemma 3 are optimal in order to obtain that $U_a f \in W^{s,p}(B)$.

Lemma 4. Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$. Assume that for some measurable function $f : \partial B \rightarrow \mathbb{R}$ we have $U_a f \in W^{s,p}(B)$. Then:

1. $f \in W^{s,p}(\partial B)$.
2. If, in addition, $U_a f$ is not a polynomial, we deduce that $(s-a)p < n$.

Proof. 1. Let $G : (1/2, 1) \times \partial B \rightarrow \mathbb{R}$, $G(r, x) := r^{-a} U_a f(rx)$. If $U_a f \in W^{s,p}(B)$, then $G \in W^{s,p}((1/2, 1) \times \partial B)$. In particular, we have $G(r, \cdot) \in W^{s,p}(\partial B)$ for a.e. r . Noting that $G(r, x) = f(x)$, we find that $f \in W^{s,p}(\partial B)$.

2. Let

$$\Omega_j := \{X \in \mathbb{R}^n; 2^{-j-1} < |X| < 2^{-j}\}, j \in \mathbb{N}.$$

We consider on each Ω_j a semi-norm as in (19)–(20). Assuming that $U_a f$ is not a polynomial, we have $|U_a f|_{W^{s,p}(\Omega_0)} > 0$. By scaling and the homogeneity of $U_a f$, we have

$$|U_a f|_{W^{s,p}(\Omega_j)}^p = 2^{j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p.$$

Assuming that $U_a f \in W^{s,p}(B)$, we find that

$$\infty > |U_a f|_{W^{s,p}(B)}^p \geq \sum_{j \geq 0} |U_a f|_{W^{s,p}(\Omega_j)}^p = \sum_{j \geq 0} 2^{j[(s-a)p-n]} |U_a f|_{W^{s,p}(\Omega_0)}^p > 0,$$

so that $(s-a)p < n$. \square \square

References

- [1] H. Brezis and P. Mironescu. On some questions of topology for S^1 -valued fractional Sobolev spaces. *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 95(1):121–143, 2001.
- [2] H. Brezis and P. Mironescu. Density in $W^{s,p}(\Omega; N)$. *J. Funct. Anal.*, 269:2045–2109, 2015.

⁽¹⁾ Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA
brezis@math.rutgers.edu

⁽²⁾ Departments of Mathematics and Computer Science, Technion - Israel Institute of Technology, Haifa, 32000, Israel

⁽³⁾ Université de Lyon; Université Lyon 1; CNRS UMR 5208 Institut Camille Jordan; 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France
mironescu@math.univ-lyon1.fr

⁽⁴⁾ Department of Mathematics, Technion - Israel Institute of Technology, 32000 Haifa, Israel
shafir@math.technion.ac.il